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MINKOWSKI CONTENT AND NATURAL PARAMETERIZATION FOR THE SCHRAMM–LOEWNER EVOLUTION

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We prove the existence and nontriviality of the d-dimensional 4 Minkowski content for the Schramm–Loewner evolution (SLE_κ) with $\kappa < 8$ and $d = 1 + \frac{\kappa}{8}$. We show that this is a multiple of the natural parameterization.

1. Introduction. A number of measures on paths or clusters on two-dimensional lattices arising from critical statistical mechanical models are believed to exhibit some kind of conformal invariance in the scaling limit. Schramm [13] introduced a one-parameter family of such processes, now called the (*chordal*) *Schramm–Loewner evolution with parameter* κ (SLE_κ) and showed that these give the only possible limits for conformally invariant processes in simply connected domains satisfying a certain "domain Markov property." He defined the process as a probability measure on curves from 0 to ∞ in $\mathbb{H}$ and then used conformal invariance to define the process in other simply connected domains.

The definition of the process in $\mathbb{H}$ uses the half-plane Loewner equation. Suppose $\gamma : (0, t] \to \overline{\mathbb{H}}$ is a curve with $\gamma(0) = 0$, and let $\gamma_t = \gamma(0, t]$. Let H_t denote the unbounded component of $\mathbb{H} \setminus \gamma_t$. We assume that γ is noncrossing in the sense that for all $s < t$, $\gamma[s, \infty) \subset \overline{H}_s$, and $\gamma[s, t] \cap H_s$ is nonempty. Let $g_t : H_t \to \mathbb{H}$ be the unique conformal transformation with $g_t(z) - z = o(1)$ as $z \to \infty$. Then for every $a > 0$, there exists a reparameterization of the curve such that the following holds:

- For $z \in \mathbb{H}$, the map $t \mapsto g_t(z)$ is a smooth flow and satisfies the Loewner differential equation

$$\partial_t g_t(z) = \frac{a}{g_t(z) - U_t}, \qquad g_0(z) = z,$$

  where U_t is a continuous function on $\mathbb{R}$. This equation is valid up to a time $T_z \in (0, \infty]$.

Under the reparameterization, the transformation g_t satisfies

$$g_t(z) = z + \frac{at}{z} + O(|z|^{-2}), \qquad z \to \infty.$$

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We say that the curve is *parameterized by* (*half-plane*) *capacity*. Schramm-defined chordal SLE_κ to be the solution to the Loewner equation with $a = 2$ and U_t a Brownian motion with variance parameter κ. An equivalent definition (up to a linear time change) is to choose U_t to be a standard Brownian motion and $a = 2/\kappa$. It has been shown that a number of discrete random models have SLE as the scaling limit provided that the discrete models are parameterized using (half-plane) capacity. Examples are loop-erased random walk for $\kappa = 2$ [7], Ising interfaces for $\kappa = 3$ [16], harmonic explorer for $\kappa = 4$ [14], percolation interfaces on the triangular lattice for $\kappa = 6$ [15] and uniform spanning trees for $\kappa = 8$ [7].

If D is a simply connected domain with distinct boundary points w_1, w_2 where ∂D is nice in neighborhoods of w_1, w_2, then chordal SLE_κ from w_1 to w_2 in D is defined by taking the conformal image of SLE_κ in the upper half plane under a transformation $F : \mathbb{H} \to D$ with $F(0) = w_1$, $F(\infty) = w_2$. The map F is not unique, but scale invariance of SLE in $\mathbb{H}$ shows that the distribution on paths is independent of the choice. This can be considered as a measure on the curves $F \circ \gamma$ with the induced parameterization or as a measure on curves modulo reparameterization.

While the capacity parameterization is useful for analyzing the curve, it is not the scaling limit of the "natural" parameterization of the discrete models. For example, for loop-erased walks, it is natural to parameterize by the length of the random walk. One can ask whether the curves parameterized by a normalized version of this "natural length" converge to SLE with a different parameterization. The Hausdorff dimension of the SLE paths [2] is $d = 1 + \min\{\frac{\kappa}{8}, 1\}$. It was conjectured in [8] that the "natural length" of an SLE path might be given by the d-dimensional Minkowski content defined as follows. Let

$$\mathrm{Cont}_d(\gamma_t; r) = e^{r(2-d)} \mathrm{Area}\{z : \mathrm{dist}(z, \gamma_t) \le e^{-r}\}.$$

Then the d-dimensional content is

$$\mathrm{Cont}_d(\gamma_t) = \lim_{r\to\infty} \mathrm{Cont}_d(\gamma_t; r),$$

provided that the limit exists. If $\kappa \ge 8$, then $d = 2$, and the two-dimensional Minkowski content is the same as the area and the limit clearly exists. If $\kappa < 8$, it is not at all obvious that the limit exists and is positive for $t > 0$. The main goal of this paper is to prove this.

Before stating the theorem, we will set some notational conventions for this paper. Let $0 < \kappa < 8$ and let

$$a = \frac{2}{\kappa} \in (1/4, \infty), \qquad d = 1 + \frac{\kappa}{8} = 1 + \frac{1}{4a} \in (1, 2).$$

Recall that $\gamma_t = \gamma(0, t]$ and we write $\gamma = \gamma_\infty = \gamma(0, \infty)$ for the entire path of the curve. The Green's function for $\kappa < 8$ is defined by

$$G(z) = \lim_{r\to\infty} e^{r(2-d)} \mathbb{P}\{\mathrm{dist}(z, \gamma) \le e^{-r}\}.$$

This limit exists (see Section 2.3) and there exists $c = c_\kappa$ such that

$$G(z) = c[\Im z]^{d-2}[\sin \arg z]^{4a-1}.$$

Our definition of the Green's function differs by a multiplicative constant from that in other papers. If $F : D \to \mathbb{H}$ is a conformal transformation with $F(w_1) = 0$, $F(w_2) = \infty$, we define

$$G_D(z; w_1, w_2) = |F'(z)|^{2-d} G(F(z)).$$

There is also a two-point Green's function (see Section 2.4)

$$G(z, w) = \lim_{r\to\infty} e^{2r(2-d)} \mathbb{P}\{\operatorname{dist}(z, \gamma) \le e^{-r}, \operatorname{dist}(w, \gamma) \le e^{-r}\}.$$

If $D \subset \mathbb{H}$, let

$$G(D) = \int_D G(z)\, dA(z), \qquad G^2(D) = \int_D \int_D G(z, w)\, dA(z)\, dA(w),$$

where dA denotes integration with respect to area. We call $\gamma(t)$ a *double point* for the SLE_κ path if there exists $s < t$ such that $\gamma(t) \in \partial H_s$. If $0 < \kappa \le 4$, the SLE path has no double points while they exist for $4 < \kappa < 8$.

THEOREM 1.1. *If $0 < \kappa < 8$ there exists $\beta > 0$, such that if $\gamma(t)$ is an SLE_κ curve from 0 to ∞ in $\mathbb{H}$ parameterized by capacity, then with probability one, the following holds*:

- *For every $t > 0$, the Minkowski content*

$$\Theta_t = \operatorname{Cont}_d(\gamma_t) = \lim_{r\to\infty} \operatorname{Cont}_d(\gamma_t; r),$$

  exists.
- *The function $t \mapsto \Theta_t$ is strictly increasing and if $s < t$,*

$$\Theta_t - \Theta_s = \operatorname{Cont}_d(\gamma[s, t]) = \operatorname{Cont}_d(\gamma(s, t] \cap H_s).$$

- *On every bounded interval $[0, t_0]$, Θ_t is Hölder continuous of order β.*

Moreover, if $D \subset \mathbb{H}$ is a bounded domain with piecewise smooth boundary, then

$$\mathbb{E}[\operatorname{Cont}_d(\gamma \cap D)] = G(D),$$
$$\mathbb{E}[\operatorname{Cont}_d(\gamma \cap D)^2] = G^2(D),$$

and if $t > 0$,

$$\mathbb{E}[\operatorname{Cont}_d(\gamma \cap D)|\gamma_t] = \operatorname{Cont}_d(\gamma_t \cap D) + \int_D G_{H_t}(z; \gamma(t), \infty)\, dA(z).$$

The proof will show that we can choose any $\beta < \alpha_0 d/2$ where

$$\beta < \frac{d}{2}\min\left\{1 - \frac{\kappa}{24 + 2\kappa - 8\sqrt{8+\kappa}}, \frac{1}{2}\right\} > 0.$$

The theorem allows us to define SLE_κ with the *natural parameterization* by letting

$$\tilde{\gamma}(t) = \gamma(\sigma_t), \qquad \sigma_t = \inf\{s : \Theta_s = t\}.$$

Under this parameterization with probability one for all t,

$$\mathrm{Cont}_d(\tilde{\gamma}_t) = t.$$

If $F : \mathbb{H} \to D$ with $F(0) = w_1$, $F(\infty) = w_2$ is a conformal transformation, then as in [4] the natural parameterization in D can be defined by saying that the time to traverse $F(\tilde{\gamma}[s,t])$ is

$$\int_s^t \left|F'\big(\tilde{\gamma}(r)\big)\right|^d dr. \tag{1}$$

If $\tilde{\gamma}[s,t] \subset \mathbb{H}$, we can see that this is the same as $\mathrm{Cont}_d[F \circ \tilde{\gamma}[s,t]]$. We expect this to be true for all nice D. The only question is the intersection of the curve with the boundary for $4 < \kappa < 8$ with D having a nonsmooth boundary, perhaps of large dimension.

As an example, let D be the unit disk $\mathbb{D}$ and let $w_1 = 1$, $w_2 = -1$. In this case, the map $F : \mathbb{H} \to \mathbb{D}$ extends analytically to $\mathbb{R}$ and there is no problem establishing that (1) equals $\mathrm{Cont}_d[F \circ \tilde{\gamma}[s,t]]$. Let $\gamma(t)$ be the SLE_κ path in $\mathbb{H}$ with the capacity parameterization, and let $\eta(t) = F(\tilde{\gamma}(t))$ which is an SLE_κ curve from 1 to -1 in $\mathbb{D}$. Let $\Theta_t = \mathrm{Cont}_d[\eta_t]$. In this case, Θ_∞ is an integrable random variable with

$$\mathbb{E}[\Theta_\infty] = \int_{\mathbb{D}} G_{\mathbb{D}}(z; 1, -1)\, dA(z) < \infty.$$

Moreover,

$$\mathbb{E}[\Theta_\infty | \eta_t] = \Theta_t + \Psi_t,$$

where

$$\Psi_t = \int_{D_t} G_{D_t}\big(z; \eta(t), -1\big)\, dA(z).$$

Since $M_t := \mathbb{E}[\Theta_\infty|\eta_t]$ is a martingale, we can see that Θ_t is the unique increasing process such that $\Psi_t + \Theta_t$ is a martingale. This is a Doob–Meyer decomposition.

In [8], the natural parameterization was *defined* to be the unique process Θ_t which makes $\Psi_t + \Theta_t$ a martingale. While this is a simple definition, it requires moment bounds in order to make sure that the process exists (uniqueness is easy). Indeed, it is not hard to see that $M_t(z) := G_{D_t}(z; \eta(t), -1)$ is a local martingale, and hence Ψ_t is an integral of positive local martingales. If Ψ_t were also a local martingale, then no nontrivial Θ_t could exist.

- In [8], it was shown that for $\kappa < 5.0, \ldots$, the process Θ_t exists in $\mathbb{H}$ (the definition has to be modified slightly in $\mathbb{H}$ because Ψ_0 as we have defined it above is infinite—this is not very difficult). The necessary second moment bounds were obtained using the reverse Loewner flow. It was shown that for this range of κ, there exists $\alpha_0 = \alpha_0(\kappa) > 0$ such that the function $t \mapsto \Theta_t$ is Hölder continuous of order α for $\alpha < \alpha_0$.
- In [10], the natural parameterization was shown to exist for all $\kappa < 8$. There the necessary two-point estimates were obtained from estimates on the two-point Green's function [2, 9]. However, the estimates were not strong enough to determine Hölder continuity of the function Θ_t.
- In [4], a new proof was given for all $\kappa < 8$ combining ideas in [8, 10] with known results about the Hölder continuity of the Schramm–Loewner evolution (with respect to the capacity parameterization). This established continuity and Hölder continuity of the natural parameterization for all κ.

Let us discuss some conclusions that we can derive. If $\Theta_t = \operatorname{Cont}_d(\gamma_t)$, then clearly Θ_t is increasing and measurable with respect to γ_t. The conditional distribution of $\operatorname{Cont}_d[\gamma(t, \infty)]$ given γ_t is the same as the distribution of the Minkowski content for SLE from $\gamma(t)$ to -1 in D_t. In particular, using the fact that $\Theta_t - \Theta_s = \operatorname{Cont}_d(\gamma(s, t] \cap D_s)$, we have

$$\mathbb{E}[\Theta_\infty - \Theta_t | \gamma_t] = \int_{D_t} \widehat{G}_t(z)\, dA(z),$$

where $\widehat{G}_t(z) = G_{D_t}(z; \gamma(t), -1)$. Therefore,

$$\Theta_t + \int_{D_t} \widehat{G}_t(z)\, dA(z)$$

is a martingale. Uniqueness of the Doob–Meyer decomposition shows that our Θ_t must be the same as the natural parameterization as discussed in [4, 8, 10]. Using the Minkowski content as the definition, we immediately get independence of domain as well as reversibility of the natural parameterization, that is, the time to traverse $\gamma[s, t]$ is the same as the time to traverse the path in the reverse direction. By independence of domain, we mean that if γ is an SLE_κ curve in $\mathbb{H}$ and $D \subset \mathbb{H}$ with $\gamma(0, \infty) \subset D$, then the natural parameterization for γ considered as an SLE curve in D is the same as that for the SLE curve in $\mathbb{H}$. While this is clearly a property that we would expect from a "natural" parameterization, it is not at all obvious using the definition in [8].

Another possible candidate for the "natural length" of an SLE curve might be the d-dimensional Hausdorff measure. However, it has been proved [12] that this is zero with probability one. It is unknown whether one can find a Hausdorff measure with a different gauge function which would give a nontrivial quantity.

1.1. *Outline of the paper.* Section 2 sets notation for the paper and reviews previous work. We define the Minkowski content in Section 2.2 and derive some simple properties. The Green's function for chordal SLE_κ is reviewed in Section 2.3. This is a normalized limit of the probability of getting near a point z. We also discuss estimates in [9] concerning the probability that an SLE_κ path gets close to two points. In the following subsection, we discuss some of the ideas used to prove two-point estimates; in particular, some precise formulations are made of the rough statement "after an SLE curve gets close to z it is unlikely to get close again." This section uses ideas from [6, 9].

The proof of the main result is in the remainder of the paper. Before going into specifics, let us outline the basic idea of the proof. For ease, let us fix a square, say $\Gamma = [0, 1) + i[1, 2)$ and consider $\gamma \cap \Gamma$. For each $z \in \Gamma$ and $r > 0$, let $\tau_r(z) = \inf\{t : |\gamma(t) - z| \le e^{-r}\}$ and let $J_r(z)$ be $e^{r(2-d)}$ times the indicator function of the event $\{\tau_r(z) < \infty\}$. Let $T_r(z)$ be the first time that the conformal radius of z in $\mathbb{H} \setminus \gamma(0, t]$ equals e^{-r+2}. The Koebe (1/4)-theorem implies that $T_r(z) < \tau_r(z)$. By comparison with "two-sided radial" (SLE conditioned to go through z), one can show that there exists c_1 such that $\mathbb{P}\{T_r(z) < \infty\} \sim c_1 G(z) e^{r(d-2)}$. If r is large, and we view the path $\gamma[0, T_r(z)]$ near z, then locally it appears like a path with the distribution of two-sided radial SLE. Using this, one can see that

$$\mathbb{P}\{J_r(z) > 0 | T_r(z) < \infty\} = \rho + o(1), \qquad r \to \infty,$$

where ρ is independent of z, and using this in turn, we get a one-point estimate

$$\mathbb{E}[J_r(z)] = c_1 \rho G(z) + o(1). \tag{2}$$

If we fix $\delta > 0$, we can see similarly that there exists ρ' such that

$$\mathbb{P}\{J_{r+\delta}(z) > 0 | T_r(z) < \infty\} = \rho' + o(1), \qquad r \to \infty,$$

and by using (2), we see that $\rho' = e^{\delta(d-2)}\rho$. In other words, $\mathbb{E}[J_{r+\delta}(z) - J_r(z)] = o(1)$. The conditional distribution of $J_{r+\delta}(z) - J_r(z)$ given $\gamma[0, T_r(z)]$ is determined (up to a small error) by the way the curve γ looks near $\gamma(T_r(z))$, and this latter distribution is understood through two-sided radial SLE_κ. If z, w are not very close together and the SLE curve gets close to both z and w, we might hope (and, indeed, this is what we show) that the local behavior of γ near $\gamma(T_r(z))$ and near $\gamma(T_r(w))$ are almost independent. The upshot of this is that if we consider the random variable

$$Y_r = \int_\Gamma [J_{r+\delta}(z) - J_r(z)]\, dA(z),$$

then $\mathbb{E}[Y_r^2]$ is small. We show that $\mathbb{E}[Y_r^2] \le c e^{-\beta r}$, from which we conclude that

$$\lim_{r\to\infty} \int_\Gamma J_r(z)\, dA(z)$$

exists as a limit in L^2 and with probability one.

