

A free stochastic partial differential equation¹

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Abstract. We get stationary solutions of a free stochastic partial differential equation. As an application, we prove equality of non-microstate and microstate free entropy dimensions under a Lipschitz like condition on conjugate variables, assuming also the von Neumann algebra R^ω embeddable. This includes an N -tuple of q -Gaussian random variables e.g. for $|q|N \leq 0.13$.

Résumé. Nous construisons des solutions stationnaires de certaines équations différentielles stochastiques libres à coefficients opérateurs non-bornés. Comme application, nous montrons l'égalité des dimensions entropiques libres microcanonique et non-microcanonique sous l'hypothèse d'une variable conjuguée Lipschitz pour les générateurs X_1, \dots, X_N d'un espace de probabilité non-commutatif inscriptible dans une ultrapuissance R^ω du facteur hyperfini. Cette hypothèse de variable conjuguée Lipschitz inclut le cas de N variables aléatoires q -Gaussiennes pour de petits q par exemple $|q|N \leq 0.13$.

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Introduction

In a fundamental series of papers, Voiculescu introduced analogs of entropy and Fisher information in the context of free probability theory. A first microstate free entropy $\chi(X_1, \dots, X_n)$ is defined as a normalized limit of the volume of sets of microstate i.e. matricial approximations (in moments) of the n -tuple of self-adjoints X_i living in a (tracial) W^* -probability space M . Starting from a definition of a free Fisher information [43], Voiculescu also defined a non-microstate free entropy $\chi^*(X_1, \dots, X_n)$, known by the fundamental work [2] to be greater than the previous microstate entropy, and believed to be equal (at least modulo Connes' embedding conjecture). For more details, we refer the reader to the survey [45] for a list of properties as well as applications of free entropies in the theory of von Neumann algebras.

Moreover in case of infinite entropy, two other invariants the microstate and non-microstate free entropy dimensions (respectively written $\delta_0(X_1, \dots, X_n)$ and $\delta^*(X_1, \dots, X_n)$) have been introduced to generalize results known for finite entropy. Surprisingly, Connes and Shlyakhtenko found in [10] a relation between those entropy dimensions and the first L^2 -Betti numbers they defined for finite von Neumann algebras. For instance, for (real and imaginary parts of) generators of finitely generated groups, δ^* has been proved in [27] to be equal to $\beta_1^{(2)}(\Gamma) - \beta_0^{(2)}(\Gamma) + 1$ (cf. e.g. [25] for L^2 -Betti numbers of groups).

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In [39], Dimitri Shlyakhtenko obtained lower bounds on microstate free entropy dimension (motivated by the goal of trying to prove equality with non-microstate free entropy dimension), in studying the following free stochastic differential equation:

$$X_t^{(i)} = X_0^{(i)} - \frac{1}{2} \int_0^t \xi_s^{(i)} ds + S_t^{(i)},$$

where $\xi_s^{(i)}$ is the i th conjugate variable of $X_s^{(i)}$'s in the sense of [43], $S_t^{(i)}$ a free Brownian motion free with respect to $X_0^{(i)}$. Let us recall that for $(M = W^*(X_1, \dots, X_N), \tau)$, if X_1, \dots, X_N are algebraically free, the i th partial free difference quotient $\partial_i : L^2(M) \rightarrow L^2(M) \otimes L^2(M)$ is the unique derivation densely defined (on non-commutative polynomials) such that $\partial_j(X_i) = 1_{i=j} 1 \otimes 1$. Then the i th conjugate variable is defined by $\partial_i^*(1 \otimes 1) \in L^2(M)$ if it exists. In [39], this equation was solved in order to get stationary solutions for analytic conjugate variable, and thus this paper proved that in case of analytic conjugate variable if moreover $W^*(X_1, \dots, X_N)$ is R^ω embeddable, then $\delta_0(X_1, \dots, X_N) = \delta^*(X_1, \dots, X_N) = N$. Of course, if we believe in the previous general equality, this should be proved in much more general cases, e.g. for L^2 conjugate variable, i.e. finite Fisher information. The goal of this paper is to prove this equality in an intermediate case, under a Lipschitz like condition on conjugate variables. Let us emphasize our definition does not involve operator Lipschitz functions and is relative to M , but it is nothing but the usual notion of being a Lipschitz function of X (for instance applied by functional calculus) in the one variable case (this is a Sobolev like definition of Lipschitzness in the one variable case):

Definition 1. $(M = W^*(X_1, \dots, X_N), \tau)$ is said to satisfy a Lipschitz conjugate variable condition if the partial free difference quotients ∂_i are defined and if the conjugate variables $\partial_i^* 1 \otimes 1$ exist in $L^2(M)$ (for all i) and moreover are in the domain of the closure $\bar{\partial}$ of $(\partial_1, \dots, \partial_N)$ with $\bar{\partial}_j \partial_i^* 1 \otimes 1 \in M \bar{\otimes} M^{\text{op}} \subset L^2(M \otimes M^{\text{op}}) \simeq L^2(M \otimes M)$ (von Neumann tensor product, M^{op} the opposite algebra).

Let us state a precise result, the main byproduct of our work in this respect (cf. Corollary 25) is the following:

Theorem. Consider $(M = W^*(X_1, \dots, X_N), \tau)$ a R^ω -embeddable finite von Neumann algebra satisfying a Lipschitz conjugate variable condition. Then the microstate entropy dimension $\delta_0(X_1, \dots, X_N) = N$.

We show in Section 4.3 that q -Gaussian variables (introduced in [7]) are a non-trivial instance of non-commutative variables having Lipschitz conjugate variables for small q (e.g. $|q|N \leq 0.13$) thus improving a computation in [39] and proving that δ_0 does not only converge to N for small q but is identically equal to N and thus equal to $\delta^*(X_1, \dots, X_N)$. One could actually prove with our techniques $\delta_0(X_1, \dots, X_N) = N$ is still valid on a slightly larger range of q 's, i.e. as soon as $|q|N < 1$ and $|q|\sqrt{N} \leq 0.13$, we will detail this elsewhere.

Let us come back to our stochastic differential equation setting. By lack of a theory of “non-commutative Lipschitz functions,” we will rather solve a dual stochastic partial differential equation with the right stationarity property to get this result.

To explain the equation we solve, let us note that if we call $\Phi_s(X) = X_s$ the automorphism we hope being able to build solving the above equation, then Ito formula implies e.g. for any non-commutative polynomials P :

$$\Phi_t(P(X_0)) = \Phi_0(P(X_0)) - \frac{1}{2} \int_0^t \Phi_s(\Delta(P(X_0))) ds + \int_0^t \Phi_s \otimes \Phi_s(\delta(P(X_0))) \# dS_s.$$

We refer the reader to the main text for reminders about free stochastic integration. Here $\delta = (\delta_1, \dots, \delta_n)$ is the free difference quotient, $\Delta = \delta^* \delta$. This is also the equation the author solved in a more recent paper [12] in a much more general context but with more limited applications to microstate free entropy dimensions, because of a lack of control on the von Neumann algebra in which we build the process in this new approach.

Here we will thus rather solve the following dual equation:

$$X_t = X_0 - \frac{1}{2} \int_0^t \Delta(X_s) ds + \int_0^t \delta(X_s) \# dS_s, \tag{1}$$

where δ will be an appropriate extension of the free difference quotient by zero on the free Brownian motion and a corresponding $\Delta = \delta^*\delta$. It is well known in quantum stochastic integration over symmetric Fock space that solving right Hudson–Parthasarathy equations instead of left HP equations enables to solve equations in a mild sense (see e.g. [18]) as in the classical stochastic differential equation case (see e.g. [13]). It is in order to use those techniques we considered this equation rather than the previous one.

Before describing the content of this paper, let us explain the relation of our work with classical stochastic partial differential equations. There are basically three main approaches to analysing SPDEs: the “martingale (or martingale measure) approach” (cf. [47]), the “semigroup (or mild solution) approach” (cf. [13]) and the “variational approach” (cf. [35]). We will mainly refer to the above monographs instead of the original enormously rich literature. Beyond those mainstream approaches, one should also mention Krylov’s L^p -theory [22] and Kostelenez’s methods [21] using limits of particle systems, and also an old approach for more concrete SPDEs using SDEs in nuclear spaces and distributions (e.g. [41]). Here we will adapt to the free SDE context a part of the semigroup approach using variational techniques. To compare with our work, we thus insist here on those two approaches.

To fix ideas a general SPDE considered in the classical literature is often of the form:

$$dX_t(\xi) = A(t, X_t(\xi), D_\xi X_t(\xi), D_\xi^2 X_t(\xi)) dt + B(t, X_t(\xi), D_\xi X_t(\xi)) dW_t.$$

Since this will be our main interest, we will mainly focus on the linear time independent case where A is thus a second order differential operator, B a first order one, let say valued in Hilbert–Schmidt operators from the noise space Y (let say W_t is a standard (with covariance id) cylindrical Brownian motion on a Hilbert space Y) to the space H where X_t lives. The linear case was also motivated in the early theory by filtering problems giving rise to linear equations suitable for the variational approach.

Both approaches share the common features of considering SPDEs as SDEs valued in infinite dimensional spaces (usually Hilbert spaces of Sobolev types), using PDE techniques often in an abstract functional analytic setting.

Let us describe first the variational approach, originating from [23,30,31] (we refer to [35], and the recent introductory [34] in the coercive case). Usually, solutions are built here by a Galerkin scheme, in first projecting the equation to finite dimensional sub-Hilbert spaces. After this transformation, the equation is an ordinary SDE solved by usual techniques. At this level, estimates (for this approximation) are proved, enabling to take a (weak) limit. The equation is first solved in a weak sense, avoiding to require $X_t \in D(A)$. The standard assumption is the so called coercivity condition (also called superparabolic case in [35] when considered for concrete differential operators).

This is roughly written:

$$2\langle x, Ax \rangle + \|B(x)\|_{HS}^2 + \delta \|x\|_U^2 \leq K \|x\|_H^2,$$

where U is another Hilbert space such that $U \subset D(B)$ continuously (often if A is time independent self-adjoint, $-A$ positive, $U = D((-A)^{1/2})$, for instance, to fix ideas). Having $\delta > 0$ then enables to get a bound on $\|X_t\|_H$ and say $\int_0^t \|X_s\|_U^2 ds$ giving sufficiently many regularity to get a weak limit, so that $B(X_t)$ makes sense, and to solve the equation weakly (i.e. after taking scalar products with $y \in D((-A)^{1/2})$ for instance in the self-adjoint case). Unfortunately, the case we are interested in is not coercive, it only satisfies the dissipative condition where $\delta = 0$ above (sometimes called degenerate parabolic case). This equation is enough to guaranty a bound on $\|X_t\|_H$ but nothing more. In this case, the usual method (for instance used in [35], Chapter 4, in a concrete differential operator setting), is to replace B by $(1 - \varepsilon)B$ (or A by $(1 + \varepsilon)A$) to get a coercive equation and get the bound on $\|X_t\|_U$ necessary to get a weak limit by another technique. In the coercive case, there are also standard ways of getting regularity results (for instance, we assume a dissipative inequality under an overall $(-A)^{1/2}$ for instance again in the self-adjoint positive case, i.e. $-2\langle Ax, Ax \rangle + \|(-A)^{1/2} B(x)\|_{HS}^2 \leq K \|(-A)^{1/2} x\|_H^2$, and not surprisingly deduce from this a bound on $\|(-A)^{1/2} X_t\|_H$, the equation being ideal to apply Gronwall’s lemma and get a bound in that way). One thus uses these standard ways of getting regularity via a priori estimates for the approximating equation to get a weak limit.

The semigroup approach uses the semigroup ϕ_t generated by A and rewrites the equation after “variation of constants,” and thus looks for a so-called mild solution, i.e. a solution of:

$$X_t = \phi_t(X_0) + \int_0^t \phi_{t-s}(B(X_s) dW_s).$$

Then the goal is to use regularization properties of this semigroup to solve this equation. For instance, to solve non-linear equations with only continuous coefficients for B , one can use compactness of the semigroup and use compactness arguments (and get a stochastically weak solution, i.e. not adapted to the filtration of the Brownian motion. Note we use in this paper only the word weak in its PDE sense as in [13]). In more standard assumptions, the semigroup is only assumed analytic, or with generator a variational or a self-adjoint operator. We will be mainly interested in the semigroup approach under the same assumptions as in the variational approach. Indeed, in our free SDE setting, it is not quite clear what kind of Galerkin's method could make us recover an ordinary free SDE setting. Moreover as we will see, we will use extensively really weak notions of being a mild solution we will call ultramild as a crucial tool to get results under really weak assumptions. Anyways, the interest of the semigroup approach for us lies in the fact it replaces Galerkin's method by a fixed point argument (for contractions) under the same coercivity assumption. Then, we can again prove a priori estimates to extend this to the degenerate parabolic case we will be interested in (since only this case can give stationary solutions in our examples).

Let us now finally describe the content of this article. In Section 1, we solve a really general stochastic partial differential equation (formally of the form (1)) with much less restrictive assumptions on δ , Δ . We find natural assumptions to get two kinds of solutions we will call mild and ultramild solutions, this second really weak sense of getting a solution has never been considered, to the best of our knowledge, in previously quoted contexts. These conditions are natural analogs of the dissipativity condition above (in case of ultramild solutions) and the dissipativity condition under $(-A)^{1/2}$ (to get regularity conditions and for us mild solutions). We have also to include in these conditions general compatibility assumptions trivially checked in our main example.

In Section 2, we prove that we can check our assumptions to get mild solutions in the free difference quotient case with a Lipschitz conjugate variable type assumption as explained above. The crucial issue here is to prove non-trivial domain properties of δ , Δ which are usually checked classically using general regularity results of PDEs not available in our non-commutative context. One of the crucial tools here is an easy boundedness criteria for $1 \otimes \tau \circ \delta$ found by the author in [11] (cf. Lemma 18 *infra*. coming from [11], Lemma 10).

In Section 3, we prove that as soon as we stick to our case of main interest of a derivation and the corresponding divergence form operator, it suffices to check $\|X_t\|_2 = \|X_0\|_2$ in order to prove any ultramild solution to be stationary, as we want in order to get lower bounds on microstate free entropy dimension. Especially, this is always true if we can get a mild solution, and this is really likely why ultramild solutions were never considered before. If we don't get an isometric map, solving those equations is not such useful.

In Section 4, we explain our main application about computation of microstate free entropy dimension under Lipschitz conjugate variable assumption. In Section 4.3, as we said above, we explain the concrete example of q -Gaussian variables, after several general preliminaries gathered in Section 4.2. Here the proof of Lipschitz conjugate variable relies heavily on Bożejko's analog of Haagerup's Inequalities [5]. We will consider elsewhere how one could use a non-coassociative derivation to compute microstate free entropy dimension of q -Gaussian variables in a slightly less small range of q 's. Of course it is possible that a better understanding of combinatorial properties of those examples may give more extended ranges of q 's with the same free SDE techniques. Finally, in Section 4.4, we explain how hard it is to get stationary solutions in an example of derivations on group von Neumann algebras coming from group cocycles valued in the left regular representation, case also considered in [39]. Here coassociativity like assumptions are not available to get "easily" mild solutions, this is why we were motivated in being able to get solutions in a really general sense like ultramild solutions under somehow an automatically verified assumption. Indeed, in such a concrete example one can easily find a necessary and sufficient condition for getting $\|X_t\|_2 = \|X_0\|_2$. However, it is expressed in terms of conservativity of a classical Markov process, well known to be hard to check. This is not such surprising since unitarity properties of left Hudson–Parthasarathy equations are also expressed in terms of conservativity of quantum Markov processes (see e.g. the survey [17]). Of course, the occurrence of a classical process is only explained by our special example on groups, anyone interested in such a criteria for more general processes may be able to generalize this to a general case using conservativity of an appropriate quantum Markov process. However since any useful (easy to check) sufficient condition for proving this conservativity is not really available (even in HP case) beyond conditions really similar to those of our Section 2 to get mild solutions (cf. [8]), we don't enter in this general question here. Let us conclude with two remarks. Having in mind those similarities with questions of unitarity of solutions of Hudson–Parthasarathy equations, we can wonder whether a duality theory analogous to (Journe) duality of left and right HP equations could be developed in our context. In the other direction, one may wonder whether an ultramild like definition of a solution may be useful for right HP equations (e.g. in order

to solve them under weaker conditions expressed in terms of conservativity assumptions similar to left HP equations) or whether the new approach of [12] could be translated in the context of left HP equations.

1. A general stochastic differential equation with unbounded coefficients

Let M_0 be a W^* -probability space (with separable predual), $S_t^{(i)}$ ($i \in \mathbb{N}$) a family of free Brownian motions. Consider $M = M_0 \star W^*(S_t^{(i)})$ the free product of W^* -probability spaces (so that $S_t^{(i)}$ are free with M_0 inside M) and consider finally the natural filtration $M_s = M_0 \star W^*(S_t^{(i)}, t \leq s)$. As a side remark, note we always use scalar products linear in the second variable.

In this part, we will be interested in the following equation:

$$X_t = X_0 - \frac{1}{2} \int_0^t \Delta(X_s) ds + \int_0^t \delta(X_s) \# dS_s, \tag{2}$$

where $\Delta : L^2(M) \rightarrow L^2(M)$ and $\delta : L^2(M) \rightarrow L^2(M) \otimes L^2(M) \oplus^{\mathbb{N}}$ are closed densely defined operators and keeping invariant for $t \in [0, T]$ $L^2(M_t)$ (resp. sending it to $L^2(M_t \otimes M_t) \oplus^{\mathbb{N}}$ and with the analog property for its adjoint. By convention we say a closed densely defined unbounded operator keep invariant a subspace or send a closed subspace S into another one S' if its restriction to the intersection $D \cap S$ of its domain D with S is valued in S' and the restricted operator is again a closed densely defined unbounded operator. See Section 1.3 for a definition of Stochastic integral). The sense in which we will solve this equation will be made precise in the 3 following sections: the first will deal with some miscellaneous results about stochastic integration in our context, the second will introduce stochastic convolution, the key tool to define mild solutions and the third one will prove in the free Brownian case some well-known (in the classical Brownian motion case) relations between mild and strong solutions, and introduce ultramild solutions (the three kinds of solutions we will be interested in getting). Let us right now state the two class of assumptions we will need to get mild (resp. ultramild) solutions in the last subsection of this section.² We also consider given another operator $\tilde{\delta}$ satisfying the assumptions for δ , i.e. $\tilde{\delta} : L^2(M) \rightarrow L^2(M) \otimes L^2(M) \oplus^{\mathbb{N}}$ are closed densely defined operators keeping invariant the corresponding filtrations. (This will be useful for further applications in the q -Gaussian variable case. We will consider them elsewhere.)

We fix a few notation before stating the assumption. We will write (for $t > s$) $U\#(S_t - S_s) = \sum_{i=0}^{\infty} U^{(i)}\#(S_t^{(i)} - S_s^{(i)})$ the Hilbert space isomorphism between the infinite direct sum of coarse correspondences $(L^2(M_s) \otimes L^2(M_s)) \oplus^{\mathbb{N}}$ and a corresponding subspace of $L^2(M_t)$, where $U^{(i)}\#(S_t^{(i)} - S_s^{(i)})$ is the linear isomorphism extending $a \otimes b\#(S_t^{(i)} - S_s^{(i)}) = a(S_t^{(i)} - S_s^{(i)})b$, for $a, b, c \in M_s$. Likewise, for 3-fold tensor products, we write $(a \otimes b \otimes c)\#_1(S_t^{(i)} - S_s^{(i)}) = a(S_t^{(i)} - S_s^{(i)})b \otimes c$, $(a \otimes b \otimes c)\#_2(S_t^{(i)} - S_s^{(i)}) = a \otimes b(S_t^{(i)} - S_s^{(i)})c$ and their corresponding L^2 extensions. On a direct sum, we write $\text{Diag}(A_i)$ the operator acting diagonally, e.g. $\text{Diag}(A_i)(b_1 \oplus b_2) = A_1b_1 \oplus A_2b_2$.

We will first use an assumption $\Gamma_0(\omega)$ to get really weak forms of solutions, we will call ultramild.

$$\Gamma_0(\omega) \left\{ \begin{array}{l} (a) \quad \Delta \text{ is a positive self-adjoint operator, } \eta_\alpha = \frac{\alpha}{\alpha + \Delta}. \\ (b.1) \quad D(\Delta^{1/2}) \subset D(\delta), \\ (c.1) \quad \text{for any } x \in D(\Delta^{1/2}) \text{ we have: } -\|\Delta^{1/2}x\|_2^2 + \|\delta(x)\|_2^2 \leq \omega\|x\|_2^2. \\ (d.1) \quad \text{There exists a closed densely defined positive operator} \\ \quad \Delta^\otimes =: \text{Diag}(\Delta_i^\otimes) : (L^2(M) \otimes L^2(M)) \oplus^{\mathbb{N}} \rightarrow (L^2(M) \otimes L^2(M)) \oplus^{\mathbb{N}} \\ \quad \text{(acting diagonally with respect to the direct sum and keeping invariant,} \\ \quad \text{for any } t, L^2(M_t \otimes M_t) \oplus^{\mathbb{N}} \text{ and) such that } \forall U \in L^2(M_s) \otimes L^2(M_s) \cap D(\Delta_i^\otimes): \\ \quad U\#(S_t^{(i)} - S_s^{(i)}) \in D(\Delta) \text{ and } \Delta(U\#(S_t^{(i)} - S_s^{(i)})) = \Delta_i^\otimes(U)\#(S_t^{(i)} - S_s^{(i)}). \\ (d.2) \quad \text{Moreover } \delta(U\#(S_t^{(i)} - S_s^{(i)})) \text{ is orthogonal to } L^2(M_s \otimes M_s) \oplus^{\mathbb{N}}. \end{array} \right.$$

²Note we will always write $A \circ B$ the closure of the composition of two closed operators if possible, and the usual composition if they are not closed, without risk of confusion. Sometimes we will even write AB for the same object.

We will use a variant $\Gamma_{0u}(\omega)$ with an extra Assumption e added to get an extra uniqueness, technically provided by checking our solution will be also what we will call an ultraweak solution. For convenience, we write $((L^2(M_0)) \otimes (L^2(M_0) \ominus \mathbb{C})^{\otimes n-1} \otimes (L^2(M_0))) = V_n$. We also consider an orthonormal basis $(e_n)_{n \in \mathbb{N}}$ of $L^2(W^*(S_t^{(i)}, t > 0, i \in \mathbb{N})) \ominus \mathbb{C}$ such that for all $i, e_i \in W^*(S_t^{(j)}, t > 0, j \in \mathbb{N})$. We also write for any $a_1, \dots, a_{n+1} \in M_0$ with $\tau(a_i) = 0$, $(i \neq 1, n+1)$ $(a_1 \otimes \dots \otimes a_{n+1})\#(e_{i_1} \otimes \dots \otimes e_{i_n}) = a_1 e_{i_1} a_2 \dots a_n e_{i_n} a_{n+1}$, and likewise $U\#(e_{i_1} \otimes \dots \otimes e_{i_n})$ the isometric extension to $U \in (L^2(M_0)) \otimes (L^2(M_0) \ominus \mathbb{C})^{\otimes n-1} \otimes (L^2(M_0))$.

Finally, we consider $\mathcal{E}_n = \text{Span}\{\alpha_0(S_{v_1}^{(k_1)} - S_{u_1}^{(k_1)}) \dots (S_{v_n}^{(k_n)} - S_{u_n}^{(k_n)})\alpha_n | \alpha_i \in M_0, k_i \in \mathbb{N}, [u_1, v_1] \times \dots \times [u_n, v_n] \subset \mathbb{R}_+^n - D^n\}$, where D^n is the full diagonal (the set of n -tuples having at least one pair of equal coordinates). We write $\bar{\mathcal{E}}_n$ the closure of \mathcal{E}_n in $L^2(M)$.

$$\Gamma_{0u}(\omega) \left\{ \begin{array}{l} \Gamma_0(\omega) \\ (e.1) \quad D(\Delta) \otimes_{\text{alg}} D(\Delta) \subset D(\delta^*). \\ (e.2) \quad \forall n \in \mathbb{N}^* \exists \Delta^{\otimes(n+1)} : V_n \rightarrow V_n \text{ closed densely defined positive operator} \\ \text{such that } \forall (i_1, \dots, i_n) \in \mathbb{N}^n, \forall U \in D(\Delta^{\otimes(n+1)}): \\ U\#(e_{i_1} \otimes \dots \otimes e_{i_n}) \in D(\Delta) \\ \text{and } \Delta(U\#(e_{i_1} \otimes \dots \otimes e_{i_n})) = \Delta^{\otimes(n+1)}(U)\#(e_{i_1} \otimes \dots \otimes e_{i_n}). \\ (e.3) \quad \mathcal{E}_n \cap D(\delta) \subset \bar{\mathcal{E}}_n \text{ dense and } \delta(\mathcal{E}_n \cap D(\delta)) \text{ orthogonal to } \bigoplus_{p+q < n} (\mathcal{E}_p \otimes \mathcal{E}_q)^{\oplus \mathbb{N}}. \\ (e.4) \quad \exists \mathcal{D} = \text{Span } \mathcal{D} \subset L^2(M_0) \ominus \mathbb{C} \text{ a dense subspace such that:} \\ (\mathcal{D} \oplus \mathbb{C}) \otimes \mathcal{D}^{\otimes(n-1)} \otimes (\mathcal{D} \oplus \mathbb{C}) \subset D(\Delta^{\otimes(n+1)}). \end{array} \right.$$

The main assumption (useful to get mild solutions) will be called $\Gamma_1(\omega, C)$ (and $\Gamma_{1u}(\omega, C)$ if we add $\Gamma_{0u}(\omega)$):

$$\Gamma_1(\omega, C) \left\{ \begin{array}{l} \Gamma_0(\omega) \\ (b.2) \quad D(\delta \circ \Delta) \subset D(\delta). \\ (c.2) \quad \text{For any } x \in D(\Delta^{1/2}): \\ -\|\Delta \eta_\alpha^{1/2} x\|_2^2 + \text{Re} \langle \delta(\Delta \eta_\alpha(x)), \delta(x) \rangle \leq \omega \|\Delta^{1/2} \eta_\alpha^{1/2} x\|_2^2. \\ (d.3) \quad D(\Delta^{1/2}) \subset D(\tilde{\delta}), D(\Delta^{1/2}) \text{ a core for } \tilde{\delta}, D(\tilde{\delta} \circ \Delta) \subset D(\tilde{\delta}) \\ \forall U \in L^2(M_s \otimes M_s) \cap D(\Delta_i^\otimes), \tilde{\delta}(U\#(S_t^{(i)} - S_s^{(i)})) \text{ orthogonal to} \\ L^2(M_s \otimes M_s)^{\oplus \mathbb{N}} \text{ and we assume we have a closed densely defined} \\ \tilde{\delta}^\otimes := \tilde{\delta}^{\otimes 1} \oplus \tilde{\delta}^{\otimes 2}, \tilde{\delta}^{\otimes i} : (L^2(M^{\otimes 2}))^{\oplus \mathbb{N}} \rightarrow (L^2(M^{\otimes 3}))^{\oplus \mathbb{N}^2} \\ \tilde{\delta}^{\otimes i}((x_j)_j) =: (\tilde{\delta}_{l,j}^{\otimes i}(x_j))_{(l,j)}, \tilde{\delta}_{l,j}^{\otimes i} : (L^2(M)^{\otimes 2}) \rightarrow (L^2(M)^{\otimes 3}) \\ \text{(keeping invariant, for any } t, \text{ the filtration induced by } M_t \text{ and)} \\ \forall U \in L^2(M_s) \otimes L^2(M_s) : U\#(S_t^{(j)} - S_s^{(j)}) \in D(\tilde{\delta}_l) \text{ if } U \in D(\tilde{\delta}_{l,j}^{\otimes 1} \oplus \tilde{\delta}_{l,j}^{\otimes 2}) \\ \text{and } \tilde{\delta}_l(U\#(S_t^{(j)} - S_s^{(j)})) = \sum_{i=1}^2 \tilde{\delta}_{l,j}^{\otimes i}(U)\#_i(S_t^{(j)} - S_s^{(j)}). \\ (d.4) \quad D(\Delta^{\otimes 1/2}) \subset D(\tilde{\delta}^\otimes), D(\Delta^{\otimes 1/2}) \text{ a core for } \tilde{\delta}^\otimes. \\ (f.1) \quad D(\tilde{\delta} \circ \Delta) \subset D(\Delta^\otimes \circ \tilde{\delta}). \\ (f.2) \quad \text{There exists a bounded operator } \mathcal{H} \text{ on } L^2(M \otimes M)^{\oplus \mathbb{N}} \\ \text{keeping invariant for any } s, L^2(M_s \otimes M_s)^{\oplus \mathbb{N}} \text{ with } \|\mathcal{H}\| \leq C^{1/2} (C \geq 1) \\ \text{such that for any } x \in D(\tilde{\delta} \circ \Delta): \\ \Delta^\otimes \circ \tilde{\delta}(x) - \tilde{\delta} \circ \Delta(x) = \mathcal{H}(\tilde{\delta}(x)). \\ (g) \quad D(\tilde{\delta}) = D(\delta) =: \tilde{\mathcal{D}}, \forall x \in \tilde{\mathcal{D}}, C^{-1/2} \|\tilde{\delta}(x)\|_2 \leq \|\delta(x)\|_2 \leq C \|\tilde{\delta}(x)\|_2, \\ \text{thus } D(\tilde{\delta} \circ \Delta) = D(\delta \circ \Delta). \\ (h) \quad D(\Delta) \subset D(\Delta^{\otimes 1/2} \circ \tilde{\delta} \oplus \tilde{\delta} \oplus \tilde{\delta}^\otimes \delta) \text{ and for } x \in D(\Delta): \\ \|\tilde{\delta}^\otimes \delta(x)\|_2^2 \leq \|\Delta^{\otimes 1/2} \circ \tilde{\delta}(x)\|_2^2 + C \|\tilde{\delta}(x)\|_2^2. \end{array} \right.$$

We will write ϕ_t the semigroup exponentiating $-1/2\Delta$ and ϕ_t^\otimes the semigroup associated to $-1/2\Delta^\otimes$.

In most cases we will be interested in the case $\tilde{\delta} = \delta$ in which case assumptions g, h will be automatic (using a variant of $c.1$ for tensor variants for the inequality in h and f for the domain assumption in h).

In the applications we have in mind, the strong assumption is f , the other ones being automatically verified and just important in this general setting.

1.1. General ideas and strategy

With those notation fixed and before entering into technical details, let us explain the intuition behind our results (in the case $\tilde{\delta} = \delta$, the general case is a slight extension following an idea of [8]). In our general setting here, the proofs will follow closely the classical case, and therefore the intuition is basically the same, namely, since we want to solve SDEs with unbounded coefficients, with Δ a kind of divergence form operator (as we will consider in the next part) the corresponding semigroup is regularizing, and we want to use this. That's why we introduce mild solutions. As explained by various equivalences in Section 2.1.3, the difference with strong solutions is only related to the domain in which we want to build the solution, we only require being in the domain of $\Delta^{1/2}$ (or even δ) for mild solutions, and as soon as it is in the domain of Δ , a mild solution is a strong solution, the converse being always true. The idea behind ultramild solutions is slightly trickier. Let us explain it in saying $\phi_{t-s}^{\otimes} \circ \delta$ may have a much huger domain again than $\Delta^{1/2}$ and we want to use this regularization effect to have solutions with almost no conditions. Indeed, condition f above will be really hard to check even with strong conditions (Section 2.2), that's why we want to have solutions in a sense as general as possible. We can also say that the current section somehow takes natural analogs of classical assumptions in the non-commutative case and check we can work with them.

