

CLT for the zeros of classical random trigonometric polynomials

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Abstract. We prove a Central Limit Theorem for the number of zeros of random trigonometric polynomials of the form $K^{-1/2} \sum_{n=1}^K a_n \cos(nt)$, being $(a_n)_n$ independent standard Gaussian random variables. In particular we show that the variance is equivalent to $V^2 K \pi$, $0 < V^2 < \infty$, as $K \rightarrow \infty$. This last result was recently proved by Su and Shao in (*Sci. China Math.* **55** (2012) 2347–2366). Our approach is based on the Hermite/Wiener Chaos decomposition for square-integrable functionals of a Gaussian process and on Rice Formula for zero counting.

Résumé. Nous montrons un Théorème de la Limite Centrale pour le nombre de racines d'un polynôme trigonométrique aléatoire de la forme $K^{-1/2} \sum_{n=1}^K a_n \cos(nt)$, ici les a_n sont des variables aléatoires Gaussiennes standard et indépendantes. En particulier, nous démontrons que la variance asymptotique du nombre de racines est équivalente à $V^2 K \pi$, pour une certaine constante $V > 0$, lorsque $K \rightarrow \infty$. Ce dernier résultat a été récemment démontré par Su and Shao dans (*Sci. China Math.* **55** (2012) 2347–2366). Notre approche utilise la décomposition dans le chaos d'Itô–Wiener d'une fonctionnelle non linéaire de carré intégrable et la formule de Rice.

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1. Introduction and main result

Consider the classical random trigonometric polynomials, that is, the polynomials defined, for $K = 1, 2, \dots$, by

$$T_K(t) := \frac{1}{\sqrt{K}} \sum_{n=1}^K a_n \cos(nt), \quad (1)$$

where the coefficients a_n are i.i.d. standard Gaussian random variables and $t \in [0, \pi]$.

One of the main questions about random polynomials concerns the random variable: *number of zeros* (level crossings in general); in our case the number of zeros of T_K on the interval $[0, \pi]$. The distribution of this random variable remains unknown, and one way to approach it is to describe it not directly but through its moments.

The asymptotic expectation of the number of zeros of classical trigonometric polynomials on $[0, \pi]$ is known since Dunnage's work [4] to be $K/\sqrt{3}$. The variance of T_K was conjectured by Farahmand and Sambandham [8] and Granville and Wigman [10] to be equivalent to $V^2 K \pi$, as K grows to infinity, for some constant $V^2 > 0$. Recently Su

and Shao in [23] give a positive answer to this conjecture. They obtain the exact asymptotic behavior of the variance of the number of roots on $[0, \pi]$ for the classical trigonometric polynomials.

In this article we revisit the above result. We exhibit a simplified proof. Furthermore, and this is the main result of our work, we establish a Central Limit Theorem (CLT) for the number of zeros of T_K on $[0, \pi]$ as $K \rightarrow \infty$.

Observe that since $\cos(t) = \cos(2\pi - t)$ the number of zeros of T_K on $[0, 2\pi]$ is exactly twice the number of zeros of T_K on $[0, \pi]$. Hence, all the results can be trivially adapted to $[0, 2\pi]$. Furthermore, it is easy to see that T_K and its derivatives are jointly non-degenerated Gaussian. Hence T_K has not multiple zeros almost surely.

Our main result is the following. Let us denote the number of zeros of a Gaussian process $Z : \mathbb{R} \rightarrow \mathbb{R}$ on the interval I by $N_Z(I) := \#\{t \in I : Z(t) = 0\}$.

Theorem 1. *The normalized number of zeros of T_K on the interval $[0, \pi]$ converges in distribution to a Gaussian random variable. More precisely*

$$\frac{N_{T_K} [0, \pi] - \mathbb{E}(N_{T_K} [0, \pi])}{\sqrt{\pi K}} \Rightarrow N(0, V^2),$$

as $K \rightarrow \infty$, being $0 < V^2 < \infty$.

In particular, we prove that

$$\lim_{K \rightarrow \infty} \frac{\text{Var}(N_{T_K} ([0, K \pi]))}{K \pi} = \lim_{K \rightarrow \infty} \frac{\text{Var}(N_{X_K} ([0, K \pi]))}{K \pi},$$

where X_K are the stationary trigonometric polynomials defined by

$$X_K(t) = \frac{1}{\sqrt{K}} \sum_{n=1}^K (a_n \cos(nt) + b_n \sin(nt)), \tag{2}$$

being a_n, b_n i.i.d. standard Gaussian random variables. The stationary trigonometric polynomials were studied by Granville and Wigman [10], Azaïs and León [2], see also the references therein. By [2], Theorem 4.2, and [10], Eq. (4), we deduce that $\text{Var}(N_X([0, \pi]))$ is equivalent to $V^2 K \pi$ with

$$V^2 \approx 0.089.$$

As said above, Su and Shao [23] show that the variance of the number of zeros of T_K is equivalent to $cK\pi$ as $K \rightarrow \infty$ where c is a complicated constant, $c \approx 0.257$.

Therefore, our result is not in concordance with this value. We checked our result by numerical simulations.

As a by-product of our proof, we obtain the following CLT for the number of zeros of a non-stationary Gaussian process on $[0, \infty)$ as the interval increases. The cardinal sine function is defined by

$$\text{sc}(x) = \frac{\sin(x)}{x}.$$

Corollary 2. *Let T be the centered Gaussian process on $[0, \infty)$ with covariance function $r(s, t) = \frac{1}{2}(\text{sc}(t - s) + \text{sc}(t + s))$. Then, the number of zeros of T on the interval $[0, K \pi]$ converges in distribution, after standardization, towards a standard Gaussian random variable.*

Background. Classical random trigonometric polynomials appears in Physics, for instance in nuclear physics (random matrix theory), statistical mechanics, quantum mechanics, theory of noise, see Granville and Wigman [10] and references therein.

Previous works. The number of zeros of different ensembles of random polynomials have attracted attention of physicists and mathematicians for at least seventy years. Consequently, there is an extensive literature on the subject, starting with Littlewood and Offord [16–18] who studied algebraic polynomials, these works were complemented by

those of Erdős and Offord [5], Kac [14] and Ibragimov and Maslova [12,13]. The final result for the mean number of roots of an algebraic polynomial of degree K , is that, for *i.i.d.* coefficients in the domain of attraction of the normal law, the mean number of roots is equivalent to $2\log(K)/\pi$ for centered a_n and half this quantity for non-centered a_n . Maslova [19,20] established the asymptotic variance and a CLT for the number of zeros of algebraic polynomials.

In the case of X_K the stationarity and the Gaussianity simplify largely the treatment of the level crossing counting problem. For this ensemble of (stationary) trigonometric polynomials Granville and Wigman [10] gave a proof of the CLT for the number of zeros, using conditions on moments of order higher than the second. After that, Azaïs and León [2] extended this result to all levels and gave a simplified proof of the CLT lying on the Wiener Chaos decomposition and Taqqu–Peccati’s method. In particular, avoiding conditions on higher moments than the second.

Finally, for Classical Trigonometric Polynomials, Wilkins [24] proved that

$$\mathbb{E}(N_{T_K}[0, 2\pi]) = \frac{1}{\sqrt{3}} \left[(2K + 1) + D_1 + \frac{D_2}{2K + 1} + \frac{D_3}{(2K + 1)^2} \right] + O\left(\frac{D_3}{(2K + 1)^3}\right)$$

with $D_1 = 0.23\dots$, etc.

The leading asymptotic term was proven to be the same for non-centered coefficients and for dependent coefficients with constant correlation, see Farahmand [6] and references therein.

Recently, Farahmand and Li [7] considered the mean number of roots of T_K and X_K allowing its coefficients to have different means and variances, but being independent and Gaussian.

The techniques. In order to obtain our results we make use of the techniques of Wiener Chaos expansion, Peccati–Tudor’s method for obtaining CLTs and Rice Formula.

More precisely, we obtain the Wiener Chaos expansion for the normalized number of zeros of the normalized version of T_K on the interval $[0, \pi]$. Then, we use this expansion in order to obtain the order of the asymptotic variance.

A key fact, is that removing the extremes of the interval, the behavior of (the covariance of the standardized version of) T_K is very similar to that of (the covariance of) its stationary counterpart X_K , so the limit variances of the respective number of zeros coincide. Our work closely follows Azaïs and León [2], but the asymptotic Gaussianity is obtained directly from Peccati–Tudor’s method rather than through the L^2 proximity to the limit process.