This outline is carried out in Section 3.1 assuming a moment bound, Theorem 3.2 which is proved later. This establishes that with probability one $\mathrm{Cont}_d[\gamma \cap \Gamma]$ exists for every dyadic square Γ. Section 3.2 uses this to prove the statements in Theorem 1.1, again leaving one fact for the last section.

The main estimates are proved in the final section. Section 4.2 analyzes the one-point estimate, that is, the estimate for getting close to a single point z. See Theorem 4.2. A key to the two-point estimate is to understand the one-point estimate very well. For ease, we consider SLE in the disk between boundary points and choose the origin to be the target point. Two-sided radial, which is an example of what are sometimes called $\mathrm{SLE}(\kappa, \rho)$ processes, describes chordal SLE "conditioned to go through z." It can be analyzed by a one-dimensional SDE. We use this to study SLE conditioned to get near z. To do the two-point estimate, we start in Section 2.4 by reviewing the basic idea that after one gets close to a point, one tends not to return to it. This statement requires care to make precise. See Lemmas 2.5 and 2.6. In the final section, we complete the proof giving a rigorous version of the rough outline above.

2. Preliminaries.

2.1. *Notations and distortion.* We fix $\kappa < 8$ and allow all constants to depend implicitly on κ. Recall that $a = 2/\kappa$ and $d = 1 + \frac{\kappa}{8}$. If γ is an SLE_κ curve from w_1 to w_2 in a simply connected domain D, we write $\gamma_t = \gamma(0, t] = \{\gamma(s) : 0 < s \le t\}$.

If n, j, k are integers, we write $\Gamma_n(j, k)$ for the dyadic square

$$\Gamma_n(j,k) = \big[j2^{-n}, (j+1)2^{-n}\big) \times i\big[k2^{-n}, (k+1)2^{-n}\big).$$

Let

$$\mathcal{Q}_n = \big\{\Gamma_n(j,k) : j \in \mathbb{Z}, k \ge 0\big\}, \qquad \mathcal{Q}_n^+ = \big\{\Gamma_n(j,k) \in \mathcal{Q}_n : k > 0\big\},$$
$$\mathcal{Q} = \bigcup_{n \in \mathbb{Z}} \mathcal{Q}_n, \qquad \mathcal{Q}^+ = \bigcup_{n \in \mathbb{Z}} \mathcal{Q}_n^+.$$

We will need the following simple distortion estimate.

LEMMA 2.1. *There exists $\delta > 0$ such that if $f : \mathbb{D} \to f(\mathbb{D})$ is a conformal transformation with $f(0) = 0$, $|f'(0)| = \lambda$ and $|z| \le \delta$*

$$|z| \exp\{-4|z|\} \le \big|f^{-1}(\lambda z)\big| \le |z| \exp\{4|z|\}.$$

PROOF. By scaling, we may assume that $f'(0) = 1$. The growth theorem (see, e.g., [5], Theorem 3.23), states that for all $|z| < 1$,

$$\frac{|z|}{(1+|z|)^2} \le \big|f(z)\big| \le \frac{|z|}{(1-|z|)^2}.$$

Since $(1 \pm |z|)^{-2} = 1 \mp 2|z| + O(|z|^2)$ and $\exp\{\pm 4|z|\} = 1 \pm 4|z| + O(|z|^2)$, we get the lemma. $\square$

2.2. *Minkowski content*. The d-dimensional Minkowski content is one way to "measure" the size of a d-dimensional fractal. We use the quotes because the content is not technically a measure. Its definition is in some ways more natural than d-dimensional Hausdorff measure; however, it has the disadvantage that it is defined in terms of a limit that does not always exist. We will restrict our consideration to $1 < d < 2$ and $V \subset \mathbb{C}$.

Let

$$\begin{aligned}\mathrm{Cont}_d(V; r) &= e^{r(2-d)} \,\mathrm{Area}\{z : \mathrm{dist}(z, V) \le e^{-r}\} \\ &= e^{r(2-d)} \int_{\mathbb{C}} 1\{\mathrm{dist}(z, V) \le e^{-r}\}\, dA(z).\end{aligned}$$

Here, and throughout this paper, dA denotes integration with respect to two-dimensional Lebesgue measure. The *upper and lower d-dimensional Minkowski contents* are defined by

$$\mathrm{Cont}_d^+(V; r) = \sup_{s \ge r} \mathrm{Cont}_d(V; s), \qquad \mathrm{Cont}_d^+(V) = \lim_{r\to\infty} \mathrm{Cont}_d^+(V; r),$$

$$\mathrm{Cont}_d^-(V; r) = \inf_{s \ge r} \mathrm{Cont}_d(V; s), \qquad \mathrm{Cont}_d^-(V) = \lim_{r\to\infty} \mathrm{Cont}_d^-(V; r).$$

The *d-dimensional Minkowski content* is defined if $\mathrm{Cont}_d^+(V) = \mathrm{Cont}_d^-(V)$ in which case

$$\mathrm{Cont}_d(V) = \lim_{r\to\infty} \mathrm{Cont}_d(V; r).$$

The following simple lemma lists the basic properties of Minkowski content that we will use.

LEMMA 2.2.

- *If* $\mathrm{Cont}_d(V), \mathrm{Cont}_d(V')$ *exist and* $\mathrm{dist}(V, V') > 0$, *then* $\mathrm{Cont}_d(V \cup V')$ *exists and*

$$\mathrm{Cont}_d(V \cup V') = \mathrm{Cont}_d(V) + \mathrm{Cont}_d(V').$$

- *If* $\mathrm{Cont}_d(V)$ *exists*, *then*

$$\mathrm{Cont}_d(V) \le \mathrm{Cont}_d^+(V \cup V') \le \mathrm{Cont}_d(V) + \mathrm{Cont}_d^+(V').$$

- *If $d > 1$ and D is a bounded domain whose boundary is a piecewise analytic curve*, *then* $\mathrm{Cont}_d(D) = 0$. *If* $V \subset \overline{D}$, *then*

$$\mathrm{Cont}_d(V) = \lim_{r\to\infty} e^{r(2-d)} \int_D 1\{\mathrm{dist}(z, V) \le e^{-r}\}\, dA(z), \tag{3}$$

provided that either side exists.

- *Suppose* $V_1, V_2, \ldots$ *are bounded sets for which* $\mathrm{Cont}_d(V_n)$ *is well defined. Let* V *be a bounded set such that*

$$\lim_{n\to\infty}\big[\mathrm{Cont}_d^+(V\setminus V_n)+\mathrm{Cont}_d^+(V_n\setminus V)\big]=0.$$

Then $\mathrm{Cont}_d(V)$ *exists and*

$$\mathrm{Cont}_d(V)=\lim_{n\to\infty}\mathrm{Cont}_d(V_n). \tag{4}$$

PROOF. We leave this to the reader. The last conclusion uses

$$\begin{aligned}\mathrm{Cont}_d(V_n;r)-\mathrm{Cont}_d(V_n\setminus V;r)&\le \mathrm{Cont}_d(V;r)\\ &\le \mathrm{Cont}_d(V_n;r)+\mathrm{Cont}_d(V\setminus V_n;r).\end{aligned}$$ □

2.3. *Green's function.* The Green's function for chordal SLE_κ is the normalized probability that the path gets near a point z. By nature, it is defined up to a multiplicative constant and we choose the constant in a way that will be convenient for us. The precise definition uses the following theorem. If D is a simply connected domain and $z\in D$ we let $\mathrm{crad}_D(z)$ denote the conformal radius of z in D, that is, if $f:\mathbb{D}\to D$ is a conformal transformation with $f(0)=z$, then $\mathrm{crad}_D(z)=|f'(0)|$.

THEOREM 2.3. *For every* $\kappa<8$, *there exists* $c'=c'(\kappa),\hat c=\hat c(\kappa),\alpha<\infty$ *such that if* $w=e^{2i\theta}\in\partial\mathbb{D}$ *and* γ *is a chordal* SLE_κ *path from* 1 *to* w *in* $\mathbb{D}$, *then*

$$\begin{aligned}\mathbb{P}\{\mathrm{crad}_A(0)\le e^{-r}\}&=c'[\sin\theta]^{4a-1}e^{r(d-2)}\big[1+O(e^{-\alpha r})\big],\\ \mathbb{P}\{\mathrm{dist}(0,\partial A)\le e^{-r}\}&=\hat c[\sin\theta]^{4a-1}e^{r(d-2)}\big[1+O(e^{-\alpha r})\big].\end{aligned}\tag{5}$$

Here, A *denotes the connected component of* $\mathbb{D}\setminus\gamma$ *containing the origin.*

PROOF. For the first expression see, for example, [9]. The proof gives an explicit form for c' but we will not need it. The second was proved in [4], but we reprove it here in Theorem 4.2. This proof does not give an explicit expression for the constant $\hat c$. □

To be precise, let $\mathbb{P}_\theta$ denote the probability distribution on paths $\gamma=\gamma[0,\infty)$ given by chordal SLE_κ from 1 to $e^{2i\theta}$ in $\mathbb{D}$. Then there exist $c',\hat c,\alpha,c$, depending only on κ, such that for all θ and all $r\ge 1/2$,

$$\begin{aligned}\big|e^{r(2-d)}[\sin\theta]^{1-4a}\mathbb{P}_\theta\{\mathrm{crad}_A(0)\le e^{-r}\}-c'\big|&\le ce^{-\alpha r},\\ \big|e^{r(2-d)}[\sin\theta]^{1-4a}\mathbb{P}_\theta\{\mathrm{dist}(0,\gamma)\le e^{-r}\}-\hat c\big|&\le ce^{-\alpha r}.\end{aligned}$$

From the previously proven (5) and the Koebe (1/4)-theorem, we can easily deduce the following estimate which we will use before deriving Theorem 4.2.

- If γ is an SLE_κ path from 0 to ∞ in $\mathbb{H}$, $\Im(z) \geq 1$ and $r \geq 0$,

$$\mathbb{P}\{\mathrm{dist}(z, \gamma) \leq e^{-r}\} \asymp [\Im(z)]^{d-2}[\sin \arg z]^{4a-1} e^{r(d-2)}. \tag{6}$$

We also use the following estimate, see [1].

- If $x > 0$,

$$\mathbb{P}\{\mathrm{dist}(x, \gamma) \leq e^{-r}\} \leq c[e^{-r}/x]^{4a-1}. \tag{7}$$

If w_1, w_2 are distinct boundary points of a simply connected domain D, let $S_D(z; w_1, w_2)$ denote the sine of the argument of z with respect to w_1, w_2, that is, if $F : D \to \mathbb{H}$ is a conformal transformation with $F(w_1) = 0$, $F(w_2) = \infty$, then $S_D(z; w_1, w_2) = \sin[\arg F(z)]$. Note that $S_D(z; w_1, w_2)$ is a conformal invariant and $\mathrm{crad}_D(z)$ is conformally covariant, $\mathrm{crad}_{f(D)}(f(z)) = |f'(z)| \,\mathrm{crad}_D(z)$. The *chordal Green's function* is defined by

$$G_D(z; w_1, w_2) = \hat{c}\, \mathrm{crad}_D(z)^{d-2} S_D(z; w_1, w_2)^{4a-1}. \tag{8}$$

Here, we choose the constant $\hat{c}$ from Theorem 3.1; our definition differs from the definition elsewhere (e.g., in [9]) by a multiplicative constant. Previously it was defined so that $G_{\mathbb{H}}(z; 0, \infty) = \Im(z)^{d-2}[\sin \arg z]^{4a-1} = [\mathrm{crad}_{\mathbb{H}}(z)/2]^{d-2} \times S_{\mathbb{H}}(z; 0, \infty)^{4a-1}$. The Green's function satisfies the conformal covariance rule

$$G_D(z; w_1, w_2) = |f'(z)|^{2-d} G_{f(D)}(f(z); f(w_1), f(w_2)).$$

We choose the definition (8) so that we do not need to keep writing the constant $\hat{c}$. Theorem 2.3 extends immediately to other simply connected domains by conformal invariance of SLE.

THEOREM 2.4. *If $\kappa < 8$, γ is a chordal* SLE_κ *path from w_1 to w_2 in a simply connected domain D, then for $z \in D$ with* $\mathrm{dist}(z, \partial D) \geq 2e^{-r}$,

$$\mathbb{P}\{\mathrm{dist}(z, \gamma) \leq e^{-r}\} = G_D(z; w_1, w_2) e^{r(d-2)}[1 + O(e^{-\alpha r})],$$

for some $\alpha > 0$ which depends only on κ.

Most of our computations will be in the upper half plane or in the disk. For notational ease, we will write

$$G(z) = G_{\mathbb{H}}(z; 0, \infty), \qquad G(z; \theta) = G_{\mathbb{D}}(z; 1, e^{2\theta i}).$$

If $V \subset \mathbb{H}$, we define

$$G(V) = \int_V G(z)\, dA(z).$$

Note that if $\Gamma \in \mathcal{Q}_n$ and z is the center point of Γ, then

$$G(\Gamma) \asymp 2^{-2n} G(z).$$

[If $\Gamma = \Gamma_n(j, 0)$, this requires a simple estimate of an integral.]

2.4. *Two-point estimates*. A basic principle in proving two-point estimates for SLE is the idea that if a path gets very close to a point z and then gets away from z, then it is unlikely to get even closer to z. While this is the heuristic, as just stated the principle is not always valid. Since this idea is important in several of our proofs, we will spend some time to formulate and prove a precise version. We are expanding on ideas in [4, 6]. Let γ be an SLE_κ curve from 0 to ∞ in $\mathbb{H}$. As before, if $z \in \mathbb{H}$, let

$$\tau_r(z) = \inf\{t : |\gamma(t) - z| = e^{-r}\}.$$

If $\tau_r = \tau_r(z) < \infty$, let $H = H_{\tau_r}$ denote the unbounded component of $\mathbb{H} \setminus \gamma_{\tau_r}$, let $\mathcal{B}_u = \mathcal{B}_u(r, z)$ denote the disk of radius e^{-ur} containing z, and let $\mathcal{B} = \mathcal{B}_1$ denote the disk of radius e^{-r} about z. Let $V_u = V_u(r, z)$ denote the connected component of $\mathcal{B}_u \cap H$ containing z. If $u \leq 1$, the intersection of ∂V_u with H is a disjoint union of open arcs in $\partial \mathcal{B}_u$ each of whose endpoints is in ∂H. There is a unique such arc l, that we denote by $l_u = l_u(r, z)$, such that z is contained in the bounded component of $H \setminus l$. Simple connectedness of H is used to see that this arc is unique. However, one may note the following facts:

- The bounded component of $H \setminus l_u$ does not need to be contained in $\mathcal{B}_u$. Indeed, we have no universal bound on the diameter of the bounded component.
- There may be other subarcs l of $\partial \mathcal{B}_u \cap H$ such that z is in the bounded component of $H \setminus l$. However, these arcs are not on ∂V_u.