It is maybe also useful to have several ideas in mind, and first how those conditions will appear in a really natural way in the proof. To get an estimate on $\|X_t\|_2^2$, or $\|\Delta^{1/2}(X_t)\|_2^2$ (first on an approximation of the solution, in the spirit of moving from a degenerate parabolic case to a superparabolic case), the common idea is to differentiate, and try to apply Gronwall's lemma. Conditions $c.1$ (called dissipativity in the classical case) and $f.2$ above correspond exactly to what we want, in order to apply this lemma respectively in those cases. The second idea is that if we replace δ by $(1 - \varepsilon)\delta$ the equation is much easier to solve, it is of superparabolic type (instead of degenerate parabolic type, said otherwise this gives a kind of coercivity, see [35] for a presentation of this point in a more concrete setting but more clearly than in [13]). First, in this case, there will be a Picard iteration argument to solve it, second we win something in terms of domains, assumption c is enough to bound $\|\Delta^{1/2}(X_t^\varepsilon)\|_2^2$, assumption f to get a bound on $\|\Delta(X_t^\varepsilon)\|_2^2$ (those bounds diverging in ε , of course). Anyways this will enable us to have respectively mild or strong solutions of an approximating equation converging to a solution of our equation, even if the solution without ε will be only a mild or ultramild solution (note that in the case $\delta \neq \tilde{\delta}$ we will lose the strong solution property but keep mild solutions, hopefully we don't use this improvement in terms of getting a strong solution). Somehow, to get later in Section 3.3 stationarity of the equation, this will be much more crucial to have an approximation by a mild solution than an ultramild solution, since Ito formula, already tricky to apply for mild solutions, seems to be completely unusable for ultramild solutions. Moreover there is the general idea that if you get a bound on $\|\Delta^{1/2}(X_t^\varepsilon)\|_2^2$ (uniform in ε), you can get a Cauchy condition in $\|\cdot\|_2$ norm and thus norm convergence, but however in general we will work only with weak convergence. Finally, we will also show in Section 1.4 that mild solutions are also weak solutions, in a usual duality sense of weak solutions, however, we don't have an analog for ultramild solutions. Thus we will also introduce a notion of ultraweak solution, mainly to get uniqueness results in applying Laplace transform techniques, our general result will be "there exists a unique ultraweak solution which is also an ultramild solution," and limit of mild solutions of approximating equations.

Before starting, we sum up the content of the next sections. In Section 1.2, we give miscellaneous definitions and results moving almost commutation properties of our assumptions to stochastic integrals. In Section 1.3, we prove an integration by parts formula for a stochastic convolution we introduce. We avoid proving a free variant of the usually used stochastic Fubini theorem in using ad hoc proofs in our really special case. In Section 1.4 we introduce our different kinds of solutions and prove relations between them (explained above). Section 1.5 contains our general theorem.

Let us prove right now an easy consequence of our main assumptions. Note we often use the bound $\|\Delta\eta_\alpha\| \leq 2\alpha$ coming from $\Delta\eta_\alpha = \alpha(1 - \eta_\alpha)$.

Lemma 2. Assume $\Gamma_1(\omega, C)$. Then, there exists for any $\alpha \in (0, \infty)$ bounded operators \mathcal{H}_α from the graph of $\Delta^{1/2}$: $G(\Delta^{1/2}) \subset L^2(M)^{\oplus 2}$ to $L^2(M \otimes M)^{\oplus \mathbb{N}}$ sending, for s , $L^2(M_s)^{\oplus 2}$ to $L^2(M_s \otimes M_s)^{\oplus \mathbb{N}}$ with $\|\mathcal{H}_\alpha\| \leq \max(1, \sqrt{\omega})C$ such that for any $x \in D(\Delta^{1/2})$ (if we write $\eta_\alpha = \frac{\alpha}{\alpha + \Delta}$ and the analog $\eta_\alpha^\otimes = \frac{\alpha}{\alpha + \Delta^\otimes}$):

$$\Delta^\otimes \eta_\alpha^\otimes \circ \tilde{\delta}(x) - \tilde{\delta} \circ \Delta \eta_\alpha(x) = \mathcal{H}_\alpha(x \oplus \Delta^{1/2}(x)).$$

Moreover, for each $x \in D(\Delta^{1/2})$, $\mathcal{H}_\alpha(x \oplus \Delta^{1/2}(x))$ converges in L^2 to $\mathcal{H}(\tilde{\delta}(x))$ when $\alpha \rightarrow \infty$.

Finally, there is also a bounded $\tilde{\mathcal{H}}_\alpha : L^2(M) \rightarrow L^2(M \otimes M)^{\oplus \mathbb{N}}$, with $\|\tilde{\mathcal{H}}_\alpha\| \leq \frac{C}{\alpha} \sqrt{\omega + 2\alpha}$ and the same invariance of filtration properties, such that for any $x \in D(\tilde{\delta})$:

$$\tilde{\delta} \eta_\alpha(x) - \eta_\alpha^\otimes \tilde{\delta}(x) = \tilde{\mathcal{H}}_\alpha(x).$$

Proof. Let \mathcal{H} be given by assumption f.2. Let us define \mathcal{H}_α . For $x \in D(\Delta^{1/2})$, $\eta_\alpha(x) \in D(\Delta^{3/2}) \subset D(\tilde{\delta} \circ \Delta)$, thus applying the equation for \mathcal{H} in f.2, we get:

$$\eta_\alpha^\otimes \Delta^\otimes \circ \tilde{\delta} \eta_\alpha(x) - \eta_\alpha^\otimes \tilde{\delta} \circ \Delta \eta_\alpha(x) = \eta_\alpha^\otimes \mathcal{H}(\tilde{\delta} \eta_\alpha(x)).$$

Thus multiplying by $\frac{1}{\alpha}$ and using $\alpha(1 - \eta_\alpha) = \Delta \eta_\alpha$, we get:

$$\tilde{\delta} \eta_\alpha(x) - \eta_\alpha^\otimes \tilde{\delta}(x) = \frac{1}{\alpha} \eta_\alpha^\otimes \mathcal{H}(\tilde{\delta} \eta_\alpha(x)).$$

Especially, defining $\tilde{\mathcal{H}}_\alpha = \frac{1}{\alpha} \eta_\alpha^\otimes \mathcal{H} \tilde{\delta} \eta_\alpha$, we get the last statement since (by assumptions f.2, g, c.1) $\|\tilde{\mathcal{H}}_\alpha\| \leq \frac{C^{1/2}}{\alpha} \|\tilde{\delta} \eta_\alpha\| \leq \frac{C}{\alpha} \sqrt{\omega + 2\alpha}$ and moreover, by d.3, $D(\Delta^{1/2})$ is a core for $\tilde{\delta}$.

Moreover we also deduce:

$$\Delta^\otimes \eta_\alpha^\otimes \tilde{\delta}(x) = -\frac{1}{\alpha} \Delta^\otimes \eta_\alpha^\otimes \mathcal{H}(\tilde{\delta} \eta_\alpha(x)) + \tilde{\delta} \Delta \eta_\alpha(x) + \mathcal{H}(\tilde{\delta} \eta_\alpha(x)),$$

thus equivalently

$$\Delta^\otimes \eta_\alpha^\otimes \tilde{\delta}(x) - \tilde{\delta} \Delta \eta_\alpha(x) = \eta_\alpha^\otimes \mathcal{H}(\tilde{\delta} \eta_\alpha(x)).$$

This suggests $\mathcal{H}_\alpha(x \oplus \Delta^{1/2}(x)) = \eta_\alpha^\otimes \mathcal{H}(\tilde{\delta} \eta_\alpha(x))$. In that way, the equation is verified, the stability properties come from the assumptions and using properties f.2, g, c.1 again, we get:

$$\|\mathcal{H}_\alpha(x \oplus \Delta^{1/2}(x))\|_2^2 \leq C \|\tilde{\delta} \eta_\alpha(x)\|_2^2 \leq C^2 (\|\Delta^{1/2}(x)\|_2^2 + \omega \|x\|_2^2).$$

Thus we get the bound on $\|\mathcal{H}_\alpha\|$, and

$$\begin{aligned} \|\mathcal{H}_\alpha(x \oplus \Delta^{1/2}(x)) - \mathcal{H}(\tilde{\delta}(x))\|_2 &\leq \|\eta_\alpha^\otimes \mathcal{H}(\tilde{\delta}(\eta_\alpha - 1)(x))\|_2 + \|(\eta_\alpha^\otimes - 1)\mathcal{H}(\tilde{\delta}(x))\|_2 \\ &\leq C^{1/2} \|\tilde{\delta}(\eta_\alpha - 1)(x)\|_2 + \|(\eta_\alpha^\otimes - 1)\mathcal{H}(\tilde{\delta}(x))\|_2, \end{aligned}$$

and the right hand side goes to zero using again assumption c.1. □

1.2. Stochastic integration in presence of δ and Δ

Following [3], except for the value in $L^2(M \otimes M)^{\oplus \mathbb{N}}$ instead of $L^2(M \otimes M)$ of bi-processes, we write $\mathcal{B}_2^a([0, T])$ the completion of the space of simple bi-processes on $[0, T]$, adapted with respect to the algebraic direct sum $(M_t \otimes M_t)^{\oplus \mathbb{N}}$, in the following norm:

$$\|U\|_{\mathcal{B}_2^a([0, T])} = \left(\int_0^T \|U_s\|_{L^2(\tau \otimes \tau)^{\oplus \mathbb{N}}}^2 ds \right)^{1/2}.$$

We may also write later \mathcal{B}_2^a instead of $\mathcal{B}_2^a([0, T])$ when T is clear from the context and similarly for variants later.

Let us remark that this space can also be seen as a subspace of $L^2([0, T], L^2(\tau \otimes \tau)^{\oplus \mathbb{N}})$ (defined, say, in Bochner's sense) and we will always see it as such a subspace.

Then, recall that the map $U \mapsto \int_0^T U_s \# dS_s = \sum_{j=0}^\infty \int_0^T U_s^{(j)} \# dS_s^{(j)}$ is an isometric linear extension from $\mathcal{B}_2^a([0, T])$ to $L^2(M, \tau)$ of the usual map, sending, for $a, b \in M_s, a \otimes b 1_{[s,t]}$ seen in the i th component of the direct sum to $a(S_t^{(i)} - S_s^{(i)})b$. Thus, we can remark for further use that weak convergence in $L^2(\tau)$ of a sequence of stochastic integrals $\int_0^T U_s^n \# dS_s$ is equivalent to weak convergence of its integrand U_s^n in $L^2([0, T], L^2(\tau \otimes \tau)^{\oplus \mathbb{N}})$.

Analogously, one can consider $\mathcal{B}_2^{a(\otimes 3)}([0, T])$ for processes adapted to the filtration $(M_t \otimes M_t \otimes M_t)^{\oplus ((1,2) \times \mathbb{N}^2)}$. We define $\int_0^T U_s \# dS_s \in L^2(M \otimes M)^{\oplus \mathbb{N}}$ in extending the definition for $a, b, c \in M_s, U_u = a \otimes b \otimes c 1_{[s,t]}(u)$ seen in the $1, j, i$ th component of the direct sum $\int_0^T a \otimes b \otimes c 1_{[s,t]}(u) \# dS_u = a(S_t^{(i)} - S_s^{(i)})b \otimes c$ in the j th component, and when seen in the $2, j, i$ th component $\int_0^T a \otimes b \otimes c 1_{[s,t]}(u) \# dS_u = a \otimes b(S_t^{(i)} - S_s^{(i)})c$ in the j th component.

We will write $\mathcal{B}_{2,\delta \circ \Delta^\beta}^a([0, T])$ (for $\beta \in \{0, 1/2, 1\}$, resp. $\mathcal{B}_{2,\Delta^\beta}^a([0, T])$ for $\beta \in \{1/2, 1, 3/2\}$) the completion with respect to the following norms of what we will call $\delta \circ \Delta^\beta$ -simple adapted processes (resp. Δ^β -simple adapted processes), i.e. processes of the form $X = \sum_{j=1}^M X_j 1_{[t_j, t_{j+1})}$ with $X_j \in D(\Delta^\beta) \cap \bigcap_{b=0}^{2\beta} D(\delta \circ \Delta^{b/2})$ (resp. $X_j \in D(\Delta^\beta)$):

$$\|X\|_{\mathcal{B}_{2,\delta \circ \Delta^\beta}^a} = \left(\int_0^T \sum_{b=0}^{2\beta} \|\delta \circ \Delta^{b/2} X_s\|_{L^2(\tau \otimes \tau)^{\oplus \mathbb{N}}}^2 + \sum_{b=0}^{2\beta} \|\Delta^{b/2} X_s\|_{L^2(\tau)}^2 ds \right)^{1/2}$$

$$\left(\text{resp. } \|X\|_{\mathcal{B}_{2,\Delta^\beta}^a} = \left(\int_0^T \sum_{b=0}^{2\beta} \|\Delta^{b/2} X_s\|_{L^2(\tau)}^2 ds \right)^{1/2} \right).$$

Of course, using g , one gets the same spaces if we replace δ by $\tilde{\delta}$. Assuming $\Gamma_0(\omega)$ (especially condition b), we have clearly continuous embeddings $\mathcal{B}_{2,\delta \circ \Delta^\beta}^a([0, T]) \rightarrow L_a^2([0, T], L^2(M))$ (space of adapted processes), $\mathcal{B}_{2,\Delta^\beta}^a([0, T]) \rightarrow L_a^2([0, T], L^2(M))$, for $\beta' \leq \beta, \beta, \beta' \in \{0, 1/2, 1\}$, $\mathcal{B}_{2,\delta \circ \Delta^\beta}^a([0, T]) \rightarrow \mathcal{B}_{2,\delta \circ \Delta^{\beta'}}^a([0, T])$ and for $\beta' \leq \beta, \beta, \beta' \in \{1/2, 1\}$, $\mathcal{B}_{2,\delta \circ \Delta^\beta}^a([0, T]) \rightarrow \mathcal{B}_{2,\Delta^{\beta'}}^a, \mathcal{B}_{2,\Delta^{\beta+1/2}}^a([0, T]) \rightarrow \mathcal{B}_{2,\delta \circ \Delta^\beta}^a([0, T])$ for $\beta \in \{0, 1/2\}$ (using assumption c).

From the assumptions on δ and Δ , we remark that we can see for any $X_s \in \mathcal{B}_{2,\delta \circ \Delta^\beta}^a([0, T])$, $\delta \circ \Delta^\beta X_s$ as an element of \mathcal{B}_2^a . Finally, let us note that if B bounded operator on $L^2(M_s \otimes M_s)^{\oplus \mathbb{N}}$, keeping invariant, for any $t, L^2(M_t \otimes M_t)^{\oplus \mathbb{N}}$, and if $U_s \in \mathcal{B}_2^a([0, T])$, then $B(U_s) \in \mathcal{B}_2^a([0, T])$.

The following lemma is the goal of these definitions:

Lemma 3. Assume $\Gamma_1(\omega, C)$ for (i) and $\Gamma_0(\omega)$ for (ii) and (iii).

- (i) Let $X_s \in \mathcal{B}_{2,\Delta^{1/2}}^a([0, T])$ then we have $\eta_\alpha(X_s) \in \mathcal{B}_{2,\delta \circ \Delta}^a([0, T])$, $\Delta^\otimes \eta_\alpha^\otimes \tilde{\delta}(X_s), \mathcal{H}_\alpha(X_s \oplus \Delta^{1/2}(X_s)) \in \mathcal{B}_2^a([0, T])$ and for $t \leq T$:

$$\int_0^t \Delta^\otimes \eta_\alpha^\otimes \tilde{\delta}(X_s) \# dS_s = \int_0^t \tilde{\delta} \circ \Delta(\eta_\alpha(X_s)) \# dS_s + \int_0^t \mathcal{H}_\alpha(X_s \oplus \Delta^{1/2}(X_s)) \# dS_s.$$

If $X_s \in \mathcal{B}_{2,\Delta^{3/2}}^a([0, T])$, then $\tilde{\delta}^\otimes \delta(X_s) = (\tilde{\delta}_{i,j}^{\otimes k} \delta_j(X_s))_{k,i,j} \in \mathcal{B}_2^{a(\otimes 3)}([0, T])$, $\int_0^t \delta(X_s) \# dS_s \in D(\tilde{\delta})$ and we have the equation:

$$\tilde{\delta} \int_0^t \delta(X_s) \# dS_s = \int_0^t \tilde{\delta}^\otimes \delta(X_s) \# dS_s.$$

- (ii) Likewise, for any $U_s \in \mathcal{B}_2^a([0, T])$ and $t, \alpha, \beta > 0, i \in \{1/2, 1\}$: $\eta_\alpha^{\otimes i}(U_s), \phi_t^\otimes(U_s) \in \mathcal{B}_2^a([0, T])$, $\Delta^\otimes \eta_\alpha^{\otimes 1/2} \times \eta_\beta^{\otimes 1/2}(U_s), \Delta^\otimes \phi_t^\otimes(U_s) \in \mathcal{B}_2^a([0, T])$ (and assuming $d.3, d.4, g$ we have also, $\tilde{\delta}^\otimes \eta_\alpha^{\otimes i}(U_s) = (\tilde{\delta}_{i,j}^{\otimes k} \eta_\alpha^{\otimes i}(U_s^j))_{k,i,j} \in$

$\mathcal{B}_2^{a(\otimes 3)}([0, T])$), and we have for $\tau \leq T$:

$$\begin{aligned}\eta_\alpha^i \left(\int_0^\tau U_s \# dS_s \right) &= \int_0^\tau \eta_\alpha^{\otimes i}(U_s) \# dS_s, & \Delta \eta_\alpha^{1/2} \eta_\beta^{1/2} \left(\int_0^\tau U_s \# dS_s \right) &= \int_0^\tau \Delta^\otimes \eta_\alpha^{\otimes 1/2} \eta_\beta^{\otimes 1/2}(U_s) \# dS_s, \\ \phi_t \left(\int_0^\tau U_s \# dS_s \right) &= \int_0^\tau \phi_t^\otimes(U_s) \# dS_s, & \Delta \phi_t \left(\int_0^\tau U_s \# dS_s \right) &= \int_0^\tau \Delta^\otimes \phi_t^\otimes(U_s) \# dS_s, \\ \tilde{\delta} \eta_\alpha^i \left(\int_0^\tau U_s \# dS_s \right) &= \int_0^\tau \tilde{\delta}^\otimes \eta_\alpha^{\otimes i}(U_s) \# dS_s \quad \text{if } d.3, d.4, g \text{ also hold.}\end{aligned}$$

Finally, for any $W \in L^2(M_t \otimes M_t)^{\oplus \mathbb{N}}$, any $V = (\int_t^\tau U_s \# dS_s)$, $t \leq \tau$, with $V \in D(\delta)$, $\langle W, \delta(V) \rangle = 0$.

(iii) For $U_s \in \mathcal{B}_2^a([0, T])$, $\int_0^T U_s \# dS_s \in D(\Delta^{1/2})$ if and only if $U_s \in D(\Delta^{\otimes 1/2})$ for almost every s and $\int_0^T ds \|\Delta^{\otimes 1/2} U_s\|_2^2 < \infty$. In this case $\Delta^{1/2} \int_0^T U_s \# dS_s = \int_0^T \Delta^{\otimes 1/2}(U_s) \# dS_s$. If $d.3, d.4, g$ also hold, then for any U_s with $\tilde{\delta}^\otimes(U_s) \in \mathcal{B}_2^{a(\otimes 3)}([0, T])$ (e.g. for $U_s = \delta(X_s)$, for $X_s \in \mathcal{B}_{2,\Delta}^a([0, T])$ if h holds), we have $\int_0^T U_s \# dS_s \in D(\tilde{\delta})$ and $\tilde{\delta} \int_0^T U_s \# dS_s = \int_0^T \tilde{\delta}^\otimes(U_s) \# dS_s$.

Proof. First of all, the statements about $\eta_\alpha^{\otimes i}(U_s), \phi_t^\otimes(U_s) \in \mathcal{B}_2^a([0, T])$, $\Delta^\otimes \eta_\alpha^{\otimes 1/2} \eta_\beta^{\otimes 1/2}(U_s), \Delta^\otimes \phi_t^\otimes(U_s) \in \mathcal{B}_2^a([0, T])$ and $\Delta^\otimes \eta_\alpha^{\otimes i} \tilde{\delta}(X_s) \in \mathcal{B}_2^a$ follow from the remark before the lemma, since e.g. $\|\Delta^\otimes \eta_\alpha^{\otimes i}\| \leq 2\alpha$. Assuming $d.3, d.4, g$, the same is true for $\tilde{\delta}^\otimes \eta_\alpha^{\otimes i}(U_s) \in \mathcal{B}_2^{a(\otimes 3)}([0, T])$.

If X_s is a $\Delta^{1/2}$ -simple process. By linearity, we can suppose $X_s = x 1_{[t_1, t_2)}(s)$, with $x \in D(\Delta^{1/2}) \cap L^2(M_{t_1})$. In that case, we have clearly $\eta_\alpha(X_s) \in D(\tilde{\delta} \circ \Delta)$ (using assumption $d.3$) and the equality stated is nothing but the one of Lemma 2. In the general case $X_s \in \mathcal{B}_{2,\Delta^{1/2}}^a([0, T])$, take by density X_s^n $\Delta^{1/2}$ -simple processes converging to X_s in $\mathcal{B}_{2,\Delta^{1/2}}^a([0, T])$. Then, since $\delta \circ \Delta^\beta \eta_\alpha(X_s^n) = \delta \circ \alpha^\beta (\text{id} - \eta_\alpha)^\beta \eta_\alpha^{1-\beta}(X_s^n)$ (and using assumption $c.1$),

$$\sum_{b=0}^2 \|\delta \circ \Delta^{b/2} \eta_\alpha(X_s^n - X_s^m)\|_2^2 \leq (1 + 2\alpha + (2\alpha)^2) (\|\Delta^{1/2}(X_s^n - X_s^m)\|_2^2 + \omega \|X_s^n - X_s^m\|_2^2).$$

Likewise

$$\sum_{b=0}^2 \|\Delta^{b/2} \eta_\alpha(X_s^n - X_s^m)\|_2^2 \leq (1 + 2\alpha + (2\alpha)^2) \|X_s^n - X_s^m\|_2^2.$$

As a consequence, $(\eta_\alpha(X_s^n))$ converges in $\mathcal{B}_{2,\delta \circ \Delta}^a([0, T])$, and by the embedding in $\mathcal{B}_{2,\Delta^{1/2}}^a([0, T])$, it converges to $\eta_\alpha(X_s)$ which is thus in $\mathcal{B}_{2,\delta \circ \Delta}^a$. Now, $\Delta^\otimes \eta_\alpha^{\otimes i} \tilde{\delta}(X_s), \tilde{\delta}(\Delta \eta_\alpha X_s) \in \mathcal{B}_2^a([0, T])$. Therefore, applying the equation of Lemma 2, $H_\alpha(X_s \oplus \Delta^{1/2}(X_s)) \in \mathcal{B}_2^a([0, T])$ and we have our equality after taking the isometric map of stochastic integration. The second statement of (i) is proved in a similar way using $d.3$ for the equation, $g, h, f.1, c.1$ for boundedness results. For (ii), the boundedness has already been discussed, and this explains the definition of the right hand sides of the equations. The equalities are clear for simple processes (easy consequences of assumptions d , and a for the semigroup. Note for $\eta_\alpha^{1/2}$ one can use $\eta_\alpha^{1/2} = \int_0^\infty \pi^{-1} \frac{t^{-1/2}}{1+t} \eta_{\alpha(1+t)/t} dt$ from [33], Lemma 2.2), this concludes by density.

For the last statement of (ii) about orthogonality, since $\delta \eta_\alpha(V) \rightarrow \delta(V)$, we can assume by the beginning of (ii) (putting η_α in the stochastic integral), $U_s \in D(\Delta^\otimes)$. Again, it suffices to prove the simple process case, and this reduces to assumption $d.2$.

Finally, for the equivalence of (iii), note that the two first equalities of (ii) give $\|\Delta^{1/2} \eta_\alpha(\int_0^T U_s \# dS_s)\|_2^2 = \int_0^T \|\Delta^{\otimes 1/2} \eta_\alpha^\otimes(U_s)\|_2^2 ds$. From this, letting $\alpha \rightarrow \infty$, the direct implication follows from monotone convergence theorem with $\Delta^{1/2} \eta_\alpha = \eta_\alpha \Delta^{1/2}$ and the reverse implication using also $\Delta^{1/2}$ is a closed operator. Let us check first for any $U \in \mathcal{B}_2^a([0, T])$, $\Delta^{1/2} \eta_\alpha(\int_0^T U_s \# dS_s) = \int_0^T \Delta^{\otimes 1/2} \eta_\alpha^\otimes U_s \# dS_s$. Again it suffices to check it on simple processes,

on which this comes from $\Delta^{1/2} = \int_0^\infty \pi^{-1} t^{-1/2} (\text{id} - \eta_t) dt$ (cf. e.g. [36] or [20]). Now, the stated result comes from $\alpha \rightarrow \infty$, the statement for $\tilde{\delta}$ is analogous. \square

1.3. A definition of free stochastic convolution

In this subsection, we assume $\Gamma_0(\omega)$. We want to give sense to the following kind of integral, for $U_s \in \mathcal{B}_2^a$: $\int_0^t \phi_{t-s}(U_s \# dS_s)$. We will define it by

$$\int_0^t \phi_{t-s}(U_s \# dS_s) = \int_0^t \phi_{t-s}^{\otimes} (U_s) \# dS_s,$$

and we want to verify the usual properties of stochastic convolution.

For this, we have to verify that $\phi_{t-s}^{\otimes}(U_s)1_{[0,t]}(s) \in \mathcal{B}_{2,\delta}^a([0,t])$, and since ϕ_{t-s}^{\otimes} is a contraction, it is sufficient to show this for U_s a simple process, and thus even for $U1_{[u,v]}(s)$, $U \in L^2(M_u \otimes M_u)$. But consider $u_{i,n} = u + i(v-u)/n$, then $U^n = \sum_{i=0}^{n-1} \phi_{t-u_{i,n}}(U)1_{[u_{i,n}, u_{i+1,n})}$ is easily shown to converge in $L^2([0,t], L^2(M \otimes M))$ to $\phi_{t-s}^{\otimes}(U)$ using strong continuity of ϕ^{\otimes} , this concludes the preliminaries for the definition.

Let us define a variant of the spaces of the previous subsection useful to define a really weak form of solutions we will call in the next subsection: “ultramild” solutions. We will write $\mathcal{B}_{2,\phi\delta}^a([0,T])$ for the completion with respect to the following norm of δ -simple adapted processes, i.e. recall this means processes of the form $X = \sum_{j=1}^M X_j 1_{[t_j, t_{j+1})}$ with $X_j \in D(\delta)$:

$$\|X\|_{\mathcal{B}_{2,\phi\delta}^a} = \left(\int_0^T \left[\int_0^t \|\phi_{t-s}^{\otimes} \delta(X_s)\|_{L^2(\tau \otimes \tau) \oplus \mathbb{N}}^2 ds + \|X_t\|_{L^2(\tau)}^2 \right] dt \right)^{1/2}.$$

We have clearly a continuous embedding $\mathcal{B}_{2,\delta}^a([0,T]) \rightarrow \mathcal{B}_{2,\phi\delta}^a([0,T])$ using Section 1.2 and the above remark defining stochastic convolution (the first space being clearly dense in the second by definition). We get thus a map $\gamma : \mathcal{B}_{2,\delta}^a([0,T]) \rightarrow L_a^2([0,T], L^2(M))$ such that $\gamma(X)_t = \int_0^t \phi_{t-s}(\delta(X_s) \# dS_s)$ and clearly $\|\gamma(X)\|_{L_a^2([0,T], L^2(M))} \leq \|X\|_{\mathcal{B}_{2,\phi\delta}^a}$ so that γ extends to a continuous map (also called) $\gamma : \mathcal{B}_{2,\phi\delta}^a([0,T]) \rightarrow L_a^2([0,T], L^2(M))$.

We also want to show that $t \mapsto \langle \int_0^t U_s \# dS_s, \zeta \rangle$ is of bounded variation so that we can define something like $\int_0^t \langle U_s \# dS_s, \zeta \rangle$ (with the same value) and see $\langle U_s \# dS_s, \zeta \rangle$ as a measure on \mathbb{R}_+ . But since stochastic integration is an isometric map onto its image we can project ζ on this space thus write its projection $\int_0^t V_s \# dS_s$, and the result is a consequence of the isometry property.

Finally, we want to define for $\zeta(\cdot) \in C^1([0,T], L^2(M))$: $\int_0^t \langle U_s \# dS_s, \zeta(s) \rangle$ and show a relation with stochastic convolution in a special case. For this, first note that the family of functions of the form $\varphi(\cdot)\zeta_0$, for $\varphi(\cdot) \in C^1([0,T], \mathbb{C})$ and $\zeta_0 \in L^2(M)$ linearly spans a dense subset of $C^1([0,T], L^2(M))$, thus consider also first $\zeta(\cdot)$ in this linear span. Consider also $\mathcal{U}_t = \int_0^t U_s \# dS_s$.

Define $\int_0^t \langle U_s \# dS_s, \varphi(s)\zeta_0 \rangle = \langle \int_0^t \overline{\varphi(s)} U_s \# dS_s, \zeta_0 \rangle$ using the previous paragraph, and consider as in this paragraph the projection of ζ_0 on the space of stochastic integrals $\int_0^t V_s \# dS_s$. Then compute using integration by parts:

$$\begin{aligned} \int_0^t \langle U_s \# dS_s, \varphi(s)\zeta_0 \rangle &= \int_0^t \varphi(s) \langle U_s, V_s \rangle ds = \varphi(t) \int_0^t \langle U_s, V_s \rangle ds - \int_0^t \varphi'(s) \langle \mathcal{U}_s, \zeta_0 \rangle ds \\ &= \langle \mathcal{U}_t, \zeta(t) \rangle - \int_0^t \langle \mathcal{U}_s, \zeta'(s) \rangle ds, \end{aligned}$$

which extends by linearity on the above mentioned linear span.

But now, we get the bound:

$$\left| \int_0^t \langle U_s \# dS_s, \varphi(s)\zeta_0 \rangle \right| \leq \|\mathcal{U}_t\|_2 \left(\|\zeta(t)\|_2 + t \sup_s \|\zeta'(s)\|_2 \right),$$

using $\|\mathcal{U}_t\|_2$ is increasing with t . We can thus extend our linear map by continuity to $\zeta(\cdot) \in C^1([0, T], L^2(M))$ and we have also the equality:

$$\int_0^t \langle U_s \# dS_s, \zeta(s) \rangle = \langle \mathcal{U}_t, \zeta(t) \rangle - \int_0^t \langle \mathcal{U}_s, \zeta'(s) \rangle ds. \tag{3}$$

Consider finally $\zeta(s) = \phi_{t-s}(\zeta)$, with $\zeta \in D(\Delta)$, writing as before $\int_0^\infty V_s \# dS_s$ the projection of ζ on the space of stochastic integrals. Using this last equality, we get:

$$\begin{aligned} & \int_0^t \langle U_s \# dS_s, \phi_{t-s}(\zeta) \rangle \\ &= \langle \mathcal{U}_t, \zeta \rangle - \int_0^t \left\langle \frac{1}{2} \Delta \phi_{t-s}(\mathcal{U}_s), \zeta \right\rangle ds = \langle \mathcal{U}_t, \zeta \rangle - \int_0^t \int_0^s \left\langle \frac{1}{2} \Delta^\otimes \phi_{t-s}^\otimes(U_u), V_u \right\rangle du ds \\ &= \int_0^t \langle U_s, V_s \rangle ds - \int_0^t \left\langle \int_u^t \frac{1}{2} \Delta^\otimes \phi_{t-s}^\otimes(U_u) ds, V_u \right\rangle du \\ &= \int_0^t \langle U_s, V_s \rangle ds - \int_0^t \langle U_u - \phi_{t-u}^\otimes(U_u), V_u \rangle du = \left\langle \int_0^t \phi_{t-s}(U_s \# dS_s), \zeta \right\rangle. \end{aligned}$$

In the first line we used our identity since $\phi_{t-s}(\zeta) \in C^1([0, T], L^2(M))$ for $\zeta \in D(\Delta)$ and then Lemma 3(ii).

Line 3 is only a computation, first with the differential equation, second, with the definition of stochastic convolution after simplification.