In that sense, it is worth to remark that whereas in the case of stationary trigonometric polynomials the CLT is inherited from that of the limit process, in our case the CLT for the limit process is a consequence of the CLT for classical trigonometric polynomials.

The article is organized as follows. It contains four sections. The present one introducing the subject and establishing the main result. Section 2 contains the proof of the main result. This section is split into two subsections. Section 2.1 deals with the expansion into the Wiener Chaos of the crossings and Section 2.2 contains the proof of the Central Limit Theorem for these crossings once centered and normalized. Section 3 contains the proof of five useful lemmas. Section 4 includes a notation table.

2. Proof

2.1. Expansion into the chaos

Let us replace t by t/K , this permits us to look at the polynomials T_K at a convenient scale and to find a limit for them. Thus, from now on we are concerned with the zeros on the interval $[0, K\pi]$ of

$$\tilde{T}_K(t) = \frac{1}{\sqrt{K}} \sum_{n=1}^K a_n \cos\left(\frac{n}{K}t\right). \quad (3)$$

Similarly, let $\tilde{X}_K(t) = X_K(t/K)$, with X_K defined in Equation (2). Clearly \tilde{T}_K and \tilde{X}_K are centered Gaussian processes in $t \in [0, K\pi]$.

Denote by c_K the covariance function of \tilde{X}_K , it is well known, see Azaïs and León [2], Eq. (1), Granville and Wigman [10], Eq. (13), that

$$c_K(t - s) = \mathbb{E}(\tilde{X}_K(s)\tilde{X}_K(t)) = \frac{1}{K} \sum_{n=1}^K \cos\left(\frac{n}{K}(t - s)\right). \tag{4}$$

Note that $c_K(\cdot)$ can be seen as a Riemann sum for $\int_0^1 \cos(u \cdot) du$. Further, c_K can be expressed in closed form using Fejér Kernel.

A direct computation shows that the covariance function of the classical trigonometric polynomials, $r_K(s, t) := \mathbb{E}(\tilde{T}_K(s)\tilde{T}_K(t))$, is given by

$$r_K(s, t) = \frac{1}{2}(c_K(t - s) + c_K(t + s)). \tag{5}$$

In particular, the variance of $\tilde{T}_K(t)$ is

$$V_K^2(t) := r_K(t, t) = \frac{1}{2}(1 + c_K(2t)).$$

Thus, the limit variance, as $t \rightarrow \infty$, is $\frac{1}{2}$ and not 1 as in the stationary case. Needless to say that the processes obtained from the \tilde{T}_K 's replacing the cosines by sines have also asymptotic variance $\frac{1}{2}$. At $t = 0$, \tilde{T}_K and \tilde{X}_K have the same variance.

From now on, we work on the case $s < t$ and denote $\tau := t - s$ and $\sigma := t + s$.

It is convenient to use the standardized version of \tilde{T}_K , namely

$$\bar{T}_K(t) := \tilde{T}_K(t)/V_K(t), \tag{6}$$

thus \bar{T}_K has unit variances and its covariance, \bar{r}_K , is given by

$$\bar{r}_K(s, t) = \frac{c_K(\tau) + c_K(\sigma)}{\sqrt{1 + c_K(2s)}\sqrt{1 + c_K(2t)}}. \tag{7}$$

Limit covariances. We collect in the following lemma some basic properties of c_K taken from Azaïs and León [2], Eqs. (2.5)–(2.8).

Lemma 1. For $\tau \in [0, K\pi]$ we have:

1. As $K \rightarrow \infty$,

$$c_K(\tau) \rightarrow \text{sc}(\tau). \tag{8}$$

Furthermore, this convergence is uniform in compacts not containing 0. Besides, the order one and order two derivatives of r_K converge in the same manner to the corresponding derivatives of r .

2. For some constant c

$$\begin{aligned} |c_K(\tau)| &\leq \frac{\pi}{\tau}, & |c'_K(\tau)| &\leq \frac{\pi}{\tau} + \frac{\pi^2}{2\tau^2}, \\ |c''_K(\tau)| &\leq c\left(\frac{1}{\tau} + \frac{1}{\tau^2} + \frac{1}{\tau^3}\right). \end{aligned} \tag{9}$$

We also need the following bounds.

Lemma 2. The variances of T and \tilde{T}_K are bounded away from zero in $[0, K\pi]$, those of \bar{T}' and \bar{T}'_K are bounded away from zero in $[t_0, K\pi - t_0]$ for t_0 large enough. These bounds are uniform in K .

From Equations (4), (5) and Lemma 1 it follows that

$$r_K(s, t) \rightarrow r(s, t) := \frac{1}{2}(\text{sc}(\tau) + \text{sc}(\sigma)), \tag{10}$$

as $K \rightarrow \infty$ for all $s, t \in [0, K\pi]$ and that this convergence is uniform in off-diagonal compacts (compacts not containing points in the diagonal $s = t$). Furthermore, the order one and order two derivatives of r_K converge in the same manner to the corresponding derivatives of r respectively. Besides the following bounds hold for $s, t \in [0, K\pi]$

$$\begin{aligned} |r_K(s, t)| &\leq \frac{\pi}{\tau} + \frac{\pi}{\sigma}, & |\partial_i r_K(s, t)| &\leq \frac{\pi}{2\tau} + \frac{\pi}{2\sigma} + \frac{\pi^2}{4\tau^2} + \frac{\pi^2}{4\sigma^2}, \\ |\partial_{ij} r_K(s, t)| &\leq c \left(\frac{1}{\tau} + \frac{1}{\sigma} + \frac{1}{\tau^2} + \frac{1}{\sigma^2} + \frac{1}{\tau^3} + \frac{1}{\sigma^3} \right), \end{aligned} \tag{11}$$

where $i, j = s, t$ and c is some constant; and

$$V_K^2(t) \leq \frac{1}{2} \left[1 + \frac{\pi}{2t} \right]. \tag{12}$$

Finally, observe that for t_0 large enough and $s, t \in [t_0, K\pi - t_0]$, $\bar{r}_K(s, t)$ and its derivatives converge to

$$\bar{r}(s, t) = \frac{\text{sc}(\tau) + \text{sc}(\sigma)}{\sqrt{1 + \text{sc}(2s)}\sqrt{1 + \text{sc}(2t)}}, \tag{13}$$

and its derivatives respectively uniformly in off-diagonal compacts. This is enough for our purposes. Therefore, \tilde{T}_K, \bar{T}_K converge (in the described sense) to centered Gaussian processes T and \bar{T} on the positive real axis having, respectively, covariances given by (10) and (13). In the sequel we need also to renormalized \bar{T}'_K in order to set its variance to 1. So we define the standard deviation of $\bar{T}'_K(s)$:

$$v_K(s) := \sqrt{\bar{r}_K^{(1,1)}(s, s)}, \tag{14}$$

where $r_K^{(i,j)}(s, t) := \frac{\partial^{i+j}}{\partial s^i \partial t^j} r_K$ and

$$\mathcal{T}'_K(s) = \bar{T}'_K(s) / v_K(s). \tag{15}$$

Remark 1. *The role of the limit process in this work is secondary, compare with Azaïs and León [2] and Granville and Wigman [10].*

Chaining and isonormal process. For the sake of readability, we write all the trigonometric polynomials on the same probability space. Note that, since we only care about distributions in the sequel, this is not really necessary.

Let $B = (B_\lambda)$ be a standard Brownian Motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We assume that \mathcal{F} is generated by B . By the isometric property of the stochastic integral, we have

$$\tilde{T}_K(t) = \int_0^1 \gamma_K(t, \lambda) dB_\lambda, \quad \gamma_K(t, \lambda) = \sum_{n=1}^K \cos\left(\frac{n}{K}t\right) \mathbf{1}_{[(n-1)/K, n/K)}(\lambda) \tag{16}$$

being $\mathbf{1}_A$ the indicator (or characteristic) function of the set A . Thus, the processes $\tilde{T}_K: K = 1, 2, \dots$ are defined on the same probability space.

Furthermore, let \mathbf{H} be the Hilbert space $L^2([0, 1], \mathcal{B}, d\lambda)$, being \mathcal{B} the Borel σ -algebra and $d\lambda$ the Lebesgue measure, with the usual inner product $\langle g, h \rangle = \int_0^1 g(\lambda)h(\lambda) d\lambda$. Thus, the map

$$h \mapsto B(h) := \int_0^1 h(\lambda) dB_\lambda,$$

defines an isometry between \mathbf{H} and $L^2(\Omega) = L^2(\Omega, \mathcal{F}, \mathbb{P})$. In this situation, $(B(h): h \in \mathbf{H})$ is called an isonormal process associated to \mathbf{H} .