For $0 < u \leq 1$, let

$$\sigma = \sigma_u(r, z) = \inf\{t \geq \tau_r : \gamma(t) \in \overline{l_u}\}.$$

Then a correct, although still imprecise, version of our heuristic principle is: if $\tau_r < \infty$, then after time σ the path is unlikely to get closer to z. We will now be more precise. Note that for fixed z, u, r, with probability one $\gamma(\sigma) \in l_u$. In this case (which we now assume), $l_u \setminus \{\gamma(\sigma)\}$ consists of two crosscuts of H_σ that we denote by l_u^* and l_u^{**}. If $z \in H_\sigma$, which is always true if $\kappa \leq 4$, we let l_u^* be the crosscut such that z lies in the bounded component of $H_\sigma \setminus l_u^*$. If $\tau_r < \infty$, define $\lambda = \lambda(r, z, u) \geq 1$ by

$$\operatorname{dist}[z, \gamma_\sigma] = e^{-\lambda r}.$$

Let l_λ^* denote the connected subarc of $\partial \mathcal{B}_\lambda \cap H_\sigma$ that separates z from infinity. (If the intersection of γ_σ with $\mathcal{B}_\lambda$ is a single point, which we expect to be the case with probability one, then l_λ^* is a circle with a single point deleted.) See Figure 1 for a figure showing illustrating these quantities.

Since we will use it in several proofs, we recall that if D is a domain and η^1, η^2 are disjoint subarcs of ∂D, then the (Brownian) *excursion measure* between η^1, η^2 is given by

$$\mathcal{E}_D(\eta^1, \eta^2) = \int_{\eta^1} \partial_n \phi(z) |dz|,$$

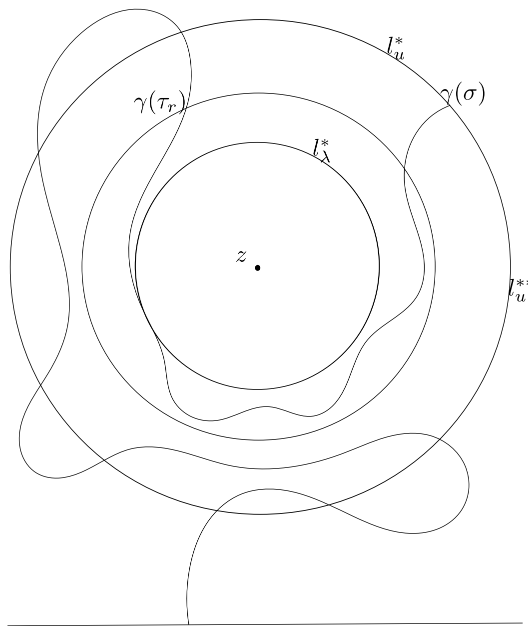

FIG. 1. *Quantities in Section 2.4.*

where ∂_n denotes the inward normal derivative and $\phi = \phi_{D,\eta^2}$ is the harmonic function on D with boundary value 1_{η^2}. The above expression assumes that η^1 is smooth, but one can check that $\mathcal{E}_D(\eta^1, \eta^2)$ is a conformal invariant and hence can be defined for all domains. Also, $\mathcal{E}_D(\eta^1, \eta^2) = \mathcal{E}_D(\eta^2, \eta^1)$. See [5], Chapter 5, for more details. When estimating excursion measures, we will use the following estimate that follows from the strong Markov property. Suppose η is a crosscut of D that separates η^1 from η^2. Then

$$\mathcal{E}_D(\eta^1, \eta^2) \le \mathcal{E}_{D\setminus\eta}(\eta^1, \eta) \sup_{z\in\eta} \phi(z). \tag{9}$$

If D is simply connected, so that (D, η^1, η^2) is a conformal rectangle, and $\mathcal{E}_D(\eta^1, \eta^2) \le 1$, then

$$\mathcal{E}_D(\eta^1, \eta^2) \asymp \max_{z\in D} \phi_1(z)\phi_2(z), \tag{10}$$

where $\phi_j = \phi_{D,\eta^j}$. One can check this by verifying it for a rectangle $[0, L] + i[0, \pi]$ by direct computation and using conformal invariance.

LEMMA 2.5. *There exists c such that for all $0 < u \le 1$,*

$$\mathbb{P}\{\text{dist}[z, \gamma_\infty] < \text{dist}[z, \gamma_\sigma] | \gamma_\sigma\} \le ce^{\alpha(u-\lambda)r},$$

where $\alpha = (4a - 1)/2 > 0$. In particular,

$$\mathbb{P}\{\text{dist}[z, \gamma_\infty] < \text{dist}[z, \gamma_\sigma] | \gamma_\sigma\} \le ce^{\alpha(u-1)r}. \tag{11}$$

PROOF. Let $g : H_\sigma \to \mathbb{H}$ be a conformal transformation with $g(\gamma(\sigma)) = 0$, $g(\infty) = \infty$. The image $\eta = g \circ l_u^*$ is a crosscut of $\mathbb{H}$ with one endpoint on the origin and one on the real line which without loss of generality we will assume is on the positive real line. The curve $\eta' = g \circ l_\lambda^*$ is a crosscut of $\mathbb{H}$ contained in the bounded component of $\mathbb{H} \setminus \eta$ with positive endpoints $x_1 \le x_2$. Let us consider the conformal rectangle given by the component of $\mathbb{H} \setminus (\eta \cup \eta')$ that contains both η and η' on its boundary and with η, η' as two of the boundary arcs of the rectangle. The excursion measure between η and η' in this rectangle is the same as the excursion measure between l_u^* and l_λ^* for the corresponding rectangle in $H_\sigma \setminus (l_u^* \cup l_\lambda^*)$. The Beurling estimate (see, e.g., [5], Theorem 3.76) implies that the latter is bounded above by $ce^{-(\lambda-u)r/2}$. Since η separates η' from the negative real line, we see that the excursion measure between η' and $(-\infty, 0]$ in the unbounded component of $\mathbb{H} \setminus \eta'$ is bounded above by $ce^{-(\lambda-u)r/2}$. By standard estimates of the Poisson kernel in $\mathbb{H}$, this shows that $\operatorname{diam}(\eta') \le ce^{-(\lambda-u)r/2} x_1$, and hence by (7), the probability that an SLE path hits it is $O(e^{-(\lambda-u)(4a-1)r/2})$. □

The next lemma strengthens (11) for $\kappa \le 4$. We do not know if it is true for $4 < \kappa < 8$. Let $\mathcal{B} = \mathcal{B}_1$ denote the disk of radius e^{-r} about z.

LEMMA 2.6. *If $\kappa \le 4$, there exists c such that if $0 < u \le 1$,*

$$\mathbb{P}\{\gamma[\sigma, \infty) \cap \mathcal{B} \neq \varnothing | \gamma_\sigma\} \le ce^{\alpha(u-1)r}, \tag{12}$$

where $\alpha = (4a - 1)/2 > 0$.

PROOF. Let V denote the unbounded component of $H_\sigma \setminus \overline{\mathcal{B}}$ and note that $l_u^*, l_u^{**} \subset V$. Let $L = \partial V \cap H_\sigma \cap \partial \mathcal{B}$ which is a disjoint (finite or countable) union of open subarcs of $\partial \mathcal{B}$, which we denote by $L_1, L_2, \ldots$. For each arc L_j, either l_u^* or l_u^{**} disconnects L_j from infinity in H_σ, that is, L_j is in the bounded component of $H_\sigma \setminus l_u^*$ or the bounded component of $H_\sigma \setminus l_u^{**}$. Write $L = L^1 \cup L^2$ where L^1, L^2 are the unions of L_j over the subarcs of the first and second type, respectively. The probability on the left-hand side of (12) is the probability that $\gamma[\sigma, \infty) \cap L \neq \varnothing$. Hence, it suffices to show that

$$\sum_{j=1}^{\infty} \mathbb{P}\{\gamma[\sigma, \infty) \cap L_j \neq \varnothing | \gamma_\sigma\} \le ce^{\alpha(u-1)r}.$$

We will give this bound for the sum over L_j of the first type; the sum over the second type is done similarly.

Let $\mathcal{R}$ denote the bounded component of $H_\sigma \setminus l_u^*$ which includes the L_j of the first type. Using the Beurling estimate, we can see that the excursion measure between l_u^* and L^1 in $\mathcal{R} \setminus L^1$, $\mathcal{E}_{\mathcal{R} \setminus L^1}(l_u^*, L^1)$, is $O(e^{(u-1)r/2})$. We claim that a stronger fact is true,

$$\sum_j \mathcal{E}_{\mathcal{R} \setminus L_j}(l_u^*, L_j) \le ce^{(u-1)r/2}, \tag{13}$$

where we are summing over L_j of the first type. Indeed, $\mathcal{E}_{\mathcal{R}\setminus L^1}(l_u^*, L^1)$ is the (integral over l_u^* of the normal derivative of the) probability that a Brownian motion starting at z hits L^1 before leaving $\mathcal{R}$. The sum on the left-hand side of (13) is the (integral over ... of the) *expected number* of crosscuts L^j visited before leaving $\mathcal{R}$. However, using the strong Markov property and simple connectedness, we can see that the probability starting on one of the crosscuts L_j of reaching another before leaving $\mathcal{R}$ is at most $1/2$, and hence the expected number of crosscuts hit given one is hit is at most 2.

As in the previous proof, we use (7) to see that

$$\sum_l \mathbb{P}\{\gamma[\sigma,\infty) \cap L_j \neq \varnothing | \gamma_\sigma\} \leq c \sum_l \mathcal{E}_{\mathcal{R}\setminus L_j}(l_u^*, L_j)^{4a-1}.$$

The argument up to this point has not used the fact that $\kappa \leq 4$. However, if $\kappa \leq 4$, we know that $4a - 1 \geq 1$, and hence (13) gives

$$\sum_l \mathcal{E}_{\mathcal{R}\setminus L_j}(l_u^*, L_j)^{4a-1} \leq \left[\sum_l \mathcal{E}_{\mathcal{R}\setminus L_j}(l_u^*, L_j)\right]^{4a-1} \leq ce^{\alpha(u-1)r}. \qquad \square$$

While we do not know if the last lemma holds for $\kappa > 4$, the next lemma will suffice for our needs.

LEMMA 2.7. *If* $4 < \kappa < 8$, *there exist* $c < \infty, \beta > 0$ *such that if* $\tau_r < \infty$ *and* $0 < u \leq 1$, *then*

$$\mathbb{P}\{\mathcal{B} \cap H_\sigma \neq \varnothing | \gamma_{\tau_r}\} \leq ce^{\beta(u-1)r}.$$

PROOF. Let $\zeta = \gamma(\tau_r)$. Let g be a conformal transformation of H_{τ_r} onto $\mathbb{H}$ with $g(\zeta) = 0$, $g(\infty) = \infty$. Let $\eta = g \circ l_u$, $\eta' = g \circ [\partial\mathcal{B} \setminus \{\zeta\}]$. Then η is a crosscut of $\mathbb{H}$ with one endpoint positive and one endpoint negative, and η' is a simple loop rooted at the origin lying in the bounded component of $\mathbb{H} \setminus \eta$. By choosing a multiple of g if necessary, we may assume that $\max\{|w| : w \in \eta'\} = 1$.

We claim that there exists c' such that $\operatorname{dist}(0, \eta) \geq c' e^{(1-u)r/2}$. To see this, let $\mathcal{R}$ denote the component of $H_\sigma \setminus (\partial\mathcal{B} \cup l_u)$ whose boundary contains both $\partial\mathcal{B}$ and l_u. Then using the Beurling estimate as in the previous lemma, we see that $\mathcal{E}_{\mathcal{R}}(\partial\mathcal{B}, l_u) \leq ce^{-(u-1)/2}$. Therefore,

$$\mathcal{E}_{g(\mathcal{R})}(\eta, \eta') = O(e^{-(u-1)r/2}).$$

But η' is a connected set containing the origin of radius 1. If $v = \operatorname{dist}(0, \eta)$, then by setting $z = i\sqrt{v}$ in (10) we get the bound

$$\mathcal{E}_{g(\mathcal{R})}(\eta, \eta') \geq cv^2.$$

By conformal invariance, $\mathbb{P}\{\mathcal{B} \cap H_\sigma \neq \varnothing | \gamma_{\tau_r}\}$ is bounded above by the probability that an SLE_κ path starting at the origin has not separated the unit circle from infinity before it reaches the circle of radius $c'e^{(1-u)r/2}$. Using scaling and the fact that SLE_κ has double points, it is not hard to show that this is $O(e^{\beta(u-1)r})$ for some β. □

COROLLARY 2.8. *If $\kappa < 8$, there exists $c < \infty$ and $\beta > 0$ such that if $|z| > e^{-r/2}$ and $0 < u \leq 1$, then if $\tau_r < \infty$,*

$$\mathbb{P}\{\gamma[\sigma, \infty) \cap \overline{\mathcal{B}} \neq \varnothing | \gamma_{\tau_r}\} \leq c e^{\beta(u-1)r},$$

$$\mathbb{P}\{\tau_r < \infty, \gamma[\sigma, \infty) \cap \overline{\mathcal{B}} \neq \varnothing\} \leq c G(z) e^{(d-2)r} e^{\beta(u-1)r}.$$

The other estimates we need will deal with upper bounds for the probabilities that the SLE curves gets close to two different points z, w. For the remainder of this section, we assume that γ is an SLE curve from 0 to ∞ in $\mathbb{H}$. If $z \in \overline{\mathbb{H}}$, we let

$$\tau_r(z) = \inf\{t : |\gamma(t) - z| \leq e^{-r}\}.$$

LEMMA 2.9. *There exists $c < \infty$ such that if $|z|, |w| \geq e^{-u}$ and $|z - w| \geq e^{-u}$, then for $0 < s < r$,*

$$\mathbb{P}\{\tau_{s+u}(w) < \infty, \tau_{r+u}(z) < \infty\} \leq c e^{(s+r)(d-2)}, \tag{14}$$

$$\mathbb{P}\{\tau_{s+u}(w) < \tau_{r+u}(z) < \tau_{r+u}(w) < \infty\} \leq c e^{2r(d-2)} e^{-\alpha s}, \tag{15}$$

where $\alpha = (4a - 1)/2$.

PROOF. By scaling, it suffices to prove the lemma for $u = 0$. The first estimate is Theorem 2 in [9]. The second estimate follows from the ideas in [9], Lemma 4.10, but we will redo the proof using some ideas from this section. Throughout this proof, we let r, s, n be integers.

Let $\gamma = \gamma_{\tau_r(z)}$ and let $A = A_{s,r}$ denote the γ-measurable event

$$A = \{\tau_s(w) \leq \tau_r(z) < \tau_{s+1}(w)\}.$$

Let $n \geq s + 1$ and let $E = E_{s,r,n}$ be the event

$$E = \{\tau_s(w) \leq \tau_r(z) \leq \tau_n(w) < \tau_{r+1}(z) < \infty\}.$$

The hard work is to show that on the event A,

$$\mathbb{P}(E|\gamma) \leq c e^{-\alpha(r+s)} e^{(d-2)(n-s)}. \tag{16}$$

The estimate (14) shows that $\mathbb{P}(A_{s,r}) \leq O(e^{(d-2)(r+s)})$ and the one-point estimate (6) shows that $\mathbb{P}\{\tau_n(z) < \infty | A \cap E\} \leq O(e^{(d-2)(n-r)})$. Hence, once we establish (16) we have

$$\mathbb{P}\{\tau_s(w) \leq \tau_r(z) \leq \tau_n(w) < \tau_{r+1}(z) \leq \tau_n(z) < \infty\} \leq c e^{-\alpha(r+s)} e^{2(d-2)n}.$$

If we sum over s, we get

$$\mathbb{P}\{\tau_{r'}(z) \le \tau_n(w) < \tau_{r'+1}(z) \le \tau_n(z) < \infty\} \le ce^{-\alpha r'} e^{2(d-2)n},$$

and if we sum this over $r' \ge r$ we get (15). We will prove (16). If $r+s \le 4$, we can estimate

$$\mathbb{P}(E|\gamma) \le \mathbb{P}\{\tau_n(w) < \infty|\gamma\},$$

and use the one-point estimate; hence we may assume that $r+s \ge 4$. We let s, r with $s+r \ge 4$ and let $\tau = \tau_r(w)$.

Let U^z (resp., U^w) denote the disk of radius $e^{-r/2}$ $[e^{-s/2}]$ centered at z $[w]$. Note that $U^z \cap U^w = \varnothing$. For each $t \ge \tau$, and $\zeta \in \{z, w\}$, let V_t^ζ denote the connected component of $H_t \cap U^\zeta$ that contains ζ. Let η_t^ζ denote the unique crosscut of H_t that is contained in $\partial V_t^\zeta \cap \partial U^\zeta$ and separates z from w in H_t. Let l_t^ζ denote the unique crosscut of H_t contained in the circle of radius $\text{dist}(\zeta, \partial H_t)$ about ζ that separates z from w in H_t. If there is a unique point in ∂H_t at minimal distance from ζ, then l_t^ζ is a circle with one point removed. We will consider four cases. Let $H = H_\tau$, $\eta = \eta_\tau^z$. Let σ be the fist time t greater than or equal to τ such that z lies in the unbounded component of $H_t \setminus \eta_t^z$.