In line 2, we have to justify application of Fubini theorem. Note that $\zeta = \eta_\alpha(z)$ (since by Hille–Yosida theory $\text{Range}(\eta_\alpha) = D(\Delta)$, see e.g. (1.3) in the proof of Chapter 1, Proposition 1.5 in [26]), if the projection of z is written $\int_0^T W_s \# dS_s$, then $V_s = \eta_\alpha^\otimes(W_s)$ a.e. by Lemma 3(ii). Thus V_s is a.e. in $D(\Delta^\otimes)$. We can now use Cauchy–Schwarz inequality:

$$\begin{aligned} \int_0^t ds \int_0^s du \left| \left\langle \frac{1}{2} \Delta^\otimes \phi_{t-s}^\otimes(U_u), V_u \right\rangle \right| &\leq \int_0^t ds \left(\int_0^s du \left\| \frac{1}{2} \phi_{t-s}^\otimes(U_u) \right\|_2^2 \right)^{1/2} \left(\int_0^s du \left\| \Delta^\otimes V_u \right\|_2^2 \right)^{1/2} \\ &\leq t \left(\int_0^t du \|U_u\|_2^2 \right)^{1/2} \|\Delta(\zeta)\|_2 < \infty. \end{aligned}$$

Starting from the second line above, applying Fubini to go upwards after a change of variable $t - s = s' - u$, we also obtain:

$$\begin{aligned} \left\langle \int_0^t \phi_{t-s}(U_s \# dS_s), \zeta \right\rangle &= \int_0^t \langle U_s, V_s \rangle ds - \int_0^t \left\langle \int_u^t \frac{1}{2} \Delta^\otimes \phi_{s-u}^\otimes(U_u) ds, V_u \right\rangle du \\ &= \langle \mathcal{U}_t, \zeta \rangle - \int_0^t \left\langle \frac{1}{2} \Delta \int_0^s \phi_{s-u}(U_u \# dS_u), \zeta \right\rangle ds. \end{aligned}$$

Proposition 4 (Integration by parts for stochastic convolution). For $\zeta \in D(\Delta)$, $U \in B_2^q([0, t])$, we have:

$$\begin{aligned} \int_0^t \phi_{t-s}(U_s \# dS_s) &= \int_0^t U_s \# dS_s - \Delta \int_0^t ds \frac{1}{2} \phi_{t-s} \left(\int_0^s U_v \# dS_v \right) \\ &= \int_0^t U_s \# dS_s - \frac{1}{2} \Delta \int_0^t ds \int_0^s \phi_{s-u}(U_u \# dS_u), \\ \int_0^t \langle U_s \# dS_s, \phi_{t-s}(\zeta) \rangle &= \left\langle \int_0^t \phi_{t-s}(U_s \# dS_s), \zeta \right\rangle. \end{aligned}$$

Note the following useful formula we will use often later:

$$\|\phi_t(x)\|_2^2 = \|x\|_2^2 - \int_0^t \|\Delta^{1/2}\phi_s(x)\|_2^2 ds. \tag{4}$$

Proposition 5. For $Y \in \mathcal{B}_{2,\delta}^a([0, T])$, define $\gamma(Y)_t = \int_0^t \phi_{t-s}(\delta(Y_s))\#dS_s$, then $\gamma(Y)_t \in D(\Delta^{1/2})$ for a.e. $t \leq T$ and moreover:

$$\int_0^T \|\Delta^{1/2}\gamma(Y)_t\|_2^2 dt = -\|\gamma(Y)_T\|_2^2 + \int_0^T dt \|\delta(Y_t)\|_2^2. \tag{5}$$

Moreover, assume $\Gamma_1(\omega, C)$ for any B among $\tilde{\delta}, \Delta^{1/2}, \alpha, \alpha' > 0$, then:

$$\begin{aligned} \|B\eta_\alpha^{1/2}\eta_{\alpha'}^{1/2}(\gamma(Y)_T)\|_2^2 &= \left\| B\eta_\alpha^{1/2}\eta_{\alpha'}^{1/2} \int_0^T \delta(Y_s)\#dS_s \right\|_2^2 \\ &\quad - \int_0^T dt \Re \langle B\Delta\eta_\alpha^{1/2}\eta_{\alpha'}^{1/2}\gamma(Y)_t, B\eta_\alpha^{1/2}\eta_{\alpha'}^{1/2}\gamma(Y)_t \rangle. \end{aligned} \tag{6}$$

Proof. By Fubini–Tonneli Theorem and the remark before the proposition, we deduce

$$\int_0^T dt \int_0^t ds \|\Delta^{\otimes 1/2}\phi_{t-s}^{\otimes}(\delta(Y_s))\|_2^2 = \int_0^T ds \|\delta(Y_s)\|_2^2 - \|\phi_{T-s}^{\otimes}(\delta(Y_s))\|_2^2.$$

Thus Lemma 3(iii) concludes to the first statement. Since $B\eta_\alpha$ is a bounded operator, and from the first statement and $\Gamma_1(\omega, C)g, c.1$ for the last term, all the terms in (6) are continuous on $\mathcal{B}_{2,\delta}^a([0, T])$. As a consequence, it suffices to prove it for simple processes of even $Y = X1_{[s,t]}$, $X \in D(\delta)$.

From Lemma 3 (and with an obvious notation B^{\otimes}), this reduces the statement to

$$\begin{aligned} &\int_s^T du \|B^{\otimes}\eta_\alpha^{\otimes 1/2}\eta_{\alpha'}^{\otimes 1/2}\phi_{T-u}^{\otimes}\delta(X)\|_2^2 \\ &= \|B^{\otimes}\eta_\alpha^{\otimes 1/2}\eta_{\alpha'}^{\otimes 1/2}(\delta(X))\|_2^2(T-s) \\ &\quad - \int_s^T dv \Re \int_s^v du \langle B^{\otimes}\Delta^{\otimes}\eta_\alpha^{\otimes 1/2}\eta_{\alpha'}^{\otimes 1/2}\phi_{v-u}^{\otimes}\delta(X), B^{\otimes}\eta_\alpha^{\otimes 1/2}\eta_{\alpha'}^{\otimes 1/2}\phi_{v-u}^{\otimes}\delta(X) \rangle. \end{aligned}$$

But this is obvious after applying Fubini on the last integral and integrating along v (using $\|\Delta^{\otimes 1/2}\phi_{v-u}^{\otimes}\| \leq \frac{cst}{\sqrt{v-u}}$ and $d.4$ in the $\tilde{\delta}$ case to bound it by the corresponding $\Delta^{1/2}$ case). □

1.4. Useful links between mild solutions and strong solutions

In this part, we will also work under assumption $\Gamma_0(\omega)$. Let us define four kinds of solutions.

Definition 6. We will call a strong solution an element $X_t \in \mathcal{B}_{2,\Delta}^a$ satisfying (2). A mild solution will be an $X_t \in \mathcal{B}_{2,\delta}^a$ satisfying:

$$X_t = \phi_t(X_0) + \int_0^t \phi_{t-s}(\delta(X_s))\#dS_s. \tag{7}$$

We call ultramild solutions, solutions of (7) in $\mathcal{B}_{2,\phi\delta}^a$. We call a weak solution an $X_t \in \mathcal{B}_{2,\delta}^a$ such that, for any $\zeta \in D(\Delta)$:

$$\langle X_t, \zeta \rangle = \langle X_0, \zeta \rangle - \frac{1}{2} \int_0^t \langle X_s, \Delta(\zeta) \rangle ds + \int_0^t \langle \delta(X_s)\#dS_s, \zeta \rangle.$$

We will first recall analogs of usual results (in classical SPDE theory) concerning the link between strong solutions and mild solutions. We mainly follow here the proofs (for a classical Brownian motion and a classical SPDE) of [13], Chapter 6.

Proposition 7. *A strong solution of (2) (in $\mathcal{B}_{2,\Delta}^a$) is also a mild solution (even a solution of (7) in $\mathcal{B}_{2,\Delta^{1/2}}^a$).*

Proof. First, note that for any $\zeta(\cdot) \in C^1([0, T]; D(\Delta))$, and any $t \in [0, T]$, we have:

$$\langle X_t, \zeta(t) \rangle = \langle X_0, \zeta(0) \rangle + \int_0^t \left\langle X_s, -\frac{1}{2} \Delta(\zeta(s)) + \zeta'(s) \right\rangle ds + \int_0^t \langle \delta(X_s) \# dS_s, \zeta(s) \rangle.$$

(To prove this, use (3) to compute the stochastic part and use an integration by parts to get the other term.)

Finally, consider $\zeta(s) = \phi_{t-s}(\zeta)$, which is in $C^1([0, T]; D(\Delta))$, if say $\zeta \in D(\Delta^2)$. The terms inside the usual integral cancel out and you get:

$$\langle X_t, \zeta \rangle = \langle X_0, \phi_t(\zeta) \rangle + \int_0^t \langle \delta(X_s) \# dS_s, \phi_{t-s}(\zeta) \rangle.$$

In as much as you can take any $\zeta \in D(\Delta^2)$ and $D(\Delta^2)$ is dense (even a core for Δ by a standard result Theorem 3.24, p. 275 in Chapter V of Kato's book [20]), you get the result (using Proposition 4). \square

Proposition 8. *A mild solution X_t is always a weak solution, and if it is also in $\mathcal{B}_{2,\Delta}^a$, then it is in fact a strong solution.*

Proof. Once we have proved that our mild solution is in fact a weak solution, we are in fact done, since under our assumption $\Delta(X_s)$ is (Lebesgue-almost surely) well defined and in $L^2([0, T], L^2(M))$, showing that the wanted equation (2) under $\langle \cdot, \zeta \rangle$, which concludes by density.

To show that we have the desired weak solution, we will merely use that the solution is in $\mathcal{B}_{2,\delta}^a$, as required for a mild solution. Consider thus $\zeta \in D(\Delta^2)$

$$\begin{aligned} & -\frac{1}{2} \int_0^t \langle X_s, \Delta(\zeta) \rangle ds \\ &= -\frac{1}{2} \int_0^t \langle X_0, \phi_s(\Delta(\zeta)) \rangle ds - \frac{1}{2} \int_0^t ds \int_0^s \langle \delta(X_u) \# dS_u, \phi_{s-u}(\Delta(\zeta)) \rangle \\ &= -\frac{1}{2} \left\langle X_0, \int_0^t \phi_s(\Delta(\zeta)) ds \right\rangle + \left(\left\langle \int_0^t \phi_{t-s}(\delta(X_s) \# dS_s), \zeta \right\rangle - \left\langle \int_0^t \delta(X_s) \# dS_s, \zeta \right\rangle \right) \\ &= \left\langle \phi_t(X_0) + \int_0^t \phi_{t-s}(\delta(X_s) \# dS_s), \zeta \right\rangle - \langle X_0, \zeta \rangle - \int_0^t \langle \delta(X_s) \# dS_s, \zeta \rangle \\ &= \langle X_t, \zeta \rangle - \langle X_0, \zeta \rangle - \int_0^t \langle \delta(X_s) \# dS_s, \zeta \rangle. \end{aligned}$$

The first line has been justified in Proposition 4 applied to the definition of mild solutions. The last line, clearly concluding to what we wanted to prove, uses nothing but again the definition of a mild solution. Of course the third line uses again the differential equation for ϕ . The second line reduces to the second equation in Proposition 4. \square

Finally, to get uniqueness results assuming only really weak conditions, we want to introduce a notion of ultraweak solution for which uniqueness will be easy to prove so that we will build unique ultraweak solutions which are also ultramild solutions. We thus assume $\Gamma_{0u}(\omega)$. This also needs some results on chaotic decomposition very similar to those of Section 5.3 in [3] but not only for the free Fock space $F(H)$ with $H = L^2(\mathbb{R}_+)$ but also for $H = L^2(\mathbb{R}_+)^{\oplus \mathbb{N}}$ and moreover with an initial condition space $L^2(M_0)$ i.e. we want to see a multiple stochastic integral variant of $L^2(M) = L^2(M_0 \star SC(H)) \simeq L^2(M_0) \star F(H)$. Since this requires a little bit of notation with nothing new, we merely state the results after introduction of notation.

For $f \in L^2(\mathbb{R}_+^n \times \mathbb{N}^n, L^2(M_0)^{n+1})$, we want to define a multiple stochastic integral $I(f) = \int f(t_1, \dots, t_n) \# dS_{t_1} \cdots dS_{t_n}$. Of course, we extend it linearly and isometrically as in [3] after defining it on appropriate multiple of characteristic function $f = 1_{\mathcal{A}} \delta_{k_1, \dots, k_n} \alpha_0 \otimes \cdots \otimes \alpha_n$, $\alpha_i \in M_0$, δ_{k_1, \dots, k_n} the function on \mathbb{N}^n taking non-zero value 1 only on the indicated support, $\mathcal{A} = [u_1, v_1] \times \cdots \times [u_n, v_n]$ with $\mathcal{A} \subset \mathbb{R}_+^n - D^n$ (D^n the usual full diagonal e.g. Definition 5.3.1 of [3]). We will call later step function any linear combination of such f 's with maybe $\alpha_0 \otimes \cdots \otimes \alpha_n$ replaced by $U \in L^2(M_0)^{\otimes(n+1)}$. We thus define:

$$I(f) := \alpha_0 (S_{v_1}^{(k_1)} - S_{u_1}^{(k_1)}) \alpha_1 \cdots (S_{v_n}^{(k_n)} - S_{u_n}^{(k_n)}) \alpha_n.$$

Recall $I(f) \in \mathcal{E}_n$ according to the notation before assumption $\Gamma_{0u}(\omega)$.

Then, we can write $f = \sum_{n=0}^\infty f_n \in L^2(M_0) \star F(H)$ so that $I(f) = \sum_{i=0}^\infty I(f_n)$ define an isometric map $I: L^2(M_0) \star F(H) \rightarrow L^2(M)$ determined by $I(f)\Omega = f(\Omega)$ (Ω the usual cyclic empty vector in Fock space), as in Proposition 5.3.2 of [3]. Recall P_Γ is the projection on adapted bi-processes. It is defined (as Γ) in Proposition 5.3.12 in [3] before free Bismut–Clark–Ocone formula. Note that this formula is also valid mutatis mutandis in our context, recall it involves ∇_s the gradient operator from Definition 5.1.1 in [3]. For instance in the really elementary case of $Y \in \mathcal{E}_n \cap L^2(M_s)$, it gives:

$$Y = E_{M_0}(Y) + \int_0^s (P_\Gamma \nabla_u Y) \# dS_u.$$

Consider a step function $f = 1_{\mathcal{A}} \delta_{k_1, \dots, k_n} \alpha_0 \otimes \cdots \otimes \alpha_n$ as above such that

$$\alpha_0 \otimes \cdots \otimes \alpha_n = \sum_{\{j_0 < \cdots < j_k\} \subset \llbracket 0, n \rrbracket} \mathcal{J}_{j_0, \dots, j_k}(U_{j_0, \dots, j_k}),$$

the sum running over (maybe empty) subsets of $\llbracket 0, n \rrbracket$, with

$$\begin{aligned} 1^{\otimes 1_{\{j_0 \neq 0\}}} \otimes U_{j_0, \dots, j_k} \otimes 1^{\otimes 1_{\{j_k \neq n\}}} &\in D(\Delta^{\otimes(k+1+1_{\{j_0 \neq 0\}}+1_{\{j_k \neq n\}})}) \\ &\subset L^2(M_0)^{\otimes 1_{\{j_0 \neq 0\}}} \otimes (L^2(M_0) \ominus \mathbb{C})^{\otimes(k+1)} \otimes L^2(M_0)^{\otimes 1_{\{j_k \neq n\}}} \end{aligned}$$

(with the notation $L^2(M_0)^{\otimes 0} = \mathbb{C}$, $1^{\otimes 0}$ the unit in this \mathbb{C}) and, for $J = \{j_0 < \cdots < j_k\}$ $\mathcal{J}_{j_0, \dots, j_k} = \mathcal{J}_J$ the unique isometric linear map $L^2(M_0)^{\otimes(k+1)} \rightarrow L^2(M_0)^{\otimes(n+1)}$ extended from $\mathcal{J}_{j_0, \dots, j_k}(a_0 \otimes \cdots \otimes a_k) = 1^{\otimes j_0} \otimes a_0 \otimes 1^{\otimes(j_1 - j_0 - 1)} \otimes a_1 \cdots \otimes a_k \otimes 1^{\otimes(n - j_k)}$. We will later write \mathcal{D}_n the space of linear combinations of such $\alpha_0 \otimes \cdots \otimes \alpha_n$'s. Since the images of $\mathcal{J}_{J \cup \{0, n\}}$ for different $J \cup \{0, n\}$ are orthogonal, there is obviously a closed densely defined positive operator $\Delta^{\otimes[n]}$ on the closure of \mathcal{D}_n in $L^2(M_0)^{\otimes(n+1)}$ defined on \mathcal{D}_n by

$$\begin{aligned} \Delta^{\otimes[n]}(\alpha_0 \otimes \cdots \otimes \alpha_n) &= \sum_{J = \{j_0 < \cdots < j_k\} \subset \llbracket 0, n \rrbracket} \mathcal{J}_{J \cup \{0, n\}}(\Delta^{\otimes(k+1+1_{\{j_0 \neq 0\}}+1_{\{j_k \neq n\}})}(1^{\otimes 1_{\{j_0 \neq 0\}}} \otimes U_{j_0, \dots, j_k} \otimes 1^{\otimes 1_{\{j_k \neq n\}}})) \end{aligned}$$

Note that by assumption e.4 $D(\Delta^{\otimes[n]}) \supset (\mathcal{D} \oplus \mathbb{C})^{\otimes(n+1)}$, this explains $\Delta^{\otimes[n]}$ densely defined. Note also that the formula $\Delta(I(f)) = I(\Delta^{\otimes[n]}(f))$ proved below first in case $f \in \mathcal{D}_n$ explains why $\Delta^{\otimes[n]}$ is positive on \mathcal{D}_n .

Lemma 9. Assume $\Gamma_{0u}(\omega)$. For any step function $f \in L^2(\mathbb{R}_+^n \times \mathbb{N}^n) \otimes_{\text{alg}} D(\Delta^{\otimes[n]})$ as above $I(f) \in D(\Delta)$ and $\Delta(I(f)) = I(\Delta^{\otimes[n]}(f))$. Moreover, if $f \in L^2(\mathbb{R}_+^n \times \mathbb{N}^n) \otimes_{\text{alg}} \mathcal{D}_n$, $P_\Gamma \nabla_s I(f) \in D(\delta^*)$ and $\delta^* P_\Gamma \nabla_s I(f) \in \overline{\text{Span}}\{h_k \in \mathcal{E}_k, k \leq n - 1\}$.

Proof. It suffices to consider $f \in L^2(\mathbb{R}_+^n \times \mathbb{N}^n) \otimes_{\text{alg}} \mathcal{D}_n$ step function, e.g. $I(f) := \alpha_0 (S_{v_1}^{(k_1)} - S_{u_1}^{(k_1)}) \cdots (S_{v_n}^{(k_n)} - S_{u_n}^{(k_n)}) \alpha_n$ as before. Now for any $J = \{j_0, \dots, j_k\} \subset \llbracket 0, n \rrbracket$ we can write

$$\begin{aligned} &(S_{v_1}^{(k_1)} - S_{u_1}^{(k_1)}) \cdots (S_{v_{j_0}}^{(k_{j_0})} - S_{u_{j_0}}^{(k_{j_0})}) \otimes (S_{v_{j_0+1}}^{(k_{j_0+1})} - S_{u_{j_0+1}}^{(k_{j_0+1})}) \cdots (S_{v_{j_1}}^{(k_{j_1})} - S_{u_{j_1}}^{(k_{j_1})}) \otimes \cdots (S_{v_n}^{(k_n)} - S_{u_n}^{(k_n)}) \\ &= \sum_{i_1, \dots, i_k \in \mathbb{N}, i_0, i_{k+1} \in \mathbb{N} \cup \{-1\}} \lambda_{i_0, \dots, i_{k+1}}^{(J)}(f) e_{i_0} \otimes \cdots \otimes e_{i_{k+1}} \end{aligned}$$

with by convention $e_{-1} = 1$ occurring only if $j_0 = 0$ and/or $j_k = n$ and in these cases the only non-vanishing λ_i is respectively for i with $i_0 = -1$ ($i_{k+1} = -1$). (Note that if J empty we have $k = -1$ and no tensor product.) This uses only the orthonormal basis introduced before $\Gamma_0(\omega)$ since any $(S_{v_{j_i+1}}^{(k_{j_i+1})} - S_{u_{j_i+1}}^{(k_{j_i+1})}) \cdots (S_{v_{j_{i+1}}}^{(k_{j_{i+1})})} - S_{u_{j_{i+1}}}^{(k_{j_{i+1})})})$ is orthogonal to \mathbb{C} when not 1.

Especially, if we write $e_i = (e_{i_0} \otimes e_{i_1} \cdots \otimes e_{i_{k+1}})$,

$$I(f) = \sum_{J=\{j_0, \dots, j_k\} \subset \llbracket 0, n \rrbracket} \sum_{i_1, \dots, i_k \in \mathbb{N}, i_0, i_{k+1} \in \mathbb{N} \cup \{-1\}} \lambda_{i_0, \dots, i_{k+1}}^{(J)}(f) (1 \otimes U_{j_0, \dots, j_k} \otimes 1) \# e_i,$$

so that using assumption e.2, writing $\lambda_i^{(J)}(f) = \lambda_{i_0, \dots, i_{k+1}}^{(J)}(f)$, $\mathbb{N}^- = \mathbb{N} \cup \{-1\}$, one gets:

$$\begin{aligned} \Delta(I(f)) &= \sum_{J \subset \llbracket 0, n \rrbracket} \sum_{i \in \mathbb{N}^- \times \mathbb{N}^k \times \mathbb{N}^-} \lambda_i^{(J)}(f) (1^{\otimes 1_{\{j_0=0\}}} \otimes (\Delta^{\otimes(k+1+1_{\{j_0 \neq 0\}}+1_{\{j_k \neq n\}})} (1^{\otimes 1_{\{j_0 \neq 0\}}} \\ &\quad \otimes U_{j_0, \dots, j_k} \otimes 1^{\otimes 1_{\{j_k \neq n\}}})) \otimes 1^{\otimes 1_{\{j_k = n\}}}) \# e_i \\ &= \sum_{J \subset \llbracket 0, n \rrbracket} I((\mathcal{J}_{J \cup \{0, n\}} (\Delta^{\otimes(k+1+1_{\{j_0 \neq 0\}}+1_{\{j_k \neq n\}})} (1^{\otimes 1_{\{j_0 \neq 0\}}} \otimes U_{j_0, \dots, j_k} \otimes 1^{\otimes 1_{\{j_k \neq n\}}}))))) \\ &= I(\Delta^{\otimes[n]}(f)). \end{aligned}$$

Recall $(P_\Gamma \nabla_s I(f))_{k_i} = 1_{(u_i, v_i]}(s) (\alpha_0 (S_{v_1}^{(k_1)} - S_{u_1}^{(k_1)}) \cdots \alpha_{i-1} \otimes \alpha_i \cdots (S_{v_n}^{(k_n)} - S_{u_1}^{(k_n)}) \alpha_n)$, where v_i is the hugest of all v_k 's ($(P_\Gamma \nabla_s I(f))_j = 0$ if $j \neq k_i$). As before, one gets $P_\Gamma \nabla_s I(f) = I \otimes I(f_{1,s} \otimes f_{2,s}) \in D(\Delta) \otimes_{\text{alg}} D(\Delta)$ so that assumption e.1 concludes to the domain statement. Note that $P_\Gamma \nabla_s I(f) \in (\mathcal{E}_{i-1} \otimes \mathcal{E}_{n-i})^{\oplus \mathbb{N}}$, and thus by assumption e.3 is orthogonal to $\delta(I(g_p))$ for any $p \geq n$ if $g_p \in L^2(\mathbb{R}_+^n \times \mathbb{N}^n) \otimes_{\text{alg}} D(\Delta^{\otimes[p]})$. Thus $\delta^* P_\Gamma \nabla_s I(f)$ is orthogonal to all such $I(g_p)$, and by density for all $h_p \in \mathcal{E}_p$, $p \geq n$. This gives the last statement. \square

We can now define:

Definition 10. An ultraweak solution (of (2)) is an L^2 -weakly continuous adapted process X_t in $L^2_{a, \text{loc}}(\mathbb{R}_+, L^2(M))$ such that, for some C and ω , $\|X_t\|_2 \leq C e^{\omega t}$ and for all finite sums $g = \sum_n g_n$, $g_n \in L^2(\mathbb{R}_+^n \times \mathbb{N}^n) \otimes_{\text{alg}} \mathcal{D}_n$ step functions as above, then a.e. in $t \in \mathbb{R}_+$:

$$\langle I(g), X_t \rangle = \langle I(g), X_0 \rangle - \frac{1}{2} \int_0^t ds \langle \Delta I(g), X_s \rangle + \int_0^t ds \langle \delta^* P_\Gamma \nabla_s I(g), X_s \rangle.$$

Note that a weak solution satisfying $\|X_t\|_2 \leq C e^{\omega t}$ is an ultraweak solution. Indeed from the free Bismut–Clark–Ocone formula and the previous lemma for domain issues, one gets:

$$\left\langle I(g), \int_0^t \delta(X_s) \# dS_s \right\rangle = \left\langle \int_0^t (P_\Gamma \nabla_u I(g)) \# dS_u, \int_0^t \delta(X_s) \# dS_s \right\rangle = \int_0^t ds \langle \delta^* P_\Gamma \nabla_s I(g), X_s \rangle.$$

1.5. Mild and ultramild solutions

Here is the main theorem in the general setting.

Theorem 11.

- (i) Let us assume $\Gamma_{0u}(\omega)$ and that $X_0 \in L^2(M_0)$, then Eq. (2) has a unique ultraweak solution. This solution is also an ultramild solution X_t and we have, for every T and a.e. in t :

$$\begin{aligned} \|X_t\|_2^2 &\leq e^{\omega t} \|X_0\|_2^2, \\ \|X\|_{\mathcal{B}_{2, \phi \delta}^a([0, T])}^2 &\leq \|X_0\|_2^2 2 \frac{e^{\omega T} - 1}{\omega} \quad (\text{or } 2T \|X_0\|_2^2 \text{ if } \omega = 0). \end{aligned}$$

Furthermore, if we write X_s^ε a solution for δ replaced by $(1 - \varepsilon)\delta$ ($\varepsilon \in (0, 1]$), then X_s^ε is a unique mild solution of this variant equation, i.e. a solution of (7\varepsilon) in $\mathcal{B}_{2,\Delta^{1/2}}^a$, and the solution built above X_t is, for every T , a weak limit ($\varepsilon \rightarrow 0$) in $\mathcal{B}_{2,\phi\delta}^a([0, T])$ and strong limit in $C^0([0, T], (L^2(M), \sigma(L^2(M), L^2(M))))$ of the solutions X_t^ε . Finally, if we assume $X_0 \in D(\Delta^{\otimes 1/2}\delta) \cap D(\Delta) \cap L^2(M_0)$ then the solution satisfies a.e.:

$$\begin{aligned} & \|X_t - X_0 - \delta(X_0)\#S_t\|_2^2 \\ & \leq \frac{t^2}{4} \|\Delta(X_0)\|_2^2 + (e^{\omega t} - 1)\|X_0\|_2^2 \\ & \quad + \frac{t^2}{2} \left(\|\Delta^{\otimes 1/2}(\delta(X_0))\|_2^2 + \frac{\pi}{4} (\|\Delta(X_0)\|_2^2 + \omega \|\Delta^{1/2}(X_0)\|_2^2)^{1/2} \|\Delta^{\otimes 1/2}(\delta(X_0))\|_2 \right) \\ & \quad + t \sup_{s \in [0,t]} (\|\Delta^{1/2}(\phi_s X_0)\|_2^2 - \|\delta(\phi_s X_0)\|_2^2) + t^2 \omega/4 \|\Delta^{1/2}(X_0)\|_2^2. \end{aligned}$$

(ii) Let us assume $\Gamma_{1u}(\omega, C)$ and that $X_0 \in D(\Delta^{1/2}) \cap L^2(M_0)$, then Eq. (2) has a unique mild solution X_t . Moreover, we have the following inequalities a.e.:

$$\begin{aligned} \|X_t\|_2^2 & \leq e^{\omega t} \|X_0\|_2^2, \\ \|\tilde{\delta}(X_t)\|_2^2 & \leq \|\tilde{\delta}(X_0)\|_2^2 e^{(6+2\omega)C^4 t}. \end{aligned}$$

If we write X_s^ε a solution for δ replaced by $(1 - \varepsilon)\delta$ ($\varepsilon \in (0, 1]$), then, if $\delta = \tilde{\delta}$, X_s^ε is a strong solution, i.e. a solution of (7\varepsilon) in $\mathcal{B}_{2,\Delta}^a([0, T])$, for every T , and otherwise, if $\delta \neq \tilde{\delta}$ a mild solution by (i). Furthermore, X_t is, for every T , the weak limit ($\varepsilon \rightarrow 0$) in $\mathcal{B}_{2,\delta}^a([0, T])$ and strong limit in $\mathcal{B}([0, T], L^2(M))$ (the space of bounded functions with uniform convergence) of the solution X_t^ε .

Proof. Let us sketch the plan of the proof. Step 0 proves uniqueness of ultraweak solutions, which is a useful preliminary. We will first find unique mild (resp strong in case (ii)) solutions after replacing δ by $(1 - \varepsilon)\delta$ with $\varepsilon > 0$ [step 1]. Then, we will prove that when $\varepsilon \rightarrow 0$ we can get some weak convergence to an ultramild (resp a mild, in case (ii)) solution of (7), mainly by showing several inequalities like the ones stated in the theorem [step 2 for part (i), step 3 for part (ii)].

Step 0: Uniqueness of ultraweak solutions in case (i).

We have to show that an ultraweak solution with $X_0 = 0$ vanishes. The proof is in the spirit of Theorem 5.6 in [17] in the symmetric Fock space context. For g_n as in Definition 10, we prove by induction on n $\langle I(g_n), X_s \rangle = 0$. By density this gives the same for a step function $g_n \in L^2(\mathbb{R}_+^n \times \mathbb{N}^n) \otimes_{\text{alg}} D(\Delta^{\otimes [n]})$. The induction hypothesis (and Lemma 9) or only the definition of ∇_s at initialization, gives the last integral in the definition of ultraweak solution vanishes, so that for a step function $g = g_n \in L^2(\mathbb{R}_+^n \times \mathbb{N}^n) \otimes_{\text{alg}} D(\Delta^{\otimes [n]})$:

$$\langle I(g), X_t \rangle = -\frac{1}{2} \int_0^t ds \langle \Delta I(g), X_s \rangle.$$

Since $\|X_t\|_2 \leq C \exp(\omega t)$, we can consider the Laplace transform for $\lambda > \omega$ so that we get:

$$\begin{aligned} \lambda \int_0^\infty dt \exp(-\lambda t) \langle I(g), X_t \rangle & = -\frac{\lambda}{2} \int_0^\infty dt \exp(-\lambda t) \int_0^t ds \langle \Delta I(g), X_s \rangle \\ & = -\frac{\lambda}{2} \int_0^\infty ds \langle \Delta I(g), X_s \rangle \int_s^\infty dt \exp(-\lambda t) \\ & = -\frac{1}{2} \int_0^\infty dt \exp(-\lambda t) \langle \Delta I(g), X_t \rangle. \end{aligned}$$

Thus $\int_0^\infty dt \exp(-\lambda t) \langle (\lambda + \Delta/2)I(g), X_t \rangle = 0$ but (using Lemma 9)

$$I((\lambda + \Delta^{\otimes [n]}/2)^{-1}(g)) = (\lambda + \Delta/2)^{-1}I(g),$$

thus applying the result above to $g_\lambda = (\lambda + \Delta^{\otimes [n]}/2)^{-1}(g)$ gives

$$\int_0^\infty dt \exp(-\lambda t) \langle I(g), X_t \rangle = 0.$$

The result of the next inductive step ($\langle I(g), X_t \rangle = 0$) follows from uniqueness of Laplace transform. Now by density of the functions of the form $I(\sum g_n)$ (as in definition of ultraweak solutions) we get $X_t = 0$.