In particular, we have

$$\tilde{T}_K(t) = B(\gamma_K(t, \cdot)), \quad \bar{T}_K(t) = B(h_K(t, \cdot)), \quad \text{with } h_K(t, \cdot) = \gamma_K(t, \cdot)/V_K(t),$$

and

$$\mathcal{T}'_K(t) := B(h'_K(t, \cdot)), \quad \text{with } h'_K(t, \cdot) = \partial_t h_K(t, \cdot) / \|\partial_t h_K(t, \cdot)\|_2.$$

In addition, $T(t) = B(\mathbf{1}_{[0,1]}(\cdot) \cos(t \cdot))$.

Chopping the extremes of the interval. The main idea of the proof of the CLT is to take advantage of the fact that the covariances of \tilde{T}_K and \tilde{X}_K are very similar one to each other for large values of s, t , even \tilde{T}_K being non-stationary. This idea is supported by the following lemma.

Denote $[0, K\pi]_{-\alpha} = [(K\pi)^\alpha, K\pi - (K\pi)^\alpha]$ for $\alpha \in (0, 1/2)$.

Lemma 3. *Let α be any value, $0 < \alpha < 1/2$ (for example $\alpha = 1/4$), we have*

$$\frac{N_{\tilde{T}_K}([0, K\pi]_{-\alpha}^c) - \mathbb{E}(N_{\tilde{T}_K}([0, K\pi]_{-\alpha}^c))}{\sqrt{K\pi}} \longrightarrow 0$$

in probability, as $K \rightarrow \infty$.

Then, it suffices to prove that

$$\frac{N_{\tilde{T}_K}([0, K\pi]_{-\alpha}) - \mathbb{E}(N_{\tilde{T}_K}([0, K\pi]_{-\alpha}))}{\sqrt{K\pi}} \Rightarrow N(0, V^2).$$

Wiener Chaos decomposition. Following Kratz and León [15] we can establish the expansion of the number of roots in the Wiener Chaos.

We need some facts about Hermite Polynomials, Multiple Wiener–Itô Integrals and the Chaotic Expansions, see Peccati and Taqqu [21] and Hiu-Hsiung Kuo [11] for details on the definitions and the results listed below.

The Hermite polynomials H_q are defined by

$$H_q(x) = (-1)^q e^{x^2/2} \left. \frac{d^q}{dt^q} e^{-t^2/2} \right|_{t=x}.$$

The family $(H_q: q \geq 0)$ is a complete orthogonal system in $L^2(\mathbb{R}, \varphi(dx))$ with $\|H_q\|_2^2 = q!$. Here φ stands for the standard Gaussian density function. Besides, for $q \geq 1$, the q -fold Multiple Wiener–Itô Integral w.r.t. B, I_q^B , can be introduced as the linear isometry, between the symmetric tensor product $L_s^2([0, 1]^q) = \mathbf{H}^{\otimes q}$ with the norm $\sqrt{q!} \|\cdot\|_{L^2([0,1]^q)}$ and its image in $L^2(\Omega)$, induced by

$$I_q^B(h^{\otimes q}) = H_q(B(h))$$

for $h \in \mathbf{H}$, with unit norm and $h^{\otimes q}(\lambda_1, \dots, \lambda_q) = \prod_{k=1}^q h(\lambda_k)$.

For $q \geq 1$, the q th Wiener Chaos \mathcal{H}_q is defined as the image of I_q^B . Furthermore, denoting the set of constants by \mathcal{H}_0 , we have

$$L^2(\Omega, \mathcal{F}, \mathbb{P}) = \bigoplus_{q=0}^{\infty} \mathcal{H}_q,$$

where \oplus indicates an orthogonal sum. In other words, for any square integrable functional, $F \in L^2(\Omega, \mathcal{F}, \mathbb{P})$, of the Brownian Motion B , there exists a unique sequence of symmetric functionals $(f_q: q \geq 1)$ with $f_q \in L^2([0, 1]^q)$, such that

$$F - \mathbb{E}(F) = \sum_{q=1}^{\infty} I_q^B(f_q),$$

where the equality holds in the L^2 sense.

We now write the Wiener Chaos expansion for the number of crossings. We need some notations. Denote

$$a_{2\ell} = \sqrt{\frac{2}{\pi}} \frac{(-1)^{\ell+1}}{2^\ell \ell! (2\ell - 1)}, \quad b_k = \begin{cases} (-1)^{k/2} (k - 1)!!, & \text{if } k \text{ is even,} \\ 0, & \text{if } k \text{ is odd,} \end{cases} \tag{17}$$

where $(2n - 1)!! = \prod_{j=1}^n (2j - 1)$;

$$f_q(x, y) = \sum_{\ell=0}^{\lfloor q/2 \rfloor} b_{q-2\ell} a_{2\ell} H_{q-2\ell}(x) H_{2\ell}(y), \tag{18}$$

$$I_q^{\bar{T}_K}([0, K\pi]_{-\alpha}) = \frac{1}{\sqrt{K\pi}} \int_{[0, K\pi]_{-\alpha}} f_q(\bar{T}_K(s), \mathcal{T}'_K(s)) v_K(s) ds. \tag{19}$$

Theorem 3. *The following expansion holds in L^2*

$$\frac{N_{\bar{T}_K}([0, K\pi]_{-\alpha}) - \mathbb{E}(N_{\bar{T}_K}([0, K\pi]_{-\alpha}))}{\sqrt{K\pi}} = \sum_{q=1}^{\infty} I_q^{\bar{T}_K}([0, K\pi]_{-\alpha}).$$

In particular, since $b_1 = 0$, $I_1^{\bar{T}_K}([0, K\pi]_{-\alpha}) = 0$ for all K .

The following two lemmas are the tools for proving the above theorem. The first one shows that the second moment of the crossings of zero of \bar{T}_K is finite.

Lemma 4. *There exists $t_0 > 0$ such that, for all $a > 0$, the variances of $N_{\bar{T}_K}([t, t + a])/\sqrt{K\pi}$ remain bounded uniformly on K for all $t \in [t_0, K\pi - t_0]$.*

Lemma 5.

1. $\mathbb{E}((N_{\bar{T}_K}([0, K\pi]_{-\alpha}))^2) < \infty$.
2. Denote by $N_{\bar{T}_K}^\eta(u, I)$ the number of crossings through the level u by \bar{T}_K in the interval I , then $\mathbb{E}(N_{\bar{T}_K}^\eta(u, I))$ is continuous in u .
3. Define

$$N_{\bar{T}_K}^\eta := \int_{[0, K\pi]_{-\alpha}} \varphi_\eta(\bar{T}_K(s)) |\mathcal{T}'_K(s)| v_K(s) ds, \tag{20}$$

where φ_η is the density of the $N(0, \eta)$ distribution. Then, $N_{\bar{T}_K}^\eta$ converges almost surely and in L^2 to $N_{\bar{T}_K}([0, K\pi]_{-\alpha})$ and $\mathbb{E}(N_{\bar{T}_K}^\eta)^2 \rightarrow \mathbb{E}(N_{\bar{T}_K}([0, K\pi]_{-\alpha}))^2$.

4. The random variable $N_{\bar{T}_K}^\eta$ has the chaotic expansion

$$N_{\bar{T}_K}^\eta = \sum_{q=0}^{\infty} \sum_{\ell=0}^{\lfloor q/2 \rfloor} b_{q-2\ell}^\eta a_{2\ell} \int_{[0, K\pi]_{-\alpha}} H_{q-2\ell}(\bar{T}_K(s)) H_{2\ell}(\mathcal{T}'_K(s)) v_K(s) ds,$$

where b_k^η are the Hermite coefficients of φ_η .

Proof of Theorem 3. We will prove now the Theorem 3. For ease of notation we will drop for a while the interval dependence in the crossings random variable.