Case 1: Let $F_1 = A \cap \{\sigma = \tau\}$. In this case, η separates w from $\gamma(\tau)$. Using the Beurling estimate, we can see that the excursion measure between η and l_τ^w is $O(e^{-(r+s)/4})$; the latter is a bound for the probability that a Brownian motion starting on l_τ^w reaches η without leaving H. The boundary estimate (7) states that the probability an SLE in H starting at $\gamma(\tau)$ hits l_τ^w is $O(e^{-\alpha(r+s)})$. Therefore, on the event F_1,

$$\mathbb{P}\{\tau_{s+1}(w) < \infty|\gamma\} \le ce^{-\alpha(r+s)},$$

and using the strong Markov property and the one point estimate (6), we see that

$$\mathbb{P}[E \cap F_1|\gamma] \le \mathbb{P}\{\tau_n(w) < \infty|\gamma\} \le ce^{-\alpha(r+s)} e^{(d-2)(n-s)}.$$

Case 2: Let $F_2 = A \cap \{\tau < \sigma < \tau_n(w)\}$. We write

$$F_2 = \bigcup_{j=s}^{n-1} F_{s,j},$$

where

$$F_{2,j} = F_2 \cap \{\sigma_j(w) \le \sigma < \sigma_{j+1}(w)\}.$$

Since the domain H_t is decreasing, in order for z to change from being in the bounded component of $H_{t'} \setminus \eta_s^z$, $s' < t$ to being in the unbounded component of $H_t \setminus \eta_t^z$, the crosscut η_t^z must be different from η_z^s for $s < t$. There are two ways that the crosscut η_t^z can change at time t; either $\gamma(t) \in \eta_{t-}^z$, or $\gamma(t) \notin \eta_{t-}^z$ but η_{t-}^z is not part of the boundary of V_t^z. In the latter case, the crosscut η_{t-}^z still separates

z from infinity and b in H_t. Also the crosscut η_t^z separates z from η_{t-}^z. Hence, in the latter case z is in the bounded component of $H_t \setminus \eta_t^z$.

Therefore, we see that $\gamma(\sigma) \in \eta_{\sigma-}^z$. One endpoint of the crosscut η_σ^z is $\gamma(\sigma)$ and it separates w from infinity. On the event $F_{2,j}$, the excursion measure between l_σ^w and η_σ^z in H_σ is bounded above by $O(e^{-(r+j)/2})$. Therefore, on the event $F_{2,j}$,

$$\mathbb{P}\{\tau_n(w) < \infty | \gamma_\sigma\} \le ce^{-\alpha(r+j)}e^{-(2-d)(n-j)}.$$

The one-point estimate shows that

$$\mathbb{P}[F_{2,j}|\gamma] \le \mathbb{P}\{\tau_j(w) < \infty | \gamma\} \le ce^{-(2-d)(j-s)}.$$

Therefore,

$$\mathbb{P}[E \cap F_{2,j}|\gamma] \le ce^{-\alpha(r+j)}e^{-(2-d)(n-s)},$$

and by summing over $j = s, s+1, \dots, n-1$, we see that

$$\mathbb{P}[E \cap F_2|\gamma] \le ce^{-\alpha(r+s)}e^{-(2-d)(n-s)}.$$

Before proceeding with the next cases, let $\tau' = \tau_n(w)$, $H' = H_{\tau'}$, and note that on $E \setminus (F_1 \cup F_2)$, we know that z is in the bounded component of $H' \setminus \eta_{\tau'}^z$.

Case 3: Let F_3 be the intersection of $A \cap \{\tau' < \tau_{s+1}(z)\}$ with the event that w is contained in the unbounded component of $H' \setminus \eta_{\tau'}^w$. (Note that this is a stronger condition than saying that z is contained in the bounded component of $H' \setminus \eta_{\tau'}^z$.) On the event F_3, the crosscut $\eta_{\tau'}^w$ separates $l_{\tau'}^z$ from $\gamma(\tau')$ in H'. The excursion measure between $l_{\tau'}^z$ and $\eta_{\tau'}^w$ in H' is bounded by $O(e^{-(r+s)/2})$, and using the boundary exponent, we see that on the event F_3,

$$\mathbb{P}\{\tau_{r+1}(z) < \infty | \tau'\} \le ce^{-(r+s)\alpha}.$$

The one-point estimate implies that on A, $\mathbb{P}\{\tau' < \infty | \gamma\} = O(e^{-(2-d)(n-s)})$, and hence

$$\mathbb{P}[E \cap F_3|\gamma] \le ce^{-\alpha(r+s)}e^{-(2-d)(n-s)}.$$

Case 4: Let F_4 be the intersection of $[A \setminus (F_1 \cup F_2)] \cap \{\tau' < \tau_{s+1}(z)\}$ with the event that w is contained in the bounded component of $H' \setminus \eta_{\tau'}^w$. As noted above, on the event F_4, z is in the bounded component of $H' \setminus \eta_{\tau'}^z$. The excursion measure between $l_{\tau'}^z$ and $\eta_{\tau'}^z$ in H' is $O(e^{-r/2})$, and as before this implies that on the event F_4,

$$\mathbb{P}\{\tau_{r+1}(z) < \infty | \gamma'\} \le ce^{-\alpha r},$$

and hence on the event A,

$$\mathbb{P}[E \cap F_4|\gamma] \le ce^{-\alpha r}\mathbb{P}[F_4|\gamma]. \tag{17}$$

Similar to case 2, let ρ be the first time $t \geq \tau_r(z)$ such that w is contained in the bounded component of $H_t \setminus \eta_t^w$, and let

$$F_{4,j} = F_4 \cap \{\tau_j(w) \leq \rho < \tau_{j+1}(w)\}.$$

Note that

$$\mathbb{P}[\tau_j(w) \leq \rho < \tau_{j+1}(w)|\gamma] \leq \mathbb{P}\{\tau_j(w) < \infty|\gamma\} \leq ce^{(j-s)(d-2)}. \tag{18}$$

As before, we can see that the crosscut η_ρ^w separates l_ρ^w from $\gamma(\rho)$ in H_ρ. [Either $\rho = \tau_r(z)$ or $\gamma(\rho)$ is an endpoint of η_ρ^w.] Since the excursion measure between η_ρ^w and l_ρ^w in H_ρ is $O(e^{-(j-(s/2))/2})$,

$$\mathbb{P}[\tau_{j+1}(w) < \infty|\gamma_\rho] \leq ce^{-(j-s)\alpha},$$

and using the one point estimate,

$$\mathbb{P}[\tau' < \infty|\gamma_\rho] \leq ce^{-(s+j)\alpha}e^{(n-j)(d-2)}.$$

Combining this with (18) and summing over $s \leq j \leq n$, we see that

$$\mathbb{P}[F_4|\gamma] \leq ce^{-s\alpha}e^{-(n-s)(2-d)}.$$

Finally, combining this with (17), we see that

$$\mathbb{P}[E \cap F_4|\gamma] \leq ce^{-\alpha r}\mathbb{P}[F_4|\gamma] \leq ce^{-(r+s)\alpha}e^{(n-s)(d-2)}. \qquad \square$$

Given this estimate one also shows that if $\Im(z), \Im(w) \geq 1$ with $|z-w| \leq 1$, then

$$\mathbb{P}\{\tau_r(z) < \infty, \tau_r(w) < \infty\} \leq ce^{2r(d-2)}|z-w|^{d-2}. \tag{19}$$

Indeed, if $\rho = \inf\{t : |\gamma(t) - z| \leq 2|z-w|\}$, then

$$\mathbb{P}\{\rho < \infty\} \leq c|z-w|^{2-d},$$

and by conformal invariance,

$$\mathbb{P}\{\tau_r(z) < \infty, \tau_r(w) < \infty|\rho < \infty\} \leq [e^{-r}/|z-w|]^{2(2-d)}.$$

In [9], it was shown that the limit,

$$\lim_{\varepsilon.\delta\downarrow 0} \varepsilon^{d-2}\delta^{d-2}\mathbb{P}\{\mathrm{crad}_{\mathbb{H}\setminus\gamma}(z_1) \leq \varepsilon, \mathrm{crad}_{\mathbb{H}\setminus\gamma}(z_2) \leq \delta\},$$

exists and defines a two-point Green's function. In Section 4.2, we show how to adapt this argument to show existence of

$$G(z_1, z_2) = \lim_{\varepsilon.\delta\downarrow 0} \varepsilon^{d-2}\delta^{d-2}\mathbb{P}\{\mathrm{dist}(z_1, \gamma) \leq \varepsilon, \mathrm{dist}(z_2, \gamma) \leq \delta\}. \tag{20}$$

In fact, we can write $G(z_1, z_2) = \widehat{G}(z_1, z_2) + \widehat{G}(z_2, z_1)$ where

$$\widehat{G}(z, w) = G(z)\mathbb{E}^*[G_{H_T}(w; z, \infty)],$$

and $\mathbb{E}^*$ denotes expectation with respect to two-sided radial SLE$_\kappa$ from 0 to z stopped at

$$T = \inf\{t : \gamma(t) = z\}.$$

See Section 4.2 for a review of two-sided radial SLE.

3. Existence of Minkowski content.

3.1. *Main theorem.* If $\Gamma \in \mathcal{Q}_n$ as defined in Section 2.1, and γ is an SLE$_\kappa$ curve from 0 to ∞ in $\mathbb{H}$, let

$$Z(\Gamma) = \mathrm{Cont}_d^+(\gamma \cap \Gamma; n \log 2),$$

$$J_r(z) = e^{(2-d)r} 1\{\tau_r(z) < \infty\}, \qquad J_r(V) = \int_V J_r(z)\, dA(z).$$

Note that if $s > 0$, then $J_{r+s}(z) \le e^{s(2-d)} J_r(z)$.

THEOREM 3.1. *Suppose $\kappa < 8$ and γ is an* SLE$_\kappa$ *curve from* 0 *to* ∞ *in* $\mathbb{H}$. *Then the following holds for all* $\Gamma \in \mathcal{Q}^+$.

- *The limit*

$$\mu(\Gamma) := \lim_{r \to \infty} J_r(\Gamma)$$

exists with probability one and in L^2.
- *With probability one*,

$$\mathrm{Cont}_d(\gamma \cap \Gamma) = \mu(\Gamma). \tag{21}$$

- *Let* $\partial_n \Gamma = \{z \in \mathbb{H} : \mathrm{dist}(z, \partial\Gamma) \le 2^{-n}\}$. *Then with probability one*,

$$\lim_{n \to \infty} \mathrm{Cont}_d(\gamma \cap \partial_n \Gamma; n \log 2) = 0. \tag{22}$$

- *The following moment relations holds*:

$$\mathbb{E}[\mu(\Gamma)] = \int_\Gamma G(z)\, dA(z), \tag{23}$$

$$\mathbb{E}[\mu(\Gamma)^2] = \int_\Gamma \int_\Gamma G(z, w)\, dA(z)\, dA(w), \tag{24}$$

$$\mathbb{E}[Z(\Gamma)^2] < \infty. \tag{25}$$

Since $\mathcal{Q}^+$ is countable, all the with probability one statements can be restated as "with probability one, for all $\Gamma \in \mathcal{Q}^+, \ldots$." The bulk of the work in proving the theorem is to prove Theorem 3.2 below. Let $0 < \delta < 1/10$. Since we want to take a limit of J_r as $r \to \infty$, we will look at

$$\begin{aligned} Q_r^\delta(z) &= J_r(z) - J_{r+\delta}(z) \\ &= e^{r(2-d)}\big[1\{\tau_r(z) < \infty\} - e^{\delta(2-d)} 1\{\tau_{r+\delta}(z) < \infty\}\big]. \end{aligned}$$

The random variable $Q_r^\delta(z)$ is normalized so that $|Q_r^\delta(z)|$ is of order 1 but $\mathbb{E}[Q_r^\delta(z)]$ is nearly zero. The main estimate shows that if r is large and z, w are not too close, then $Q_r(z)$ and $Q_r(w)$ are almost independent.

THEOREM 3.2. *Suppose $\kappa < 8$ and γ is* SLE_κ *from* 0 *to* ∞ *in* $\mathbb{H}$. *There exists* $c < \infty$, $\beta > 0$ *such that if* $0 < \delta < 1/10$, $\Im(z), \Im(w) \geq 1$ *and* $r \geq 0$, *then*

$$\mathbb{E}\big[Q_r^\delta(z) Q_r^\delta(w)\big] \leq c e^{-\beta r} |z-w|^{\beta-2}. \tag{26}$$

We will not try to find the optimal c, β in our proof.

PROOF OF THEOREM 3.1 GIVEN THEOREM 3.2. By scaling, we may assume that $\Gamma = [j, j+1) \times i[k, k+1) \in \mathcal{Q}_0^+$ with $k \geq 1$. Suppose that $0 < \delta < 1/10$. Let $J_r = J_r(\Gamma)$ and

$$Q_r = Q_r^\delta = Q_r^\delta(\Gamma) = J_r - J_{r+\delta} = \int_\Gamma Q_r^\delta(z)\, dA(z).$$

By integrating (26), we see that if $r \geq 0$, then $\mathbb{E}[Q_r^2] \leq c e^{-\beta r}$. Let

$$X_n = X_n^\delta = J_0 + \sum_{j=1}^n |Q_{j\delta}|.$$

Then X_n converges in L^2 to a random variable X_∞. For each positive integer n, $|J_{n\delta}| \leq X_\infty$, and hence

$$\sup_{r\geq 0} J_r \leq e^{\delta(2-d)} \sup_n J_{n\delta} \leq e^{1/10} X_\infty. \tag{27}$$

Also, if $n \leq m$,

$$|J_{n\delta} - J_{m\delta}| \leq X_\infty - X_n.$$

Therefore, $\{J_{n\delta}\}$ is a Cauchy sequence in L^2 and has an L^2-limit which we call J_∞. If $r\delta \leq s < (r+1)\delta$, we similarly have

$$\mathbb{E}\big[(J_s - J_{r\delta})^2\big] = \mathbb{E}\big[\big(Q_{r\delta}^{s-r\delta}\big)^2\big] \leq c e^{-\beta r\delta}, \tag{28}$$

so we see that

$$\lim_{s\to\infty} \mathbb{E}\big[(J_s - J_\infty)^2\big] \leq \lim_{s\to\infty} \mathbb{E}\big[(J_s - J_{r\delta})^2\big] + \lim_{s\to\infty} \mathbb{E}\big[(J_{r\delta} - J_\infty)^2\big] = 0.$$

Hence, $J_s \to J_\infty$ in L^2; in particular, J_∞ does not depend on δ.

Chebyshev's inequality shows that

$$\sum_{n=1}^\infty \mathbb{P}\big\{|Q_{n\delta}| \geq e^{-\beta n\delta/4}\big\} \leq \sum_{n=1}^\infty \frac{\mathbb{E}[Q_{n\delta}^2]}{e^{-\beta n\delta/2}} < \infty.$$

Hence, for each δ, by the Borel–Cantelli lemma, with probability one for all n sufficiently large,

$$|J_{n\delta} - J_{(n+1)\delta}| \leq 2e^{-\beta n\delta/4}.$$

This shows that with probability one, the sequence $\{J_{n\delta}\}$ is a Cauchy sequence, and hence with probability one, for all $\delta = 2^{-m}$,

$$\lim_{n\to\infty} J_{n\delta} = J_\infty.$$

If $n\delta \le r \le (n+1)\delta$, then

$$e^{\delta(d-2)} J_{(n+1)\delta} \le J_r \le e^{\delta(2-d)} J_{n\delta}, \tag{29}$$

from which we conclude that with probability one, for all $\delta = 2^{-m}$,

$$e^{\delta(d-2)} J_\infty \le \liminf_{r\to\infty} J_r \le \limsup_{r\to\infty} J_r \le e^{\delta(2-d)} J_\infty.$$

Since this holds for all δ, $J_r \to J_\infty$.