Step 1: Assume $\Gamma_0(\omega)$. For any $\varepsilon \in (0, 1]$ and $X_0 = X_0^\varepsilon \in L^2(M_0)$ there exists a unique mild solution (even in $\mathcal{B}_{2,\Delta^{1/2}}^a([0, T])$ for any T) to $X_t^\varepsilon = \phi_t(X_0) + (1 - \varepsilon)\gamma(X^\varepsilon)_t$.

Assume now $\Gamma_1(\omega, C)$ and $\delta = \tilde{\delta}$. For any $\varepsilon \in (0, 1]$ and $X_0 = X_0^\varepsilon \in D(\Delta^{1/2}) \cap L^2(M_0)$ there exists a unique strong solution (i.e. in $\mathcal{B}_{2,\Delta}^a([0, T])$ for any T) to $X_t^\varepsilon = \phi_t(X_0) + (1 - \varepsilon)\gamma(X^\varepsilon)_t$.

For each statement we can be content with proving for a small $T > 0$ to be fixed later. Then, using the fact that Γ_0, Γ_1 are translation invariant in time, if we consider the same problem starting at kT , this gives the same result on any $[0, T]$.

The first statement is easy and a consequence of (5) in Proposition 5. If $Y \in \mathcal{B}_{2,\Delta^{1/2}}^a([0, T])$, define an element at least in $L^2(M_t) \cap D(\Delta^{1/2})$ (for a.e. $t \in (0, T]$, by Proposition 5 and since $\mathcal{B}_{2,\Delta^{1/2}}^a([0, T]) \hookrightarrow \mathcal{B}_{2,\delta}^a([0, T])$):

$$\Gamma(Y)_t = \phi_t(X_0) + (1 - \varepsilon) \int_0^t \phi_{t-s}(\delta(Y_s) \# dS_s).$$

First of all, $\Gamma(Y)$ is in $\mathcal{B}_{2,\Delta^{1/2}}^a$ for Y in this space. Indeed, first $\phi_t(X_0)$ is in this space, as a limit (coming from (4)) of $\phi_t(\eta_\alpha(X_0))$, continuous function in $C^0([0, T], D(\Delta^{1/2})) \hookrightarrow \mathcal{B}_{2,\Delta^{1/2}}^a([0, T])$ (a usual $\Delta^{1/2}$ -simple-process approximation giving this). Second, since, if Y_n is a $\Delta^{1/2}$ -simple process converging to Y , $\gamma(Y_n)$ converge to $\gamma(Y)$ (a priori in $L^2([0, T], D(\Delta^{1/2}))$ from Proposition 5 (5)), it suffices to note $\gamma(Y_n)$ is itself in $\mathcal{B}_{2,\Delta^{1/2}}^a([0, T])$.

Finally it suffices to check Γ is a contraction (after moving to an equivalent norm) on $\mathcal{B}_{2,\Delta^{1/2}}^a([0, T])$. Indeed note from Proposition 5 and the definition:

$$\begin{aligned} \int_0^T ds \|\Delta^{1/2}(\Gamma(Y)_s - \Gamma(Z)_s)\|_2^2 &\leq (1 - \varepsilon)^2 \int_0^T ds \|\delta(Y - Z)_s\|_2^2, \\ \int_0^T ds \|\Gamma(Y)_s - \Gamma(Z)_s\|_2^2 &\leq (1 - \varepsilon)^2 T \int_0^T ds \|\delta(Y - Z)_s\|_2^2. \end{aligned} \tag{8}$$

Thus, fix $0 < T < \varepsilon/(2 \max(1, \omega))$, so that one can take $K = \varepsilon/2T$ to get $(1 - \varepsilon + KT) \max(1, \omega) < (1 - \varepsilon/2)K$ and define the equivalent norm on $\mathcal{B}_{2,\Delta^{1/2}}^a$:

$$\|Y\|_{\mathcal{B}_{2,\Delta^{1/2},K}^a}^2 = \int_0^T ds \|\Delta^{1/2}(Y)_s\|_2^2 + K \|Y_s\|_2^2.$$

We deduce from $\Gamma_0(\omega)$ c.1 and the previous inequalities that

$$\begin{aligned} &\|\Gamma(Y) - \Gamma(Z)\|_{\mathcal{B}_{2,\Delta^{1/2},K}^a}^2 \\ &\leq ((1 - \varepsilon)^2 + KT) \int_0^T ds (\|\Delta^{1/2}(Y - Z)_s\|_2^2 + \max(1, \omega) \|(Y - Z)_s\|_2^2) \\ &\leq (1 - \varepsilon/2) \|Y - Z\|_{\mathcal{B}_{2,\Delta^{1/2},K}^a}^2. \end{aligned}$$

This concludes to the first statement.

For the second statement, we want to show Γ is a contraction on $\mathcal{B}_{2,\Delta}^a([0, T])$ after taking an equivalent norm again. We thus now take $Y \in \mathcal{B}_{2,\Delta}^a([0, T])$.

We can apply Proposition 5 (6) to get:

$$\begin{aligned} \int_0^T dt \|\Delta \eta_\alpha^{1/2} \eta_{\alpha'}^{1/2} \gamma(Y)_t\|_2^2 &= -\|\Delta^{1/2} \eta_\alpha^{1/2} \eta_{\alpha'}^{1/2} (\gamma(Y)_T)\|_2^2 + \int_0^T dt \|\Delta^{\otimes 1/2} \eta_\alpha^{\otimes 1/2} \eta_{\alpha'}^{\otimes 1/2} (\delta(Y_t))\|_2^2 \\ &\leq \left\langle \Delta \eta_\alpha \int_0^T \delta(Y_t) \# dS_t, \int_0^T \delta(Y_t) \# dS_t \right\rangle, \end{aligned}$$

where we have used in the second line Lemma 3(ii) and contractivity of $\eta_{\alpha'}^{1/2}$. But now (in the case we assume Γ_1 and $\delta = \tilde{\delta}$), we can use Lemma 3(i) and then Γ_1 c.2 and the bound in Lemma 2 for \mathcal{H}_α to get:

$$\begin{aligned} &\left\langle \Delta \eta_\alpha \int_0^T \delta(Y_t) \# dS_t, \int_0^T \delta(Y_t) \# dS_t \right\rangle \\ &= \left\langle \int_0^T \delta \circ \Delta (\eta_\alpha(Y_s)) \# dS_s + \int_0^T \mathcal{H}_\alpha(Y_s \oplus \Delta^{1/2}(Y_s)) \# dS_s, \int_0^T \delta(Y_t) \# dS_t \right\rangle \\ &\leq \int_0^T \|\Delta(Y_t)\|_2^2 dt + (\max(1, \omega)C + \omega) \int_0^T \|Y_t\|_2^2 + \|\Delta^{1/2}(Y_t)\|_2^2 dt. \end{aligned}$$

But better, we can write $\|\Delta \eta_\alpha^{1/2} \eta_{\alpha'}^{1/2} \gamma(Y)_t\|_2^2 = \langle \Delta \eta_\alpha (\Delta)^{1/2} \eta_{\alpha'}^{1/2} \gamma(Y)_t, (\Delta)^{1/2} \eta_{\alpha'}^{1/2} \gamma(Y)_t \rangle$ to show that this increases to $\|\Delta \eta_{\alpha'}^{1/2} \gamma(Y)_t\|_2^2$ in α and then to $\|\Delta \gamma(Y)_t\|_2^2$ in α' , with the inequality bellow and as a consequence (recall $C \geq 1$) $\gamma(Y)_t \in D(\Delta)$ a.e. and we got:

$$\int_0^T dt \|\Delta \gamma(Y)_t\|_2^2 \leq \int_0^T \|\Delta(Y_t)\|_2^2 dt + 2 \max(1, \omega)C \int_0^T \|Y_t\|_2^2 + \|\Delta^{1/2}(Y_t)\|_2^2 dt. \tag{9}$$

Time has gone to choose T small enough and introduce the equivalent norm on $\mathcal{B}_{2,\Delta}^a([0, T])$ for which Γ will be a contraction under the assumptions of (ii).

First choose T such that $T\omega < 1 - (1 - \varepsilon)^2$ so that $T\omega < \frac{1}{(1-\varepsilon)^2} - 1$. Second, let $L > \frac{2C \max(1,\omega)(1+T)}{1-(1-\varepsilon)^2-\omega T} > 0$, and $K = L\omega + 2C \max(1, \omega) > 0$ thus:

$$\begin{aligned} L > L\eta &:= L(1 - \varepsilon)^2 + (2C \max(1, \omega)(1 + T) + \omega TL)(1 - \varepsilon)^2 \\ &= L(1 - \varepsilon)^2 + (2C \max(1, \omega) + TK)(1 - \varepsilon)^2. \end{aligned}$$

We get also:

$$K > K\eta' := (1 - \varepsilon)^2 K(1 + T\omega) = (1 - \varepsilon)^2 (L\omega + 2C \max(1, \omega) + KT\omega).$$

Finally, define the clearly equivalent norm: $\|X\|_{L,K,T}^2 = \int_0^T (L\|\Delta^{1/2}(X_s)\|_{L^2(\tau)}^2 + K\|X_s\|_{L^2(\tau)}^2 + \|\Delta(X_s)\|_{L^2(\tau)}^2) ds$. We get, using (8) and (9) in the first line, and then assumption c.1 in the second line:

$$\begin{aligned} &(1 - \varepsilon)^2 \|\gamma(Y)\|_{L,K,T}^2 \\ &\leq (1 - \varepsilon)^2 \int_0^T \|\Delta(Y_t)\|_2^2 dt \\ &\quad + (1 - \varepsilon)^2 2C \max(1, \omega) \int_0^T \|\Delta^{1/2}(Y_t)\|_2^2 + \|Y_t\|_2^2 dt + (L + KT)(1 - \varepsilon)^2 \int_0^T \|\delta(Y_t)\|_2^2 dt \end{aligned}$$

$$\begin{aligned} &\leq (1 - \varepsilon)^2 \int_0^T \|\Delta(Y_t)\|_2^2 dt + L\eta \int_0^T \|\Delta^{1/2}(Y_t)\|_2^2 dt + K\eta' \int_0^T \|Y_t\|_2^2 dt \\ &\leq \max((1 - \varepsilon)^2, \eta, \eta') \|Y\|_{L,K,T}^2. \end{aligned}$$

First of all, this shows that $\Gamma(Y)$ is indeed in $\mathcal{B}_{2,\Delta}^a$ for Y in this space: first, since $\phi_t(X_0)$ is in this space as before and second, since, if Y_n is a Δ -simple process converging to Y , $\gamma(Y_n)$ converge to $\gamma(Y)$ (a priori in $L^2([0, T], D(\Delta))$, and $\gamma(Y_n)$ is itself in $\mathcal{B}_{2,\Delta}^a$.

Then, we can say that Γ is a contraction on $\mathcal{B}_{2,\Delta}^a([0, T])$ equipped of the norm $\|\cdot\|_{L,K,T}$, this concludes.

Step 2: Conclusion of the proof of (i).

Applying orthogonality (via Lemma 3(iii)) and equation (5) to $(1 - \varepsilon)\gamma(X^\varepsilon)_t = X_t^\varepsilon - \phi_t(X_0)$, we know that for any T :

$$\int_0^T dt \|\Delta^{1/2} X_t^\varepsilon\|_2^2 = \int_0^T dt \|\Delta^{1/2} \phi_t(X_0)\|_2^2 - (1 - \varepsilon)^2 \|\gamma(X^\varepsilon)_T\|_2^2 + (1 - \varepsilon)^2 \int_0^T dt \|(\delta(X_t^\varepsilon))\|_2^2. \quad (10)$$

Using Eq. (4) and orthogonality and then assumption c.1 we deduce:

$$\begin{aligned} (1 - \varepsilon)^2 \|\gamma(X^\varepsilon)_t\|_2^2 &\leq \|X_0\|_2^2 - \|\phi_t(X_0)\|_2^2 + \omega \int_0^t \|X_s^\varepsilon\|_2^2 ds, \\ \|X_t^\varepsilon\|_2^2 &= \|X_0^\varepsilon\|_2^2 + (1 - \varepsilon)^2 \left\| \int_0^t \delta(X_s^\varepsilon) \# dS_s \right\|_2^2 - \int_0^t \|\Delta^{1/2}(X_s^\varepsilon)\|_2^2 ds \leq \|X_0^\varepsilon\|_2^2 + \omega \int_0^t \|X_s^\varepsilon\|_2^2 ds. \end{aligned}$$

Note this second inequality works for $\varepsilon = 0$ as soon as we have a solution in this case. We can use Gronwall's lemma on this second inequality. It proves the first inequality of the theorem (for X^ε instead of X). Combining this with the first inequality, we get, after integration, the second inequality in part (i), showing that X^ε is bounded in $\mathcal{B}_{2,\phi\delta}^a$.

$$\begin{aligned} \|X^\varepsilon\|_{\mathcal{B}_{2,\phi\delta}^a} &= \left(\int_0^T \int_0^t \|\phi_{t-s}^{\otimes 2} \delta(X_s)\|_{L^2(\tau \otimes \tau) \oplus \mathbb{N}}^2 ds + \|X_t\|_{L^2(\tau)}^2 dt \right)^{1/2} \\ &\leq \frac{1}{(1 - \varepsilon)} \left(\int_0^T \left(\|X_0\|_{L^2(\tau)}^2 + \omega \int_0^t ds e^{\omega s} \|X_0\|_{L^2(\tau)}^2 \right) + e^{\omega t} \|X_0\|_{L^2(\tau)}^2 dt \right)^{1/2} \\ &\leq \frac{1}{(1 - \varepsilon)} \left(\int_0^T 2e^{\omega t} \|X_0\|_{L^2(\tau)}^2 dt \right)^{1/2}. \end{aligned}$$

Modulo extraction, we get a *-weak limit in $\mathcal{B}_{2,\phi\delta}^a([0, T])$ by compactness. As a consequence, since γ is a linear continuous map as recalled in the part on stochastic convolution, $\gamma(X^\varepsilon)$ (or at least the image of the previous extraction) converges in $L^2([0, T], L^2(M))$ weakly. Since $\phi_t(X_0)$ is a constant in this space we can take the limit and verify the equation in this space, thus a.e., we especially get an ultramild solution. Since we deduce any such *-weak limit point is also an ultraweak solution (since X^ε is a mild thus weak thus ultraweak solution of the ε variant) we get *-weak convergence from uniqueness proved in step 0.

Moreover, taking $\xi \in L^2(M)$, with, say, the projection of ξ on the space of stochastic integrals given by $\int_0^T \eta_s \# dS_s$, let us prove that $\langle \xi, X_t^\varepsilon \rangle$ is an equicontinuous and uniformly bounded family (for $\varepsilon \in (0, 1]$) on $[0, T]$. From what we obtained above, only equicontinuity need to be proved, but (for $t \leq \tau$) we have (using the equation for X_τ^ε and Cauchy-Schwarz):

$$\begin{aligned} \langle \xi, X_\tau^\varepsilon - X_t^\varepsilon \rangle &\leq \|\xi\|_2 \|\phi_{\tau-t}(X_0) - X_0\|_2 \\ &\quad + (1 - \varepsilon) \int_t^\tau ds \langle \eta_s, \phi_{\tau-s}^{\otimes 2}(\delta(X_s^\varepsilon)) \rangle + (1 - \varepsilon) \int_0^t ds \langle \phi_{\tau-t}^{\otimes 2} \eta_s - \eta_s, \phi_{t-s}^{\otimes 2}(\delta(X_s^\varepsilon)) \rangle \\ &\leq \|\xi\|_2 \|\phi_{\tau-t}(X_0) - X_0\|_2 + \left(\int_t^\tau ds \|\eta_s\|_2^2 \right)^{1/2} \|(1 - \varepsilon)\gamma_\tau(X^\varepsilon)\|_2 \end{aligned}$$

$$\begin{aligned}
 & + \left(\int_0^t \|\phi_{\tau-t}^\otimes \eta_s - \eta_s\|_2^2 \right)^{1/2} \|(1-\varepsilon)\gamma_t(X^\varepsilon)\|_2 \\
 & \leq \|\xi\|_2 \|\phi_{\tau-t}(X_0) - X_0\|_2 + e^{\omega\tau/2} \|X_0\|_2 \left(\left(\int_t^\tau ds \|\eta_s\|_2^2 \right)^{1/2} + \|\phi_{\tau-t}\xi - \xi\|_2 \right).
 \end{aligned}$$

This concludes using strong continuity of ϕ_t (and using Heine–Cantor Theorem). As a consequence, using Arzela–Ascoli Theorem (and separability assumption on $L^2(M)$), we get via diagonal extraction, X_t is weakly continuous, and limit of a subsequence of X_t^ε in $C^0([0, T], (L^2(M), \sigma(L^2(M), L^2(M))))$. As a consequence, this easily enables us to pass to the limit $\varepsilon \rightarrow 0$ in the first inequality of the theorem. From this we get also that any limit point is an ultraweak solution, so that from uniqueness we get the stated limit without extraction.

We now establish the supplementary inequality.

First by orthogonality and assumption *d.2* of $\Gamma_0(\omega)$, we have:

$$\begin{aligned}
 & \|X_T^\varepsilon - X_0 - (1-\varepsilon)\delta(X_0)\#S_T\|_2^2 \\
 & = \|\phi_T(X_0) - X_0\|_2^2 + (1-\varepsilon)^2 \int_0^T dt \|\phi_{T-t}^\otimes \delta(\phi_t(X_0)) - \delta(X_0)\|_2^2 + (1-\varepsilon)^4 \|\gamma \circ \gamma(X^\varepsilon)_T\|_2^2.
 \end{aligned}$$

Moreover, the same kind of orthogonality and relations (5) and (4) imply that:

$$\begin{aligned}
 & (1-\varepsilon)^4 \|\gamma \circ \gamma(X^\varepsilon)_T\|_2^2 \\
 & = \int_0^T dt \left((1-\varepsilon)^2 \|\delta(X_t^\varepsilon)\|_2^2 - \|\Delta^{1/2}(X_t^\varepsilon)\|_2^2 \right) \\
 & \quad + \int_0^T dt \left(\|\Delta^{1/2}(\phi_t(X_0))\|_2^2 - (1-\varepsilon)^2 \|\delta(\phi_t(X_0))\|_2^2 \right) + (1-\varepsilon)^2 \|\Delta^{1/2}(\gamma(\phi_t(X_0)))_t\|_2^2 \\
 & \leq (e^{\omega T} - 1) \|X_0\|_2^2 + T \sup_{[0, T]} \left(\|\Delta^{1/2}(\phi_t X_0)\|_2^2 - (1-\varepsilon)^2 \|\delta(\phi_t X_0)\|_2^2 \right) \\
 & \quad + (1-\varepsilon)^2 \int_0^T dt \|\delta(\phi_t(X_0))\|_2^2 - \|\phi_{T-t}^\otimes \delta(\phi_t(X_0))\|_2^2,
 \end{aligned}$$

where we used, in the inequality, the first inequality of our theorem.

Our first line is in our estimate (once added a $(1-\varepsilon)$ where needed for our ε variant). It only remains to get the other terms by several elementary computations, only involving X_0 .

$$\begin{aligned}
 & \int_0^T dt \|\phi_{T-t}^\otimes \delta(\phi_t(X_0)) - \delta(X_0)\|_2^2 + \|\delta(\phi_t(X_0))\|_2^2 - \|\phi_{T-t}^\otimes \delta(\phi_t(X_0))\|_2^2 \\
 & = \int_0^T dt \|\delta(\phi_t X_0 - X_0)\|_2^2 + 2\Re \int_0^T dt \langle \delta(X_0) - \phi_{T-t}^\otimes \delta(X_0), \delta(X_0) \rangle \\
 & \quad + 2\Re \int_0^T dt \langle (\phi_{T-t}^\otimes - \text{id})\delta(X_0), \delta(X_0 - \phi_t(X_0)) \rangle \\
 & \leq \int_0^T dt \|\Delta^{1/2}(\phi_t X_0 - X_0)\|_2^2 + T^2 \omega/4 \|\Delta^{1/2}(X_0)\|_2^2 \\
 & \quad + \int_0^T dt (T-t) \|\Delta^{\otimes 1/2}(\delta(X_0))\|_2^2 \\
 & \quad + \int_0^T dt \sqrt{t(T-t)} \left(\|\Delta(X_0)\|_2^2 + \omega \|\Delta^{1/2}(X_0)\|_2^2 \right)^{1/2} \|\Delta^{\otimes 1/2}(\delta(X_0))\|_2
 \end{aligned}$$

$$\begin{aligned}
&= \int_0^T dt \|\Delta^{1/2}(\phi_t X_0 - X_0)\|_2^2 + T^2 \omega/4 \|\Delta^{1/2}(X_0)\|_2^2 \\
&\quad + \frac{T^2}{2} \left(\|\Delta^{\otimes 1/2}(\delta(X_0))\|_2^2 + \frac{\pi}{4} (\|\Delta(X_0)\|_2^2 + \omega \|\Delta^{1/2}(X_0)\|_2^2)^{1/2} \|\Delta^{\otimes 1/2}(\delta(X_0))\|_2 \right).
\end{aligned}$$

The inequality comes from $\Gamma_0(\omega)c.1$ and several uses of the spectral theorem applied in the form $\langle (\text{id} - \phi_t)^i x, x \rangle \leq \langle \frac{t}{2} \Delta x, x \rangle$ ($i = 1$ or 2).

Finally, it remains to compute the last integral using the spectral theorem for Δ :

$$\begin{aligned}
\int_0^T dt \|\Delta^{1/2}(\phi_t(X_0) - X_0)\|_2^2 &= 4\|\phi_{T/2}(X_0)\|_2^2 - 3\|X_0\|^2 - \|\phi_T(X_0)\|_2^2 + T\|\Delta^{1/2}(X_0)\|_2^2 \\
&= 2\langle (\phi_T - \text{id} + T\Delta/2)(X_0), X_0 \rangle - \|\phi_T(X_0) - X_0\|_2^2 \\
&\leq \frac{T^2}{4} \|\Delta(X_0)\|_2^2 - \|\phi_T(X_0) - X_0\|_2^2.
\end{aligned}$$

Putting everything together this concludes to:

$$\begin{aligned}
&\|X_t^\varepsilon - X_0 - (1 - \varepsilon)\delta(X_0)\#S_t\|_2^2 \\
&\leq \frac{t^2}{4} \|\Delta(X_0)\|_2^2 + (e^{\omega t} - 1)\|X_0\|_2^2 + (1 - (1 - \varepsilon)^2)\|\phi_t(X_0) - X_0\|_2^2 \\
&\quad + \frac{t^2}{2} \left(\|\Delta^{\otimes 1/2}(\delta(X_0))\|_2^2 + \frac{\pi}{4} (\|\Delta(X_0)\|_2^2 + \omega \|\Delta^{1/2}(X_0)\|_2^2)^{1/2} \|\Delta^{\otimes 1/2}(\delta(X_0))\|_2 \right) \\
&\quad + t \sup_{[0,t]} (\|\Delta^{1/2}(\phi_s X_0)\|_2^2 - (1 - \varepsilon)^2 \|\delta(\phi_s X_0)\|_2^2) + t^2 \omega/4 \|\Delta^{1/2}(X_0)\|_2^2.
\end{aligned}$$

We easily obtain the limit case $\varepsilon = 0$ using the limit in $C^0([0, T], (L^2(M), \sigma(L^2(M), L^2(M))))$.

Step 3: Under the assumptions of (ii), with $\mathcal{B}_{2,\delta}^a$ depending on our fixed $T > 0$, and for $X_0 \in D(\Delta)$, there exists a unique mild solution X_t of (2) which is the weak limit in $\mathcal{B}_{2,\delta}^a$ and strong limit in $\mathcal{B}([0, T], L^2(M))$ of the solution X_t^ε of step one. Moreover, this solution satisfies the two first inequalities of (ii) in the theorem.

Consider $\varepsilon > 0$ like in step 1. In case $\delta \neq \tilde{\delta}$, we don't know $X_t^\varepsilon \in D(\Delta)$, since we have only a mild solution, we have to circumvent this trouble for computational purposes.

Applying the first part of step 1 with δ replaced by $\eta_\beta^\otimes \delta$, we get a solution $X_t^{\varepsilon,\beta}$ in $\mathcal{B}_{2,\Delta^{1/2}}^a$ and since by Proposition 5 (5) and the argument in step one, $\gamma(X_t^{\varepsilon,\beta}) \in \mathcal{B}_{2,\Delta^{1/2}}^a$ we deduce $\eta_\beta \gamma(X_t^{\varepsilon,\beta}) \in \mathcal{B}_{2,\Delta^{3/2}}^a$. As a consequence, if $X_0 \in D(\Delta)$ we get as in step 1, $X_t^{\varepsilon,\beta} \in \mathcal{B}_{2,\Delta^{3/2}}^a$.

We can now compute for our solution $X_t^{\varepsilon,\beta}$. We can apply Proposition 5 (6) in case $B = \tilde{\delta}$, $\alpha = \alpha' = \beta$ and the variant of (4) valid for $x = X_0 \in D(\Delta^{1/2})$: $\|\tilde{\delta}\phi_t(x)\|_2^2 = \|\tilde{\delta}x\|_2^2 - \int_0^t \Re\langle \tilde{\delta}\Delta\phi_s(x), \tilde{\delta}\phi_s(x) \rangle ds$. Using also orthogonality from Lemma 3 but for $\tilde{\delta}$, we get:

$$\|\tilde{\delta}(X_t^{\varepsilon,\beta})\|_2^2 = \|\tilde{\delta}(X_0^\varepsilon)\|_2^2 + (1 - \varepsilon)^2 \left\| \tilde{\delta}\eta_\beta \int_0^t \delta(X_s^{\varepsilon,\beta})\#dS_s \right\|_2^2 - \int_0^t \Re\langle \tilde{\delta}\Delta X_s^{\varepsilon,\beta}, \tilde{\delta}X_s^{\varepsilon,\beta} \rangle ds.$$

We have thus shown:

$$\begin{aligned}
\|\tilde{\delta}(X_t^{\varepsilon,\beta})\|_2^2 &= \|\tilde{\delta}(X_0^\varepsilon)\|_2^2 \\
&\quad + (1 - \varepsilon)^2 \left\| (\eta_\beta^\otimes \tilde{\delta} + \tilde{\mathcal{H}}_\beta) \int_0^t \delta(X_s^{\varepsilon,\beta})\#dS_s \right\|_2^2 - \int_0^t \Re\langle \Delta^\otimes \circ \tilde{\delta}(X_s^{\varepsilon,\beta}) - \mathcal{H}(\tilde{\delta}(X_s^{\varepsilon,\beta})), \tilde{\delta}X_s^{\varepsilon,\beta} \rangle ds.
\end{aligned}$$

We used the identities of assumption *f.2* and of Lemma 2 about $\tilde{\delta}\Delta$ and $\tilde{\delta}\eta_\beta$ justified since almost surely in s $X_s^{\varepsilon,\beta} \in D(\Delta^{3/2})$ and because via Lemma 3(i) we know $\int_0^t \delta(X_s^{\varepsilon,\beta}) \# dS_s \in D(\tilde{\delta})$. We deduce:

$$\begin{aligned} & \|\tilde{\delta}(X_t^{\varepsilon,\beta})\|_2^2 \\ & \leq \|\tilde{\delta}(X_0^\varepsilon)\|_2^2 + (1 - \varepsilon)^2 \int_0^t \|\tilde{\delta}^\otimes \delta(X_s^{\varepsilon,\beta})\|_2^2 ds + 2\Re\left\langle \tilde{\delta}\eta_\beta \int_0^t \delta(X_s^{\varepsilon,\beta}) \# dS_s, \tilde{\mathcal{H}}_\beta \int_0^t \delta(X_s^{\varepsilon,\beta}) \# dS_s \right\rangle \\ & \quad - \int_0^t \Re\langle \Delta^\otimes \circ \tilde{\delta}(X_s^{\varepsilon,\beta}) - \mathcal{H}(\tilde{\delta}(X_s^{\varepsilon,\beta})), \tilde{\delta}X_s^{\varepsilon,\beta} \rangle ds \\ & \leq \|\tilde{\delta}(X_0^\varepsilon)\|_2^2 + \int_0^t 2C \|\tilde{\delta}(X_s^{\varepsilon,\beta})\|_2^2 ds + 2\Re\left\langle \tilde{\delta}\eta_\beta \int_0^t \delta(X_s^{\varepsilon,\beta}) \# dS_s, \tilde{\mathcal{H}}_\beta \int_0^t \delta(X_s^{\varepsilon,\beta}) \# dS_s \right\rangle \\ & \leq \|\tilde{\delta}(X_0^\varepsilon)\|_2^2 + \int_0^t \left(2C + 2\frac{C^4}{\beta}(\omega + 2\beta) \right) \|\tilde{\delta}(X_s^{\varepsilon,\beta})\|_2^2 ds. \end{aligned}$$

In the first line, we used η_β contractive after computing the first scalar product. In the second line, we used assumption *h* to cancel one term and the bound $\|\mathcal{H}\| \leq C$. In the last line, we used assumption *g*, the bound on $\tilde{\mathcal{H}}_\beta$ from Lemma 2 and $\|\tilde{\delta}\eta_\beta\| \leq C\sqrt{\omega + 2\beta}$ already used there.

Applying Gronwall’s lemma, we got (for $\beta \geq 1$):

$$\|\tilde{\delta}(X_t^{\varepsilon,\beta})\|_2^2 \leq \|\tilde{\delta}(X_0^{\varepsilon,\beta})\|_2^2 e^{(6+2\omega)C^4 t}.$$

As a consequence, we get a weak limit point $X_t^{\varepsilon,\infty}$ in $\mathcal{B}_{2,\delta}^a$. Let us show such a limit point is a solution of (2ε) in $\mathcal{B}_{2,\delta}^a$. This gives by uniqueness $X_t^{\varepsilon,\infty} = X_t^\varepsilon$, and the fact that the weak limit point is a limit. Of course, it suffices to show the equation weakly, the only non-trivial limit is the stochastic integral, but since $\delta X_t^{\varepsilon,\beta}$ is bounded it is easy to remove η_β on the other side of the scalar product, and then to use weak convergence of $X_t^{\varepsilon,\beta}$ in $\mathcal{B}_{2,\delta}^a$. We also get a corresponding inequality a.e. for the limit by seeing the inequality weakly in $L^2([0, T])$.

As is usual, if we are able to prove bounds in $D(\delta)$, we can also deduce $\|\cdot\|_2$ Cauchy property. Using (5) after using the SDE and the common initial conditions, we also get (for $0 < \varepsilon, \eta < 1$):

$$\begin{aligned} \|X_t^\varepsilon - X_t^\eta\|_2^2 & = \|\gamma((1 - \varepsilon)X_t^\varepsilon - (1 - \eta)X_t^\eta)\|_2^2 \\ & = - \int_0^t \|\Delta^{1/2} \gamma((1 - \varepsilon)X_s^\varepsilon - (1 - \eta)X_s^\eta)\|_2^2 ds + \int_0^t \|\delta((1 - \varepsilon)X_s^\varepsilon - (1 - \eta)X_s^\eta)\|_2^2 ds \\ & \leq - \int_0^t \|\Delta^{1/2}(X_s^\varepsilon - X_s^\eta)\|_2^2 ds + \int_0^t \|\delta(X_s^\varepsilon - X_s^\eta)\|_2^2 ds \\ & \quad + 12 \max(\varepsilon, \eta) \max(\|\delta(X_s^\varepsilon)\|_2, \|\delta(X_s^\eta)\|_2)^2 ds. \end{aligned}$$

In the last line, we used an elementary bound on the second integral expanding the scalar products with $(1 - \varepsilon)X_s^\varepsilon - (1 - \eta)X_s^\eta = (X_s^\varepsilon - X_s^\eta) + (\eta X_s^\eta - \varepsilon X_s^\varepsilon)$ and again the SDE with same initial condition on the first integral. Using assumption *c.1*, *g* and our bound on $\|\delta(X_s^\varepsilon)\|_2$, one gets:

$$\begin{aligned} \|X_t^\varepsilon - X_t^\eta\|_2^2 & \leq \int_0^t \omega \|(X_s^\varepsilon - X_s^\eta)\|_2^2 + 12 \max(\varepsilon, \eta) C^2 \|\tilde{\delta}(X_0^{\varepsilon,\beta})\|_2^2 e^{(6+2\omega)C^4 s} ds \\ & \leq 12 \max(\varepsilon, \eta) C^2 \|\tilde{\delta}(X_0^{\varepsilon,\beta})\|_2^2 \frac{e^{(6+2\omega)C^4 t + \omega t}}{(6 + 2\omega)C^4}. \end{aligned}$$

As noted at the beginning of step 2, we know any (mild) solution of the case $\varepsilon = 0$, if it exists satisfies: $\|X_t\|_2^2 \leq e^{\omega t} \|X_0\|_2^2$, giving especially uniqueness.