Since the zeros are isolated, formally, we can write the Kac Formula:

$$N_{\tilde{T}_K} = N_{\bar{T}_K} = N_{\bar{T}_K}([0, K\pi]_{-\alpha}) = \int_{[0, K\pi]_{-\alpha}} \delta_0(\bar{T}_K(s)) |\mathcal{T}'_K(s)| v_K(s) ds.$$

In order to give it a precise meaning, we approximate the delta function at 0 by Gaussian kernels φ_η . Consider the random variables $N_{\bar{T}_K}^\eta$. By Part 4 of Lemma 5

$$N_{\bar{T}_K}^\eta = \sum_{q=0}^{\infty} \sum_{\ell=0}^{\lfloor q/2 \rfloor} b_{q-2\ell}^\eta a_{2\ell} \int_{[0, K\pi]_{-\alpha}} H_{q-2\ell}(\bar{T}_K(s)) H_{2\ell}(\mathcal{T}'_K(s)) v_K(s) ds. \tag{21}$$

Now, the idea is to pass to the limit this expansion as $\eta \rightarrow 0$ in order to obtain the expansion for $N_{\bar{T}_K}$.

First, observe that $b_k^\eta \rightarrow b_k$ (non-random), and that this is the only ingredient depending on η . Hence $N_{\bar{T}_K}^\eta \rightarrow N_{\bar{T}_K}$ almost surely. Now consider the sum of the first Q terms in the right-hand side of (21). By Fatou's Lemma we have

$$\begin{aligned} & \sum_{q=0}^Q \mathbb{E} \left(\left[\sum_{\ell=0}^{\lfloor q/2 \rfloor} b_{q-2\ell} a_{2\ell} \int_{[0, K\pi]_{-\alpha}} H_{q-2\ell}(\bar{T}_K(s)) H_{2\ell}(\mathcal{T}'_K(s)) v_K(s) ds \right]^2 \right) \\ & \leq \liminf_{\eta \rightarrow 0} \mathbb{E} \left([N_{\bar{T}_K}^\eta]^2 \right) = \mathbb{E} \left(N_{\bar{T}_K}^2 \right). \end{aligned}$$

In the last equality we used Part 3 of Lemma 5. Therefore, the right-hand side of (21) has a limit, say \mathcal{N} , in L^2 (with $b_{q-2\ell}$ instead of $b_{q-2\ell}^\eta$). It remains to show that this limit is, effectively, $N_{\bar{T}_K}$. The result follows writing

$$\|N_{\bar{T}_K} - \mathcal{N}\|_{L^2}^2 \leq 2[\|N_{\bar{T}_K} - N_{\bar{T}_K}^\eta\|_{L^2}^2 + \|N_{\bar{T}_K}^\eta - \mathcal{N}\|_{L^2}^2].$$

The first term in the right-hand side tends to zero by Part 3 of Lemma 5. To show that the second term tends to zero, consider its chaotic expansion

$$N_{\bar{T}_K}^\eta - \mathcal{N} = \sum_{q=0}^{\infty} \sum_{\ell=0}^{\lfloor q/2 \rfloor} (b_{q-2\ell}^\eta - b_{q-2\ell}) a_{2\ell} J_q,$$

where we denote

$$J_q = \int_{[0, K\pi]_{-\alpha}} H_{q-2\ell}(\bar{T}_K(s)) H_{2\ell}(\mathcal{T}'_K(s)) v_K(s) ds. \tag{22}$$

Note that J_q does not depend on η . Then, for each Q we have

$$\begin{aligned} \|N_{\bar{T}_K}^\eta - \mathcal{N}\|_{L^2}^2 & \leq 3 \left[\sum_{q=Q+1}^{\infty} \mathbb{E} \left(\left[\sum_{\ell=0}^{\lfloor q/2 \rfloor} b_{q-2\ell} a_{2\ell} J_q \right]^2 \right) \right. \\ & \quad \left. + \sum_{q=0}^Q \mathbb{E} \left(\left[\sum_{\ell=0}^{\lfloor q/2 \rfloor} (b_{q-2\ell}^\eta - b_{q-2\ell}) a_{2\ell} J_q \right]^2 \right) + \sum_{q=Q+1}^{\infty} \mathbb{E} \left(\left[\sum_{\ell=0}^{\lfloor q/2 \rfloor} b_{q-2\ell}^\eta a_{2\ell} J_q \right]^2 \right) \right]. \end{aligned}$$

Now, take limit as $\eta \rightarrow 0$, the first term does not depend on η ; the second one tends to 0 since it is a finite sum and $b_{q-2\ell}^\eta \rightarrow b_{q-2\ell}$; the third term is $\|P_Q(N_{\bar{T}_K}^\eta)\|_{L^2}^2$, where P_Q is the orthogonal projection of the L^2 random variable

$N_{\overline{T}_K}^\eta$ on the subspace $\bigoplus_{Q+1}^\infty \mathcal{H}_q$, thus by Part 3 of Lemma 5 it converges with $\eta \rightarrow 0$ to $\|P_Q(N_{\overline{T}_K})\|^2$ and tends to 0 when $Q \rightarrow \infty$.

This proves the theorem. \square

Remark 2. As shown in the chaining paragraph, $\overline{T}_K(t)$ and $\overline{T}'_K(t)$ can be written as $B(h_K(t, \cdot))$ and $B(h'_K(t, \cdot))$ for $h(t, \cdot), h'(t, \cdot) \in \mathbf{H}$. Since $\overline{T}_K(t)$ and $\overline{T}'_K(t)$ are orthogonal, so are by the isometry $h(t, \cdot)$, and $h'(t, \cdot)$, then, using the multiplication formula [21], Eq. (6.4.17), we get

$$\begin{aligned} H_{q-2\ell}(\overline{T}_K(t))H_{2\ell}(\overline{T}'_K(t)) &= H_{q-2\ell}(B(h_K(t, \cdot)))H_{2\ell}(B(h'_K(t, \cdot))) \\ &= I_q^B(h_K^{\otimes q-2\ell}(t, \cdot) \otimes h'_K{}^{\otimes 2\ell}(t, \cdot)). \end{aligned}$$

Therefore, by the Stochastic Fubini's Theorem, see Peccati and Taqqu [21], Section 5.13, we have $I_q^{\overline{T}_K}([0, K\pi]_{-\alpha}) = I_q^B(g_q)$ for $g_q \in L^2([0, 1]^q)$ given by

$$g_q(\boldsymbol{\lambda}, K) = \frac{1}{\sqrt{K\pi}} \int_{[0, K\pi]_{-\alpha}} \sum_{\ell=0}^{\lfloor q/2 \rfloor} b_{q-2\ell} a_{2\ell} (h_K^{\otimes q-2\ell}(s, \cdot) \otimes h'_K{}^{\otimes 2\ell}(s, \cdot))(\boldsymbol{\lambda}) v_K(s) \, ds, \quad (23)$$

with $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_q)$. That is, the variable $I_q^{\overline{T}_K}([0, K\pi]_{-\alpha})$ belongs to the q th Wiener Chaos \mathcal{H}_q . In particular, they are orthogonal for different values of q .

Asymptotic variance of $I_q^{\overline{T}_K}([0, K\pi]_{-\alpha})$. Following the arguments in Azaïs and León [2] we obtain the asymptotic variance of the number of zeros of \overline{T}_K on $[0, K\pi]_{-\alpha}$ as $K \rightarrow \infty$. Furthermore, in Proposition 1 we show that the limit variance of $N_{\overline{T}_K}([0, K\pi]_{-\alpha})$ coincides with that of the number of zeros, $N_X([0, K\pi]_{-\alpha})$, of the stationary Gaussian process X .

The following lemma gives a uniform upper bound (on K) for these variances.

Lemma 6. *There exists $t_0 > 0$ such that the variances of the normalized number of roots of \overline{T}_K on the interval $[t_0, K\pi - t_0]$*

$$\text{Var}\left(\frac{N_{\overline{T}_K}[t_0, K\pi - t_0] - \mathbb{E}(N_{\overline{T}_K}[t_0, K\pi - t_0])}{\sqrt{K\pi}}\right)$$

are uniformly bounded on K .

We are ready to compare the limit variances of $N_{\overline{T}_K}([0, K\pi]_{-\alpha})/\sqrt{K\pi}$ and $N_X([0, K\pi]_{-\alpha})/\sqrt{K\pi}$.