Note that for $r > 0$,

$$\mathbb{E}[J_r] = \int_\Gamma e^{(2-d)r}\mathbb{P}\{\tau_r(z) < \infty\}\, dA(z),$$
$$\mathbb{E}[J_r^2] = \int_\Gamma \int_\Gamma e^{2(2-d)r}\mathbb{P}\{\tau_r(z), \tau_r(w) < \infty\}\, dA(z)\, dA(w).$$

Since $J_r \to J_\infty$ in L^2, we know that

$$\mathbb{E}[J_\infty] = \lim_{r\to\infty} \mathbb{E}[J_r], \qquad \mathbb{E}[J_\infty^2] = \lim_{r\to\infty} \mathbb{E}[J_r^2].$$

Hence, (23) and (24) follow from Theorem 2.3 and (20). Indeed, the definition of the Green's function (including the choice of multiplicative constant) was made in order for these equalities to hold.

Note that if $n \log 2 \le r \le (n+1)\log 2$,

$$\left|\mathrm{Cont}_d[\gamma \cap \Gamma; r] - J_r(\Gamma)\right| \le c J_{n\log 2}(\partial_n \Gamma).$$

Using (6), we see that $\mathbb{E}[\mathrm{Cont}_d(\gamma \cap \partial_n\Gamma; n\log 2)] \le c\,\mathrm{Area}(\partial_n\Gamma) \le c2^{-n}$. Hence, using the Markov inequality and the Borel–Cantelli lemma, we see that with probability one for all n sufficiently large $\mathrm{Cont}_d(\gamma \cap \partial_n\Gamma; n\log 2) \le 2^{-n/2}$. This gives (22). Also,

$$\begin{aligned}\mathrm{Cont}_d[\gamma \cap \Gamma; n\log 2] &\le J_{n\log 2}\\ &\le \mathrm{Cont}_d[\gamma \cap \Gamma; n\log 2] + \mathrm{Cont}_d(\gamma \cap \partial_n\Gamma; n\log 2).\end{aligned}$$

This gives (21).

Given $\Gamma \in \mathcal{Q}_0$, let $\Gamma_1, \dots, \Gamma_{12}$ denote the twelve squares in $\mathcal{Q}_1$ whose interior does not intersect Γ but whose boundary does. Note that these squares are in $\mathcal{Q}_1^+$. Any point within distance $1/2$ of $\gamma \cap \Gamma$ is contained in $\Gamma \cup \Gamma_1 \cup \cdots \cup \Gamma_{12}$, and hence for $r \ge 1$,

$$\mathrm{Cont}_d(\gamma \cap \Gamma; r) \le J_r(\Gamma) + J_r(\Gamma_1) + \cdots + J_r(\Gamma_{12}).$$

This implies that

$$\mathrm{Cont}_d^+(\gamma \cap \Gamma; 0) \le c + \sup_{r \ge 1}\big[J_r(\Gamma) + J_r(\Gamma_1) + \cdots + J_r(\Gamma_{12})\big].$$

Since $\Gamma_j \in \mathcal{Q}_1^+$, the argument as in (27), we see that for each j,

$$\sup_{r \ge 1} J_r(\Gamma_j)$$

is square integrable. Hence, $\mathrm{Cont}_d^+(\gamma \cap \Gamma; 0)$ is square integrable which gives (25). $\square$

3.2. *Natural length.* By Theorem 3.1, with probability one we can define a function on $\mathcal{Q}$ by

$$\mu(\Gamma) = \mathrm{Cont}_d(\gamma \cap \Gamma) = \mathrm{Cont}_d\big(\gamma \cap \mathrm{int}(\Gamma)\big) = \mathrm{Cont}_d(\gamma \cap \overline{\Gamma}).$$

PROPOSITION 3.3. *On this event, μ extends to be a Borel measure.*

PROOF. For each $\Gamma \in \mathcal{Q}^+$ and positive integer r, we can define a Borel measure μ_r by stating that the Radon–Nikodym derivative with respect to Lebesgue measure is $J_r(z)$. Since $\mu_r(\Gamma) \to \mu(\Gamma)$ and Γ is compact, for each subsequence $\{r_j\}$, there is a sub-subsequence $\{r_{j_k}\}$ that converges to a measure μ' with total mass $\mu(\Gamma)$. By using a diagonalization argument, we can find a single sub-subsequence such that the convergence holds for all $\Gamma \in \mathcal{Q}^+$. By (22), we see that $\mu'(\partial\Gamma) = 0$. Any open set U can be written as a countable union of squares $\Gamma \in \mathcal{Q}^+$ such that the interiors of the squares are disjoint. Hence, we can determine $\mu'(U)$ for any open set, and hence we can see that $m' = \mu$ is unique. $\square$

We call μ the (*natural*) *occupation measure* for the SLE curve γ. If D is an open set, then we can find $D_n \in \mathcal{S}_{\mathbb{H}}$ increasing to D, and hence

$$\mathbb{E}\big[\mu(D)\big] = \int_D G(z)\, dA(z), \qquad \mathbb{E}\big[\mu(D)^2\big] = \int_{D\times D} G(z,w)\, dA(z)\, dA(w).$$

It is not immediately obvious, but we will now show that, with probability one, for all $0 \le s < t < \infty$,

$$\mu\big(\gamma[s,t]\big) = \mu(\gamma_t \setminus \gamma_s) = \mathrm{Cont}_d\big(\gamma[s,t]\big). \tag{30}$$

The bulk of the work is in the following lemma. Recall that H_s is the unbounded component of $\mathbb{H} \setminus \gamma_s$, and let

$$\partial_n H_s = \big\{z \in \overline{H}_s : \mathrm{dist}(z, \partial H_s) \le 2^{-n}\big\}.$$

LEMMA 3.4. *There exists $\alpha > 0$ such that the following holds with probability one.*

- *For each* t_0, *there exists* $n_0 < \infty$ *such that if* $0 \le s \le t_0$ *and* $n \ge n_0$, *then*

$$\text{Cont}_d^+(\gamma[s, s + 2^{-n}]) \le 2^{-n\alpha}. \tag{31}$$

- *Suppose that* $0 \le s < t$, *and* $u > 0$. *Then*

$$\lim_{n\to\infty} \text{Cont}_d^+[\gamma[s + u, t] \cap \partial_n H_s] = 0. \tag{32}$$

The limit (32) is immediate for $\kappa \le 4$ since $\gamma[s + u, t] \cap \partial_n H_s$ is empty if n is large. Before proving the lemma, we will show how to deduce (30) from the lemma. We approximate $\gamma[s, t]$ by intersections of γ with finite unions of dyadic squares. If $s < t$ and n is a positive integer, let $V_n(s, t)$ denote the union of all $\Gamma \in \mathcal{Q}_n$ satisfying $\Gamma \subset H_s \setminus \partial_n H_s$ and $\gamma[s, t] \cap \Gamma \ne \varnothing$. Let $O_n(s, t) = \gamma \cap V_n(s, t)$. Note that $O_n(s, t) \subset \gamma \setminus \gamma_s$, but it is possible for $\gamma(t, \infty) \cap O_n(s, t)$ to be nonempty. Note that if $u > 0$, then

$$O_n(s, t) \setminus \gamma[s, t] \subset \gamma[t, t + u] \cup (\gamma[t + u, \infty) \cap \partial_{n-1} H_t),$$
$$\gamma[s, t] \setminus O_n(s, t) \subset \gamma[s, s + u] \cup [\gamma(s + u, \infty) \cap \partial_{n-2} H_s].$$

Here, we use the simple geometric facts that $O_n(s, t) \cap \overline{H}_t \subset \partial_{n-1} H_t$, and that if $\Gamma \in \mathcal{Q}_n$, then either $\Gamma \subset H_s \setminus \partial_n H_s$ or $\Gamma \subset \partial_{n-2} H_s$. The lemma implies that

$$\lim_{n\to\infty} \text{Cont}_d^+[\gamma[s, t] \setminus O_n(s, t)] + \lim_{n\to\infty} \text{Cont}_d^+[O_n(s, t) \setminus \gamma[s, t]] = 0.$$

Then (30) follows from (4). The remainder of this subsection will be devoted to proving the lemma. There is some technical work involved here and are the basic reasons why the lemma holds.

- For (31), we use the Hölder continuity of an SLE path to say that that the diameter of $\gamma[s, s + 2^{-n}]$ is not very big. We also use moment estimates to show that the Minkowski content is not very big on any set of small diameter.
- For (32), we use the fact that $\gamma[s + u, t] \cap \partial_n H_s$ consists of points of the curve that are either near the real line or are nearly double points of the curve. We estimate moments for the content of such paths.

We start by using the following lemma.

LEMMA 3.5. *Let* $Z(\Gamma)$ *be defined as before Theorem* 3.1. *There exists* $c < \infty$ *such that if* $\Gamma \in \mathcal{Q}_n^+$, *then*

$$\mathbb{E}[Z(\Gamma)] \le cG(\Gamma), \qquad \mathbb{E}[Z(\Gamma)^2] \le c2^{-dn} G(\Gamma).$$

PROOF. If $\tilde{\Gamma} \in \mathcal{Q}_n^+$, then $\Gamma = 2^n \tilde{\Gamma} \in \mathcal{Q}_0^+$ with $G(\Gamma) = 2^{dn} G(\tilde{\Gamma})$. Also, the distribution of $Z(\Gamma)$ is the same as that of $2^{dn} Z(\tilde{\Gamma})$. Hence, we may assume that $\Gamma \in \mathcal{Q}_0^+$.

If $\mathrm{dist}(0, \Gamma) \le 10$, then $G(\Gamma) \asymp 1$ and we can use (25). Otherwise, let τ be the first time that $\mathrm{dist}(\Gamma, \gamma(t)) = 8$. By (6), $\mathbb{P}\{\tau < \infty\} \asymp G(\Gamma)$, and by distortion estimates we can see that

$$\mathbb{E}[Z(\Gamma)|\tau < \infty] \le c, \qquad \mathbb{E}[Z(\Gamma)^2|\tau < \infty] \le c. \qquad \square$$

COROLLARY 3.6. *With probability one, if* $R < \infty$, *then for* n *sufficiently large*, $\Gamma \in \mathcal{Q}_n$ *with* $\mathrm{dist}(0, \Gamma) \le R$,

$$Z(\Gamma) \le n 2^{-dn/2}.$$

PROOF. By Chebyshev's inequality, if $\Gamma \in \mathcal{Q}_n$,

$$\mathbb{P}\{Z(\Gamma) \ge n2^{-nd/2}\} \le n^{-2} 2^{nd} \mathbb{E}[Z(\Gamma)^2] \le cn^{-2} G(\Gamma).$$

Hence, if V is any bounded set,

$$\sum_{n=0}^{\infty} \sum_{\Gamma \in \mathcal{Q}_n, \Gamma \subset V} \mathbb{P}\{Z(\Gamma) \ge n2^{-nd/2}\} \le c \int_V G(z)\, dA(z) < \infty.$$

The result follows from the Borel–Cantelli lemma. $\square$

Note that

$$\partial_n \mathbb{H} = \{z \in \mathbb{H} : \Im(z) \le 2^{-n}\}.$$

LEMMA 3.7. *With probability one, if* $R < \infty$ *and* $u > 0$, *then for all* n *sufficiently large*

$$\mathrm{Cont}_d^+(\gamma \cap \partial_n \mathbb{H} \cap \{|z| \le R\}) \le u 2^{-n}.$$

In particular, for each t_0, *for all* n *sufficiently large*,

$$\mathrm{Cont}_d^+(\gamma[0, t_0] \cap \partial_n \mathbb{H}) \le u 2^{-n}.$$

PROOF. The argument is the same for all R; for ease, we let $R = 1$ and write $V_n = \partial_n \mathbb{H} \cap \{|z| \le 1\}$. We will first show that

$$\sum_{n=1}^{\infty} \mathbb{P}\{\mathrm{Cont}_d(\gamma \cap V_n; n \log 2) \ge 2^{-n}\} < \infty. \tag{33}$$

Since $V_n \subset \bigcup_{|j| \le 2^n} \Gamma_n(j, 0)$,

$$\begin{aligned} \mathrm{Cont}_d(\gamma \cap V_n; n\log 2) &\le \sum_{|j| \le 2^n} \mathrm{Cont}_d(\gamma \cap \Gamma_n(j,0); n \log 2) \\ &\le 6 \cdot 2^{-2n} 2^{(2-d)n} \sum_{|j| \le 2^n} 1\{\gamma \cap \Gamma_n(j, 0) \ne \varnothing\}. \end{aligned}$$

The estimate (7) implies that $\mathbb{P}\{\gamma \cap \Gamma_n(j,0) \neq \emptyset\} \leq c j^{1-4a}$. If we choose β with $1 < \beta < d - (2-4a)_+$, we can see that

$$\mathbb{E}[\mathrm{Cont}_d(\gamma \cap V_n; n \log 2)] \leq c 2^{-n\beta},$$

$$\mathbb{P}\{\mathrm{Cont}_d(\gamma \cap V_n; n \log 2) \geq 2^{-n}\} \leq c 2^{-n(\beta-1)}.$$

This gives (33), and by the Borel–Cantelli lemma with probability one for all n sufficiently large,

$$\mathrm{Cont}_d(\gamma \cap V_n; n \log 2) \leq 2^{-n}.$$

It follows that for n sufficiently large, if $m \geq n$,

$$\mathrm{Cont}_d(\gamma \cap V_n; m \log 2) \leq 2^{-m} + \mathrm{Cont}_d(\gamma \cap (V_n \setminus V_m); m \log 2),$$

and hence

$$\mathrm{Cont}_d^+(\gamma \cap V_n) \leq 2^{2-d} \sup_{m \geq n} \mathrm{Cont}_d(\gamma \cap (V_n \setminus V_m); m \log 2),$$

where the supremum on the right is restricted to integers m. Let $\mathcal{A}_n$ denote the set of all squares of the form $\Gamma_l(j,1), l, j \in \mathbb{Z}$, that intersect V_n. These squares are disjoint and

$$\mathrm{Cont}_d(\gamma \cap (V_n \setminus V_m); m \log 2) \leq \sum_{\Gamma \in \mathcal{A}_n} Z(\Gamma).$$

Hence,

$$\mathrm{Cont}_d^+(\gamma \cap V_n) \leq 2^{2-d} \sum_{\Gamma \in \mathcal{A}_n} Z(\Gamma).$$

By Lemma 3.5,

$$\sum_{\Gamma \in \mathcal{A}_n} \mathbb{E}[Z(\Gamma)] \leq c \sum_{\Gamma \in \mathcal{A}_n} G(\Gamma) \leq c G(V_n).$$

As above, we find $\beta > 1$ such that $G(V_n) \leq c 2^{-n\beta}$, and hence

$$\begin{aligned}\mathbb{P}\left\{2^{2-d} \sum_{\Gamma \in \mathcal{A}_n} Z(\Gamma) \geq u 2^{-n}\right\} &\leq u^{-1} 2^n \mathbb{E}\left[2^{2-d} \sum_{\Gamma \in \mathcal{A}_n} Z(\Gamma)\right] \\ &\leq c u^{-1} 2^{n(1-\beta)}.\end{aligned}$$

Hence, by the Borel–Cantelli lemma, with probability one, for all n sufficiently large and all $m \geq n$,

$$2^{2-d} \sum_{\Gamma \in \mathcal{A}_n} Z(\Gamma) \leq u 2^{-n}.$$

□

The next proposition establishes the Hölder continuity of the function $t \mapsto \mathrm{Cont}_d(\gamma(0,t])$ and completes the proof of (31).