We have thus obtained strong convergence on X_t^ε in $\mathcal{B}([0, T], L^2(M))$ by Cauchy property. We have also boundedness of X_t^ε in $\mathcal{B}_{2,\delta}^a$, which gives by weak compactness a limit up to extraction when $\varepsilon \rightarrow 0$. Once we will have proved that any such limit point is a mild solution with $\varepsilon = 0$, uniqueness (of the solution thus of the limit point) will get that in fact X_t^ε weakly converges in $\mathcal{B}_{2,\delta}^a$ to the newly found solution X_t . Since we have already noticed weak continuity of Stochastic convolution, we are in fact done for proving that any limit point is a mild solution.

Finally, we conclude the proof of the part (ii) of our theorem, by considering $\eta_\alpha(X_0)$ as initial condition of a solution $X_{t,\alpha}$, in case we have only $X_0 \in D(\Delta^{1/2})$ (and not anymore $D(\Delta)$) and letting go $\alpha \rightarrow \infty$. With the same weak limit arguments, we show that $X_{t,\alpha}$ converges weakly in $\mathcal{B}_{2,\delta}^a$ to X_t . Moreover, note for further use that we have also strong convergence of $X_{t,\alpha}$ to X_t in $\mathcal{B}([0, T], L^2(M))$ by the following inequality (proved as above for the Cauchy property, except we don't have the same initial conditions anymore, but, however, more cancellations):

$$\begin{aligned} \|X_{t,\alpha} - X_{t,\beta}\|_2^2 &= \|\phi_t(X_{0,\alpha} - X_{0,\beta})\|_2^2 + \|\gamma(X_{t,\alpha} - X_{t,\beta})\|_2^2 \\ &= \|\phi_t(X_{0,\alpha} - X_{0,\beta})\|_2^2 - \int_0^t \|\Delta^{1/2}\gamma(X_{s,\alpha} - X_{s,\beta})\|_2^2 ds + \int_0^t \|\delta(X_{s,\alpha} - X_{s,\beta})\|_2^2 ds \\ &\leq \|X_{0,\alpha} - X_{0,\beta}\|_2^2 - \int_0^t \|\Delta^{1/2}(X_{s,\alpha} - X_{s,\beta})\|_2^2 - \|\Delta^{1/2}\phi_s(X_{0,\alpha} - X_{0,\beta})\|_2^2 ds \\ &\quad + \int_0^t \|\Delta^{1/2}(X_{s,\alpha} - X_{s,\beta})\|_2^2 + \omega\|X_{s,\alpha} - X_{s,\beta}\|_2^2 ds \\ &\leq e^{\omega t} (\|\eta_\alpha(X_0) - \eta_\beta(X_0)\|_2^2 + T\|\eta_\alpha(\Delta^{1/2}X_0) - \eta_\beta(\Delta^{1/2}X_0)\|_2^2). \end{aligned} \quad \square$$

2. Our main example: Derivation-generator of a Dirichlet form

As explained in the [Introduction](#), our main case of interest will be when δ is a derivation and $\Delta = \delta^* \delta$ the corresponding generator of a Dirichlet form. Note that in that case it is well known (cf e.g. [9]) ϕ_t and η_α are completely positive contractions on M .

2.1. Preliminaries and notation around zero extensions of a derivation on free Brownian motions

2.1.1. Setting and extension

Recall $M = W^*(M_0; S_s^{(j)}, 0 \leq s \leq \infty, 0 \leq j \leq N)$ (we will consider only here the case of finitely many derivations and thus free Brownian motions) and $M_t = W^*(M_0; S_s^{(j)}, 0 \leq s \leq t, 0 \leq j \leq N)$.

Let us assume we are given $\partial : D(\partial) \rightarrow \text{HS}(M_0)^N \simeq (L^2(M_0) \otimes L^2(M_0))^N \simeq_{1 \otimes O} (L^2(M_0) \otimes L^2(M_0^{\text{op}}))^N$ a derivation valued in a direct sum of Hilbert–Schmidt operators over $L^2(M_0)$. As usual the identification of $L^2(M_0) \otimes L^2(M_0)$ with Hilbert–Schmidt operators sends $a \otimes b$ to the finite rank operator $x \mapsto a\tau(bx)$. As real bimodules, they are considered with bimodule structure induced by $a(b \otimes c)d = ab \otimes cd$, and real structure $\mathcal{J}(a \otimes b) = b^* \otimes a^*$ corresponding to adjointness of Hilbert–Schmidt operators. We will emphasize the isomorphism $1 \otimes O$ with $L^2(M_0) \otimes L^2(M_0^{\text{op}})$ (coming from traciality) with corresponding bimodule structure when necessary (it is induced by the identity map for $a, b \in M$ $(1 \otimes O)(a \otimes b) = a \otimes b$ with b seen in M^{op}).

We write $Z_j = (0, \dots, 0, 1 \otimes 1, 0, \dots, 0)$ in $\text{HS}(M_0)^N$ the non-zero term lying on the j th component. We also write ∂_j for the j th component in $\text{HS}(M_0)^N$ (and we will use freely later this kind of notation). For $U \in L^2(M) \otimes L^2(M^{\text{op}})$, $K \in M \overline{\otimes} M^{\text{op}}$, we write consistently with our previous notation $U\#K$ the map induced by multiplication in $M \overline{\otimes} M^{\text{op}}$. If $U \in L^2(M) \otimes L^2(M)$, we write in this way the map induced by the previous isomorphism: $(1 \otimes O)(U\#K) := ((1 \otimes O)(U))\#K$.

Domains of closures will be considered in this L^2 setting, $D(\partial) \subset M_0$ is a weakly dense $*$ -subalgebra. We will really soon impose conditions making ∂ closable as an unbounded operator from $L^2(M_0, \tau) \rightarrow \text{HS}(M_0)^N$, and real (i.e. we have the relation $\partial(x)^* = \partial(x^*)$ with the adjoint of Hilbert–Schmidt operators in each component and as a consequence $\langle \partial(x), y\partial(z) \rangle = \langle \partial(z^*)y^*, \partial(x^*) \rangle, \forall x, y, z \in D(\partial)$). After extending it to a closed derivation $\bar{\delta}$ on M we will be interested in the corresponding generator of a Dirichlet form $\Delta = \delta^* \bar{\delta}$. This part will find realistic assumptions on ∂ to get $\Gamma_{1u}(\omega, C)$ and thus to be able to apply our general theory.

Suppose also that $\mathcal{J}_j := \partial^*(Z_j) \in L^2(M_0)$ is well defined for all $j \in [1, N]$. We have a well-known lemma (identical to Proposition 4.1 in [43] which is valid for any real derivation of the kind considered above, as pointed out after Proposition 6.2 in [44]):

Lemma 12. *Consider ∂ real densely defined derivation with $\mathcal{J}_j := \partial^*(Z_j) \in L^2(M_0)$, then $(D(\partial) \otimes_{\text{alg}} D(\partial))^N$ is contained in $D(\partial^*)$ (as a consequence assumption e.1 is satisfied) and:*

$$\partial_j^*(a \otimes b) := \partial^*(aZ_jb) = a\mathcal{J}_jb - (1 \otimes \tau)[\partial_j(a)]b - a(\tau \otimes 1)[\partial_j(b)].$$

Moreover (see e.g. [11], Remark 7, using mainly [14]), $\bar{\partial}|_{M_0 \cap D(\bar{\partial})}$ defines a derivation (noted ∂^∞ on the $*$ -algebra $M_0 \cap D(\bar{\partial})$), closed as an unbounded operator $M_0 \rightarrow \text{HS}(M_0)^N$. Finally (see e.g. Proposition 6 in [11]), for any $Z \in D(\bar{\partial}) \cap M_0$, there exists a sequence $Z_n \in D(\partial)$ with $\|Z_n\| \leq \|Z\|$, $\|Z_n - Z\|_2, \|\partial(Z_n) - \bar{\partial}(Z)\|_2 \rightarrow 0$.

Consider also $D(\delta) = D(\partial) * \mathbb{C}\langle S_s^{(j)}, 0 \leq j \leq N, 0 \leq s \leq \infty \rangle \subset M$, the algebra generated by $S_s^{(j)}$ and $D(\partial)$ (thus $D(\delta)$ is a weakly dense $*$ -subalgebra of M). Define $\delta : D(\delta) \rightarrow \text{HS}(L^2(M))^N$ the unique derivation such that $\delta(x) = \partial(x)$ if $x \in D(\partial)$ and $\delta(S_t^{(j)}) = 0$ for all t . Then, clearly $\mathcal{J}_j = \delta^*(Z_j) \in L^2(M_0) \subset L^2(M)$ (see e.g. [37], Example 2.4), and using the lemma above, δ is also closable (since δ^* is densely defined). δ is thus a closable real derivation, like ∂ , satisfying e.1. We may sometimes write $\delta^\infty : M \cap D(\bar{\delta}) \rightarrow \text{HS}(L^2(M))^N$ the analog derivation defined in the previous lemma (when we want to emphasize the domain). We will write $\Delta = \delta^*\bar{\delta}$ the associated generator of a completely Dirichlet form, ϕ_t the semigroup generated by $-1/2\Delta$, $\eta_\alpha = \frac{\alpha}{\alpha+\Delta}$ the “resolvent map” associated, as before. As we already pointed out, η_α and ϕ_t induce completely positive contractions on M .

We thus only assumed in this section Assumption 0:

Assumption 0. (a) $\partial : D(\partial) \rightarrow \text{HS}(M_0)^N$ real derivation $D(\partial) \subset M_0$ weakly dense $*$ -subalgebra.

(b) $\mathcal{J}_j := \partial^*(Z_j) \in L^2(M_0)$ is well defined for all $j \in [1, N]$, and δ is an extension by 0 on free Brownian motions: $\delta(x) = \partial(x)$ if $x \in D(\partial)$ and $\delta(S_t^{(j)}) = 0$ for all t .

This subsection will mainly develop general consequences of this Assumption 0, giving at the end Γ_{0u} .

2.1.2. Useful L^1 -closures

Here we assume Assumption 0.

We will also define following [32], Section 1.4, an analog of $\Delta, \Delta^1 : M \rightarrow L^1(M, \tau)$ (there noted Ψ), by

$$D(\Delta^1) = \{x \in D(\bar{\delta}) \cap M | y \mapsto \langle \bar{\delta}(x), \bar{\delta}(y) \rangle \text{ extends to a normal linear functional on } M\}. \tag{11}$$

$\Delta^1(x)$ is defined as the adjoint of the Radon–Nikodym derivative of the preceding linear functional $y \mapsto \langle \bar{\delta}(x), \bar{\delta}(y) \rangle$, i.e. $\langle \Delta^1(x), y \rangle := \tau(\Delta^1(x)^*y) = \langle \bar{\delta}(x), \bar{\delta}(y) \rangle$ (note the anti-linear duality bracket consistent with scalar products).

Likewise, we can define $\delta^{*1} : (L^2(M) \otimes L^2(M))^N \rightarrow L^1(M, \tau)$, by

$$D(\delta^{*1}) = \{U \in (L^2(M) \otimes L^2(M))^N | y \mapsto \langle U, \bar{\delta}(y) \rangle \text{ extends to a normal linear functional on } M\}.$$

$\delta^{*1}(U)$ is defined as the adjoint of the Radon–Nikodym derivative of the preceding linear functional $y \mapsto \langle U, \bar{\delta}(y) \rangle$. By the very definition, we see that for any $x \in D(\Delta^1), \bar{\delta}(x) \in D(\delta^{*1})$ and $\Delta^1(x) = \delta^{*1}\bar{\delta}(x)$. Moreover, we see obviously that δ^{*1} is a closed densely defined operator (using $D(\delta^*) \subset D(\delta^{*1})$ and $\bar{\delta}|_{M \otimes D(\bar{\delta})}$ is a densely defined formal adjoint). Note the following elementary lemma, using mainly the fact that δ^∞ is a derivation:

Lemma 13. $D(\Delta^1)$ is a $*$ -subalgebra of M containing $D(\Delta) \cap M$, and for any $x, y \in D(\Delta^1)$:

$$\Delta^1(xy) = \Delta^1(x)y + x\Delta^1(y) - 2 \sum_{i=1}^N m \circ (1 \otimes (\tau \circ m) \otimes 1)(\bar{\delta}_i(x) \otimes \bar{\delta}_i(y)),$$

where m denote the multiplication map $L^2(M) \hat{\otimes} L^2(M) \rightarrow L^1(M)$. Finally, for any $x, y \in D(\Delta^1)$: $\langle \Delta^1(x), y \rangle = \langle x, \Delta^1(y) \rangle$.

Proof. Take $x, y \in D(\Delta^1)$, $z \in M \cap D(\delta)$, thus

$$\begin{aligned} \langle \delta(xy), \delta(z) \rangle &= \langle \delta(x)y, \delta(z) \rangle + \langle x\delta(y), \delta(z) \rangle \\ &= \langle \delta(x), \delta(z)y^* \rangle + \langle \delta(y), x^*\delta(z) \rangle \\ &= \langle \delta(x), \delta(zy^*) \rangle + \langle \delta(y), \delta(x^*z) \rangle - \langle \delta(x), z\delta(y^*) \rangle - \langle \delta(y), \delta(x^*)z \rangle \\ &= \langle \Delta^1(x)y, z \rangle + \langle x\Delta^1(y), z \rangle - \langle \delta(y)z^*, \delta(x^*) \rangle - \langle \delta(y), \delta(x^*)z \rangle \\ &= \langle \Delta^1(x)y, z \rangle + \langle x\Delta^1(y), z \rangle - 2 \sum_i \text{Tr}(\delta_i(y)^* \circ \delta_i(x)^*z) \\ &= \langle \Delta^1(x)y, z \rangle + \langle x\Delta^1(y), z \rangle - 2\tau \left(zm \left(\sum_i \delta_i(x) \circ \delta_i(y) \right)^* \right). \end{aligned}$$

In the fourth line, we used the definition of Δ^1 and the fact δ is a real derivation. We used at the next to last line the identification of $L^2 \otimes L^2$ with Hilbert–Schmidt operators and the Trace on trace class, and the relation $\delta_i(x)^* = \delta_i(x^*)$ with the adjoint of Hilbert–Schmidt operators coming from the fact we have a real derivation. At the last line, we used the multiplication map to $L^1(M)$, induced by $m(a \otimes b) = ab$.

This proves the domain property and the equation. □

We will also need an extension $\overline{\Delta^1} : L^2(M) \rightarrow L^1(M)$. But the last equality of the previous lemma especially shows that $\Delta|_{D(\Delta) \cap M} : M \rightarrow L^2(M)$ is a $(\sigma$ -weakly) densely defined formal adjoint of $\Delta^1 : L^2(M) \rightarrow L^1(M)$, thus this operator is closable. And moreover, for any $x \in D(\Delta) \cap M$, $y \in D(\overline{\Delta^1})$, $\langle \Delta(x), y \rangle = \langle x, \overline{\Delta^1}(y) \rangle$.

Note the following elementary lemma, using $M \cap D(\Delta)$ is a core for Δ (thanks to stability of M by ϕ_t):

Lemma 14. For any $x, y \in D(\Delta)$ with either x or y in M , then $xy \in D(\overline{\Delta^1})$:

$$\overline{\Delta^1}(xy) = \Delta(x)y + x\Delta(y) - 2 \sum_{i=1}^N m \circ (1 \otimes \tau \circ m \otimes 1) (\bar{\delta}_i(x) \otimes \bar{\delta}_i(y)),$$

where m denotes the multiplication map $L^2(M) \hat{\otimes} L^2(M) \rightarrow L^1(M)$.

2.1.3. Lemmas about the extension

Here again we only assume 0.

We can consider $(\delta \otimes 1) \oplus (1 \otimes \delta) : L^2(M) \otimes L^2(M) \rightarrow (L^2(M) \otimes L^2(M) \otimes L^2(M))^{2N}$ (later abbreviated $\delta \otimes 1 \oplus 1 \otimes \delta$ or $\delta^{\otimes 2}$), which is easily seen to be densely defined on $D(\delta) \otimes_{\text{alg}} D(\delta)$, and closable (with an explicit densely defined adjoint coming from Lemma 12 in case of assumption 0). We will write $\Delta^{\otimes 2} := (\delta \otimes 1 \oplus 1 \otimes \delta)^* (\delta \otimes 1 \oplus 1 \otimes \delta) = \Delta \otimes 1 + 1 \otimes \Delta$, which is thus a densely defined closed self-adjoint positive operator. It can be seen, as stated above, to be equal to the closure of $\Delta \otimes 1 + 1 \otimes \Delta$ (defined on $D(\Delta) \otimes_{\text{alg}} D(\Delta)$), using the stability of this space by $\phi_t \otimes \phi_t$, or rather more the regularization effect, implying this is a core of the previous closed operator).

Likewise, we define $\Delta^{\otimes(n+1)}$ on $D(\Delta)^{\otimes_{\text{alg}}(n+1)} \cap V_n$ (with the notation before Γ_{0u}), i.e. for $a_i \in D(\Delta) \cap L^2(M_0)$, by

$$\Delta^{\otimes(n+1)}(a_0 \otimes \dots \otimes a_n) = \sum_{i=0}^n a_0 \otimes \dots \otimes a_{i-1} \otimes \Delta(a_i) \otimes a_{i+1} \otimes \dots \otimes a_n.$$

It clearly extends to a positive densely defined self-adjoint operator on V_n . Assumption e.4 is obvious with $\mathcal{D} = D(\Delta) \ominus \mathbb{C}$.

Recall $\cdot\#(S_t^i - S_s^i)/\sqrt{t-s} : L^2(M_s) \otimes L^2(M_s) \rightarrow L^2(M)$ is the standard isometric map extending $(a \otimes b)\#(S_t^i - S_s^i) = a(S_t^i - S_s^i)b$. Likewise, we define $\#_j$ extending $(a \otimes b \otimes c)\#_1(S_t^i - S_s^i) = a(S_t^i - S_s^i)b \otimes c$ and $(a \otimes b \otimes c)\#_2(S_t^i - S_s^i) = a \otimes b(S_t^i - S_s^i)c$, $a, b, c \in M_s$.

Corollary 15. *For any $U \in D(\Delta^\otimes) \cap L^2(M_s) \otimes L^2(M_s)$, then $U\#(S_t^i - S_s^i) \in D(\Delta)$ ($t \geq s$) and:*

$$\Delta(U\#(S_t^i - S_s^i)) = \Delta^\otimes(U)\#(S_t^i - S_s^i).$$

Moreover, for any $U \in L^2(M_s) \otimes L^2(M_s)$, $U\#(S_t^i - S_s^i) \in D(\bar{\delta})$ if and only if $U \in D(\bar{\delta} \otimes 1 \oplus 1 \otimes \bar{\delta})$, and we also have

$$\bar{\delta}(U\#(S_t^i - S_s^i)) = \bar{\delta} \otimes 1(U)\#_2(S_t^i - S_s^i) + 1 \otimes \bar{\delta}(U)\#_1(S_t^i - S_s^i).$$

As a consequence, such an element is orthogonal to any $L^2(M_0 \otimes M_0)$ (as claimed in assumption d.2). Finally, assumptions e.2, e.4 are verified by $\Delta^{\otimes(n+1)}$, and e.3 by δ .

Proof. Consider $U \in (D(\Delta) \cap M_s) \otimes_{\text{alg}} (D(\Delta) \cap M_s)$, and by linearity even $U = a \otimes b$, then by Lemma 13, we have $U\#(S_t^i - S_s^i) \in D(\Delta)$ and the formula comes from the formula there (applied twice and using freeness to cancel the other terms). The density remark before the proof and the isometric map $\cdot\#(S_t^i - S_s^i)/\sqrt{t-s} : L^2(M_s) \otimes L^2(M_s) \rightarrow L^2(M)$ conclude the general case. Assumption e.2 follows similarly.

The second property comes from δ a derivation starting with the case $U \in D(\delta) \otimes_{\text{alg}} D(\delta) \cap M_s \otimes M_s$, and using δ closed for the if part and in order to extend the formula. The only if part uses δ is defined first on $D(\delta) * \mathbb{C}\langle S_s^{(j)} \rangle$, $0 \leq j \leq N$, $0 \leq s \leq \infty$, and the fact we can take the approximation of $U\#(S_t^i - S_s^i)$ in the image of $M_s \otimes_{\text{alg}} M_s$ by $\cdot\#(S_t^i - S_s^i)$ (using freeness and the derivation property on the free product above to get the projection of a first approximation on the set above is dominated for the norm of δ by the first one). Assumption e.3 is also checked using the derivation property (the density statement is obvious). On $\mathcal{E}'_n = \mathcal{E}_n \cap (D(\delta) * \mathbb{C}\langle S_s^{(j)} \rangle, 0 \leq j \leq N, 0 \leq s \leq \infty)$, we can apply the derivation property to show $\delta(\mathcal{E}'_n) \subset \overline{\bigoplus_{p+q=n} \mathcal{E}_p \otimes \mathcal{E}_q}$ (closure in L^2), implying the orthogonality statement. \square

2.1.4. *Summary of results under Assumption 0*

We summarize the easy results obtained at this stage:

Lemma 16. *With this assumption 0, $\bar{\delta}$ and Δ satisfy the stability of filtration properties and $\Gamma_{0u}(\omega = 0)$ and also b.2, c.2 (i.e. assumptions a, b, c, e and d.1, d.2 of $\Gamma_{1u}(\omega = 0, C)$, and also d.3, d.4 in case $\delta = \bar{\delta}$).*

2.2. *Sufficient conditions for the main assumption*

2.2.1. *Statement of result*

Let us sum up right now the assumptions we will use and our result. We consider here an exact coassociativity assumption even if an almost coassociativity (considered in a previous preprint version of this paper) would be enough. This will limit the applications of this section essentially to free difference quotients. We will also consider the case $\delta = \bar{\delta}$ and consider elsewhere the case where we need and use two derivations.

Assumption 1. (a) $\partial : D(\delta) \rightarrow (D(\delta) \otimes_{\text{alg}} D(\delta))^N \subset (L^2(M_0) \otimes L^2(M_0^{\text{op}}))^N$ is coassociative i.e. $\forall i, j \forall x \in D(\delta)$:

$$(\partial_j \otimes 1) \circ \partial_i(x) - (1 \otimes \partial_i) \circ \partial_j(x) = 0.$$

(a') ∂ satisfy Assumption 0 and $\partial_j^* 1 \otimes 1 \in M_0$.

Moreover, we suppose that (b) $\partial_j^* 1 \otimes 1 \in D(\bar{\delta})$ and

$$(1 \otimes O) \bar{\delta}_i \partial_j^* 1 \otimes 1 \in M_0 \overline{M_0^{\text{op}}}.$$

Theorem 17. *Under Assumption 1, $\bar{\delta}$ and Δ satisfy the stability of filtration properties and assumption $\Gamma_{1u}(\omega = 0, C)$ for some finite constant C in the context $\delta = \bar{\delta}$.*

2.2.2. Boundedness for $(1 \otimes \tau) \circ \delta_k$ under Assumption 1

We first recall Lemma 10 in [11], which is stated there for the free difference quotient, but the coassociative case is identical. We can and will also extend it elsewhere to an almost coassociative case.

Lemma 18. *Assume Assumption 1. Let $Z \in M \cap D(\bar{\delta})$, then the following inequality holds:*

$$\|(1 \otimes \tau)(\bar{\delta}_i(Z))\|_2 \leq \|Z\|_2 \left[(2\|\delta_i^*(1 \otimes 1)\|) + (\|\delta_i^*(1 \otimes 1)\|^2 + \|\bar{\delta}_i \delta_i^*(1 \otimes 1)\|_{M \bar{\otimes} M^{\text{op}}})^{1/2} \right].$$

As a consequence, $(1 \otimes \tau) \circ \bar{\delta}_i$ extends as a bounded map $L^2(M, \tau) \rightarrow L^2(M, \tau)$.

2.2.3. Almost commutation of δ and Δ on an extended domain

We are now ready to solve our main domain issues (to get f) in the next:

Lemma 19. *Assume Assumption 1.*

(i) *For any $x \in D(\delta)$ we have $x \in D(\Delta_j)$, $\delta_i(x) \in D(\Delta_j \otimes 1 + 1 \otimes \Delta_j)$, $x \in D(\Delta^{3/2})$ and:*

$$\delta_i \Delta_j(x) = (1 \otimes \Delta_j + \Delta_j \otimes 1) \delta_i(x) + \delta_j(x) \# ((1 \otimes O) \bar{\delta}_i \partial_j^*(1 \otimes 1)).$$

(ii) *If $x \in D(\bar{\delta})$ (resp. $x \in D(\Delta)$) then so is $1 \otimes \tau(\bar{\delta}_i(x))$.*

(iii) *$D(\Delta^{3/2}) \subset D(\Delta \otimes 1 + 1 \otimes \Delta \circ \bar{\delta})$ and moreover we have for any $x \in D(\Delta^{3/2})$*

$$\bar{\delta}_i \Delta(x) = \Delta^{\otimes} \bar{\delta}_i(x) + \sum_{j=1}^N \bar{\delta}_j(x) \# ((1 \otimes O) \bar{\delta}_i \partial_j^*(1 \otimes 1)).$$

Proof. (i) Consider $x \in D(\delta)$, by Assumption 1 $\delta(x) \in (D(\delta) \otimes_{\text{alg}} D(\delta))^N \subset (M \otimes_{\text{alg}} M)^N$ (the extension from ∂ to δ is easy), thus using Lemma 12, $\delta_j(x) \in D(\delta_j^*)$, i.e. $x \in D(\Delta_j)$ for all j and

$$\Delta_j(x) = \delta_j(x) \# \delta_j^*(1 \otimes 1) - m \circ (1 \otimes \tau \otimes 1) \circ [\delta_j \otimes 1] \circ \delta_j(x) - m \circ (1 \otimes \tau \otimes 1) \circ [1 \otimes \delta_j] \circ \delta_j(x).$$

Recall $\delta_j^*(1 \otimes 1) = \partial_j^* 1 \otimes 1 \in D(\delta)$, and for any j, k $(\delta_j \otimes 1) \delta_k(x) \in D(\delta) \otimes_{\text{alg}} D(\delta) \otimes_{\text{alg}} D(\delta)$ so that one gets $\Delta_j(x) \in D(\delta_j)$ (and also the statement $\delta_i(x) \in D(\Delta_j \otimes 1)$ using again Lemma 12) and applying the derivation property for δ_i , we get (recall the notation for $\#$ before Lemma 12 and $\#_i$ similar to the one used before Corollary 15, $(a \otimes b \otimes c) \#_2 d = a \otimes bdc$, $(a \otimes b \otimes c) \#_1 d = adb \otimes c$):

$$\begin{aligned} \bar{\delta}_i \Delta_j(x) &= ((\delta_i \otimes 1) \delta_j(x)) \#_2 \delta_j^*(1 \otimes 1) + ((1 \otimes \delta_i) \delta_j(x)) \#_1 \delta_j^*(1 \otimes 1) \\ &\quad + \delta_j(x) \# ((1 \otimes O) \bar{\delta}_i \partial_j^*(1 \otimes 1)) \\ &\quad - (1 \otimes m \circ (1 \otimes \tau \otimes 1)) \circ [\delta_i \otimes 1 \otimes 1] \circ [\delta_j \otimes 1] \circ \delta_j(x) \\ &\quad - (m \circ (1 \otimes \tau \otimes 1) \otimes 1) \circ [1 \otimes 1 \otimes \delta_i] \circ [\delta_j \otimes 1] \circ \delta_j(x) \\ &\quad - (1 \otimes m \circ (1 \otimes \tau \otimes 1)) \circ [\delta_i \otimes 1 \otimes 1] \circ [1 \otimes \delta_j] \circ \delta_j(x) \\ &\quad - (m \circ (1 \otimes \tau \otimes 1) \otimes 1) \circ [1 \otimes 1 \otimes \delta_i] \circ [1 \otimes \delta_j] \circ \delta_j(x). \end{aligned}$$

Now, one can easily extend coassociativity to δ (the coassociator $(\delta_j \otimes 1) \circ \delta_i - (1 \otimes \delta_i) \circ \delta_j$ being a derivation, coassociativity is checked on generators). Thus one can rewrite:

$$\begin{aligned} -((\delta_i \otimes 1) \delta_j(x)) \#_2 \delta_j^*(1 \otimes 1) &= -((1 \otimes \delta_j) \delta_i(x)) \#_2 \delta_j^*(1 \otimes 1), \\ (1 \otimes m \circ (1 \otimes \tau \otimes 1)) \circ [\delta_i \otimes 1 \otimes 1] \circ [\delta_j \otimes 1] \circ \delta_j(x) \\ &= (1 \otimes [m \circ (1 \otimes \tau \otimes 1) \circ [\delta_j \otimes 1] \circ \delta_j]) \circ \delta_i(x), \\ (1 \otimes m \circ (1 \otimes \tau \otimes 1)) \circ [\delta_i \otimes 1 \otimes 1] \circ [1 \otimes \delta_j] \circ \delta_j(x) \\ &= (1 \otimes [m \circ (1 \otimes \tau \otimes 1) \circ [1 \otimes \delta_j] \circ \delta_j]) \circ \delta_i(x), \end{aligned}$$

and similar results for other lines in our previous sum. Using the formula in Lemma 12, the three previous lines sum up to $-(1 \otimes \Delta_j) \circ \delta_i(x)$. Doing the same for the other lines, we thus proved the expected formula.

(ii) Let $x \in D(\bar{\delta})$ and take $x_n \in D(\delta)$ converging to x in $D(\bar{\delta})$. We can compute (using coassociativity again):

$$\bar{\delta}_j(1 \otimes \tau)(\bar{\delta}_i(x_n)) = (1 \otimes [(1 \otimes \tau)(\bar{\delta}_i)]) (\bar{\delta}_j(x_n)).$$

By the boundedness result of Lemma 18, the right hand side converges and this gives the result since $\bar{\delta}$ is closed.

For the second statement, consider the equation of (i) applied via scalar product to $U \in (D(\bar{\delta}) \cap M) \otimes_{\text{alg}} (D(\bar{\delta}) \cap M)$:

$$\langle \Delta_j(x_n), \delta_i^*(U) \rangle = \langle (1 \otimes \Delta_j + \Delta_j \otimes 1) \delta_i(x_n), U \rangle + \langle \delta_j(x_n) \# ((1 \otimes O) \bar{\delta}_i \partial_j^*(1 \otimes 1)), U \rangle \quad (12)$$

with $U = V \otimes 1$, $V \in D(\delta)$, x_n above (with $x \in D(\Delta)$). Note that using Lemma 12, Assumption 1 and our first result in (ii), $\delta_i^*(V \otimes 1) \in D(\bar{\delta})$ with $\bar{\delta}_j \delta_i^*(V \otimes 1) = \delta_j(V) \delta_i^* 1 \otimes 1 + V \bar{\delta}_j \delta_i^*(1 \otimes 1) - \delta_j 1 \otimes \tau \delta_i(V) \in L^2(M \otimes M)$. Thus $\langle \Delta_j(x_n), \delta_i^* U \rangle = \langle \delta_j(x_n), \bar{\delta}_j \delta_i^*(V \otimes 1) \rangle$. Note also that $\langle 1 \otimes \tau \delta_i(x_n), \Delta V \rangle = \langle \bar{\delta}(1 \otimes \tau \delta_i(x_n)), \bar{\delta}(V) \rangle$ converges to the analog with x by what we have just proved. Since the resulting terms in (12) are bounded with respect to $\|\delta(x_n)\|_2$, we can get the equation at the limit $x_n \rightarrow x$.