Proposition 1. *For $q \geq 2$, the variances of $I_q^{\overline{T}_K}([0, K\pi]_{-\alpha})$ and $I_q^X([0, K\pi]_{-\alpha})$ have the same limit, denoted by $\sigma_q^2(0)$.*

Remark 3. *The value of $V^2 = \sum_{q=2}^\infty \sigma_q^2(0)$ is shown to be finite in Azaïs and León [2]. Furthermore, in Granville and Wigman it is shown to be approximately 0.089. Furthermore, by Azaïs and León [2] this also coincides with the asymptotic variance (after the same standardization) of N_{X_K} .*

Proof of Proposition 1. By Remark 2 $I_q^{\overline{T}_K}([0, K\pi]_{-\alpha})$ is a Wiener–Itô Integral, then it is a centered random variable, moreover recalling that $\tau = t - s$ and $\sigma = t + s$, we have

$$\begin{aligned} \text{Var}(I_q^{\overline{T}_K}([0, K\pi]_{-\alpha})) \\ = \mathbb{E}([I_q^{\overline{T}_K}([0, K\pi]_{-\alpha})]^2) \end{aligned}$$

$$\begin{aligned} &= \frac{1}{K\pi} \int_{(K\pi)^\alpha}^{K\pi - (K\pi)^\alpha} \int_{(K\pi)^\alpha}^{K\pi - (K\pi)^\alpha} \mathbb{E}[f_q(\overline{T}_K(s), \overline{T}'_K(s)) f_q(\overline{T}_K(t), \overline{T}'_K(t))] v_K(s) v_K(t) ds dt \\ &= \frac{1}{2} \int_{-K\pi + 2(K\pi)^\alpha}^{K\pi - 2(K\pi)^\alpha} g(\tau) d\tau, \end{aligned}$$

where

$$g(\tau) = \frac{1}{K\pi} \int_{2(K\pi)^\alpha + |\tau|}^{2K\pi - 2(K\pi)^\alpha - |\tau|} E_K(\tau, \sigma) v_K((\sigma - \tau)/2) v_K((\sigma + \tau)/2) d\sigma$$

and $E_K(\tau, \sigma)$ is the expectation written in terms of τ and σ , namely $E_K(\tau, \sigma) = \mathbb{E}[f_q(\overline{T}_K((\sigma - \tau)/2), \overline{T}'_K((\sigma - \tau)/2)) f_q(\overline{T}_K((\sigma + \tau)/2), \overline{T}'_K((\sigma + \tau)/2))]$.

Applying the mean value theorem, there exists $\tilde{\sigma}_K \in [2(K\pi)^\alpha - |\tau|, 2K\pi - 2(K\pi)^\alpha + |\tau|]$, such that

$$g(\tau) = \frac{2K\pi - 4(K\pi)^\alpha - 4|\tau|}{K\pi} E_K(\tau, \tilde{\sigma}_K) v_K((\tilde{\sigma}_K - \tau)/2) v_K((\tilde{\sigma}_K + \tau)/2).$$

A direct computation shows that

$$v_K^2(u) = \frac{1}{1 + c_K(2u)} \left[c_K''(2u) - c_K''(0) - \frac{(c_K'(2u))^2}{1 + c_K(2u)} \right]. \tag{24}$$

Note that by the inequalities in (11) $c_K(2u)$ and its derivatives tend to 0 when $u \rightarrow \infty$. Besides, $c_K''(0) = -\frac{(K+1)(2K+1)}{6K^2}$. Therefore, since $\tilde{\sigma}_K \rightarrow_K \infty$ we deduce that $v_K(\tilde{\sigma}_K) \rightarrow \sqrt{-sc''(0)} = 1/\sqrt{3}$ as $K \rightarrow \infty$.

By Mehler formula, see Azaïs and Wschebor [3], Lemma 10.7, $E_K(\tau, \sigma)$ is a polynomial on the covariances of $(\overline{T}_K(s), \overline{T}'_K(s))$ and $(\overline{T}_K(t), \overline{T}'_K(t))$; thus, $E_K(\tau, \tilde{\sigma}_K)$ is a polynomial on

$$(c_K(\tau) \pm c_K(\tilde{\sigma})) / \sqrt{(1 + c_K(\tilde{\sigma} - \tau))(1 + c_K(\tilde{\sigma} + \tau))},$$

and its derivatives. Let $K \rightarrow \infty$, then, $\tilde{\sigma}_K \rightarrow \infty$, $c_K(\tilde{\sigma}_K), c_K(\tilde{\sigma}_K \pm \tau) \rightarrow 0$, $c_K(\tau) \rightarrow sc(\tau)$ and $E_K(\tau, \tilde{\sigma})$ tends to the same polynomial but with $c_K(\tau)$ replaced by $sc(\tau)$ and $c_K(\tilde{\sigma}_K)$ replaced by 0.

Consider now the centered stationary Gaussian process X with covariance function at (s, t) given by $sc(\tau)$. For $\text{Var}(I_q^X)$, analogous computations to the preceding ones show that

$$\lim_{K \rightarrow \infty} \text{Var}(I_q^X([0, K\pi]_{-\alpha})) = \lim_{K \rightarrow \infty} \text{Var}(I_q^{\overline{T}^K}([0, K\pi]_{-\alpha})),$$

provided the necessary domination given by Lemma 6. The result follows. □

2.2. CLT for the number of roots of Classical Trigonometric Polynomials

This section is devoted to prove our main result: Theorem 1.

Recall that for $q = 1$, the random variables $I_1^{\overline{T}^K}([0, K\pi]_{-\alpha}) = 0$ for all K because $b_1 = 0$.

Asymptotic Gaussianity of $I_q^{\overline{T}^K}$ for $q > 1$. We use the Fourth Moment Theorem by Peccati and Tudor.

Theorem 4 (Theorem 1 of [22]). Assume that for $q_1 < q_2 < \dots < q_m$ it holds that $\mathbb{E}[I_{q_i}(f_i^{(k)})]^2 \rightarrow_k \sigma_{ii}^2$ then the following conditions are equivalent:

1. The vector

$$(I_{q_1}(f_1^{(k)}), \dots, I_{q_m}(f_m^{(k)})) \xrightarrow[k \rightarrow \infty]{} N(0, D_m),$$

where D_m is a diagonal matrix with entries σ_{ii}^2 .

2. For $i = 1, \dots, m$ and $p = 0, 1, \dots, q_i - 1$

$$\|f_i^{(k)} \otimes_p f_i^{(k)}\|_{L^2([0,1]^{2q_i-2p})}^2 \xrightarrow{k \rightarrow \infty} 0,$$

where \otimes_p denotes the p -order contraction, that is

$$\begin{aligned} & f_{q_i}^{(k)} \otimes_p f_{q_i}^{(k)}(x_1, \dots, x_{q_i-p}, y_1, \dots, y_{q_i-p}) \\ &= \int_{[0,1]^p} f_{q_i}^{(k)}(x_1, \dots, x_{q_i-p}, z_1, \dots, z_p) \\ & \quad \cdot f_{q_i}^{(k)}(y_1, \dots, y_{q_i-p}, z_1, \dots, z_p) dz_1 \cdots dz_p. \end{aligned}$$

Proof. By Equation (23) we know that $I_q^{\bar{T}K}([0, K\pi]_{-\alpha}) = I_q^B(g_q)$ with

$$g_q(\lambda, K) = \frac{1}{\sqrt{K\pi}} \int_{[0, K\pi]_{-\alpha}} \sum_{\ell=0}^{\lfloor q/2 \rfloor} b_{q-2\ell} a_{2\ell} (h(s, \cdot)^{\otimes q-2\ell} \otimes h'(s, \cdot)^{\otimes 2\ell})(\lambda) v_K(s) ds,$$

and $\lambda = (\lambda_1, \dots, \lambda_q)$.