PROPOSITION 3.8. *There exists* $\alpha > 0$ *such that with probability one for every* $t < \infty$ *for all* n *sufficiently large and all* $s \le t$,

$$\mathrm{Cont}_d^+\big(\gamma\big[s, s + 2^{-n}\big]\big) \le 2^{-n\alpha}.$$

PROOF. It is known [3, 11] that for $\kappa \ne 8$, the SLE_κ curve is Hölder continuous with respect to the capacity parameterization. That is to say, there exists $\beta = \beta_\kappa > 0$ such that with probability one, if $t < \infty$, then for n sufficiently large, and all $0 \le s \le t$,

$$\mathrm{diam}\big(\gamma\big[s, s + 2^{-n}\big]\big) \le 2^{-n\beta}.$$

Let m be the largest integer less than βn. Then $\gamma[s, s + 2^{-n}]$ is contained in the union of four rectangles $\Gamma_1, \dots, \Gamma_4 \in \mathcal{Q}_m$. For n sufficiently large, if $\Gamma_j \in \mathcal{Q}_m^+$, then Corollary 3.6 implies that $\mathrm{Cont}_d^+(\Gamma_j \cap \gamma) \le m 2^{-dm/2}$. If $\Gamma_j \in \mathcal{Q}_n \setminus \mathcal{Q}_n^+$, then Lemma 3.7 implies that $\mathrm{Cont}_d^+(\Gamma_j) \le c 2^{-m}$. The result follows for $\alpha < \beta d/2$. $\square$

In the remainder of this section, we prove (32) which is

$$\lim_{n\to\infty} \mathrm{Cont}_d^+\big[\gamma[s + u, t] \cap \partial_n H_s\big] = 0.$$

Let $U = U_{j,k} = \{x + iy : -2^k \le x < 2^k, y \ge 2^{-j}\}$. Using Lemma 3.7 and compactness of $\gamma[t, u]$, we see that it suffices to prove that with probability one for every $s < u$ and all positive integers j, k,

$$\lim_{n\to\infty} \mathrm{Cont}_d^+\big[\gamma[u, \infty) \cap \partial_n H_s \cap U_{j,k}\big] = 0. \tag{34}$$

It suffices to consider rational s, u, and hence we need to show that for fixed s, u, j, k, (34) holds with probability one. By scaling, it suffices to prove this for $j = 0$ which we now assume. So we have

$$U = U_{0,k} = \big\{x + iy : -2^k \le x < 2^k, y \ge 1\big\}.$$

We fix integer $k > 0$ and allow constants to depend on k. We only consider $n \ge k + 4$. Let $U = U_{0,k}$, and let $\mathcal{Q}_n(U)$ denote the set of $\Gamma \in \mathcal{Q}_n$ with $\Gamma \subset U$. Note that $G(\Gamma) \le c 2^{-2n}$ if $\Gamma \in \mathcal{Q}_n(U)$.

We will now define a quantity $\widehat{Z}(\Gamma)$ for $\Gamma \in \mathcal{Q}$ that is an upper bound for the Minkowski content of the intersection of the path with Γ "after it has gotten close to the square and then gotten away from the square." To be precise, suppose that $\Gamma \in \mathcal{Q}_n$ with center point z, and define the following quantities:

- $\xi_1 = \xi_1(\Gamma)$ is the first time t such that $|z - \gamma(t)| = 2^{-n+3}$. If $\xi_1 < \infty$, let $l = l(\Gamma)$ denote a subarc of the circle of radius $2^{-n/2}$ about z such that z is in the bounded component of $H_{\xi_1} \setminus l$. See Section 2.4 where a particular such arc l was selected. To be specific, we will make that choice here.
- $\xi_2 = \xi_2(\Gamma)$ is the first time $t > \xi_1$ such that $\gamma(t) \in \bar{l}$.
- $\xi_3 = \xi_3(\Gamma)$ is the first time $t > \xi_1$ such that $|z - \gamma(t)| = 2^{-n+1}$.

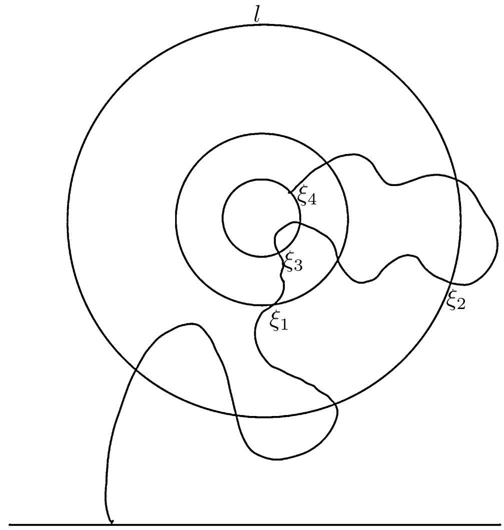

FIG. 2. *The quantities in Proposition 3.8 in the case* $\xi_3 < \xi_2$.

- $\xi_4 = \xi_4(\Gamma)$ is the first time $t > \xi_2$ such that $|z - \gamma(t)| = 2^{-n+1}$.

We think of time ξ_4 as the time of the "second return" to the (neighborhood of the) square, see Figure 2.

LEMMA 3.9. *There exists n_0 such that if $n \geq n_0$, $\Gamma \in \mathcal{Q}_n$, $\Gamma \cap U \neq \varnothing$, and $\xi_1 \leq s$, then $\xi_2 < u$.*

PROOF. The curve γ is parameterized so that $\mathrm{hcap}(\gamma[s_1, s_2]) = a(s_2 - s_1)$ where hcap is the half-plane capacity which can be defined by

$$\mathrm{hcap}(V) = \lim_{y\to\infty} y\mathbb{E}^{iy}[\Im(B_\tau)],$$

where B_t is a standard Brownian motion and $\tau = \tau_V = \inf\{t : B_t \in \mathbb{R} \cup V\}$. In particular, if $V_1 \subset V_2$,

$$\mathrm{hcap}(V_2) - \mathrm{hcap}(V_1) \leq \lim_{y\to\infty} y\mathbb{E}^{iy}[\Im(B_{\tau_2}); \tau_2 < \tau_1], \qquad \tau_j = \tau_{V_j}.$$

Since the half-plane capacity is monotone,

$$\mathrm{hcap}(\gamma[0, \xi_2]) \leq \mathrm{hcap}(\gamma[0, \xi_1] \cup l).$$

Using the Beurling estimate and the fact that $\Gamma \cap U \neq \varnothing$, we can see that if $V_1 = \gamma(0, \xi_1]$, $V_2 = \gamma(0, \xi_1] \cap l$, then

$$\lim_{y\to\infty} y\mathbb{E}^{iy}[\Im(B_{\tau_2}); \tau_2 < \tau_1] \leq \lim_{y\to\infty} y\mathbb{P}^{iy}\{\tau_2 < \tau_1\} \leq c\,\mathrm{diam}[l]^{1/2} \leq c2^{-n/4}.$$

If n_0 is chosen sufficiently large, then the right-hand side is less than $u - s$ and hence $\xi_2 - \xi_1 < u - s$. $\square$

We let $E_1(\Gamma)$ be the event $\{\xi_1 < \xi_3 < \xi_2 < \xi_4 < \infty\}$, $E_2(\Gamma)$ the event $\{\xi_1 < \xi_2 < \xi_3 = \xi_4 < \infty\}$, and $E(\Gamma) = E_1(\Gamma) \cup E_2(\Gamma) = \{\xi_4 < \infty\}$. We define $\widehat{Z}(\Gamma)$ as follows:

$$\widehat{Z}(\Gamma) = 0 \qquad \text{on the complement of } E(\Gamma),$$
$$\widehat{Z}(\Gamma) = 2^{(n+1)(2-d)} \qquad \text{on the event } E_1(\Gamma),$$
$$\widehat{Z}(\Gamma) = \mathrm{Cont}_d^+(\Gamma \cap \gamma; n \log 2) \qquad \text{on the event } E_2(\Gamma).$$

Take n_0 as in the previous lemma and recall that $U = U_{0,k} = \{x + iy : -2^k \le x < 2^k, y \ge 1\}$. The definition is such that the following holds:

- If $m > n_0$ and $\Gamma \in \mathcal{Q}_m$ with $\Gamma \cap U \ne \varnothing$, then on the event $E(\Gamma)$,
$$\mathrm{Cont}_d\big[\gamma[u,\infty) \cap \Gamma; m \log 2\big] \le \widehat{Z}(\Gamma).$$
- If $m \ge n > n_0$ and $\Gamma \in \mathcal{Q}_n$ with $\Gamma \cap U \ne \varnothing$, then on the event $E_2(\Gamma)$,
$$\mathrm{Cont}_d\big[\gamma[u,\infty) \cap \Gamma; m \log 2\big] \le \widehat{Z}(\Gamma).$$

We will use the following fact which states that once one gets close to z and then leaves, one is unlikely to return. It is a quantitative expression of the fact that the double points of SLE$_\kappa$ curve have strictly smaller fractal dimension than the curve itself. It is an immediate corollary of Corollary 2.8.

LEMMA 3.10. *There exist c, β such that if $\Gamma \in \mathcal{Q}_n$ with $\Gamma \subset \{\Im(z) \ge 1\}$, then* $\mathbb{P}[E(\Gamma)] \le c2^{n(d-2)}2^{-n\beta}$.

Arguing as in the proof of Lemma 3.5, we have on the event $E_2(\Gamma)$,
$$\mathbb{E}\big[\widehat{Z}(\Gamma) | \gamma_{\xi_4}\big] \le c2^{n(2-d)}.$$

Then we see that
$$\mathbb{E}\big[\widehat{Z}(\Gamma) | \xi_1(\Gamma) < \infty\big] \le c2^{n(d-2)},$$
$$\mathbb{E}\big[\widehat{Z}(\Gamma)\big] \le \mathbb{P}\{\xi_1(\Gamma) < \infty\}\mathbb{E}\big[\widehat{Z}(\Gamma) | \xi_1(\Gamma) < \infty\big] \le cG(\Gamma)2^{-n\beta}.$$

Let
$$\widehat{Z}_n = \widehat{Z}_n(U) = \sum_{m=n}^{\infty} \sum_{\Gamma \in \mathcal{Q}_m, \Gamma \subset U} \widehat{Z}(\Gamma).$$

Then $\mathbb{E}[\widehat{Z}_n] \le c2^{-\beta n}$, and hence using the Borel–Cantelli lemma, with probability one for all n sufficiently large, $\widehat{Z}_n \le 2^{-\beta n/2}$. To establish (34), it therefore suffices to show that there exists c such that for $n > n_0$,
$$\mathrm{Cont}_d^+\big[(\gamma \setminus [\gamma_u \cup \partial_n H_s]) \cap U\big] \le c\widehat{Z}_n.$$

To show this it suffices to show for all integers $m \geq n$

$$\mathrm{Cont}_d\big[(\gamma \setminus [\gamma_u \cup \partial_n H_s]) \cap U; m \log 2\big] \leq c\widehat{Z}_n.$$

For each $m \geq n > n_0$, will cover $\gamma \setminus [\gamma_u \cup \partial_n H_s]$ by squares $\Gamma \in \mathcal{Q}_j$ with $n \leq j \leq m$. We will choose all squares $\Gamma \in \mathcal{Q}_m$ that intersect H_s and are within distance 2^{-m+2} of ∂H_s. This includes squares that intersect ∂H_s. However, for $n \leq j < m$, we only choose squares whose distance from ∂H_s is comparable to 2^{-j}. In particular, these squares do not intersect ∂H_s.

To be precise, let $s < u$ and assume that $n > n_0$. For fixed $n < m$, let $\mathcal{A} = \mathcal{A}_{s,m,n}$ denote the set of $\Gamma \in \mathcal{Q}_j(U)$, $j = n, \dots, m$, that satisfy $2^{-j+1} \leq \mathrm{dist}(\Gamma, \partial H_s) \leq 2^{-j+3}$. Let $\mathcal{C} = \mathcal{C}_{s,m}$ denote the set of $\Gamma \in \mathcal{Q}_m(U)$ that satisfy $\mathrm{dist}(\Gamma, \partial H_s) \leq 2^{-m+2}$. We claim that for each m, the squares in $\mathcal{A} \cup \mathcal{C}$ cover $U \cap \partial_n H_s$. To see this, suppose that $z \in U \cap \partial_n H_s$. Then $\mathrm{dist}(z, \partial H_s) \leq 2^{-n}$. If $\mathrm{dist}(z, \partial H_s) \leq 2^{-m+2}$, then the unique $\Gamma \in \mathcal{Q}_m(U)$ containing z is in $\mathcal{C}$. If $\mathrm{dist}(z, \partial H_s) > 2^{-m+2}$ find j such that $2^{-j+2} < \mathrm{dist}(z, \partial H_s) \leq 2^{-j+3}$. Let Γ be the unique square in $\mathcal{Q}_j(U)$ that contains z and note that $2^{-j+1} \leq \mathrm{dist}(\Gamma, \partial_n H_s) \leq 2^{-j+3}$.

If $\Gamma \in \mathcal{A} \cup \mathcal{C}$, then $\xi_1 \leq s$. Since $n > n_0$, by Lemma 3.9 $\xi_2 < u$. Hence, $\gamma[u, \infty) \cap \Gamma \subset \gamma[\xi_4, \infty)$. If $\Gamma \in \mathcal{A}$ with $\gamma[u, \infty) \cap \Gamma \neq \varnothing$, then $\xi_2 < \xi_3$ which means that the event $E_2(\Gamma)$ has occurred. If $\Gamma \in \mathcal{C}$ and $\gamma[u, \infty) \cap \Gamma \neq \varnothing$, we know that $E(\Gamma)$ has occurred. Either way, we see that

$$\mathrm{Cont}_d\big[\gamma[u, \infty) \cap \Gamma; m \log 2\big] \leq \widehat{Z}_n(\Gamma).$$

4. Proof of Theorem 3.2.

Throughout this section, $0 < \delta \leq 1/10$, but constants are independent of δ.

4.1. *Some reductions.*

Suppose $\Im(z), \Im(w) \geq 1$, and let $J_r(z)$, $Q_r(z) = Q_r^\delta(z)$ as in Section 3.1. Since $Q_r(z)Q_r(w) = 0$ if $\tau_r(z) = \infty$ or $\tau_r(w) = \infty$, in order to prove (26) it suffices by symmetry to prove that

$$\mathbb{E}\big[Q_r(z)Q_r(w); \tau_r(z) < \tau_r(w) < \infty\big] \leq ce^{-\beta r}|z - w|^{\beta - 2}. \tag{35}$$

By (19), we know that if $|z - w| \leq e^{-ur}$,

$$\begin{aligned}\mathbb{E}\big[Q_r(z)Q_r(w)\big] &\leq ce^{2r(2-d)}\mathbb{P}\{\tau_r(z) < \infty, \tau_r(w) < \infty\} \\ &\leq c|z - w|^{d-2} \\ &\leq c|z - w|^{(d-1)-2}e^{-ur}.\end{aligned}$$

Hence, it suffices to find $u > 0$ and c, β such that (35) holds for $|z - w| \geq e^{-ur}$. Let α be as in (15), and suppose that $s > 0$. Choose $u > 0$ with $u[2(2-d)+\alpha] \leq \alpha s/2$.

Then if $|z - w| \geq e^{-ur}$,

$$\begin{aligned}&\mathbb{E}\big[Q_r(z)Q_r(w); \tau_{sr}(w) \leq \tau_r(z) \leq \tau_r(w) < \infty\big] \\ &\quad \leq ce^{2r(2-d)}\mathbb{P}\{\tau_{sr}(w) \leq \tau_r(z) \leq \tau_r(w) < \infty\}\end{aligned}$$

$$\leq ce^{2r(2-d)}\mathbb{P}\{\tau_{sr-ur+ur}(w) \leq \tau_{r-ur+ur}(z) \leq \tau_{r-ur+ur}(w) < \infty\}$$
$$\leq ce^{2r(2-d)}e^{2(r-ur)(d-2)}e^{-\alpha(sr-ur)}$$
$$\leq ce^{ru[2(2-d)+\alpha]}e^{-\alpha sr} \leq ce^{-\alpha sr/2}.$$

From this, we see that in order to prove (35) it suffices to prove the following. There exist $u > 0, s > 0, \beta > 0, c < \infty$ such that if $\Im(z), \Im(w) \geq 1$ and $|z-w| \geq e^{-ur}$, then

$$\mathbb{E}[Q_r(z)Q_r(w); \tau_r(z) < \tau_{sr}(w) < \tau_r(w) < \infty] \leq ce^{-\beta r}|z-w|^{\beta-2}. \tag{36}$$

This is what we will establish in this section.

4.2. *One-point estimate*. As is often the case, an important step in getting a two-point estimate is to get a sharp one-point estimate with good control on the error terms. Much of the necessary analysis has been done for SLE, and we review some of the methods here.