We thus got:

$$\langle (1 \otimes \tau \delta_i) \Delta(x), V \rangle = \langle \bar{\delta}(1 \otimes \tau \delta_i(x)), \bar{\delta}(V) \rangle + \sum_{j=1}^N \langle \bar{\delta}_j(x) \# ((1 \otimes O) \bar{\delta}_i \partial_j^*(1 \otimes 1)), V \otimes 1 \rangle.$$

Now we can extend this from $V \in D(\delta)$ to $V \in D(\bar{\delta})$ and thus we obtain our result by definition of Δ .

(iii) Consider again this time the variant of Eq. (12) with $U \in (M \cap D(\Delta)) \otimes_{\text{alg}} (M \cap D(\Delta))$, and $x \in D(\delta)$. Everything reduces to $U = a \otimes b$. Using Lemma 12, we have $\delta_i^*(U) = a \delta_i^*(1 \otimes 1) b - (1 \otimes \tau) \delta_i(a) b - a(\tau \otimes 1) \delta_i(b)$. But now, $a, b \in M \cap D(\Delta)$, $(1 \otimes \tau) \delta_i(a)$, $(\tau \otimes 1) \delta_i(b) \in D(\Delta)$ by (ii) thus Lemma 14 proves $-(1 \otimes \tau) \delta_i(a) b - a(\tau \otimes 1) \delta_i(b) \in D(\bar{\Delta}^1)$. Then, let us write, for any $U \in (M \cap D(\Delta)) \otimes_{\text{alg}} (M \cap D(\Delta))$ (with the notation $(a \otimes b) \# c = acb$), $\delta_i^*(U) = U \# \delta_i^*(1 \otimes 1) - V$ with $V \in D(\bar{\Delta}^1)$. We can now rewrite our equation (using $\bar{\delta}_i$ is a derivation on $M \cap D(\bar{\delta}_i)$) to see $U \# \delta_i^*(1 \otimes 1) \in D(\bar{\delta}_i)$:

$$\begin{aligned} & \sum_{j=1}^N \langle \delta_j(x), \bar{\delta}_j(U \# \delta_i^*(1 \otimes 1)) \rangle - \langle \bar{\delta}_j(x) \# ((1 \otimes O) \bar{\delta}_i \partial_j^*(1 \otimes 1)), U \rangle \\ &= \langle x, \bar{\Delta}^1(V) \rangle + \langle \bar{\delta}_i(x), [1 \otimes \Delta + \Delta \otimes 1](U) \rangle. \end{aligned}$$

Now, once again using the second part of Lemma 12, we get this for any $x \in D(\Delta^{3/2}) \cap M \subset D(\bar{\delta}) \cap M$. Using the remark before Lemma 14, we can rewrite $\langle x, \bar{\Delta}^1(V) \rangle = \langle \Delta(x), V \rangle$, and thus, finally coming back to our original notation:

$$\langle \bar{\delta}_i \Delta(x), U \rangle = \langle \bar{\delta}_i(x), [1 \otimes \Delta + \Delta \otimes 1](U) \rangle + \sum_{j=1}^N \langle \bar{\delta}_j(x) \# ((1 \otimes O) \bar{\delta}_i \partial_j^*(1 \otimes 1)), U \rangle.$$

Finally, (using stability by $\phi_t \otimes \phi_t$) it is easily seen that $(M \cap D(\Delta)) \otimes_{\text{alg}} (M \cap D(\Delta))$ is a core for $\overline{\Delta \otimes 1 + 1 \otimes \Delta}$, and thus we can take U in the domain of that operator and finally, since this operator is closed, we get our result. The extension from $x \in D(\Delta^{3/2}) \cap M$ to $x \in D(\Delta^{3/2})$ is easy. \square

2.2.4. Proof of Theorem 17

Using Lemma 16, it only remains to check assumption f, h . Lemma 19(iii) proves $f.1$ and $f.2$ with $\mathcal{H}: L^2(M \otimes M)^N \rightarrow L^2(M \otimes M)^N$ given by $(\mathcal{H}(C))_i = \sum_{j=1}^N C_j \# ((1 \otimes O) \bar{\delta}_i \partial_j^*(1 \otimes 1))$ so that $\|\mathcal{H}\| \leq \|((1 \otimes O) \bar{\delta}_i \partial_j^*(1 \otimes 1))_{(i,j)}\|_{M_N(M \otimes M^{\text{op}})}$.

It remains to check h . Consider $x \in D(\Delta)$ then $\eta_\alpha(x) \in D(\Delta^{3/2})$, thus we can apply Lemma 19(iii) to get:

$$\begin{aligned} & \langle \Delta(\eta_\alpha(x)), \Delta(\eta_\alpha(x)) \rangle \\ &= \sum_{i=1}^N \langle \bar{\delta}_i \Delta(\eta_\alpha(x)), \bar{\delta}_i(\eta_\alpha(x)) \rangle \\ &= \sum_{i=1}^N \langle \Delta^\otimes \bar{\delta}_i(\eta_\alpha(x)), \bar{\delta}_i(\eta_\alpha(x)) \rangle + \langle \mathcal{H}(\bar{\delta}(\eta_\alpha(x)))_i, \bar{\delta}_i(\eta_\alpha(x)) \rangle \\ &= \sum_{i,j=1}^N \overline{\langle \bar{\delta}_j \otimes 1 \oplus 1 \otimes \bar{\delta}_j \bar{\delta}_i(\eta_\alpha(x)), \bar{\delta}_j \otimes 1 \oplus 1 \otimes \bar{\delta}_j \bar{\delta}_i(\eta_\alpha(x)) \rangle} + \sum_{i=1}^N \langle \mathcal{H}(\bar{\delta}(\eta_\alpha(x)))_i, \bar{\delta}_i(\eta_\alpha(x)) \rangle. \end{aligned}$$

Since we assume $x \in D(\Delta)$, the left hand side and the second term in the right hand side converge when $\alpha \rightarrow \infty$ showing that $x \in D(\bar{\delta} \otimes 1 \oplus 1 \otimes \bar{\delta} \otimes \bar{\delta})$ as expected. Now, the inequality stated in h is a tensor variant of the one stated in c and already checked.

3. Complementary properties of our main example

3.1. An Ito formula for resolvent operators under weak assumptions

Let us consider an integral of the form:

$$X_t = X_0 + \int_0^t K_s \, ds + \int_0^t U_s \# dS_s,$$

where $X_0 \in M_0 = W$, $s \mapsto K_s$ weakly measurable with $K_s \in L^1(M_s)$, $\int_0^T \|K_s\|_1 \, ds < \infty \, \forall T > 0$ and $U \in \mathcal{B}_2^a$. We also assume $K_s = K_s^*$, $U_s = U_s^*$ (in this section we use the involution induced by $\text{HS}(M)$, i.e. $(a \otimes b)^* = b^* \otimes a^*$), $X_0 = X_0^*$ so that $X_t = X_t^*$.

We would like to find a formula for $(z + X_t)^{-1}$, $z \in \mathbb{C}$, $\Im z > 0$, to compute the Cauchy-transform of X_t with this unbounded $X_t \in L^1(M_t)$. If we supposed $K_s \in M_s$, $U_s \in \mathcal{B}_\infty^a$ Proposition 4.3.4 of [3] would conclude (see this article for the notation, the case with N free Brownian motions as in our case is similar to their case, especially we write in this section also $\#$ for multiplication in $M \otimes M^{\text{op}} \otimes M$ without confusion with the previous notation for multiplication in $M \otimes M^{\text{op}}$) since $f(x) = \frac{1}{z+x} = \int_{\mathbb{R}} e^{ixy} \mu(dy)$ with $\mu(dy) = -i1_{(0,\infty)} e^{izy} dy$ (which satisfy $\mathcal{I}_2(f) < \infty$, and thus their results apply).

But we are not in such a bad position because all the terms of their expression in the Ito formula for $(z + X_t)^{-1}$ make sense, this almost only requires applying a standard density argument left to the reader.

Proposition 20. *With the previous assumptions we have:*

$$\begin{aligned} & (z + X_t)^{-1} \\ &= (z + X_0)^{-1} - \int_0^t [(z + X_s)^{-1} \otimes (z + X_s)^{-1}] \# U_s \# dS_s \\ &\quad - \int_0^t [(z + X_s)^{-1} \otimes (z + X_s)^{-1}] \# K_s \, ds \\ &\quad + \sum_{i=1}^N \int_0^t m \circ (1 \otimes \tau \otimes 1) ((1 \otimes U_s^{(i)}) \# ((z + X_s)^{-1} \otimes (z + X_s)^{-1} \otimes (z + X_s)^{-1}) \# (U_s^{(i)} \otimes 1)) \, ds. \end{aligned}$$

The two next lemmas are also left to the reader.

Lemma 21. *Let*

$$X_t = X_0 + \int_0^t K_s \, ds + \int_0^t U_s \# dS_s,$$

where $X_0 \in M_0$, $s \mapsto K_s$ weakly measurable with $K_s \in L^1([0, T], L^1(M_s))$, and $U \in \mathcal{B}_2^a$. We also assume $X_t \in M$ (in a bounded way in t). Let say $\|X_t\| < 1$.

Then, there exists $X_t^n = X_0 + \int_0^t K_s^n \, ds + \int_0^t U_s^n \# dS_s$ with $s \mapsto K_s^n$ weakly measurable with $K_s^n \in L^\infty([0, T]) \overline{\otimes} M_s$, K^n converging to K in $L^1([0, T], L^1(M_s))$, and $U^n \in \mathcal{B}_\infty^a$, U^n converging to U in \mathcal{B}_2^a . Moreover, we have $\|X_t^n\| \leq 1$.

The following variant of the Ito product formula (Proposition 4.3.2 in [3]) is now obvious:

Lemma 22. *Let*

$$X_t = X_0 + \int_0^t K_s \, ds + \int_0^t U_s \# dS_s,$$

$$Y_t = Y_0 + \int_0^t L_s \, ds + \int_0^t V_s \# dS_s,$$

where $X_0, Y_0 \in M_0$, $s \mapsto K_s, s \mapsto L_s$ weakly measurable with $K_s, L_s \in L^1([0, T], L^1(M_s))$, and $U, V \in \mathcal{B}_2^a$. We also assume $X_t, Y_t \in M$ (in a bounded way in t).

Then, for any $t \leq T$:

$$\begin{aligned} X_t Y_t &= X_0 Y_0 + \int_0^t (X_s L_s + K_s Y_s) \, ds + \int_0^t m \circ (1 \otimes (\tau \circ m) \otimes 1)(U_s \otimes V_s) \, ds \\ &\quad + \int_0^t (X_s V_s + U_s Y_s) \# dS_s. \end{aligned}$$

3.2. Boundedness

In this subsection, we are interested in the example of part 2. Under Assumption 0, we write $X_t \in \mathcal{B}_{2, \phi \bar{\delta}}^a, X_t^\varepsilon \in \mathcal{B}_{2, \Delta^{1/2}}^a$ the solutions given by Theorem 11(i) and Lemma 16. We moreover consider an initial condition $X_0 \in M_0 \cap D(\bar{\delta})$.

Proposition 23. *With those assumptions, for any complex number z with $\Im z > 0, \frac{1}{z + X_s^\varepsilon}$ is in $\mathcal{B}_{2, \Delta^{1/2}}^a$ and*

$$\begin{aligned} (z + X_t^\varepsilon)^{-1} &= \phi_t((z + X_0^\varepsilon)^{-1}) + (1 - \varepsilon) \int_0^t \phi_{t-s}(\bar{\delta}((z + X_s^\varepsilon)^{-1}) \# dS_s) \\ &\quad + ((1 - \varepsilon)^2 - 1) \sum_{i=1}^N \int_0^t \phi_{t-s}(m \circ (1 \otimes \tau \otimes 1)((z + X_s^\varepsilon)^{-1} \bar{\delta}_i(X_s^\varepsilon)(z + X_s^\varepsilon)^{-1} \bar{\delta}_i(X_s^\varepsilon)(z + X_s^\varepsilon)^{-1})) \, ds. \end{aligned}$$

As a consequence, if we assume moreover $\|X_t\|_2 = \|X_0\|_2$ (a.e. t , this is the case e.g. for a mild solution given by Theorem 11(ii)) then $X_t \in M$ for all t (recall we supposed $X_0 \in M$) and we also have $\|X_t\| \leq \|X_0\|$ (actually equal a.e.), and likewise for any $\varepsilon > 0, X_t^\varepsilon \in M$. Finally, if $\|X_t\|_2 = \|X_0\|_2$ a.e. in t , then $\|X_t - X_t^\varepsilon\|_2 \rightarrow 0$ a.e. in t .

Proof. Let $\varepsilon > 0$. Since we have a mild solution at ε level, by Theorem 11 and since a mild solution is a weak solution as seen in Proposition 8, we get by self-adjointness of Δ for characterizing its domain that $\int_0^t X_s^\varepsilon \, ds \in D(\Delta)$ and:

$$X_t^\varepsilon = X_0 - \frac{1}{2} \Delta \int_0^t X_s^\varepsilon \, ds + (1 - \varepsilon) \int_0^t \delta(X_s^\varepsilon) \# dS_s.$$

Thus, applying a resolvent, using Lemma 3(ii), we deduce for any $\alpha > 0$:

$$\eta_\alpha(X_t^\varepsilon) = \eta_\alpha(X_0) - \frac{1}{2} \int_0^t \Delta \eta_\alpha(X_s^\varepsilon) ds + (1 - \varepsilon) \int_0^t \eta_\alpha^\otimes \delta(X_s^\varepsilon) \# dS_s,$$

where $X_0 \in M_0$, $s \mapsto K_s = -\frac{1}{2} \Delta \eta_\alpha(X_s^\varepsilon)$ weakly measurable with $K_s \in L^2(M_s)$, $\int_0^T \|K_s\|_2^2 ds < \infty \forall T > 0$ (all this using the definition of $\mathcal{B}_{2, \Delta^{1/2}}^a$) and $U_s = \eta_\alpha^\otimes \delta(X_s^\varepsilon) \in \mathcal{B}_2^a$. Recall that $\eta_\alpha(X_0) \in M_0$ by the general Dirichlet form theory implying η_α is a completely positive contraction on M . We are in position to apply Proposition 20, thus we have:

$$\begin{aligned} & (z + \eta_\alpha(X_t^\varepsilon))^{-1} \\ &= (z + \eta_\alpha(X_0^\varepsilon))^{-1} \\ & \quad - (1 - \varepsilon) \int_0^t [(z + \eta_\alpha(X_s^\varepsilon))^{-1} \otimes (z + \eta_\alpha(X_s^\varepsilon))^{-1}] \# \eta_\alpha^\otimes \bar{\delta}(X_s^\varepsilon) \# dS_s \\ & \quad + \frac{1}{2} \int_0^t [(z + \eta_\alpha(X_s^\varepsilon))^{-1} \otimes (z + \eta_\alpha(X_s^\varepsilon))^{-1}] \# \Delta \eta_\alpha(X_s^\varepsilon) ds \\ & \quad + (1 - \varepsilon)^2 \sum_{i=1}^N \int_0^t ds m \circ (1 \otimes \tau \otimes 1) ((z + \eta_\alpha(X_s^\varepsilon))^{-1} \\ & \quad \times \eta_\alpha^\otimes \bar{\delta}_i(X_s^\varepsilon) (z + \eta_\alpha(X_s^\varepsilon))^{-1} \eta_\alpha^\otimes \bar{\delta}_i(X_s^\varepsilon) (z + \eta_\alpha(X_s^\varepsilon))^{-1}). \end{aligned}$$

But, note that for any $x \in D(\bar{\delta})$, $(z + x)^{-1} \in D(\bar{\delta})$, and $\bar{\delta}((z + x)^{-1}) = -(z + x)^{-1} \bar{\delta}(x) (z + x)^{-1}$. Indeed, we check this easily on $D(\bar{\delta}) \subset M$ by Leibniz rule, and taking $x_n \in D(\bar{\delta})$ converging to x in $D(\bar{\delta})$, a usual formula on resolvent operators $(z + x_n)^{-1} - (z + x)^{-1} = (z + x_n)^{-1} (x - x_n) (z + x)^{-1}$ gives convergence of $(z + x_n)^{-1}$ to $(z + x)^{-1}$ in $L^2(M)$, and thus of $\bar{\delta}((z + x_n)^{-1})$ in $L^1(M \otimes M)$ to $(z + x)^{-1} \bar{\delta}(x) (z + x)^{-1}$. A fortiori, we have weak convergence in $L^2(M \otimes M)$. Since a convex set in $L^2(M) \oplus L^2(M \otimes M)$ is closed if and only if it is weakly closed by Hahn–Banach theorem, we get $(z + x)^{-1} \in D(\bar{\delta})$ and the result.

Analogously, we have for any $x \in D(\Delta)$, $(z + x)^{-1} \in D(\Delta^1)$ (cf. the paragraph before Lemma 13 for a definition) and moreover:

$$\begin{aligned} -\Delta^1((z + x)^{-1}) &= (z + x)^{-1} \Delta(x) (z + x)^{-1} \\ & \quad + 2 \sum_{i=1}^N m \circ (1 \otimes \tau \otimes 1) (1 \otimes \bar{\delta}_i(x) \# (z + x)^{-1} \otimes (z + x)^{-1} \otimes (z + x)^{-1} \# \bar{\delta}_i(x) \otimes 1). \end{aligned}$$

Let us write $R_{t,z,\alpha,\varepsilon} = (z + \eta_\alpha(X_t^\varepsilon))^{-1}$. Thus, we have obtained, if we apply this formula to our previous equation in making appear terms by emphasizing “commutators” of η_α^\otimes and δ . We also write $Y_{s,z,\alpha,\varepsilon,i} := (R_{s,z,\alpha,\varepsilon} (\eta_\alpha^\otimes \bar{\delta}_i(X_s^\varepsilon) - \bar{\delta}_i \eta_\alpha(X_s^\varepsilon)) R_{s,z,\alpha,\varepsilon}$:

$$\begin{aligned} R_{t,z,\alpha,\varepsilon} &= R_{0,z,\alpha,\varepsilon} + (1 - \varepsilon) \int_0^t (\bar{\delta}(R_{s,z,\alpha,\varepsilon}) - Y_{s,z,\alpha,\varepsilon}) \# dS_s - \frac{1}{2} \int_0^t \Delta^1((z + \eta_\alpha(X_s^\varepsilon))^{-1}) ds \\ & \quad + \sum_{i=1}^N \int_0^t m \circ (1 \otimes \tau \otimes 1) (Y_{s,z,\alpha,\varepsilon,i} \eta_\alpha^\otimes \bar{\delta}_i(X_s^\varepsilon) R_{s,z,\alpha,\varepsilon} + R_{s,z,\alpha,\varepsilon} \bar{\delta}_i \eta_\alpha(X_s^\varepsilon) Y_{s,z,\alpha,\varepsilon,i}) ds \\ & \quad + ((1 - \varepsilon)^2 - 1) \sum_{i=1}^N \int_0^t m \circ (1 \otimes \tau \otimes 1) (R_{s,z,\alpha,\varepsilon} \eta_\alpha^\otimes \bar{\delta}_i(X_s^\varepsilon) R_{s,z,\alpha,\varepsilon} \eta_\alpha^\otimes \bar{\delta}_i(X_s^\varepsilon) R_{s,z,\alpha,\varepsilon}) ds. \end{aligned}$$

As in the proof of Proposition 7 showing that a strong solution is a mild solution, but take here $\zeta \in D(\Delta) \cap M$ in the proof, we have:

$$\begin{aligned}
 R_{t,z,\alpha,\varepsilon} &= \phi_t(R_{0,z,\alpha,\varepsilon}) + (1 - \varepsilon) \int_0^t \phi_{t-s}^{\otimes} (\bar{\delta}(R_{s,z,\alpha,\varepsilon}) - Y_{s,z,\alpha,\varepsilon}) \# dS_s \\
 &+ \sum_{i=1}^N \int_0^t \phi_{t-s} (m \circ (1 \otimes \tau \otimes 1)(Y_{s,z,\alpha,\varepsilon,i} \eta_\alpha^{\otimes} \bar{\delta}_i(X_s^\varepsilon) R_{s,z,\alpha,\varepsilon}) + R_{s,z,\alpha,\varepsilon} \bar{\delta}_i \eta_\alpha(X_s^\varepsilon) Y_{s,z,\alpha,\varepsilon,i}) ds \\
 &+ ((1 - \varepsilon)^2 - 1) \sum_{i=1}^N \int_0^t \phi_{t-s} m \circ (1 \otimes \tau \otimes 1)(R_{s,z,\alpha,\varepsilon} \eta_\alpha^{\otimes} \bar{\delta}_i(X_s^\varepsilon) R_{s,z,\alpha,\varepsilon} \eta_\alpha^{\otimes} \bar{\delta}_i(X_s^\varepsilon) R_{s,z,\alpha,\varepsilon}) ds.
 \end{aligned}$$

We now want to make α tend to ∞ . The three terms with Y tend to zero by dominated convergence theorem (domination modulo constant by $\|\bar{\delta}(X_s^\varepsilon)\|_2^2$ since $X^\varepsilon \in \mathcal{B}_{2,\Delta^{1/2}}^a$). In the last line we can remove η_α^{\otimes} in the same way and we get weak convergence in L^1 to the expected limit (of course we have to use ϕ bounded on M). Clearly, the two resolvent operators in the first line converge in L^2 and the same kind of reasoning already made shows that $\bar{\delta}((z + \eta_\alpha(X_s^\varepsilon))^{-1})$ weakly converges in L^2 to $\bar{\delta}((z + X_s^\varepsilon)^{-1})$.³ A dominated convergence theorem concludes as above for the corresponding stochastic integral. At the end, we have got weak convergence in L^1 of all terms so that:

$$\begin{aligned}
 (z + X_t^\varepsilon)^{-1} &= \phi_t((z + X_0^\varepsilon)^{-1}) \\
 &+ (1 - \varepsilon) \int_0^t \phi_{t-s} (\bar{\delta}((z + X_s^\varepsilon)^{-1}) \# dS_s) + ((1 - \varepsilon)^2 - 1) \\
 &\times \sum_{i=1}^N \int_0^t ds \phi_{t-s} (m \circ (1 \otimes \tau \otimes 1)((z + X_s^\varepsilon)^{-1} \bar{\delta}_i(X_s^\varepsilon) (z + X_s^\varepsilon)^{-1} \bar{\delta}_i(X_s^\varepsilon) (z + X_s^\varepsilon)^{-1})).
 \end{aligned}$$

We now want to make ε tend to 0, after taking the trace, to get the second statement. Note that in our context of Section 2 where $\|\delta(x)\|_2 = \|\Delta^{1/2}(x)\|_2$, (5) gives:

$$(1 - (1 - \varepsilon)^2) \int_0^t \|\bar{\delta}(X_s^\varepsilon)\|_2^2 ds = \|X_0\|_2^2 - \|X_t^\varepsilon\|_2^2.$$

Incidentally, this proves the statement that $\|X_0\|_2^2 = \|X_t\|_2^2$ in case (ii) of Theorem 11 since we proved there convergence of X_t in L^2 and boundedness of $\|\bar{\delta}(X_s^\varepsilon)\|_2$.

But (modulo extraction) the weak limit defining X_t gives $\|X_t\|_2 \leq \liminf \|X_t^\varepsilon\|_2$ and thus

$$\limsup_{\varepsilon \rightarrow 0} (1 - (1 - \varepsilon)^2) \int_0^t \|\bar{\delta}(X_s^\varepsilon)\|_2^2 ds \leq \|X_0\|_2^2 - \|X_t\|_2^2. \tag{13}$$

And the last term is almost everywhere 0 under our assumption. As a consequence, since we already know X_t^ε converges to X_t weakly in L^2 by Theorem 11(i), we deduce the stated $\|\cdot\|_2$ convergence of X_t^ε to X_t on the a.e. set where $\|X_t\|_2 = \|X_0\|_2$. Moreover, the trace of the second line of the equality of Proposition 23 is bounded up to the cube of an inverse of $\Im(z)$ by this quantity, and thus we get almost everywhere (in t independent of z) equality of the Cauchy transforms of X_0 and X_t , giving a.e. boundedness (and equality of von Neumann algebra norms). Now we can use the weak continuity proved in Theorem 11 to extend boundedness everywhere.

Second, to prove that $X_t^\varepsilon \in M$, consider $S_t^{(i,J)}$ $1 \leq i \leq N$, $J \in \{a, b\}$ a family of free Brownian motions, on which we extend δ by 0. We can always write $S_s^{(i)} = (1 - \varepsilon)S_s^{(i,a)} + \sqrt{1 - (1 - \varepsilon)^2} S_s^{(i,b)}$.

³Remark that this second term is already known to exist; by boundedness in L^2 of the convergent $\delta \eta_\alpha(X_s^\varepsilon)$, we get that $(z + \eta_\alpha(X_s^\varepsilon))^{-1} \delta(\eta_\alpha(X_s^\varepsilon))(z + \eta_\alpha(X_s^\varepsilon))^{-1}$ is close in $\|\cdot\|_1$ of $(z + X_s^\varepsilon)^{-1} \delta(\eta_\alpha(X_s^\varepsilon))(z + X_s^\varepsilon)^{-1}$, and finally, with convergence in L^2 of $(z + X_s^\varepsilon)^{-1} \delta \eta_\alpha(X_s^\varepsilon)(z + X_s^\varepsilon)^{-1}$ to $(z + X_s^\varepsilon)^{-1} \delta(X_s^\varepsilon)(z + X_s^\varepsilon)^{-1}$; we have thus obtained the convergence in L^1 , using that the two terms are known to be in L^2 and the sequence bounded in this space, you get the result.

We have thus

$$X_t = \phi_t(X_0) + (1 - \varepsilon) \int_0^t \phi_{t-s}(\delta(X_s) \# dS_s^{(a)}) + \sqrt{1 - (1 - \varepsilon)^2} \int_0^t \phi_{t-s}(\delta(X_s) \# dS_s^{(b)}).$$

We want to prove that, if we apply E_a , the conditional expectation on the von Neumann algebra $M^{(a)}$ generated by M_0 and $S_s^{(a)}$, we get:

$$E_a(X_t) = \phi_t(X_0) + (1 - \varepsilon) \int_0^t \phi_{t-s}(\delta(E_a(X_s)) \# dS_s^{(a)}),$$

which says nothing but by changing S_s in $S_s^{(a)}$, $E_a(X_t)$ is an instance of (the unique solution) X_t^ε . As a consequence, this gives the stated boundedness.

Since $E_a(\int_0^t \phi_{t-s}(\delta(\frac{1}{z+X_s}) \# dS_s^{(b)})) = 0$ is a consequence of freeness between $\{S_s^{(a)}\}$ and $\{S_s^{(b)}\}$, we just have to show several commutations of E_a with several operations, more precisely: $E_a \phi_t = \phi_t E_a$, $E_a(\cdot \# (S_t^{(a)} - S_s^{(a)})) = (E_a \overline{\otimes} E_a(\cdot)) \# (S_t^{(a)} - S_s^{(a)})$ on $L^2(M_s)$ and $E_a \overline{\otimes} E_a \circ \bar{\delta} = \bar{\delta} \circ E_a$. With that and obvious lemmas about stochastic integrals, we will have what we want. The first equation is nothing but an instance of the preservation (contained in the preliminaries of Section 2.1 with this new case of zero extension) by Δ of $M^{(a)}$ (and characterization of conditional expectation). The second is proved in using also the characterization of conditional expectation once noted that we can use instead of someone in $L^2(M^{(a)})$, someone in $L^2(M_s^{(a)} \otimes M_s^{(a)}) \# (S_t^{(a)} - S_s^{(a)})$ by orthogonality. The third one is verified by using the fact that $\delta^* : L^2(M^{(a)} \otimes M^{(a)}) \rightarrow L^2(M^{(a)})$ (and characterization of conditional expectation). \square

3.3. Stationarity

Proposition 24. *Let us call $\Phi_t : X_0 \in M_0 \cap D(\bar{\delta}) \mapsto X_t \in M_t$ the previous ultramild solution of Theorem 11(i) assuming $\|X_t\|_2 = \|X_0\|_2$ a.e. for all $X_0 \in M_0 \cap D(\bar{\delta})$. Then, $\Phi_t(X_0 Y_0) = \Phi_t(X_0) \Phi_t(Y_0)$ if $X_0, Y_0 \in D(\bar{\delta}) \cap M_0$.*

Proof. Since $\Phi_t(X^*) = \Phi_t(X)^*$, $\Phi_t(1) = 1$ and τ is faithful, $D(\bar{\delta}) \cap M_0$ a $*$ -algebra, it suffices to prove that for any $X_0, Y_0, Z_0, T_0 \in D(\bar{\delta}) \cap M_0$ $\tau(\Phi_t(X_0) \Phi_t(Y_0) \Phi_t(Z_0) \Phi_t(T_0)) = \tau(X_0 Y_0 Z_0 T_0)$. For notational convenience, we prove only the case $Z_0 = T_0 = 1$ (even if this case is also a direct consequence of the assumed isometry by polarization), the general similar case being left to the reader.

Let also $\Phi_t^\varepsilon : X_0 \in M_0 \cap D(\bar{\delta}) \mapsto X_t^\varepsilon \in M_t$.