Using the isometric property of the stochastic integral, we have

$$h(s, \cdot)^{\otimes p} \otimes_n h(t, \cdot)^{\otimes p} = \bar{r}_K(s, t)^n \cdot h(s, \cdot)^{\otimes p-n} \otimes h(t, \cdot)^{\otimes p-n}.$$

Similar formulas hold for the terms which include h' . More precisely, by the isometric property, they give factors that are powers of $\text{cov}(\bar{T}_K(s), \bar{T}_K(t)) =: \tilde{r}_K'(s, t)$ or $\text{cov}(\bar{T}'_K(s), \bar{T}'_K(t)) =: \tilde{r}_K''(s, t)$. Therefore

$$\begin{aligned} \|g_q(K)\|_2^2 &= \frac{1}{K\pi} \int_{[0, K\pi]_{-\alpha}} \int_{[0, K\pi]_{-\alpha}} v_K(s) v_K(t) \sum_{\ell=0}^{q/2} \sum_{\ell'=0}^{q/2} b_{q-2\ell} b_{q-2\ell'} a_{2\ell} a_{2\ell'} \\ & \quad \cdot \bar{r}_K(s, t)^{q-2(\ell \vee \ell')} \tilde{r}_K''(s, t)^{2(\ell \wedge \ell')} \tilde{r}_K'(s, t)^{2(\ell \vee \ell')-2(\ell \wedge \ell')} ds dt, \end{aligned}$$

where $x \vee y = \sup(x, y)$ and $x \wedge y = \inf(x, y)$. It is quite tedious to write down the contractions and their norms, but the resulting integrals are quite similar. Let us do it for the case $n = 1$ and q odd (so that $q - 2\ell > 0$), we have

$$\begin{aligned} \|g_q \otimes_1 g_q(K)\|_2^2 &= \frac{1}{(K\pi)^2} \int_{[0, K\pi]_{-\alpha}} \int_{[0, K\pi]_{-\alpha}} \int_{[0, K\pi]_{-\alpha}} \int_{[0, K\pi]_{-\alpha}} ds dt ds' dt' \\ & \quad \cdot v_K(s) v_K(t) v_K(s') v_K(t') \sum_{\ell, \ell'=0}^{q/2} \sum_{k, k'=0}^{q/2} b_{q-2\ell} b_{q-2\ell'} a_{2\ell} a_{2\ell'} b_{q-2k} b_{q-2k'} a_{2k} a_{2k'} \\ & \quad \cdot \bar{r}_K(s, t) \bar{r}_K(s', t') \bar{r}_K(s, s')^{q-2(\ell \vee \ell')-1} \tilde{r}_K''(s, s')^{2(\ell \wedge \ell')} \tilde{r}_K'(s, s')^{2(\ell \vee \ell')-2(\ell \wedge \ell')} \\ & \quad \cdot \bar{r}_K(t, t')^{q-2(k \vee k')-1} \tilde{r}_K''(t, t')^{2(k \wedge k')} \tilde{r}_K'(t, t')^{2(k \vee k')-2(k \wedge k')}. \end{aligned}$$

Observe that in the general case, the exponent of $\bar{r}_K(s, t)$ and $\bar{r}_K(s', t')$ is n , but the sum of the exponents in the factors involving (s, t) , (s, s') , and in those involving (s', t') , (t, t') , is q (the total sum of the exponents is $2q$).

Now, we have to take the limit as $K \rightarrow \infty$.

In the case of $\|g_q\|_2^2$, since I_q^B is an isometry, it follows that

$$\|g_q\|_2^2 = \text{Var}(I_q^{\bar{T}K}([0, K\pi]_{-\alpha})) \xrightarrow{K \rightarrow \infty} \sigma_q^2(0) > 0.$$

Now, we consider the limit of $\|g_q \otimes_n g_q\|_2^2$ as $K \rightarrow \infty$. Since, the v_K are bounded by Equation (24) and the inequalities in (11), the sums have a finite fixed number of terms and the a, b coefficients are constant, the important

ingredients are the covariances. We split the domain of integration into two parts: a tubular neighborhood of radio η around the diagonal $s = t = s' = t'$ and its complement. Therefore, we have:

Close to the diagonal. We assume that $|t - s| < \eta$, $|t' - s'| < \eta$, $|t' - t| < \eta$ and $|s' - s| < \eta$.

The covariances are bounded, in absolute value, from above by constants, for instance

$$|\text{cov}(\overline{T}_K(s), \mathcal{T}'_K(t))| = \frac{1}{V_K(s)V_K(t)v_K(t)} \left| \partial_t r_K(s, t) - \frac{c'_K(2t)}{2V_K^2(t)} r_K(s, t) \right|,$$

with $r_K(s, t) = \frac{1}{2}(c_K(\tau) + c_K(\sigma))$. At $\tau = 0$ we have $c_K(0) = 1$, $c'_K(0) = 0$ and $c''_K(0) = -\frac{(K+1)(2K+1)}{6K^2}$, thus, by continuity the terms involving τ are bounded. The remaining quantities are easily seen to be bounded by Lemma 2 and the inequalities in (11) and (12).

Therefore, the integral is bounded by a constant times the volume of the tubular neighborhood of the diagonal. Such volume is proportional to $K\pi$.

In conclusion, the $1/(K\pi)^2$ is not compensated by the integral, so the upper bound for the integral in the formula for the norm of the contraction (restricted to the neighborhood of the diagonal) tends to zero.

Far from the diagonal. In the rest of the proof of the CLT, C stands for some constant whose actual value is meaningless, but not depending on K .

We may assume that at least one of the following $|t - s| > \eta$, $|t' - s'| > \eta$, $|t' - t| > \eta$, $|s' - s| > \eta$ holds true.

By the inequalities in (11) and (12), the (absolute value of the) covariances \overline{r} , \overline{r}' and \overline{r}'' at x, y are bounded from above by $C(1/|x - y| + 1/|x + y|)$ when $|x - y| > \eta$. Therefore, the product in the integrand of $\|g_q \otimes_n g_q\|_2^2$ is bounded by the product of $1/|t - s| + 1/|t + s|$, $1/|t' - s'| + 1/|t' + s'|$, $1/|s - s'| + 1/|s + s'|$ and $1/|t - t'| + 1/|t + t'|$. Each one of these factors appears in the bound if the distance between the corresponding variables is larger than η . Otherwise, we bound them by constant as in the previous case.

Furthermore, the exponents of the covariances involving s, t and s', t' is $n = 1, \dots, q - 1$, the sum of the exponents in the factors involving s, s' is $q - n = 1, \dots, q - 1 \geq 1$, and the sum of the exponents in the factors involving t, t' also is $q - n \geq 1$.

Let us consider one of the possible cases, the others are similar. Say that $n = 1$, $|t - s| > \eta$, $|t' - s'| < \eta$, $|t' - t| < \eta$ and $|s' - s| < \eta$, call A the set of points verifying these inequalities. We bound the covariances involving the variables $t', s'; t', t$; and s', s in the integrand by constants. Then, the integrand is bounded by

$$\begin{aligned} & C \int_A \left[\frac{1}{\tau} + \frac{1}{\sigma} \right] ds dt ds' dt' \\ &= C \int_A \frac{1}{\tau} dt ds' dt' + C \int_A \frac{1}{\sigma} dt ds' dt' \\ &\leq C \log(K\pi) \int_{A_{-1}} dt ds' dt', \end{aligned}$$

where $\tau = t - s$, $\sigma = t + s$ and $A_{-1} = \{(t, s', t') : |t' - s'| < \eta, |t' - t| < \eta, |s' - s| < \eta\}$. Now, the volume of A_{-1} is bounded by the volume of the tubular neighborhood of the diagonal of radius $\sqrt{3}\eta$, its volume is bounded by constant times $K\pi$, the result follows. Thus, the integral is bounded by a constant times $\log(K\pi)K\pi$, divided by $(K\pi)^2$, it tends to zero.

Putting together both parts we conclude that $\|g_q \otimes_n g_q\|_2^2 \rightarrow 0$ as $K \rightarrow \infty$ for $n = 1, \dots, q - 1$. Therefore, $I_q^{\overline{T}K}$ converges in distribution to a Gaussian random variable as $K \rightarrow \infty$ for all q .

Asymptotic Gaussianity of the sum. Since, the $I_q^{\overline{T}K}([0, K\pi]_{-\alpha})$ are orthogonal, by Theorem 4 and Lemma 1, their Q th partial sum converges in distribution, as K grows to infinity, to a Gaussian random variable with variance $\sum_1^Q \text{Var}(I_q^X([0, K\pi]_{-\alpha}))$, therefore

$$\frac{N_{\overline{T}K}([0, K\pi]_{-\alpha}) - \mathbb{E}(N_{\overline{T}K}([0, K\pi]_{-\alpha}))}{\sqrt{K\pi}} = \sum_{q=1}^{\infty} I_q^{\overline{T}K}([0, K\pi]_{-\alpha})$$

converges in distribution to a Gaussian random variable with variance $V^2 = \sum_{q=1}^{\infty} \text{Var}(I_q^X([0, K\pi]_{-\alpha}))$. (It is well known that if $\mu_n \rightarrow \mu_{\infty}, \sigma_n \rightarrow \sigma_{\infty}, X_n \sim N(\mu_n, \sigma_n^2)$ then $X_n \Rightarrow X_{\infty}$.) Finally, reasoning as in Lemma 3, note that

$$\lim_{K \rightarrow \infty} \text{Var}(I_q^X([0, K\pi]_{-\alpha})) = \lim_{K \rightarrow \infty} \text{Var}(I_q^X([0, K\pi])) = \sigma_q^2(0).$$

This proves the theorem. □

Proof of Corollary 2. We do not fill in the details. The scheme of the proof is the following.