We will consider chordal SLE_κ from 1 to $w = e^{2i\theta}$ in the unit disk $\mathbb{D}$, and we will study how close the path gets to the origin. We parameterize the SLE_κ path γ using the radial parameterization. To be specific, we let D_t denote the component of $\mathbb{D} \setminus \gamma_t$ containing the origin and $g_t : D_t \to \mathbb{D}$ the unique conformal transformation with $g_t(0) = 0, g_t(\gamma(t)) = 1$. The radial parameterization is defined so that $|g_t'(0)| = e^t$. The total lifetime of the curve in this parameterization, T, is finite with probability one. (If $\kappa > 4$, T is not the time that the curve reaches w, but rather the time at which the curve disconnects the origin from w. Although the SLE curves continues after this time, the domain D_t does not change so we do not need to consider the path after time T.) We write $w_t = e^{2i\theta_t} = g_t(w)$. The path γ_t, and hence the transformations g_t, are determined by $\theta_s, 0 \leq s \leq t$. If $t > T$, then $g_t = g_T$. We let $\mathbb{P}_\theta, \mathbb{E}_\theta$ denote probabilities and expectations given by chordal SLE_κ from 1 to $e^{2i\theta}$. The angle θ_t satisfies a simple one-dimensional SDE. Its form is a little nicer if we consider a linear time change. If $\hat{\theta}_t = \theta_{2at}$, then $\hat{\theta}_t$ satisfies the "radial Bessel equation"

$$d\hat{\theta}_t = (1-2a)\cot\hat{\theta}_t\,dt + dB_t,$$

where B_t is a standard Brownian motion. This equation is valid until the time $\widehat{T} = T/2a$ at which $\hat{\theta}_{\widehat{T}} = \theta_T \in \{0, \pi\}$.

The Koebe (1/4)-theorem and Schwarz lemma implies that for $0 \leq t \leq T$,

$$e^{-t-\log 4} \leq \mathrm{dist}[0, \gamma_t] \leq e^{-t}. \tag{37}$$

Let $S_t = S_{D_t}(0; w, \gamma(t)) = S_{\mathbb{D}}(0; w_t, 1) = \sin\theta_t$. Itô's formula shows that

$$M_t = 1\{T > t\}e^{t(2-d)}S_t^{4a-1}$$

is a local martingale; more precisely, $\widehat{M}_t = M_{2at}$ satisfies

$$d\widehat{M}_t = (4a-1)[\cot\hat{\theta}_t]\widehat{M}_t\,dB_t, \qquad t < \widehat{T}.$$

In fact, M_t is a continuous martingale with $\mathbb{P}\{M_T = 0\} = 1$.

Let $\mathbb{D}_r$ denote the open disk of radius e^{-r} about the origin with closure $\overline{\mathbb{D}}_r$, and

$$\tau_r = \inf\{t : \mathrm{dist}[0, \gamma_t] = e^{-r}\} = \inf\{t : \gamma(t) \in \overline{\mathbb{D}}_r\}.$$

Note that (37) implies that

$$r - 2 < r - \log 4 \le \tau_r \le r.$$

The measure obtained by tilting by the martingale M_t is called *two-sided radial* SLE$_\kappa$ (*from* 0 *to* $e^{2i\theta}$ *in* $\mathbb{D}$ *going through the origin stopped when it reaches the origin*). We will write $\mathbb{P}^*, \mathbb{E}^*$ for probabilities and expectations with respect to this measure. These measures depend on the initial angle θ and we will write $\mathbb{P}^*_\theta, \mathbb{E}^*_\theta$ if we wish to make this explicit. The quantity $\mathbb{E}^*_\theta$ is defined by saying that if X is a random variable that depends only on γ_t, then

$$\mathbb{E}^*_\theta(X) = M_0^{-1}\mathbb{E}_\theta[XM_t] = [\sin\theta]^{1-4a} e^{t(2-d)} \mathbb{E}_\theta[X S_t^{4a-1} 1\{T > t\}],$$

or equivalently,

$$\mathbb{E}_\theta[X 1\{T > t\}] = e^{(d-2)t}[\sin\theta]^{4a-1} \mathbb{E}^*_\theta[X S_t^{1-4a}]. \tag{38}$$

The Girsanov theorem shows that under the measure $\mathbb{E}^*_\theta$,

$$d\hat{\theta}_t = 2a \cot\hat{\theta}_t\, dt + dW_t, \tag{39}$$

where, as before, $\hat{\theta}_t = \theta_{2at}$ and W_t is a standard Brownian motion with respect to the tilted measure. This equation has an invariant probability density

$$\phi(\theta) = C_{4a}^{-1}[\sin\theta]^{4a}, \qquad C_{4a} = \int_0^\pi \sin^{4a}\theta\, d\theta.$$

Moreover, the rate of convergence to equilibrium is exponential (see, e.g., [10], Section 2.1.1). To be more explicit, there exists $\alpha > 0$ such that if $\phi_t(\theta; \theta_0)$ is the density at time t given initial condition θ_0, then

$$\phi_t(\theta; \theta_0) = \phi(\theta)[1 + O(e^{-\alpha t})]. \tag{40}$$

Implicit in this formulation is the fact that for every $t_0 > 0$ there exists $C = C(t_0) < \infty$ such that if $t \ge t_0$, $C^{-1}\phi(\theta) \le \phi_t(\theta; \theta_0) \le C\phi(\theta)$.

If we apply this to (38) with $X \equiv 1$, we get

$$\mathbb{P}_\theta\{T > t\} = c_* e^{(d-2)t}[\sin\theta]^{4a-1}[1 + O(e^{-\alpha t})],$$

where

$$c_* = \int_0^\pi [\sin\theta]^{1-4a}\phi(\theta)\, d\theta = 2C_{4a}^{-1}.$$

In particular, we see that for $r \ge 1/10$,

$$\mathbb{P}_\theta\{T > \tau_r\} \asymp [\sin\theta]^{4a-1} e^{(d-2)r},$$

and if $r \geq 3$, $0 \leq s \leq 1/10$, and $T > r - 2$, conformal invariance implies that

$$\mathbb{P}_\theta\{T > \tau_{r+s} \neq \varnothing | \gamma_{r-2}\} = \mathbb{P}_{\theta_{r-2}}\{\gamma \cap g_{r-2}(\mathbb{D}_{r+s}) \neq \varnothing\} \asymp [\sin \theta_{r-2}]^{4a-1}.$$

We can also phrase this in terms of the quasi-stationary distribution for θ_t. Let $\psi(\theta) = \frac{1}{2} \sin \theta$. Under the measure $\mathbb{P}$, the random variable $\theta_t 1\{T > t\}$ has an atom at 0 and has a density $\psi_t(\theta)$ for $0 < \theta < \pi$ satisfying

$$\mathbb{P}\{T > t\} = \int_0^\pi \psi_t(\theta)\, d\theta.$$

The results of the previous paragraph show that ψ is a quasi-stationary density in the sense that if $\phi_0 \equiv \psi$, then

$$\psi_t(\theta) = e^{t(d-2)} \psi(\theta).$$

Moreover, if $\psi_t(\theta; \theta_0)$ denotes the density assuming initial condition θ_0,

$$\begin{aligned} \psi_t(\theta; \theta_0) &= \mathbb{P}_{\theta_0}\{T > t\} \psi(\theta)\big[1 + O\big(e^{-t\alpha}\big)\big] \\ &= c_* e^{t(d-2)} \psi(\theta)\big[1 + O\big(e^{-t\alpha}\big)\big]. \end{aligned} \tag{41}$$

We write $\mathbb{P}_\psi$ for probabilities assuming the initial density ψ. We can see

$$\mathbb{P}_\psi\{T > r\} = e^{r(d-2)}.$$

PROPOSITION 4.1. *There exists* $0 < c_1 < \infty$ *such that*

$$\mathbb{P}_\psi\{\tau_r < \infty\} = c_1 e^{r(d-2)}\big[1 + O\big(e^{-r}\big)\big].$$

PROOF. If $r > 0$, $u > 2$, then since $\tau_r > r - 2$,

$$\mathbb{P}_\psi\{\tau_{r+u} < \infty\} = \mathbb{P}_\psi\{T > r\} \mathbb{P}_\psi\{\tau_{r+u} < \infty | T > r\}.$$

The conformal Markov property implies that if $T > r$, then

$$\mathbb{P}_\psi\{\tau_{r+u} < \infty | \gamma_r\} = \mathbb{P}_{\theta_r}\{\gamma[0, \infty) \cap g_r(\mathbb{D}_{r+u}) \neq \varnothing\},$$

where θ_r started according to ψ. By Lemma 2.1, there exists u_0 such that if $u > u_0$, then on the event $T > r$, if $|z| = e^{-r-u}$,

$$e^{-u} \exp\{-4e^{-u}\} \leq |g_r(z)| \leq e^{-u} \exp\{4e^{-u}\}.$$

Combining the last two expressions, we see that if $r > 0$ and $u > u_0$, then if $T > r$,

$$\begin{aligned} \mathbb{P}_{\theta_r}\{\tau_{u+4e^{-u}} < \infty\} &\leq \mathbb{P}_{\theta_r}\{\gamma[0, \infty) \cap g_r(\mathbb{D}_{r+u}) \neq \varnothing\} \\ &\leq \mathbb{P}_{\theta_r}\{\tau_{u-4e^{-u}} < \infty\}. \end{aligned} \tag{42}$$

Since ψ is the quasi-stationary density, the conditional density on θ_r given $T > r$ is ψ. Therefore,

$$e^{r(d-2)}\mathbb{P}_\psi\{\tau_{u+4e^{-u}} < \infty\} \le \mathbb{P}_\psi\{\tau_{r+u} < \infty\} \le e^{r(d-2)}\mathbb{P}_\psi\{\tau_{u-4e^{-u}} < \infty\}.$$

If we replace r with $s = r - 5e^{-u}$ and u with $v = u + 5e^{-u}$, we get for u sufficiently large so that $e^{-v} \ge (4/5)e^{-u}$,

$$\begin{aligned}\mathbb{P}_\psi\{\tau_{r+u} < \infty\} &= \mathbb{P}_\psi\{\tau_{s+v} < \infty\}\\ &\le e^{s(d-2)}\mathbb{P}_\psi\{\tau_{v-4e^{-v}} < \infty\}\\ &\le e^{s(d-2)}\mathbb{P}_\psi\{\tau_u < \infty\}\\ &= e^{r(d-2)}\mathbb{P}_\psi\{\tau_u < \infty\}[1 + O(e^{-u})].\end{aligned}$$

We get a bound in the other direction by choosing $s = r + 5e^{-u}$ and $v = r = e^{-5u}$. Hence,

$$\mathbb{P}_\psi\{\tau_{r+u} < \infty\} = e^{r(d-2)}\mathbb{P}_\psi\{\tau_u < \infty\}[1 + O(e^{-u})],$$

where the error term is bounded uniformly independent of r. If we define $L_r = \log[e^{r(2-d)}\mathbb{P}_\psi\{\tau_r < \infty\}]$, then the above expression can be written as

$$\sup_{r \ge u} |L_r - L_u| = O(e^{-u}),$$

which implies that the limit $L_\infty = \lim_{u\to\infty} L_u \in (-\infty, \infty)$ exists, and $|L_u - L_\infty| = O(e^{-u})$. The proposition follows with $c_1 = e^{L_\infty}$. □

THEOREM 4.2. *There exists* $0 < \hat{c} < \infty$ *and* $\beta > 0$, *such that*

$$\mathbb{P}_\theta\{\tau_r < \infty\} = \hat{c}[\sin\theta]^{4a-1}e^{r(d-2)}[1 + O(e^{-r\beta})]. \tag{43}$$

PROOF. As in (42),

$$\mathbb{P}_{\theta_r}\{\tau_{r+4e^{-r}} < \infty\} \le \mathbb{P}_{\theta_r}\{\gamma[0,\infty) \cap g_r(\mathbb{D}_{2r}) \ne \varnothing\} \le \mathbb{P}_{\theta_r}\{\tau_{r-4e^{-r}} < \infty\}.$$

By Proposition 4.1, if ψ is the invariant distribution,

$$\mathbb{P}_\psi\{\tau_{r\pm 4e^{-r}} < \infty\} = c_1 e^{r(d-2)}[1 + O(e^{-r})].$$

Combining this with (41), we see that

$$\begin{aligned}\mathbb{P}_\theta\{\tau_{2r} < \infty\} &= \mathbb{P}_\theta\{T > r\}\mathbb{P}_\theta\{\tau_{2r} < \infty | T > r\}\\ &= c_1 c_* e^{2r(d-2)}[1 + O(e^{-\alpha r})].\end{aligned}$$ □

With this theorem we could *define* the chordal Green's function on $\mathbb{D}$ by

$$G_\mathbb{D}(0; 1, e^{2i\theta}) = \hat{c}[\sin\theta]^{4a-1},$$

and define it for other simply connected domains by

$$G_D(z; w_1, w_2) = |f'(z)|^{2-d} G_{\mathbb{D}}(0; 1, e^{2i\theta}),$$

where $f : D \to \mathbb{D}$ is a conformal transformation with $f(z) = 0$, $f(w_1) = 1$, $f(w_2) = e^{2i\theta}$. In fact,

$$G_D(z; w_1, w_2) = \hat{c}\, \mathrm{crad}_D(z)^{d-2} S_D(z; w_1, w_2)^{4a-1}.$$

PROPOSITION 4.3. *Let $0 < \kappa < 8$. There exists $c < \infty, \alpha > 0$ such that the following is true. Suppose that D is a simply connected domain and γ is a chordal* SLE$_\kappa$ *path from w_1 to w_2 in D. Suppose that $z \in D$, $R = \mathrm{dist}(z, \partial D)$ and $G = G_D(z; w_1, w_2)$. Then if $e^{-r} \le R/2$,*

$$\left|G^{-1} e^{r(2-d)} \mathbb{P}\{\mathrm{dist}(\gamma, z) \le e^{-r}\} - 1\right| \le c[e^{-r}/R]^{\alpha}.$$

In particular, there exists $c < \infty$ such that if $0 < r < s$, then

$$\begin{aligned} &\left|\mathbb{P}\{\mathrm{dist}(\gamma, z) \le e^{-r}\} - e^{(s-r)(2-d)} \mathbb{P}\{\mathrm{dist}(\gamma, z) \le e^{-s}\}\right| \\ &\quad \le c[e^{-r}/R]^{2-d+\alpha}. \end{aligned} \tag{44}$$

PROOF. Without loss of generality, we assume $z = 0$, and by scaling we may assume that $R = 1$. Let $F : D \to \mathbb{D}$ be the conformal transformation with $F(0) = 0$, $F(w_1) = 1$, $F(w_2) = e^{2i\theta}$ where $\sin\theta = S_D(z; w_1, w_2)$. The Schwarz lemma and Koebe (1/4)-theorem imply that $1/4 \le |F'(0)| \le 1$. Note that $G = \hat{c}|F'(0)|^{2-d}[\sin\theta]^{4a-1}$. Proposition 2.1 implies that there exists universal r_0 such that if $r > r_0$,

$$|F'(0)||z| \exp\{-4|z|\} \le |F(z)| \le |F'(0)||z| \exp\{4|z|\}.$$

Therefore, by conformal invariance, if $q = -\log|F'(0)|$

$$\mathbb{P}_\theta\{\tau_{r+4e^{-r}+q} < \infty\} \le \mathbb{P}\{\mathrm{dist}(\gamma, z) \le e^{-r}\} \le \mathbb{P}_\theta\{\tau_{r-4e^{-r}+q} < \infty\}.$$

But (43) tells us that

$$\begin{aligned} \mathbb{P}_\theta\{\tau_{r\pm 4e^{-r}+q} < \infty\} &= \hat{c}[\sin\theta]^{4a-1} e^{(r+q)(d-2)}[1 + O(e^{-\alpha r})] \\ &= G e^{(r+q)(d-2)}[1 + O(e^{-\alpha r})]. \end{aligned}$$

□

With these results, we can follow the proof in [9], Section 3, which proves the corresponding result with distance replaced by conformal radius, to conclude (20). We need to replace Lemma 2.16 of [9], with the corresponding result for the distance. The necessary lemma, written in the notation of this paper, is the following.

LEMMA 4.4. *There exist $\alpha > 0$, $c < \infty$ such that if $0 < s < u < 1$ and $r \ge 3$,*

$$\mathbb{P}_\theta\{\tau_r < \infty, \gamma(\tau_{ur}, \tau_r) \not\subset \mathbb{D}_{sr}\} \le c[\sin\theta]^{4a-1} e^{r(d-2)} e^{-\alpha t r},$$

where $t = \min\{1 - u, u - s\}$.