Apply Ito's formula (assumptions of Lemma 22) to $\eta_\alpha(X_t^\varepsilon)$, and $\eta_\alpha(Y_t^\varepsilon)$ (using the result of Proposition 23 they are valued in M):

$$\begin{aligned} \eta_\alpha(X_t^\varepsilon) \eta_\alpha(Y_t^\varepsilon) &= \eta_\alpha(X_0^\varepsilon) \eta_\alpha(Y_0^\varepsilon) + (1 - \varepsilon) \int_0^t \eta_\alpha^\otimes(\bar{\delta}(X_s^\varepsilon)) \eta_\alpha(Y_s^\varepsilon) + \eta_\alpha(X_s^\varepsilon) \eta_\alpha^\otimes(\bar{\delta}(Y_s^\varepsilon)) \# dS_s \\ &\quad - \frac{1}{2} \int_0^t \eta_\alpha(X_s^\varepsilon) \Delta \eta_\alpha(Y_s^\varepsilon) + \Delta \eta_\alpha(X_s^\varepsilon) \eta_\alpha(Y_s^\varepsilon) ds \\ &\quad + (1 - \varepsilon)^2 \sum_{i=1}^N \int_0^t m \circ (1 \otimes (\tau \circ m) \otimes 1) (\eta_\alpha^\otimes(\bar{\delta}_i(X_s^\varepsilon)) \otimes \eta_\alpha^\otimes(\bar{\delta}_i(Y_s^\varepsilon))) ds. \end{aligned}$$

We can now use Lemma 13 to get:

$$\begin{aligned} \eta_\alpha(X_t^\varepsilon) \eta_\alpha(Y_t^\varepsilon) &= \eta_\alpha(X_0^\varepsilon) \eta_\alpha(Y_0^\varepsilon) \\ &\quad + (1 - \varepsilon) \int_0^t \bar{\delta}(\eta_\alpha(X_s^\varepsilon) \eta_\alpha(Y_s^\varepsilon)) \# dS_s - \frac{1}{2} \int_0^t \Delta^1(\eta_\alpha(X_t^\varepsilon) \eta_\alpha(Y_t^\varepsilon)) ds \\ &\quad + ((1 - \varepsilon)^2 - 1) \sum_{i=1}^N \int_0^t m \circ (1 \otimes (\tau \circ m) \otimes 1) (\bar{\delta}_i(\eta_\alpha(X_s^\varepsilon)) \otimes \bar{\delta}_i(\eta_\alpha(Y_s^\varepsilon))) ds \end{aligned}$$

$$\begin{aligned}
 &+ (1 - \varepsilon) \int_0^t [(\eta_\alpha^\otimes(\bar{\delta}(X_s^\varepsilon)) - \bar{\delta}(\eta_\alpha(X_s^\varepsilon)))\eta_\alpha(Y_s^\varepsilon) + \eta_\alpha(X_s^\varepsilon)(\eta_\alpha^\otimes(\bar{\delta}(Y_s^\varepsilon)) - \bar{\delta}(\eta_\alpha(Y_s^\varepsilon)))] \# dS_s \\
 &+ (1 - \varepsilon)^2 \sum_{i=1}^N \int_0^t m \circ (1 \otimes (\tau \circ m) \otimes 1) ((\eta_\alpha^\otimes(\bar{\delta}_i(X_s^\varepsilon)) - \bar{\delta}_i(\eta_\alpha(X_s^\varepsilon))) \otimes \eta_\alpha^\otimes(\bar{\delta}_i(Y_s^\varepsilon))) \, ds \\
 &+ (1 - \varepsilon)^2 \sum_{i=1}^N \int_0^t m \circ (1 \otimes (\tau \circ m) \otimes 1) (\bar{\delta}_i(\eta_\alpha(X_s^\varepsilon))) \otimes (\eta_\alpha^\otimes(\bar{\delta}_i(Y_s^\varepsilon)) - \bar{\delta}_i(\eta_\alpha(Y_s^\varepsilon))) \, ds.
 \end{aligned}$$

Using once again the trick of Proposition 7 to pass to something which looks like a mild solution, then we can take the limit $\alpha \rightarrow \infty$ as in Proposition 23 and finally we get (using that $\Phi_t^\varepsilon(X_0 Y_0)$ is a mild solution since $X_0 Y_0 \in D(\bar{\delta}) \cap M$):

$$\begin{aligned}
 &\Phi_t^\varepsilon(X_0)\Phi_t^\varepsilon(Y_0) - \Phi_t^\varepsilon(X_0 Y_0) \\
 &= (1 - \varepsilon) \int_0^t \phi_{t-s}(\bar{\delta}(\Phi_s^\varepsilon(X_0)\Phi_s^\varepsilon(Y_0) - \Phi_s^\varepsilon(X_0 Y_0))) \# dS_s \\
 &+ ((1 - \varepsilon)^2 - 1) \sum_{i=1}^N \int_0^t \phi_{t-s}(m \circ (1 \otimes (\tau \circ m) \otimes 1) (\bar{\delta}_i(\Phi_s^\varepsilon(X_0)) \otimes \bar{\delta}_i(\Phi_s^\varepsilon(Y_0)))) \, ds.
 \end{aligned}$$

Since $\Phi_t^\varepsilon(X_0)$ converges in $\|\cdot\|_2$ -norm to $\Phi_t(X_0)$ (a.e.) by the last statement of Proposition 23, we can show that, after taking the trace, this equation converges to the relation $\tau(\Phi_t(X_0)\Phi_t(Y_0)) = \tau(X_0 Y_0)$, using also the fact that the last term goes to zero via (13) as in Proposition 23. □

4. Applications

4.1. Free difference quotient

Corollary 25. *Assume Assumption 1 and $X_1, \dots, X_n \in D(\Delta) \cap M_0$. Then, for any $t \geq 0$, there exists an embedding $\Phi_t : M_0 = W^*(X_1, \dots, X_n) \rightarrow M_0 * L(F(\infty))$ and $S_1, \dots, S_N \in L(F(\infty))$ a free $(0, 1)$ -semicircular family (depending on t), free from M_0 and such that:*

$$\left\| \Phi_t(X_j) - X_j - \sqrt{t} \sum_{i=1}^N \bar{\delta}_i(X_j) \# S_i \right\|_2 \leq c_j t,$$

for a fixed constant

$$c_j^2 = \frac{1}{4} \|\Delta(X_j)\|_2^2 + \frac{1}{2} \left(\|\Delta^{\otimes 1/2}(\delta(X_j))\|_2^2 + \frac{\pi}{4} \|\Delta(X_j)\|_2 \|\Delta^{\otimes 1/2}(\delta(X_j))\|_2 \right).$$

Moreover, $\Phi_t(X_j) \in W^*(X_1, \dots, X_n, S_1, \dots, S_N, \{S'_j\}_{j=0}^\infty)$ where $\{S'_j\}_{j=0}^\infty$ is a free semicircular family free with $\{X_1, \dots, X_n, S_1, \dots, S_N\}$.

As a consequence, if we define $c^2 = \sum c_j^2$, we have the following inequality for the Wasserstein–Biane–Voiculescu distance ([4]):

$$d_W(\mu_{X_1, \dots, X_n}, \mu_{X_1 + \sqrt{t}\delta(X_1) \# S, \dots, X_n + \sqrt{t}\delta(X_n) \# S}) \leq ct.$$

As another consequence, using [39], Theorem 16, any R^ω -embeddable von Neumann algebra generated by X_1, \dots, X_n with Lipschitz conjugate variable have $\delta_0(X_1, \dots, X_n) = n$.

Remark 26. This result is analogous to Proposition 2 in [39] and to an inequality in [4]. But the latter is for the free difference quotient for $n = 1$ with only finite Fisher information. And the former deals with any derivation, for a general n , assuming $\partial(X_j)$ and $\partial^* \partial(X_j)$ can be written in terms of non-commutative power series. Compared to these results, our result can be applied for a general n but for coassociative (or even as we will see elsewhere also “almost coassociative”) derivations, and for the free difference quotient with only the assumption Lipschitz conjugate variable (i.e. $\bar{\partial} \partial_j^* 1 \otimes 1 \in (M \bar{\otimes} M^{\text{op}})^n$, which corresponds to Lipschitz conjugate variable in the $n = 1$ case, cf. also [48] for a more general justification of this terminology). Note also that, in this case, the constant is expressed in terms of free Fisher information $\Phi^*(X_1, \dots, X_n) = \sum_i \|\Delta(X_i)\|_2^2$, it becomes the expected $c = \Phi^*(X_1, \dots, X_n)^{1/2}/2$, so that for instance if X_1, \dots, X_n is such that the associated Orstein–Ühlenbeck process $Y_i(t) = e^{-t/2} X_i + (1 - e^{-t})^{1/2} S_i$ satisfy $X_1(t), \dots, X_n(t)$ have Lipschitz conjugate variable (in the above sense for all $t > 0$, which is by no means a trivial assumption) then the argument of [4], variant of [29], gives the corresponding free Talagrand transportation cost inequality:

$$d_W((X_1, \dots, X_N), (S_1, \dots, S_N)) \leq \sqrt{2} \left(\chi^*(S_1, \dots, S_n) - \chi^*(X_1, \dots, X_n) - \frac{n}{2} + \frac{1}{2} \sum_{i=1}^N \tau(X_i^2) \right)^{1/2}.$$

We prove in [12] this result in full generality using another way of solving stochastic differential equations.

We give a concrete non-trivial example of Lipschitz conjugate variable in Section 4.3.

Sketch of proof. For the reader’s convenience, we outline how this follows from the beginning of the paper. Using Assumption 1, Theorem 17 gives the conditions to apply Theorem 11(ii) with $\omega = 0$. Then $\Phi_t(X) = X_t$ is given by the mild solution of the SDE from (ii) and the stated inequality is the one coming from (i) in Theorem 11 (the inequality on Wasserstein distance is then an obvious consequence, note that $\delta(X_0) \in D((\Delta \otimes 1 + 1 \otimes \Delta)^{1/2})$ follows from Lemma 19(iii) as in the proof of Assumption 1 h). The fact that Φ_t gives a $*$ -homomorphism comes from Proposition 24. Since it preserves the trace by the SDE it satisfies, we can extend it at the von Neumann algebraic level. (S_i, S'_j are produced from the free Brownian motion of the SDE.) Assumption 1 is true in case of Lipschitz conjugate variable as follows. First, the free difference quotient being coassociative, (a) is true in choosing non-commutative polynomials as $D(\partial)$. (a’) is true because having Lipschitz conjugate variables imply the conjugate variables are in M (using e.g. the equality (1) in [11]). (b) is valid directly by Lipschitz conjugate variable assumption.

As stated, the equality on microstate free entropy dimension then comes from [39], Theorem 16. □

4.2. Preliminaries and relations of three natural derivations on q -Gaussian factors

Our goal is to study three derivations on q -Gaussian factors: the free difference quotient, the commutator with right creation operators and the one giving the number operator as generator of the associated Dirichlet form. Especially, we want to find values of q ’s for which they can be seen as closed derivations with value in the coarse correspondence, with the same domain and equivalent norms.

We will use this preliminaries to apply our results in the next subsection and give an interesting example of Lipschitz conjugate variables.

4.2.1. Preliminaries on q -Gaussian factors

We recall the construction of q -Gaussian variables given by Bożejko and Speicher in [7].

Let $N < \infty$ be an integer, $\mathcal{H} = \mathbb{R}^N$, $\mathcal{H}_C = \mathbb{C}^N$ its complexification, and $-1 < q < 1$. Consider the vector space

$$F_{\text{alg}}(\mathcal{H}) = \mathbb{C}\Omega \oplus \bigoplus_{n \geq 1} \mathcal{H}_C^{\otimes n}$$

(algebraic direct sum and tensor products). This vector space is endowed with a positive definite inner product given by

$$\langle \xi_1 \otimes \dots \otimes \xi_n, \zeta_1 \otimes \dots \otimes \zeta_m \rangle_q = \delta_{n=m} \sum_{\pi \in S_n} q^{i(\pi)} \prod_{j=1}^n \langle \xi_j, \zeta_{\pi(j)} \rangle = \delta_{n=m} \langle \xi_1 \otimes \dots \otimes \xi_n, P_q^{(n)} \zeta_1 \otimes \dots \otimes \zeta_n \rangle_0,$$

where $i(\pi) = \#\{(i, j) : i < j \text{ and } \pi(i) > \pi(j)\}$, and $P_q^{(n)} = \sum_{\pi \in S_n} q^{i(\pi)} \pi$ where π acts via $\pi^{-1}(\zeta_1 \otimes \dots \otimes \zeta_n) = \zeta_{\pi(1)} \otimes \dots \otimes \zeta_{\pi(n)}$. Denote by $F_q(\mathcal{H})$ the completion of $F_{\text{alg}}(\mathcal{H})$ with respect to this inner product.

For $h \in \mathcal{H}$, define $\ell(h) : F_q(\mathcal{H}) \rightarrow F_q(\mathcal{H})$ by extending continuously the map

$$\begin{aligned} \ell(h)h_1 \otimes \dots \otimes h_n &= h \otimes h_1 \otimes \dots \otimes h_n, \\ \ell(h)\Omega &= h. \end{aligned}$$

The adjoint is given by

$$\begin{aligned} \ell^*(h)h_1 \otimes \dots \otimes h_n &= \sum_{k=1}^n q^{k-1} \langle h_k, h \rangle h_1 \otimes \dots \otimes \hat{h}_k \otimes \dots \otimes h_n, \\ \ell^*(h)\Omega &= 0, \end{aligned}$$

where $\hat{\cdot}$ denotes omission. $\omega(h) = \ell(h) + \ell^*(h)$ are q -Gaussian variables. $\Gamma_q(\mathcal{H})$ is the von Neumann algebra generated by $\omega(h) \ h \in \mathcal{H}$, acting as bounded operators on $F_q(\mathcal{H})$. We use on it the faithful trace $\tau_q(X) = \langle X\Omega, \Omega \rangle$. It is well-known that $L^2(\Gamma_q(\mathcal{H}), \tau_q) \simeq F_q(\mathcal{H})$. For $\xi \in F_{\text{alg}}(\mathcal{H})$ we write $\psi(\xi)$ the element in $\Gamma_q(\mathcal{H})$ such that $\psi(\xi)\Omega = \xi$, associated to this identification (since it is easy to see that $F_{\text{alg}}(\mathcal{H}) \subset F_q(\mathcal{H})$ is identified with a subspace of $\Gamma_q(\mathcal{H}) \subset L^2(\Gamma_q(\mathcal{H}), \tau_q)$ corresponding to polynomials in $\omega(h)$'s).

Consider also $r(h)$ given by

$$\begin{aligned} r(h)h_1 \otimes \dots \otimes h_n &= h_1 \otimes \dots \otimes h_n \otimes h, \\ r(h)\Omega &= h. \end{aligned}$$

Finally, let $P_n : F_q(\mathcal{H}) \rightarrow F_q(\mathcal{H})$ be the orthogonal projection onto tensors of rank n . Let $\mathcal{E}_q = \sum_{n \geq 0} q^n P_n$. It is obvious that \mathcal{E}_q is an Hilbert–Schmidt operator as soon as $q^{2N} < 1$. We also introduce a natural finite rank approximation $\mathcal{E}_q^{\mathcal{O}} = \sum_{n=0}^{\mathcal{O}} q^n P_n$.

4.2.2. Three natural derivations on q -Gaussian factors

Fix an orthonormal basis $\{h_i\}_{i=1}^N \subset \mathbb{R}^N$ and let $X_i = \omega(h_i)$. Thus $\Gamma_q(\mathcal{H}) = W^*(X_1, \dots, X_N)$, $N = \dim \mathcal{H}_{\mathbb{R}}$. We may also write for $\underline{i} = (i_1, \dots, i_n) \in \mathbb{N}^n$ $\psi_{\underline{i}} = \psi(h_{i_1} \otimes \dots \otimes h_{i_n})$. Finally, for a von Neumann algebra M , M^{op} will be as usual the opposite algebra. Later, I will consider $M = \Gamma_q(\mathcal{H})$.

The following lemma is proven in [37] (and stated exactly in that way in [39], Lemma 10, cf. also [38], Theorem 1).

Lemma 27 ([37]). *For $j = 1, \dots, N$, $q^{2N} < 1$, let $\partial_j^{(q)} : \mathbb{C}\langle X_1, \dots, X_N \rangle \rightarrow \text{HS}$ be the derivation given by $\partial_j^{(q)}(X_i) = \delta_{i=j} \mathcal{E}_q = [X_i, r(h_j)] = [r(h_j)^*, X_i]$. Let $\partial : \mathbb{C}\langle X_1, \dots, X_N \rangle \rightarrow \text{HS}^N$ be given by $\partial^{(q)} = \partial_1^{(q)} \oplus \dots \oplus \partial_N^{(q)}$ and regard ∂ as an unbounded operator densely defined on $L^2(\Gamma_q(\mathcal{H}))$. Then:*

- (i) $\partial^{(q)}$ is closable.
- (ii) If we denote by Z_j the vector $0 \oplus \dots \oplus P_{\Omega} \oplus \dots \oplus 0 \in \text{HS}^N$ (non-zero entry in j th place, P_{Ω} is the orthogonal projection onto $\mathbb{C}\Omega \in F_q(\mathcal{H})$), then Z_j is in the domain of ∂^* and $\partial^{(q)*}(Z_j) = h_j$.
- (iii) $1 \otimes \tau(\partial_j^{(q)}(X)) = \partial_j^{(q)}(X)\Omega = r(h_j)^*(X.\Omega)$ (in the first equality we identify isometrically HS with $L^2(\Gamma_q(\mathcal{H}) \otimes \Gamma_q(\mathcal{H})^{\text{op}})$ as usual via $a \otimes b$ with the rank one operator $a\tau(b)$)

Let us recall the following crucial result of Bożejko ([5]) giving an Haagerup like inequality for q -Gaussian variables.

Theorem 28 (Haagerup–Bożejko Inequality [5]). *If $C_q^{-1} = \prod_{m=1}^{\infty} (1 - q^m)$ then for any $\xi \in \mathcal{H}^{\otimes n} \subset F_{\text{alg}}(\mathcal{H})$:*

$$\|\psi(\xi)\|_{L^2(\Gamma_q(\mathcal{H}), \tau_q)} \leq \|\psi(\xi)\|_{\Gamma_q(\mathcal{H})} \leq C_{|q|}^{3/2} (n + 1) \|\psi(\xi)\|_{L^2(\Gamma_q(\mathcal{H}), \tau_q)}.$$

Moreover, for any $\eta \in \mathcal{H}^{\otimes n} \otimes \mathcal{H}^{\otimes m} \subset F_{\text{alg}}(\mathcal{H}) \otimes_{\text{alg}} F_{\text{alg}}(\mathcal{H})$ (ε either *op* or *nothing*)

$$\|\psi \otimes \psi(\eta)\|_{\Gamma_q(\mathcal{H}) \overline{\otimes} \Gamma_q(\mathcal{H})^\varepsilon} \leq C_{|q|}^3 (n+1)(m+1) \|\psi \otimes \psi(\eta)\|_{L^2(\Gamma_q(\mathcal{H}) \otimes \Gamma_q(\mathcal{H})^{\text{op}}, \tau_q \otimes \tau_q)}.$$

A short proof of the first part can be found in [28] (basically a variant of [5] without writing the computations), the argument obviously giving the second part too. Alternatively, as pointed out by our referee, we can apply to $u_i = \psi$ (and a variant with right multiplication in the case $\varepsilon = \text{op}$) the following fact. If $u_i : H_i \rightarrow B(K_i)$ are bounded maps from Hilbert spaces to bounded maps on a Hilbert space (nothing but trilinear forms on Hilbert spaces), then their tensor product $u_1 \otimes u_2$ is bounded from $H_1 \otimes H_2$ to $B(K_1 \otimes K_2)$ with $\|u_1 \otimes u_2\| \leq \|u_1\| \|u_2\|$.

From now on, ψ may not be written explicitly, no more than identifications between $L^2(\Gamma_q(\mathcal{H}) \otimes \Gamma_q(\mathcal{H})^{\text{op}})$ and Hilbert–Schmidt operators (following Section 2, but here the adjoint being the one coming from $\Gamma_q(\mathcal{H}) \otimes \Gamma_q(\mathcal{H})^{\text{op}}$ if not specified explicitly).

As a consequence, for any $\xi \in \bigoplus_{p \leq n} \mathcal{H}^{\otimes p}$ of component ξ_p , we also have by Cauchy–Schwarz:

$$\begin{aligned} \|\psi(\xi)\|_{\Gamma_q(\mathcal{H})} &\leq C_{|q|}^{3/2} \sum (p+1) \|\psi(\xi_p)\|_{L^2(\Gamma_q(\mathcal{H}), \tau_q)} \\ &\leq C_{|q|}^{3/2} (n+1)^{3/2} \left(\sum \|\psi(\xi_p)\|_{L^2(\Gamma_q(\mathcal{H}), \tau_q)}^2 \right)^{1/2} \\ &= C_{|q|}^{3/2} (n+1)^{3/2} \|\psi(\xi)\|_{L^2(\Gamma_q(\mathcal{H}), \tau_q)}. \end{aligned} \tag{14}$$

Likewise, for any $\eta \in \bigoplus_{p+q \leq n} \mathcal{H}^{\otimes p} \otimes \mathcal{H}^{\otimes q}$, we also have:

$$\|\psi(\eta)\|_{\Gamma_q(\mathcal{H}) \overline{\otimes} \Gamma_q(\mathcal{H})^{\text{op}}} \leq C_{|q|}^3 (n+1)^3 \|\psi(\eta)\|_{L^2(\Gamma_q(\mathcal{H}) \otimes \Gamma_q(\mathcal{H})^{\text{op}}, \tau_q \otimes \tau_q)}. \tag{15}$$

In order to state the next result, let us fix several notation about tensor products (similar to those of [42], Section 3.1). M is a given finite II_1 factor with faithful normal trace τ . $M \hat{\otimes} M^{\text{op}}$ is the projective tensor product of M with its opposite algebra, with the corresponding $*$ -Banach algebra structure. Let $\alpha : M \hat{\otimes} M^{\text{op}} \rightarrow B(M)$ be the contractive homomorphism given by $\alpha(a \otimes b) = L_a R_b$, where L_a and R_b are respectively the left and right multiplication operators by a and b . We will denote $\text{LR}(M)$ the algebra $\alpha(M \hat{\otimes} M^{\text{op}})$. It is easily seen that $\|\alpha(x)m\|_p \leq \|x\|_{M \hat{\otimes} M^{\text{op}}} \|m\|_p$, for $1 \leq p \leq \infty$ so that $\text{LR}(M)$ acts in a bounded way on $L^p(M, \tau)$ (the completion of M with respect to $\|x\|_p = \tau(|x|^p)^{1/p}$). Consistently with our previous notation, we will write $x \# m$ any of those actions (and several others we are about to discuss). For $p = 2$, this gives a map $\beta : \text{LR}(M) \rightarrow C^*(M, M')$ where M , and M' are with respect to the standard form of M on $L^2(M)$. Further, we have a $*$ -homomorphism $\gamma : C^*(M, M') \rightarrow M \overline{\otimes} M^{\text{op}}$ with value in the von Neumann algebra tensor product given by the general C^* tensor product theory. We will of course see $M \overline{\otimes} M^{\text{op}}$ as a II_1 factor with canonical trace $\tau \otimes \tau$. Finally, we will write $\#$ any “side multiplication” when defined. For instance, $a \otimes b \# a' \otimes b' \# a'' \otimes b'' = aa'a'' \otimes b''b'b$ so that $\#$ may be in this case multiplication in $M \overline{\otimes} M^{\text{op}}$, or any of its induced actions on $L^2(M \otimes M^{\text{op}})$. More generally, for $i \in [1, p-1]$, $a_i, b_j \in M$, we write

$$(a_1 \otimes a_2 \otimes \cdots \otimes a_p) \#_i (b_1 \otimes \cdots \otimes b_n) = a_1 \otimes \cdots \otimes a_i b_1 \otimes b_2 \otimes \cdots \otimes b_n a_{i+1} \otimes \cdots \otimes a_p$$

(if $p = 2$, $\#_1 = \#$), and likewise the corresponding extension for instance $M^{\overline{\otimes} i} \hat{\otimes} M^{\overline{\otimes} p-i} \times M^{\overline{\otimes} n} \rightarrow M^{\overline{\otimes} n+p-2}$ (or any analogues containing M^{op} the multiplication being then consistently defined to get what expected above in M as if there where everywhere M , for instance if $a \otimes a', c \otimes c' \in M \otimes M^{\text{op}}$, $b \otimes b' \otimes b'' \in (M \otimes M) \hat{\otimes} M^{\text{op}}$ we have $(a \otimes a') \# ((b \otimes b' \otimes b'') \#_2 (c \otimes c')) = ((a \otimes a') \# (b \otimes b' \otimes b'')) \#_2 (c \otimes c') = (ab \otimes b'c \otimes c'b'a') \in M \otimes M \otimes M^{\text{op}}$ (all multiplications written in M , if we were more consistent with M^{op} we would have written $a'b''c'$). However, we won't use this notation if $c \otimes c'$ is thought of in $M \otimes M$, but everything would be the same if also $b \otimes b' \otimes b'' \in (M \otimes M^{\text{op}}) \hat{\otimes} M^{\text{op}}$ except for the value in this space in $M \otimes M^{\text{op}} \otimes M^{\text{op}}$).

We will often use the following assumption and give an easy sufficient condition deduced from Bożejko inequality in the next corollary.

Assumption I_q . $q\sqrt{N} < 1$ and Ξ_q is invertible in $M \overline{\otimes} M^{\text{op}}$.

Even if we will scarcely use it, for $R > 1$ and a non-commutative power series (of radius of convergence larger than R with value in a tensor product) $F(Y_1, \dots, Y_n) = \sum a_{i_1, \dots, i_n, p} Y_{i_1} \cdots Y_{i_p} \otimes Y_{i_{p+1}} \cdots Y_{i_n}$ we write the usual norm $\|F\|_R = \sum |a_{i_1, \dots, i_n, p}| R^n$. We will use the same notation with less or more tensors in the space of value.

Corollary 29. *When the right hand side in the inequalities below is finite, \mathcal{E}_q comes from an element in $M \hat{\otimes} M^{\text{op}}$, and respectively $M \bar{\otimes} M^{\text{op}}$ via $\iota\gamma\beta\alpha$ or $\iota: M \bar{\otimes} M^{\text{op}} \rightarrow L^2(M \bar{\otimes} M^{\text{op}})$ and with an obvious notation:*

$$\begin{aligned} \|\mathcal{E}_q - 1 \otimes 1\|_{M \hat{\otimes} M^{\text{op}}} &\leq (C_{|q|})^3 \left[\frac{4|q|N}{1 - |q|N} + \frac{5(|q|N)^2}{(1 - |q|N)^2} + \frac{2(|q|N)^3}{(1 - |q|N)^3} \right] =: \nu(q, N), \\ \|\mathcal{E}_q - 1 \otimes 1\|_{M \bar{\otimes} M^{\text{op}}} &\leq (C_{|q|})^3 \left[\frac{4|q|\sqrt{N}}{1 - |q|\sqrt{N}} + \frac{5(|q|\sqrt{N})^2}{(1 - |q|\sqrt{N})^2} + \frac{2(|q|\sqrt{N})^3}{(1 - |q|\sqrt{N})^3} \right] =: \rho(q, N). \end{aligned}$$

Especially ($N \geq 2$) if q is such that $\nu(q, N) < 1$, e.g. for $|q|N \leq 0.13$, then \mathcal{E}_q is invertible in $M \hat{\otimes} M^{\text{op}}$ (resp. if q is such that $\rho(q, N) < 1$ e.g. when $|q|\sqrt{N} \leq 0.13$ then I_q holds). Moreover, if $q\sqrt{N} < 1$, $\|\mathcal{E}_q^Q - \mathcal{E}_q\|_{M \bar{\otimes} M^{\text{op}}} \rightarrow_{Q \rightarrow \infty} 0$ and $\mathcal{E}_q \in C^*(X_1 \otimes 1, 1 \otimes X_1, \dots, 1 \otimes X_N) \subset M \bar{\otimes} M^{\text{op}}$ is positive so that $\mathcal{E}_q^{1/2}$ is well defined.

Moreover, if $\varepsilon > 0$ and $((3 + \varepsilon)^2 N + 2)|q| < 1$ there exists a non-commutative power series $\mathcal{E}_q(Y_1, \dots, Y_N)$ with radius of convergence greater than $R = (1 + \varepsilon/2) \frac{2}{1 - |q|} > \|X_i\|$ such that $\mathcal{E}_q(X_1, \dots, X_N) = \mathcal{E}_q$, and

$$\|\mathcal{E}_q - 1 \otimes 1\|_R \leq \frac{(3 + \varepsilon)^2 N |q|}{1 - (2 + (3 + \varepsilon)^2 N) |q|} =: \pi(q, N),$$

and likewise,

$$\max \left(\sum_{i=1}^N \|\partial_i \otimes 1(\mathcal{E}_q)\|_R, \sum_{i=1}^N \|1 \otimes \partial_i(\mathcal{E}_q)\|_R \right) \leq \frac{(3 + \varepsilon)^2 N |q| (1 - 2|q|)}{(1 - (2 + (3 + \varepsilon)^2 N) |q|)^2}.$$

Proof. Since P_n can be seen as a finite rank operator written as $\sum_{\xi} \xi \otimes \xi^*$ with the usual identification (the sum running over an orthonormal basis of $\mathcal{H}^{\otimes n}$), the previous theorem gives: $\|P_n\|_{M \hat{\otimes} M^{\text{op}}} \leq \sum_{\xi} \|\xi\|^2 \leq C_{|q|}^3 (n + 1)^2 \sum_{\xi} \|\xi\|_2^2 = C_{|q|}^3 (n + 1)^2 N^n$. The inequality follows from a standard computation.

Likewise

$$\begin{aligned} \|\mathcal{E}_q - 1 \otimes 1\|_{M \bar{\otimes} M^{\text{op}}} &\leq \sum_{n \geq 1} q^n \left\| \sum_{\xi} \xi \otimes \xi^* \right\| \\ &\leq C_{|q|}^3 \sum_{n \geq 1} q^n (n + 1)^2 \left\| \sum_{\xi} \xi \otimes \xi^* \right\|_2 = C_{|q|}^3 \sum_{n \geq 1} q^n (n + 1)^2 N^{n/2}. \end{aligned}$$

Let us call $f(|q|N) = \nu(q, N)/(C_{|q|})^3$. To get $f(|q|N) < C_{|q|}^{-3}$, it suffices to have $f(|q|N) < (1 + |q|)^3 \prod_{m=1}^{\infty} (1 - |q|^m)^3 / (1 + |q|^m)^3 = (1 + |q|)^3 (\sum_{n \in \mathbb{Z}} (-1)^n |q|^{n^2})^3$, and again keeping only the smallest order it suffices to have $f(|q|N) < (1 - |q| - 2|q|^2)^3$ and solving numerically $f(|q|N) < (1 - |q|N/2 - |q|^2 N^2/2)^3$ (sufficient since $N \geq 2$) one gets $|q|N < 0.1386 \dots$

For the last statement, we only improve an estimate in [39]. We write $p_{\underline{l}}$ the polynomials giving, by evaluation on X_1, \dots, X_N , the orthonormalization of $\psi_{\underline{l}}$ defined in Lemma 13 in [39]. More specifically, we consider Γ_n the Gram matrix of q -scalar products in the space of tensors of length n given (for $|\underline{j}| = |\underline{l}| = n$) by: $(\Gamma_n)_{(\underline{j}, \underline{l})} = \langle \psi_{j_1, \dots, j_n}, \psi_{l_1, \dots, l_n} \rangle_q$. This is an $N^n \times N^n$ matrix known to be positive and invertible (with real coefficients), and we consider $B = \Gamma_n^{-1/2}$. Note that by definition Γ_n is given by the image of the element of $P_q^{(n)} = \sum_{\pi \in S_n} q^{i(\pi)} \pi$ in the algebra of the symmetric group S_n by the obvious representation $\pi_{q, N, n}$ of S_n on (the formal basis of the \mathbb{C}^{N^n}) $\xi_{\underline{l}}$ and it is known from [16] and [49] a formula for $P_q^{(n)-1}$ given by the inductive relation $P_q^{(n)} = \pi_{n-1, n}(P_q^{(n-1)})M_n, \pi_{n-1, n}$ the usual embedding of S_{n-1} in S_n with image leaving 1 invariant and $M_n = \sum_{k=1}^n q^{k-1} (1 \rightarrow k)$ (with the notation

($k \rightarrow l$) the cycle sending $k + i$ to $k + i + 1$ for $0 \leq i \leq l - k - 1$ and sending l to k) via $M_n^{-1} = \prod_{j=n-1}^1 (1 - q^j (1 \rightarrow j + 1)) \prod_{j=n-2}^0 (1 - q^{n-j} (2 \rightarrow n - j))^{-1}$. We will use it through $B^2 = \pi_{q,N,n}(P_q^{(n)})^{-1}$. We also write $\psi_{\underline{j}}(Y_1, \dots, Y_N)$ the non-commutative polynomial defined inductively by ($\psi_{\varepsilon} = 1$ for the empty word ε):

$$\psi_{i_1, \dots, i_n} = Y_{i_1} \psi_{i_2, \dots, i_n} - \sum_{j \geq 2} q^{j-2} \delta_{i_1=j} \psi_{i_2, \dots, \hat{i}_j, \dots, i_n}. \tag{16}$$

As in the proof of Proposition 2.7 in [6], we use the following identity for $\psi_{\underline{l}} = \psi_{\underline{l}}(X_1, \dots, X_N)$ for $\psi_{\underline{l}}$ introduced before.

Then, by definition, $p_{\underline{l}}(Y_1, \dots, Y_N) = \sum_{j, |j|=\underline{l}} B_{\underline{l}, j} \psi_{\underline{j}}(Y_1, \dots, Y_N)$ so that (as checked in Lemma 13 in [39]) $\{p_{\underline{l}}(X_1, \dots, X_N) \Omega\}_{|l|=n}$ is obviously an orthonormal basis of $\mathcal{H}^{\otimes n}$.