First compute the cross correlation $\rho_K(s, t) = \mathbb{E}(\overline{T}_K(s)\overline{T}(t))$ between \overline{T}_K and \overline{T} :

$$\begin{aligned} \rho_K(s, t) &:= \mathbb{E}\overline{T}_K(s)\overline{T}(t) \\ &= \frac{1}{V_K(s)V(t)} \mathbb{E} \int_0^1 \sum_{n=1}^K \cos\left(\frac{n}{K}s\right) \mathbf{1}\left\{\left[\frac{n-1}{K}, \frac{n}{K}\right]\right\}(\lambda) dB_{\lambda} \cdot \int_0^1 \cos(\lambda't) dB_{\lambda'} \\ &= \frac{1}{V_K(s)V(t)} \sum_{n=1}^K \int_{(n-1)/K}^{n/K} \cos\left(\frac{n}{K}s\right) \cos(\lambda t) d\lambda \\ &= \frac{1}{V_K(s)V(t)} \frac{1}{2} \sum_{n=1}^K \int_{(n-1)/K}^{n/K} \left[\cos\left(\frac{n}{K}s - \lambda t\right) + \cos\left(\frac{n}{K}s + \lambda t\right) \right] d\lambda \\ &= \frac{1}{\sqrt{1+c_K(2s)}\sqrt{1+sc(2t)}} \sum_{n=1}^K \int_0^{1/K} \left[\cos\left(\frac{n}{K}\tau - vt\right) + \cos\left(\frac{n}{K}\sigma - vt\right) \right] dv, \end{aligned}$$

where we made the change of variable $\lambda \mapsto \frac{n}{K} - \lambda$ in the last equality. It follows that, see Azais and León [2], $\rho_K(s, t) \rightarrow \overline{r}(s, t)$ uniformly on off-diagonal compacts and so do the derivatives of $\rho_K(s, t)$ to the respective derivatives of $\overline{r}(s, t)$. Furthermore, these functions are bounded by $c(1/\tau + 1/\sigma)$.

Now

$$\mathbb{E}((I_q^{\overline{T}_K} - I_q^{\overline{T}})^2) = \mathbb{E}((I_q^{\overline{T}_K})^2) - 2\mathbb{E}(I_q^{\overline{T}_K} \cdot I_q^{\overline{T}}) + \mathbb{E}((I_q^{\overline{T}})^2).$$

Repeating the arguments in Proposition 1 and using these convergences we deduce that each expectation converges to $\sigma_q^2(0)$. Hence, $\|I_q^{\overline{T}_K} - I_q^{\overline{T}}\|_{L^2} \rightarrow_K 0$.

Finally, repeating the arguments in the proof of Theorem 1 for the Gaussianity of the sum we deduce that the asymptotic distributions of $N_{\overline{T}}([0, K\pi]_{-\alpha})$ and $N_{\overline{T}_K}([0, K\pi]_{-\alpha})$ (after standardization) coincide. □

3. Proofs of the lemmas

Proof of Lemma 2. Let us start with the proof for T and \overline{T}' .

The variance of T is equal to $V(t) := (1 + sc(2t))/2$, hence it suffices to prove that $sc(2t) > -2/\pi$ for $t \in [0, K\pi]$. Observe that the critical points of sc are the roots of the equation $sc(x) = \cos(x)$, then, it is easy to see that in the interval $[-\pi/2, \pi/2]$ the only root of this equation is $x = 0$, which is a maximum since $sc(0) = 1$. Thus, the minimum of sc lies outside this interval, so, as $|sc(x)| \leq 1/|x|$ the minimum is greater than $-2/\pi$, thus the result follows.

Let us look now at the variance of $\overline{T}'(t)$

$$v^2(t) = \frac{1}{1 + sc(2t)} \left[sc''(2t) - sc''(0) - \frac{sc'(2t)^2}{1 + sc(2t)} \right]. \tag{25}$$

Then

$$v^2(t) \geq \frac{1}{2} \left[\frac{1}{3} - sc''(2t) - \frac{sc'(2t)^2}{1 + sc(2t)} \right]$$

and from the inequalities in (11), which hold since we are looking at compact intervals not containing 0, it is easy to see that the latter expression is bounded away from zero for large t .

Now, let us consider the variances of T_K and \overline{T}'_K . By Lemma 1 (see also Equation (37) in [10]) we know that as $K \rightarrow \infty$

$$c_K^{(j)}(2s) = sc^{(j)}(2s) + O(1/K^j),$$

where the constants involved in the O notation do not depend on K and $j = 0, 1, 2$ stands for functional value, first and second derivative respectively. From this asymptotic and the result for T and \overline{T}' , the result follows also for T_K and \overline{T}'_K . \square

Proof of Lemma 3. Let us look at the interval $[0, (K\pi)^\alpha]$, the other one is analogous. First, we use Markov inequality to bound the probability by an expression involving the expectation of the number of roots, that is, for given $\varepsilon > 0$ we have

$$\mathbb{P}\left(\left|\frac{N_{\overline{T}_K} - \mathbb{E}(N_{\overline{T}_K})}{\sqrt{K\pi}}\right| > \varepsilon\right) \leq \frac{\mathbb{E}|N_{\overline{T}_K} - \mathbb{E}(N_{\overline{T}_K})|}{\varepsilon\sqrt{K\pi}} \leq \frac{2\mathbb{E}(N_{\overline{T}_K})}{\varepsilon\sqrt{K\pi}}$$

(note that above we have eliminated the dependence of the interval in the notation of crossings for ease of notation) thus, it is enough to show that $\mathbb{E}(N_{\overline{T}_K}([0, (K\pi)^\alpha]))/\sqrt{K\pi} \rightarrow 0$.

With this aim, we use the first order Rice Formula

$$\mathbb{E}(N_{\overline{T}_K}([0, (K\pi)^\alpha])) = \int_0^{(K\pi)^\alpha} \mathbb{E}[|\overline{T}'_K(t)| | \overline{T}_K(t) = 0] p_{\overline{T}_K(t)}(0) dt.$$

By a standard Gaussian regression

$$\mathbb{E}[|\overline{T}'_K(t)| | \overline{T}_K(t) = 0] = \mathbb{E}\left[\left|\overline{T}'_K(t) - \frac{\text{cov}(\overline{T}'_K(t), \overline{T}_K(t))}{\text{Var}(\overline{T}_K(t))} \overline{T}_K(t)\right|\right].$$

The random variable within the expectation is a centered Gaussian one, hence the expectation of its modulus equals $\sqrt{2/\pi}$ times its standard deviation, so

$$\begin{aligned} &\mathbb{E}(N_{\overline{T}_K}([0, (K\pi)^\alpha])) \\ &= \frac{1}{\pi} \int_0^{(K\pi)^\alpha} \left[c''_K(2t) - c''_K(0) - \frac{(c'_K(2t) - c'_K(0))^2}{1 + c_K(2t)} \right]^{1/2} \frac{dt}{\sqrt{1 + c_K(2t)}}. \end{aligned}$$

Let us look at the integrand. By Lemmas 1 and 2 we know that the denominator is bounded away from zero uniformly in K and that the first factor is bounded uniformly in K in any compact interval not containing 0. Finally taking Taylor expansions at $t = 0$, taking into account the particular values of the derivatives of c_K at 0 obtained from Equation (4), it follows that the first factor is bounded uniformly in K in a neighborhood of 0. Hence, the result follows. \square

Proof of Lemma 4. We start from (4) that shows that c_K has derivatives of any order uniformly bounded.

Let t_0 to be as in Lemma 2. We know that denominator of (6) is bounded away from zero for $t, s \in [t_0, \pi - t_0]$, this implies that $\overline{r}_K(s, t)$ has bounded derivatives of any order.