PROOF. A corresponding result was proved for two-sided radial SLE_κ in [6]. In particular, there exist c, α such that

$$\mathbb{P}^*_\theta\{\gamma(\tau_{ur}, \tau_r) \not\subset \mathbb{D}_{sr}\} \le ce^{\alpha(s-u)r}.$$

In particular, since $\tau_r > r - 2$,

$$\mathbb{P}^*_\theta\{\gamma(\tau_{ur}, r-2) \not\subset \mathbb{D}_{sr}\} \le ce^{\alpha(s-u)r}.$$

Using the definition of the measure $\mathbb{P}^*_\theta$ we see that this implies that

$$\begin{aligned}\mathbb{E}_\theta\big[[\sin\theta_{r-2}]^{1-4a}; T > r-2, \gamma(\tau_{ur}, r-2) \not\subset \mathbb{D}_{sr}\big]\\ \le c[\sin\theta]^{4a-1}e^{r(d-2)}e^{\alpha(s-u)r}.\end{aligned}$$

However, if $T > r-2$, $\mathbb{P}\{\tau_r < \infty | \gamma_{r-2}\} \asymp [\sin\theta_{r-2}]^{4a-1}$. Hence,

$$\mathbb{P}_\theta\{\tau_r < \infty, \gamma(\tau_{ur}, r-2) \not\subset \mathbb{D}_{sr}\} \le c[\sin\theta]^{4a-1}e^{r(d-2)}e^{\alpha(s-u)r}. \tag{45}$$

On the event $E := \{T > r-2, \gamma(\tau_{ur}, r-2) \subset \mathbb{D}_{sr}\}$, topological considerations (see [6], Lemma 2.3) imply that there is a unique subarc l of $\partial\mathbb{D}_{ur} \cap D_{r-2}$ such that removal of l disconnects 0 from $\partial\mathbb{D}_{sr}$ in D_{r-2}. The point $\gamma(r-2)$ may be in l or in either of the connected components of $D_{r-2} \setminus l$. In any of these cases, if $\sigma = \inf\{t \ge r-2 : \gamma(t) \in \bar{l}\}$, then the event $E \cap \{\tau_r < \infty, \gamma(\tau_{ur}, \tau_r) \not\subset \mathbb{D}_{sr}\}$ is contained in the event $E \cap \{\sigma < \tau_r < \infty\}$. As in (11), on the event $E \cap \{\sigma < \infty, \sigma < \tau_r\}$,

$$\mathbb{P}\{\tau_r < \infty | \gamma_\sigma\} \le ce^{\alpha(u-1)r}.$$

Hence,

$$\begin{aligned}&\mathbb{P}_\theta\{\tau_r < \infty, \gamma(\tau_{ur}, \tau_r) \not\subset \mathbb{D}_{sr}, \gamma(\tau_{ur}, r-2) \subset \mathbb{D}_{sr}\}\\ &\quad\le \mathbb{P}_\theta(E)\mathbb{P}_\theta\{\tau_r < \infty, \gamma(\tau_{ur}, \tau_r) \not\subset \mathbb{D}_{sr} | E\}\\ &\quad\le \mathbb{P}_\theta\{T > r-2\}\mathbb{P}_\theta\{\tau_r < \infty, \gamma(\tau_{ur}, \tau_r) \not\subset \mathbb{D}_{sr} | E\}\\ &\quad\le c[\sin\theta]^{4a-1}e^{r(d-2)}e^{\alpha r(u-1)}.\end{aligned} \tag{46}$$

The lemma follows from (45) and (46). □

The proof of (20) follows that in [9], Section 3. We will not give all the details, but we sketch the argument using the notation of this paper. We need to prove the existence of the limit

$$G(z, w) = \lim_{r,s\to\infty} e^{r(2-d)}e^{s(2-d)}\mathbb{P}\{\tau_r(z), \tau_s(w) < \infty\}.$$

PROPOSITION 4.5. *For every $z, w \in \mathbb{H}$,*

$$\lim_{r,s\to\infty} e^{r(2-d)}e^{s(2-d)}\mathbb{P}\{\tau_r(z) < \tau_s(w) < \infty\} = G(z)\mathbb{E}^*[G_{H_T}(w; z, \infty)],$$

where $\mathbb{E}^*$ denotes expectation with respect to two-sided radial SLE to z.

PROOF (SKETCH). Let $\tau_r = \tau_r(z)$. Arguing as in (15), we see that

$$\lim_{r,s\to\infty} e^{r(2-d)} e^{s(2-d)} \mathbb{P}\{\tau_{s/2}(w) < \tau_r < \tau_s(w) < \infty\} = 0.$$

Also, by (15) and Proposition 4.3,

$$\lim_{r,s\to\infty} e^{r(2-d)} \mathbb{P}\{\tau_r < \tau_{s/2}(w)\} = \lim_{r\to\infty} e^{r(2-d)} \mathbb{P}\{\tau_r < \infty\} = G(z).$$

By Proposition 4.3, there exists α such that if $\tau < \tau_{s/2}(w)$,

$$\mathbb{P}\{\tau_s(w) < \infty | \gamma_{\tau_r}\} = e^{s(d-2)} G_{H_{\tau_r}}(w; \gamma(\tau_r), \infty)[1 + O(e^{-\alpha s})].$$

Therefore,

$$\begin{aligned} &\lim_{r,s\to\infty} e^{r(2-d)} e^{s(2-d)} G(z)^{-1} \mathbb{P}\{\tau_r < \tau_s(w) < \infty\} \\ &\quad = \lim_{r,s\to\infty} \mathbb{E}[G_{H_{\tau_r}}(w; \gamma(\tau_r), \infty) 1\{\tau_r < \tau_{s/2}(w)\} | \tau_r < \infty]. \end{aligned}$$

Hence, we need to show that the right-hand side equals $\mathbb{E}^*[G_{H_T}(w; z, \infty)]$.

We assume that the curve has the radial parameterization heading to z. We use Lemma 4.4 to see that as $r, s \to \infty$,

$$\begin{aligned} &\mathbb{E}[G_{H_{\tau_r}}(w; \gamma(\tau_r), \infty) 1\{\tau_r < \tau_{s/2}(w)\} | \tau_r < \infty] \\ &\quad \sim \mathbb{E}[G_{H_{r/2}}(w; \gamma(r/2), \infty) 1\{\tau_r < \tau_{s/2}(w)\} | \tau_r < \infty] \\ &\quad \sim \mathbb{E}[G_{H_{r/2}}(w; \gamma(r/2), \infty) \mathbb{P}\{\tau_r < \tau_{s/2}(w) | \gamma_{r/2}\} | \tau_r < \infty]. \end{aligned}$$

We now use Proposition 4.3 to see that the weighting by $\mathbb{P}\{\tau_r < \tau_{s/2}(w)|\gamma_{r/2}\}$ is the same up to small error as weighting by $G_{H_{s/2}}(z; \gamma(s/2), \infty)$ which is the weighting which defines two-sided SLE going to z. The arguments for justifying this are the same whether one uses conformal radius or τ_r as the stopping time, so the proof in [9], Section 3, works here. □

REMARK. The same method shows that we can define n-point Green's function and we expect that

$$\mathbb{E}[\Theta(D)^n] = \int_{D^n} G(z_1, \ldots, z_n)\, dA(z_1) \cdots dA(z_n).$$

At the moment, we cannot prove it because we have no upper bound for $G(z_1, \ldots, z_n)$.

4.3. *Proof of* (36). We will now prove (36) for an appropriate $0 < u \le 1/4$ that we will define below. We assume that $\Im(z), \Im(w) \ge 1$ and $|z - w| > e^{-r/4}$. It suffices to prove the result for $r > 4$, and hence $|z - w| > e^{-(r-2)/2}$.

Let $0 < q < 1/8$ be a parameter that we will choose later. Let $\tau_r = \tau_r(z)$, $H = H_{\tau_r}(z)$, $l_{3/4} = l_{3/4}(r, z)$, $\lambda = \lambda(r, z, 3/4)$, $\mathcal{B}_u = \mathcal{B}_u(r, z)$, $V_u = V_u(r, z)$ be as in Section 2.4. Recall that we are assuming that $|z - w| > e^{-r/2}$ and hence $w \notin \mathcal{B}_{1/2}$.

Let $\mathcal{I}_r(z, w)$ be the indicator function of the event that $\tau_r < \tau_{qr}(w)$ and w is in the unbounded component of $H \setminus l_{3/4}$ and let $\mathcal{J}_r(z, w)$ be the indicator function of the event that w is in the bounded component of $H \setminus l_{3/4}$.

We consider two cases. First, suppose that $\mathcal{J}_r(z, w) = 1$. Then w, z are both in the bounded component of $H \setminus l_{3/4}$. Since $w \notin \mathcal{B}_{1/2}$, there is a unique subarc l' of $\partial V_{3/4} \cap H$ such that z, w are in different components of $H \setminus l'$. Since w is in the bounded component of $H \setminus l_{3/4}$, $l' \neq l_{3/4}$. In particular, z is in the unbounded component of $H \setminus l'$. The Beurling estimate implies that the probability that a Brownian motion starting at w reaches l' without leaving H is bounded above by $ce^{-r/8}$. Hence, $S_{\tau_r}(w) \leq ce^{-r/8}$ and, therefore,

$$\begin{aligned}\mathbb{P}\{\tau_r(w) < \infty | \gamma_{\tau_r}\} &\leq cG_H(w; \gamma(\tau_r), \infty)e^{r(d-2)} \\ &\leq cS_{\tau_r}(w)^{4a-1} \operatorname{dist}(w, \gamma_{\tau_r})^{d-2} e^{r(d-2)} \\ &\leq ce^{-pr} \operatorname{dist}(w, \gamma_{\tau_r})^{d-2} e^{r(d-2)},\end{aligned}$$

where $p = (4a - 1)/8 > 0$. We know from (15) that

$$\mathbb{P}\{\operatorname{dist}(w, \gamma_{\tau_r}) \leq e^s, \tau_r < \infty\} \leq c|w - z|^{d-2} e^{r(d-2)} e^{s(d-2)}.$$

By summing over positive integers $s \leq r$, we get

$$\mathbb{P}\{\tau_r < \tau_r(w) < \infty, \mathcal{J}_r(z, w) = 1\} \leq cr|w - z|^{d-2} e^{2r(d-2)} e^{-pr}.$$

In particular, if $|z - w| \geq e^{-ur}$, where $u = p/[3(2 - d)]$,

$$\mathbb{P}\{\tau_r(z) < \tau_r(w) < \infty, \mathcal{J}_r(z, w) = 1\} \leq ce^{2r(d-2)} e^{-pr/2},$$

which implies that if $|z - w| \geq e^{-ur}$,

$$\mathbb{E}[Q_r(z)Q_r(w)\mathcal{J}_r(z, w)] \leq ce^{-pr/2}.$$

For the remainder, we will assume that $\mathcal{I}_r(z, w) = 1$. Let $\sigma = \sigma_{3/4}(r - 2, z)$ as in Section 2.4. Let $\widetilde{Q}_r(z)$ be the analogue of $Q_r(z)$ for the curve stopped at time σ,

$$\widetilde{Q}_r(z) = e^{r(2-d)}[1\{\tau_r(z) < \sigma\} - e^{\delta(2-d)} 1\{\tau_{r+\delta}(z) < \sigma\}].$$

To establish our estimate, we will show that

$$|\mathbb{E}[\widetilde{Q}_r(z)Q_r(w)\mathcal{I}_r(z, w)]| \leq ce^{-\beta r}, \tag{47}$$

and

$$\mathbb{E}[|Q_r(z) - \widetilde{Q}_r(z)||Q_r(w)|\mathcal{I}_r(z, w)] \leq ce^{-\beta r}, \tag{48}$$

which together imply that

$$|\mathbb{E}[Q_r(z)Q_r(w)\mathcal{I}_r(z, w)]| \leq ce^{-\beta r}.$$

To prove (47), note that since $\widetilde{Q}_r(z)\mathcal{I}_r(z, w)$ is γ_σ measurable,

$$\mathbb{E}[\widetilde{Q}_r(z)Q_r(w)\mathcal{I}_r(z, w)] = \mathbb{E}[\widetilde{Q}_r(z)\mathcal{I}_r(z, w)\mathbb{E}(Q_r(w)|\gamma_\sigma)],$$

and hence,

$$\left|\mathbb{E}\left[\widetilde{Q}_r(z)Q_r(w)\mathcal{I}_r(z,w)\right]\right| \le \mathbb{E}\left[\left|\widetilde{Q}_r(z)\right|\mathcal{I}_r(z,w)\left|\mathbb{E}\left(Q_r(w)|\gamma_\sigma\right)\right|\right].$$

We appeal to (44) to see that

$$\begin{aligned}\left|\mathbb{E}\left(Q_r(w)|\gamma_\sigma\right)\right| &\le ce^{(2-d)r}\left[e^{-r}/\operatorname{dist}(w,\partial H_\sigma)\right]^{2-d+\alpha}\\ &\le c\exp\left\{\left[(2-d)+(q-1)(2-d+\alpha)\right]r\right\}.\end{aligned}$$

In particular, if q is chosen sufficiently small so that $q(2-d)\le\alpha(1-q)/2$,

$$\left|\mathbb{E}\left(Q_r(w)|\gamma_\sigma\right)\right| \le ce^{-\alpha r/2},$$

and hence

$$\begin{aligned}\left|\mathbb{E}\left[\widetilde{Q}_r(z)Q_r(w)\mathcal{I}_r(z,w)\right]\right| &\le ce^{-r\alpha/2}\mathbb{E}\left[\left|\widetilde{Q}_r(z)\right|\mathcal{I}_r(z,w)\right]\\ &\le ce^{-r\alpha/2}e^{r(2-d)}\mathbb{P}\left[\mathcal{I}_r(z,w)\right] \le ce^{-r\alpha/2}.\end{aligned}$$

For (48), we observe that if $Q_r(z)\ne\widetilde{Q}_r(z)$, then $\operatorname{dist}(z,\gamma)<\operatorname{dist}(z,\gamma_\sigma)$. In other words,

$$\begin{aligned}&\mathbb{E}\left[\left|Q_r(z)-\widetilde{Q}_r(z)\right|\left|Q_r(w)\right|\mathcal{I}_r(z,w)\right]\\ &\quad\le ce^{2r(2-d)}\mathbb{P}\left\{\mathcal{I}_r(z,w)=1,\operatorname{dist}(z,\gamma)<\operatorname{dist}(z,\gamma_\sigma),\tau_r(w)<\infty\right\}.\end{aligned}$$

We know that $\mathbb{P}\{\mathcal{I}_r(z,w)=1\}\le ce^{r(d-2)}$. Hence, it suffices to show that we can find $q,\beta>0$ and $c<\infty$ such that that on the event $\{\mathcal{I}_r(z,w)=1\}$,

$$\mathbb{P}\left\{\rho<\infty,\tau_r(w)<\infty|\gamma_\sigma\right\}\le ce^{r(2-d)}e^{-r\beta},$$

where $\rho=\inf\{t\ge\sigma:|\gamma(t)-z|=\operatorname{dist}(z,\gamma_\sigma)\}$. For every integer k with $qr+1<k<r-1$, we consider the event

$$E_k=\left\{\tau_k(w)<\rho<\tau_{k+1}(w)<\tau_r(w)<\infty\right\}.$$

We claim that there exists c,α such that on the event $\{\mathcal{I}_r(z,w)=1\}$

$$\mathbb{P}(E_k|\gamma_\sigma)\le ce^{(2-d)(q-1)r}e^{-\alpha r}. \tag{49}$$

Indeed, recall that on the event on the event $\{\mathcal{I}_r(z,w)=1\}$, $\operatorname{dist}(w,\partial H_\sigma)\ge e^{-qr}$. If we use $\widetilde{\mathbb{P}}$ to denote conditional probabilities given γ_σ, then

$$\begin{aligned}\widetilde{\mathbb{P}}\left\{\tau_k(w)<\infty\right\}&\le ce^{(d-2)(k-qr)},\\ \widetilde{\mathbb{P}}\left\{\rho<\infty|\tau_k(w)<\infty\right\}&\le ce^{-\alpha r},\\ \widetilde{\mathbb{P}}\left\{\tau_r(w)<\infty|\tau_k(w)<\rho<\tau_{k+1}(w)<\infty\right\}&\le ce^{(d-2)(r-k)}.\end{aligned}$$

By summing (49) over k, and then choosing q sufficiently small, we see that

$$\widetilde{\mathbb{P}}\left\{\rho<\infty,\tau_r(w)<\infty\right\}\le cre^{(2-d)(q-1)r}e^{-\alpha r}\le ce^{r(d-2)}e^{-\alpha r/2}.$$

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