On $[t_0, \pi - t_0]$ let d_K the distance induced by the sixth derivative $\overline{T}_K^{(6)}$ of \overline{T}_K i.e.

$$\begin{aligned} d_K^2(s, t) &:= \mathbb{E}(\overline{T}_K^{(6)}(s) - \overline{T}_K^{(6)}(t))^2 \\ &= \overline{r}_K^{(6)}(s, s) + \overline{r}_K^{(6)}(t, t) - 2\overline{r}_K^{(6)}(s, t) \\ &= \frac{(s-t)^2}{2} \overline{r}_K^{(7)}(\zeta, \zeta) \leq C(s-t)^2. \end{aligned}$$

This implies by the Dudley Theorem, see for example Theorem 2.40 of [3], that $\mathbb{E}(\sup_{s \in (t_0, \pi - t_0]} (\overline{T}_K^{(6)}(s)))$, is bounded and by consequence $\mathbb{E} \|\overline{T}_K^{(6)}\|_\infty$, where the norm is taken on $[t_0, \pi - t_0]$.

Then applying Theorem 3.6 of [3] with $m = 2$ and $p = 5$ we get that

$$\mathbb{E}((N_{\overline{T}_K}[t_0, t_0 + a])^2) \leq c_{5,2}[A + \mathbf{C} + \mathbb{E} \|\overline{T}_K^{(6)}\|_\infty],$$

where \mathbf{C} is a bound for the density of $T_K(s)$ which is finite because the variance is bounded away from zero. Finally observe that these bounds are uniform in K since the covariances and its derivatives are uniformly bounded in K . This concludes the proof. \square

Proof of Lemma 5. We drop the dependence of the intervals in the definition of the crossings.

1. This is a consequence of Lemma 4, which bounds the integral near the diagonal (the difficult part), or of Corollary 3.7 of Azaïs and Wschebor [3].
2. To show this statement we begin with the following inequality that is a consequence of the Rolle's Theorem

$$N_{\overline{T}_K}(u) \leq N_{\overline{T}_K'}(0) + 1,$$

where $N_{\overline{T}_K}(u)$ are the crossings of \overline{T}_K of level u and $N_{\overline{T}_K'}(0)$ are the zeros of the derivative of \overline{T}_K . But when u is in a neighborhood of zero we have that $\lim_{u \rightarrow 0} N_{\overline{T}_K}(u) = N_{\overline{T}_K}(0)$ a.s. (in fact the random variables are locally constant). Thus by convergence dominated theorem and given that $\mathbb{E}(N_{\overline{T}_K}^2(0)) < \infty$, fact that can be proven similarly that the result of Lemma 4, we have

$$\lim_{u \rightarrow 0} \mathbb{E}((N_{\overline{T}_K}(u))^2) = \mathbb{E}((N_{\overline{T}_K}(0))^2) < \infty.$$

3. First, note that $N_{\overline{T}_K}^\eta \rightarrow N_{\overline{T}_K}$ almost surely.

Therefore, it suffices to state the convergence of the second moment.

Fatou's Lemma implies that

$$\mathbb{E}((N_{\overline{T}_K}^\eta)^2) = \mathbb{E}\left(\lim_{\eta} (N_{\overline{T}_K}^\eta)^2\right) \leq \lim_{\eta} \mathbb{E}((N_{\overline{T}_K}^\eta)^2).$$

On the other hand, Area-Formula (see Federer [9] applied for $d = 1$, $B = [0, t]$, $g = \varphi_\eta$ and $f = \overline{T}_K \in C^1$), almost surely, permits us to write

$$N_{\overline{T}_K}^\eta = \int_{-\infty}^{\infty} N_{\overline{T}_K}(u) \varphi_\eta(u) du = \mathbb{E}_Z(N_{\overline{T}_K}(Z)),$$

where Z has density φ_η . Thus, applying Jensen's Inequality in the inner expectation and Tonelli's Theorem, we have

$$\begin{aligned} \mathbb{E}((N_{\overline{T}_K}^\eta)^2) &= \mathbb{E}((\mathbb{E}_Z(N_{\overline{T}_K}(Z)))^2) \leq \mathbb{E}(\mathbb{E}_Z(N_{\overline{T}_K}^2(Z))) \\ &= \int_{-\infty}^{\infty} \mathbb{E}(N_{\overline{T}_K}^2(u)) \varphi_\eta(u) du. \end{aligned}$$

Since, φ_η approximates the unity, since $\mathbb{E}(N_{\overline{T}_K}^2(u))$ is continuous in u (Part 2 of this lemma), passing to the limit, with $\eta \rightarrow 0$, in the latter inequality gives the desired result.

4. This fact follows from a straightforward adaptation of the proof of Lemma 2 in Kratz and León [15]. In fact, let ξ be a standard Gaussian random variable. Then, since φ_η and $|\cdot|$ are functions in $L^2(\varphi(dx))$ they have Hermite expansions

$$|\xi| = \sum_{k=0}^{\infty} a_k H_k(\xi); \quad \varphi_\eta(\xi) = \sum_{k=0}^{\infty} b_k H_k(\xi),$$

where the Hermite coefficients are defined by $a_k = 1/\sqrt{k!} \int |x|H(x)\varphi(x) dx$ and $b_k = 1/\sqrt{k!} \int \varphi_\eta(x)H(x)\varphi(x) dx$. Next, since $\overline{T}_K(s)$ and $\mathcal{T}'_K(s)$ are standard Gaussian random variables for each $s \in [0, K\pi]_{-\alpha}$ we can apply these expansions pointwise in s and replace them in the integral in the right-hand side of (20).

The rest of the proof (i.e.: passing to the limit under the integral sign) follows exactly the same lines as in [15], Lemma 2, we just have to add the bound for $v_K(s)$ which is easy. \square

Proof of Lemma 6. The proof follows the same lines of the proof of the similar claim in Azaïs and León [2], p. 7, with minor changes. The idea is to divide the integral into two parts, near the diagonal the bound is obtained using Cauchy–Schwarz Inequality and Lemma 4, for the off-diagonal part, the bound is obtained applying Arcones Inequality [1], Lemma 1, for the vectors $(\overline{T}_K(s), \mathcal{T}'_K(s))$ and $(\overline{T}_K(t), \mathcal{T}'_K(t))$: if

$$\psi_K(\tau, \sigma) := \max\{|\text{cov}(\overline{T}_K(s), \overline{T}_K(t))| + |\text{cov}(\overline{T}_K(s), \mathcal{T}'_K(t))|; |\text{cov}(\mathcal{T}'_K(s), \overline{T}_K(t))| + |\text{cov}(\mathcal{T}'_K(s), \mathcal{T}'_K(t))|\} < 1$$

then

$$\mathbb{E}[f_q(\overline{T}_K(s), \mathcal{T}'_K(s))f_q(\overline{T}_K(t), \mathcal{T}'_K(t))] \leq \|f_q\|_2^2 \psi_K^q.$$

Observe that the inequalities in (11) allow to choose τ, σ large enough in order to have Arcones coefficient $\psi_K < \rho < 1$, for fixed ρ . Besides $\|f_q\|_2^2$ is easily seen to be uniformly bounded using Hermite polynomials properties. \square

4. Notation table

- $T_K(t)$ see (1),
- $N_Z(I)$ number of zeroes of process Z on I ,
- V limit variance in Theorem 1,
- $T(t)$ see Corollary 2,
- $r(s, t)$ covariance of $T(\cdot)$,
- $sc(x)$ $\sin(x)/x$,
- $X_k(t)$ see (2),
- $\tilde{T}_K(t), \tilde{X}_K(t)$, see (3),
- $C_K(t)$ see (4),
- $r_K(s, t)$ see (5),
- $V_K^2(t)$ $r_K(t, t)$,
- τ $s - t$,
- σ $s + t$,
- $\overline{T}_K(t)$ see (6),
- $\overline{r}_K(s, t)$ see (7),
- $\overline{r}(s, t)$ see (13),
- B Brownian Motion,
- $\gamma_K(t, \lambda)$ see (16),
- $h_K(t, \cdot)$ $\gamma_K(t, \cdot)/V_K(t)$,
- $\mathcal{T}'_K(s)$ see (15),
- a_k, b_k see (17),
- $f_q(x, y)$ see (18),
- $I_q^{\overline{T}_K}([0, K\pi]_{-\alpha})$ see (19),
- φ, φ_η density of the $N(0, 1), N(0, \eta)$,
- $N_{\overline{T}_K}(u, I)$ number of crossings of u by \overline{T}_K in the interval I ,
- $N_{\overline{T}_K, \alpha}^\eta$ see (20),
- C some constant with meaningless value,
- $v^2(t)$ see (25).

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