$\mathcal{E}_q(Y_1, \dots, Y_N) = \sum_n q^n \sum_i p_{\underline{i}}(Y_1, \dots, Y_N) \otimes p_{\underline{i}}^*(Y_1, \dots, Y_N)$ will be the power series we are looking for, once proved an estimate on its norm. It suffices to bound (using symmetry of the matrix B):

$$\begin{aligned} \left\| \sum_{\underline{l}} p_{\underline{l}}(Y_1, \dots, Y_N) \otimes p_{\underline{l}}^*(Y_1, \dots, Y_N) \right\|_R &= \left\| \sum_{\underline{l}, j, \underline{l}} B_{\underline{l}, j} \psi_{\underline{j}}(Y_1, \dots, Y_N) \otimes B_{\underline{l}, \underline{l}} \psi_{\underline{l}}^*(Y_1, \dots, Y_N) \right\|_R \\ &\leq \sum_{\underline{l}} \left\| \sum_j B_{\underline{l}, j}^2 \psi_{\underline{j}}(Y_1, \dots, Y_N) \right\|_R \left\| \psi_{\underline{l}}^*(Y_1, \dots, Y_N) \right\|_R. \end{aligned}$$

Now using the expression for B^2 expanded from the inverse coming from the action of the symmetric group algebra, it involves only $\|\psi_{\sigma(j)}(Y_1, \dots, Y_N)\|_R$ and from the bound in [16], Lemma 4.1, one gets

$$\left\| \sum_{\underline{j}} B_{\underline{l}, \underline{j}}^2 \psi_{\underline{j}}(Y_1, \dots, Y_N) \right\|_R \leq \left((1 - |q|) \prod_{k=1}^{\infty} \frac{1 + |q|^k}{1 - |q|^k} \right)^n \sup_{\sigma \in \mathcal{S}_n} \|\psi_{\sigma(j)}(Y_1, \dots, Y_N)\|_R.$$

Finally, using (16), if we call $C_n = \sup_{i_1, \dots, i_n} \|\psi_{i_1, \dots, i_n}\|_R$, then $C_n \leq RC_{n-1} + C_{n-2}/(1 - |q|)$, thus $C_n \leq (R + \frac{1}{1-|q|})^n$. Likewise, if $D_n = \sup_{i_1, \dots, i_n} \sum_j \|\partial_j \psi_{i_1, \dots, i_n}(Y_1, \dots, Y_N)\|_R$ then $D_n \leq C_{n-1} + RD_{n-1} + D_{n-2}/(1 - |q|)$ thus one checks by induction: $D_n \leq n(R + \frac{1}{1-|q|})^{n-1}$.

Finally, we proved:

$$\begin{aligned} \left\| \sum_{\underline{l}} p_{\underline{l}}(Y_1, \dots, Y_N) \otimes p_{\underline{l}}^*(Y_1, \dots, Y_N) \right\|_R &\leq \left((1 - |q|) \prod_{k=1}^{\infty} \frac{1 + |q|^k}{1 - |q|^k} \right)^n \left(R + \frac{1}{1 - |q|} \right)^{2n} N^n \\ &\leq \left(\frac{(1 - |q|)^2}{1 - 2|q|} \right)^n \left(\frac{(3 + \varepsilon)}{1 - |q|} \right)^{2n} N^n. \end{aligned}$$

The last rough estimate is as above, in this proof, for the estimate on $f(|q|N)$ and detailed in Lemma 13 in [39], and it concludes. Likewise, we have:

$$\sum_j \left\| \sum_{\underline{l}} \partial_j p_{\underline{l}}(Y_1, \dots, Y_N) \otimes p_{\underline{l}}^*(Y_1, \dots, Y_N) \right\|_R \leq n \left(\frac{(1 - |q|)^2}{1 - 2|q|} \right)^n \left(\frac{(3 + \varepsilon)}{1 - |q|} \right)^{2n} N^n.$$

Note that positivity comes from the identification of $\sum_n q^n P_n = \Gamma_q(q \text{ id})$ with the second quantization. □

As a consequence, for q such that I_q holds (e.g. $\rho(q, N) < 1$), if ∂_j is the j th free difference quotient with respect to X_1, \dots, X_N we have $\partial_j = \partial_j^{(q)} \# \mathcal{E}_q^{-1}$ since $\partial_j(X_i) = 1_{i=j} \mathcal{E}_q \# \mathcal{E}_q^{-1} = \delta_{i=j} 1 \otimes 1$. Recall $\#$ in this context is multiplication in $M \overline{\otimes} M^{\text{op}}$.

Finally, we want to introduce a derivation giving the number operator as generator of the corresponding Dirichlet form. We define first the $\hat{\delta}_j^{(q)} := \partial_k \# X_q^k$ valued in $\Gamma_q(\mathcal{H} \oplus \mathcal{H})$ where X_q^k is the q -Gaussian variable corresponding to

the second copy of the eigenvector h_k in the second term of the direct sum. Said otherwise this is the only derivation sending X_q^k to $X_q^{k'}$. This derivation is defined for any q . We also want to compare this derivation to another derivation valued in the coarse correspondence. For q such that $q\sqrt{N} < 1$, we can define $\tilde{\partial}_j^{(q)} := \partial_j \# \mathcal{E}_q^{1/2}$.

Proposition 30. *For any $\xi \in \mathcal{H}^{\otimes n}$, $\eta \in \mathcal{H}^{\otimes m}$, any $q \in (-1, 1)$, $\psi(\xi) \in D(\hat{\partial}_k^{(q)})$ and we have:*

$$\sum_k \langle \hat{\partial}_k^{(q)}(\psi(\xi)), \hat{\partial}_k^{(q)}(\psi(\eta)) \rangle_q = n \langle \xi, \eta \rangle_q.$$

As a consequence, $\hat{\partial}^{(q)} = (\hat{\partial}_1^{(q)}, \dots, \hat{\partial}_n^{(q)})$ is a closable derivation, with $\hat{\partial}^{(q)*} \hat{\partial}^{(q)} = \tilde{\Delta}$ the number operator satisfying $\tilde{\Delta}(\xi) = n\xi$, $\xi \in \mathcal{H}^{\otimes n}$.

Moreover if $q\sqrt{N} < 1$, for any polynomial $P, Q, R, S \in \mathbb{C}\langle X_1, \dots, X_n \rangle$,

$$\langle R \hat{\partial}_k^{(q)}(P), S \hat{\partial}_k^{(q)}(Q) \rangle = \langle R \tilde{\partial}_k^{(q)}(P), S \tilde{\partial}_k^{(q)}(Q) \rangle.$$

Thus, one can see $\hat{\partial}_k^{(q)}$ as valued in a bimodule included in the coarse correspondence.

Finally, if I_q holds, $\tilde{\partial}_k^{(q)}$, $\partial_k^{(q)}$, and ∂_k are all closable and their closures share the same domain, with, for any x in their common domain:

$$\begin{aligned} \|\partial_k^{(q)}(x)\|_2 &\leq \|\mathcal{E}_q^{1/2}\|_{M \overline{\otimes} M^{\text{op}}} \|\tilde{\partial}_k^{(q)}(x)\|_2 \leq \|\mathcal{E}_q^{1/2}\|_{M \overline{\otimes} M^{\text{op}}}^2 \|\partial_k(x)\|_2, \\ \|\partial_k(x)\|_2 &\leq \|\mathcal{E}_q^{-1/2}\|_{M \overline{\otimes} M^{\text{op}}} \|\tilde{\partial}_k^{(q)}(x)\|_2 \leq \|\mathcal{E}_q^{-1/2}\|_{M \overline{\otimes} M^{\text{op}}}^2 \|\partial_k^{(q)}(x)\|_2. \end{aligned}$$

Proof. The domain property stated is obvious since $\psi(\xi)$ is a non-commutative polynomial in X_1, \dots, X_N . Moreover, by linearity, we need to check the first equality only for $\psi(\xi) = \psi_{j_1, \dots, j_n}$ and $\psi(\eta) = \psi_{l_1, \dots, l_p}$.

As in the proof of Proposition 2.7 in [6], we use the following identity:

$$\psi_{i_1, \dots, i_n} = X_{i_1} \psi_{i_2, \dots, i_n} - \sum_{j \geq 2} q^{j-2} \delta_{i_1=j} \psi_{i_2, \dots, \hat{i}_j, \dots, i_n}.$$

Applying ∂_k , we find:

$$\partial_k(\psi_{i_1, \dots, i_n}) = 1_{i_1=k} \otimes \psi_{i_2, \dots, i_n} + X_{i_1} \partial_k(\psi_{i_2, \dots, i_n}) - \sum_{j \geq 2} q^{j-2} \delta_{i_1=j} \partial_k(\psi_{i_2, \dots, \hat{i}_j, \dots, i_n}).$$

As a consequence, we deduce by an immediate induction:

$$\partial_k(\psi_{i_1, \dots, i_n}) \# X_q^{k'} = \sum_j 1_{i_j=k} \psi_{i_1, \dots, i'_j, \dots, i_n}$$

(where the prime indicates we have to consider the i_j of the second copy of \mathcal{H}).

We can thus compute (using the definition of the scalar product in the second and fourth lines, and removing properly summations and Kronecker functions $1_{a=b}$ in the third and fifth lines):

$$\begin{aligned} \sum_k \langle \hat{\partial}_k^{(q)}(\psi_{j_1, \dots, j_n}), \hat{\partial}_k^{(q)}(\psi_{l_1, \dots, l_m}) \rangle_q &= \sum_k \sum_{i, t} 1_{j_i=k=l_t} \langle \psi_{j_1, \dots, j'_i, \dots, j_n}, \psi_{l_1, \dots, l'_t, \dots, l_m} \rangle_q \\ &= 1_{n=m} \sum_k \sum_{i, t} 1_{j_i=k=l_t} \sum_{\pi \in S_n} q^{i(\pi)} 1_{\pi(i)=t} \prod_{p=1}^n 1_{j_p=l_{\pi(p)}} \\ &= 1_{n=m} \sum_k \sum_i 1_{j_i=k} \sum_{\pi \in S_n} q^{i(\pi)} \prod_{p=1}^n 1_{j_p=l_{\pi(p)}} \end{aligned}$$

$$\begin{aligned}
 &= 1_{n=m} \sum_k \sum_i 1_{j_i=k} \langle \psi_{j_1, \dots, j_n}, \psi_{l_1, \dots, l_m} \rangle_q \\
 &= n \langle \psi_{j_1, \dots, j_n}, \psi_{l_1, \dots, l_m} \rangle_q.
 \end{aligned}$$

We now assume $q\sqrt{N} < 1$. To explain the second equality, note that we can rewrite (since \mathcal{E}_q self-adjoint) $\langle a \otimes b\#\mathcal{E}_q^{1/2}, a' \otimes b'\#\mathcal{E}_q^{1/2} \rangle = \langle a \otimes b, a' \otimes b'\#\mathcal{E}_q \rangle$ and then:

$$\langle a \otimes b, a' \otimes b'\#\mathcal{E}_q \rangle = \sum_n q^n \sum_{\xi} \tau(a^* a' \xi) \tau(\xi^* b' b^*) = \sum_n q^n \tau(a^* a' P_n(b' b^*)) = \tau(a^* a' \Gamma_q(q \text{id})(b' b^*)),$$

where $\Gamma_q(q \text{id})$ is the second quantization. Then, our claim follows for instance from Theorem 3.2 in [15] which implies $\tau(a^* a' \Gamma_q(q \text{id})(b' b^*)) = \langle a \otimes b\#X'_k, a' \otimes b'\#X'_k \rangle$ (Theorem 3.2 is a variant of Ito formula, one can apply it after identifying the first copy of \mathcal{H} with $\text{Span}\{\sqrt{n}1_{[k/2n, (k+1)/2n)}, k = 1, \dots, n\}$ in $L^2([0, 1])$ and the second with $\text{Span}\{\sqrt{n}1_{[k/2n, (k+1)/2n)}, k = n + 1, \dots, 2n\}$).

The last inequalities in the proposition on non-commutative polynomials follow from Corollary 29 and Assumption I_q . It implies closability since $\partial^{(q)}$ is closable (by Lemmas 12 and 27) and the result extended to the closures. \square

Remark 31. Even if we won't use significantly later the analytic bound we got in Corollary 29, it is worth noting it can enable us using our last derivation $\tilde{\partial}^{(q)}$ to prove complete metric approximation property for $\Gamma_q(\mathcal{H})$ with small q , or (reprove) absence of non-trivial projections for the corresponding C^* -algebras following the lines of [19]. Indeed, first note that using the analytic expansion $\mathcal{E}_q^{1/2}(Y_1, \dots, Y_N) = 1 \otimes 1 + \sum_{k=1}^{\infty} \binom{1/2}{k} (\mathcal{E}_q(Y_1, \dots, Y_N) - 1 \otimes 1)^k$ so that we get a Lipschitz bound

$$\begin{aligned}
 &\| \mathcal{E}_q^{1/2}(Y_1, \dots, Y_N) - \mathcal{E}_q^{1/2}(Z_1, \dots, Z_N) \| \\
 &\leq \sum_{k=1}^{\infty} \left| \binom{1/2}{k} \right| k \| \mathcal{E}_q - 1 \otimes 1 \|_R^{k-1} \sum_i (\| \partial_i \otimes 1(\mathcal{E}_q) \|_R + \| 1 \otimes \partial_i(\mathcal{E}_q) \|_R) \| X_i - Z_i \| \\
 &\leq \frac{1}{2\sqrt{1 - \|(\mathcal{E}_q - 1 \otimes 1)\|_R}} \sup_i (\| X_i - Z_i \|) 2 \frac{(3 + \varepsilon)^2 N |q| (1 - 2|q|)}{(1 - (2 + (3 + \varepsilon)^2 N) |q|)^2} \\
 &\leq \frac{1}{\sqrt{1 - (2 + 2(3 + \varepsilon)^2 N) |q|}} \sup_i (\| X_i - Z_i \|) \frac{(3 + \varepsilon)^2 N |q| (1 - 2|q|)}{(1 - (2 + (3 + \varepsilon)^2 N) |q|)^{3/2}} \\
 &\leq \kappa \sup_i (\| X_i - Z_i \|),
 \end{aligned}$$

the last inequality being true for $\kappa < 1/2$ if $(3 + \varepsilon)^2 N |q| (1 - 2|q|) \leq (3 + \varepsilon)^2 N |q| < (1 - 2(2 + 2(3 + \varepsilon)^2 N) |q|) / 2 \leq (1 - (2 + 2(3 + \varepsilon)^2 N) |q|)^2 / 2$, i.e. e.g. for $(4 + 6(3 + \varepsilon)^2 N) |q| < 1$.

As in [19], one can consider the solutions (given by Picard iteration) $X_{i,t} = X_i - \frac{1}{2} \int_0^t ds X_{i,s} + \int_0^t \mathcal{E}_q^{1/2}(X_{1,s}, \dots, X_{N,s}) \# dS_s^i$, $Y_{i,t} = 0 - \frac{1}{2} \int_0^t ds Y_{i,s} + \int_0^t \mathcal{E}_q^{1/2}(Y_{1,s}, \dots, Y_{N,s}) \# dS_s^i$. From [39] or [12] (and the above Proposition 30), $X_{i,t}$ is stationary so that $\alpha_t(X_i) = X_{i,t}$ ($i = 1, \dots, N$) defines a trace preserving homomorphism. By variation of constants, one gets:

$$\begin{aligned}
 (X_{i,t} - Y_{i,t}) &= X_i - \frac{1}{2} \int_0^t ds (X_{i,s} - Y_{i,s}) + \int_0^t (\mathcal{E}_q^{1/2}(X_{1,s}, \dots, X_{N,s}) - \mathcal{E}_q^{1/2}(Y_{1,s}, \dots, Y_{N,s})) \# dS_s^i \\
 &= e^{-1/2t} X_i + \int_0^t e^{-1/2(t-s)} (\mathcal{E}_q^{1/2}(X_{1,s}, \dots, X_{N,s}) - \mathcal{E}_q^{1/2}(Y_{1,s}, \dots, Y_{N,s})) \# dS_s^i.
 \end{aligned}$$

And from our inequality above and Biane–Speicher's L^∞ version of Burkholder–Gundy inequality, we deduce:

$$\sup_i \| X_{i,t} - Y_{i,t} \| \leq e^{-t/2} \sup_i \| X_i \| + \left(\int_0^t ds e^{-(t-s)} \kappa^2 \sup_i \| X_{i,s} - Y_{i,s} \|^2 \right)^{1/2}$$

so that from Gronwall's lemma (in line 2 after using a trivial bound on squares and $\kappa < 1/2$):

$$\begin{aligned} \sup_i \|X_{i,t} - Y_{i,t}\|^2 &\leq 2e^{-t} \sup_i \|X_i\|^2 + 1/2 \left(\int_0^t ds e^{-(t-s)} \sup_i \|X_{i,s} - Y_{i,s}\|^2 \right) \\ &\leq 2e^{-t/2} \sup_i \|X_i\|^2 \rightarrow 0. \end{aligned}$$

Thus, since $Y_{i,s} \in C^*(S_i^j)$ we got the property of Corollary 4.1 in [19] and by the reasoning of Theorem 4.2 there, $C^*(X_1, \dots, X_N)$ has no non-trivial projections (remember this applies when $(4 + 6(3 + \varepsilon)^2 N)|q| < 1$). Likewise, by the reasoning of Theorem 4.3 in [19] we get complete metric approximation property in the way they get Haagerup property. This last result has been recently extended by Stephen Avsec [1] to all $q \in (-1, 1)$. Of course, in the smaller range of q we consider we have almost inclusion in $L(\mathbb{F}_\infty)$ too.

4.2.3. Regularity for \mathcal{E}_q

Let us write $\partial_i^{(k,j)} = 1^{\otimes(k-1)} \otimes \partial_i \otimes 1^{\otimes(j)} : L^2(M)^{\otimes(k+j)} \rightarrow L^2(M)^{\otimes(k+j+1)}$ and the corresponding L^2 closure $\overline{\partial_i^{(k,j)}}$. We start by noting the following consequence of Proposition 30:

Lemma 32. *If I_q holds and for $\xi \in \mathcal{H}^{\otimes n}$, for \mathcal{D} any among $\overline{\partial_{m_p}^{(k_p, p-k_p)}} \circ \dots \circ \overline{\partial_{m_1}^{(k_1, 1-k_1)}}$ $p \in [1, n], k_l \in [1, l], m_l \in [1, N], l = 1, \dots, p, \|\mathcal{D}(\xi)\|_2^2 \leq (n \|\mathcal{E}_q^{-1}\|_{M \overline{\otimes} M^{\text{op}}})^p \|\xi\|_2^2$.*

Lemma 33. *Assume I_q and $|q|N < 1$, then $\|\partial_k \otimes 1 \mathcal{E}_q\|_{(M \overline{\otimes} M^{\text{op}}) \hat{\otimes} M} < \infty$. Likewise, with U, V any among $\partial_j, \partial_i^{(q)}$, we have $U \otimes V(\mathcal{E}_q) \in (M \overline{\otimes} M^{\text{op}}) \hat{\otimes} (M^{\text{op}} \overline{\otimes} M), U \otimes V(\mathcal{E}_q) \in (M^{\text{op}} \overline{\otimes} M) \hat{\otimes} (M \overline{\otimes} M^{\text{op}}), (U \otimes 1 \otimes 1)(V \otimes 1)(\mathcal{E}_q) \in (M \overline{\otimes} M^{\text{op}} \overline{\otimes} M) \hat{\otimes} M^{\text{op}}, (1 \otimes 1 \otimes U)(1 \otimes V)(\mathcal{E}_q) \in M \hat{\otimes} (M^{\text{op}} \overline{\otimes} M \overline{\otimes} M^{\text{op}})$.*

Proof. We compute (the first inequality bellow is obvious from Lemma 29, the second equality comes from Proposition 30):

$$\|\partial_k(p_i)\|_{L^2(M, \tau_q) \otimes L^2(M, \tau_q)}^2 \leq \|\mathcal{E}_q^{-1/2}\|^2 \langle \partial_k(p_i) \# \mathcal{E}_q, \partial_k(p_i) \rangle = \|\mathcal{E}_q^{-1/2}\|^2 |i|.$$

Now, one can use Theorem 28 and (15) in the second line and our previous inequality in the third to conclude to the first result.

$$\begin{aligned} \|\partial_k \otimes 1 \mathcal{E}_q\|_{(M \overline{\otimes} M^{\text{op}}) \hat{\otimes} M} &\leq \sum_n q^n \sum_{|i|=n} \|\partial_k(p_i)\|_{M \overline{\otimes} M^{\text{op}}} \|p_i\|_M \\ &\leq C_{|q|}^{9/2} \sum_n |q|^n \sum_{|i|=n} (n+1)^4 \|p_i\|_2 \|\partial_k(p_i)\|_{L^2(M \overline{\otimes} M)} \\ &\leq C_{|q|}^{9/2} \sum_n |q|^n (n+1)^4 \|\mathcal{E}_q^{-1/2}\| n^{1/2} N^n. \end{aligned}$$

As stated, as soon as $\|\mathcal{E}_q^{-1/2}\| < \infty$ and $|q|N < 1$, this sum is finite. The proof of $\partial_j \otimes \partial_i^{(q)}(\mathcal{E}_q) \in (M \overline{\otimes} M^{\text{op}}) \hat{\otimes} (M^{\text{op}} \overline{\otimes} M)$ is really similar. For the last statement, we need likewise a bound e.g. on $\|\partial_j \otimes 1 \partial_i^{(q)}(p_j)\|_{M \overline{\otimes} M^{\text{op}} \overline{\otimes} M} \leq \|\partial_j \otimes 1(\mathcal{E}_q)\|_{(M \overline{\otimes} M^{\text{op}}) \hat{\otimes} M} \|\partial_i(p_j)\|_{M \overline{\otimes} M} + \|\mathcal{E}_q\|_{M^{\text{op}} \overline{\otimes} M} \|\partial_j \otimes 1 \partial_i(p_j)\|_{M \overline{\otimes} M^{\text{op}} \overline{\otimes} M}$ coming from the previous lemma (with a 3 tensor product variant of Bożejko inequality and Lemma 30). \square

4.3. An example of Lipschitz conjugate variable: q -Gaussian families for small q

We now want to play with the three previous derivations to get regularity results for conjugate variables.

Theorem 34. Assume $\rho(q, N) < 1$ as defined in Corollary 29 (e.g. $|q|\sqrt{N} \leq 0.13$) and $|q|N < 1$ then q -Gaussian variables have finite free Fisher information $\Phi^*(X_1, \dots, X_N) < \infty$ (and actually the conjugate variable is in the domain of the L^2 -closure of the free difference quotient).

Furthermore assume also condition $\nu(q, N) < 1$ in Corollary 29 (e.g. $|q|N \leq 0.13$), in that case the conjugate variables are in $\Gamma_q(\mathcal{H})$ and X_1, \dots, X_N have even Lipschitz conjugate variables. As a consequence, under condition $\nu(q, N) < 1$ we have $\delta_0(X_1, \dots, X_N) = N$.

Finally, if we assume $\pi(q, N) < 1$, then there exists a non-commutative power series ξ_j of radius $R = (1 + \varepsilon/2) \frac{2}{1-|q|} > \|X_i\|$ such that $\xi_j(X_1, \dots, X_N)$ are the conjugate variables of X_1, \dots, X_N . Moreover, there exists a self-adjoint potential V which is also a non-commutative power series of radius R such that its cyclic gradient is $D_i V = \xi_i$.

Remark 35. In [39], Shlyakhtenko proved $\delta_0(X_1, \dots, X_N) \rightarrow N$ when $q \rightarrow 0$, we can prove this value is identically equal to N on a small neighborhood of 0. Actually, he proved $\delta_0(X_1, \dots, X_N) \geq N(1 - \frac{q^2 N}{1-q^2 N})$ for $|q| < (4N^3 + 2)^{-1}$. Here the improvement in terms of value of δ_0 mainly comes from using a better derivation in that respect (the free difference quotient). The improvement in terms of values of q comes from the fact we only need a Lipschitz condition instead of an analyticity condition on the conjugate variable. However, in considering like us the free difference quotient and with a better estimate of the domain of analyticity, one would also get a range of order $|q| < 1/CN$ in inverse of the number of generators (with a huger C than ours, cf. Remark 31). Note finally that Corollary 2.11 in [37] implies $\delta^*(X_1, \dots, X_N) = N$ as soon as I_q holds, thus e.g. assuming only $\rho(q, N) < 1$.

Proof of Theorem 34. Let $M = \Gamma_q(\mathcal{H})$.

Step 1: Finite Fisher Information under $|q|N < 1$ and $\rho(q, N) < 1$.

Recall the notation introduced before (and in) Corollary 29 so that $\iota\gamma\beta\alpha$ is the natural map from $M \hat{\otimes} M^{\text{op}}$ to $L^2(M \otimes M^{\text{op}})$ (we may use later implicitly).

Claim. $\iota\gamma\beta\alpha(M \hat{\otimes} M^{\text{op}}) \subset D(\partial_j^{(q)*})$ and for any $a, b \in M$

$$\partial_j^{(q)*}(a \otimes b) = aX_j b - r(h_j)^*(a)b - a(l(h_j)^*(b)).$$

Proof. As reminded in Lemma 27, $1 \otimes \tau \partial_j^{(q)} = r(h_j)^*$. Moreover, since $\partial_j^{(q)}$ is a real derivation for any $x \in D(\partial_j^{(q)})$, we have $1 \otimes \tau \partial_j^{(q)}(x^*) = (\tau \otimes 1)(\partial_j^{(q)}(x))^*$. Thus if J denotes the antilinear isometry extending $J(x) = x^*$ to $L^2(M)$, we have $\tau \otimes 1 \partial_j^{(q)} = J1 \otimes \tau \partial_j^{(q)} J = Jr(h_j)^* J = l(h_j)^*$. The last equality follows from formulas for annihilation operators and $J\psi_{i_1, \dots, i_n} = \psi_{i_n, \dots, i_1}$.

From Lemma 12 and $\partial_j^{(q)*}(1 \otimes 1) = X_j$, one deduces for $a, b \in D(\partial_j^{(q)}) \cap M$, $\partial_j^{(q)*}(a \otimes b) = aX_j b - r(h_j)^*(a)b - a(l(h_j)^*(b))$, so that

$$\begin{aligned} \|\partial_j^{(q)*}(a \otimes b)\|_2 &\leq \|a\| \|b\| \|X_j\| + \|r(h_j)^*\|_{B(L^2(M))} \|a\|_2 \|b\| + \|l(h_j)^*\|_{B(L^2(M))} \|b\|_2 \|a\| \\ &\leq 4\|a\| \|b\| / \sqrt{1-|q|}. \end{aligned}$$

Now for any $a, b \in M$, if $\eta_\alpha = \alpha(\alpha + \partial_j^{(q)*} \partial_j^{(q)})^{-1}$ the completely positive (thus contractive on M) resolvent associated to the generator of the corresponding Dirichlet form, we have for any $x \in M$, $\eta_\alpha(x) \in D(\partial_j^{(q)}) \cap M$ and $\|\eta_\alpha(x) - x\|_2 \rightarrow 0$ when $\alpha \rightarrow \infty$. Since $\|\partial_j^{(q)*}(\eta_\alpha(a) \otimes \eta_\alpha(b))\|_2 \leq 4\|a\| \|b\| / \sqrt{1-|q|}$ we have weak convergence in L^2 up to extraction and as $\eta_\alpha(a) \otimes \eta_\alpha(b) \rightarrow a \otimes b \in L^2(M \otimes M)$, we get $a \otimes b$ in the domain of the closed operator $\partial_j^{(q)*}$ with the formula and inequality above remaining true. This concludes. \square

Note that assuming $\nu(q, N) < 1$, one thus deduces $\mathcal{E}_q^{-1} \in D(\partial_j^{(q)*})$ with the formula:

$$\partial_j^{(q)*}(\mathcal{E}_q^{-1}) = \mathcal{E}_q^{-1} \# X_j - m(r(h_j)^* \otimes 1 + 1 \otimes l(h_j)^*)(\mathcal{E}_q^{-1}).$$

Since $(\mathcal{E}_q)^* = \mathcal{E}_q \in M \overline{\otimes} M^{\text{op}}$, we have thus shown our first result about finite Fisher information in this case.

First recall $\{h_i\}_{i=1}^N \subset \mathbb{R}^N$ is an orthonormal basis. We write for $\underline{i} = (i_1, \dots, i_n) \in N^n$ $\psi_{\underline{i}} = \psi(h_{i_1} \otimes \dots \otimes h_{i_n})$. We define the length $|\underline{i}| = n$.

We now want to prove finite Fisher information under the less restrictive condition $\rho(q, N) < 1, |q|N < 1$. We need to show $\mathcal{E}_q^{-1} \in D(\partial_i^{(q)*})$ and we only know from Lemma 33: $\mathcal{E}_q \in M \hat{\otimes} M^{\text{op}}, \partial_k \otimes 1 \mathcal{E}_q \in (M \overline{\otimes} M^{\text{op}}) \hat{\otimes} M, (1 \otimes \partial_k) \mathcal{E}_q \in M^{\text{op}} \hat{\otimes} (M \overline{\otimes} M^{\text{op}}), (1 \otimes \partial_i^{(q)}) \mathcal{E}_q \in M \overline{\otimes} M \overline{\otimes} M^{\text{op}}, (\partial_i^{(q)} \otimes 1) \mathcal{E}_q \in M \overline{\otimes} M^{\text{op}} \overline{\otimes} M^{\text{op}}, \mathcal{E}_q^{-1} \in M \overline{\otimes} M^{\text{op}}, (\partial_j \otimes \partial_i^{(q)}) (\mathcal{E}_q) \in (M \overline{\otimes} M^{\text{op}}) \hat{\otimes} (M^{\text{op}} \overline{\otimes} M), (\partial_i^{(q)} \otimes \partial_j) (\mathcal{E}_q) \in (M^{\text{op}} \overline{\otimes} M) \hat{\otimes} (M \overline{\otimes} M^{\text{op}}), ((\partial_j \otimes 1 \otimes 1 \partial_i^{(q)}) \otimes 1) (\mathcal{E}_q) \in (M \overline{\otimes} M^{\text{op}} \overline{\otimes} M) \hat{\otimes} M^{\text{op}}, (1 \otimes ((1 \otimes 1 \otimes \partial_j) \partial_i^{(q)})) (\mathcal{E}_q) \in M \hat{\otimes} (M^{\text{op}} \overline{\otimes} M \overline{\otimes} M^{\text{op}})$ (the norms of those quantities bellow are always taken in those spaces if not otherwise specified).

Let us call $U_n = \sum_{i=0}^n (-1)^i (\mathcal{E}_q - 1 \otimes 1)^i$ (power in $M \hat{\otimes} M^{\text{op}}$) so that we know $U_n \rightarrow \mathcal{E}_q^{-1}$ in $L^2, U_n \in M \hat{\otimes} M^{\text{op}}$ and by our first claim $U_n \in D(\partial_i^{(q)*})$. Since $\partial_i^{(q)*}$ is closed it suffices to show $\partial_i^{(q)*}(U_n)$ bounded in L^2 to get a weak limit up to extraction and $\mathcal{E}_q^{-1} \in D(\partial_i^{(q)*})$ and to get also $\mathcal{E}_q^{-1} \in D(\partial_j \partial_i^{(q)*})$, it suffices to bound $\partial_j \partial_i^{(q)*}(U_n)$ (since such a bound gives also a bound on $\|\partial_i^{(q)*}(U_n)\|_2^2 = \langle \partial_i^{(q)} \partial_i^{(q)*}(U_n), U_n \rangle$, we only sketch the proof of both at once).

This is mainly a computation using U_n is almost an inverse and thus will behave almost as inverse when computing derivatives coming from application of ∂ . The second key point will be that, apart from a bunch of terms we can gather in something of the form $\partial_i^{(q)*}(U_n)$, the ∂_j will enable us to use only a bound on terms coming from U_n in von Neumann norm. Recall notation $\#_i$ was introduced before Corollary 29. We get (after using our formula for $\partial_i^{(q)*}$, we mainly use derivation property of ∂_j and changes of summation):

$$\begin{aligned} \partial_j \partial_i^{(q)*}(U_n) &= \partial_j(U_n \# X_i - m \circ (1 \otimes \tau \otimes 1)(\partial_i^{(q)} \otimes 1(U_n) + 1 \otimes \partial_i^{(q)}(U_n))), \\ \partial_j(U_n \# X_i) &= 1_{i=j} U_n + \sum_{i=1}^n (-1)^i \sum_{k=0}^{i-1} (\mathcal{E}_q - 1 \otimes 1)^k \# (\partial_j \otimes 1(\mathcal{E}_q) \#_2 ((\mathcal{E}_q - 1 \otimes 1)^{i-k-1} \# X_i)) \\ &\quad + \sum_{i=1}^n (-1)^i \sum_{k=0}^{i-1} (\mathcal{E}_q - 1 \otimes 1)^k \# (1 \otimes \partial_j(\mathcal{E}_q) \#_1 ((\mathcal{E}_q - 1 \otimes 1)^{i-k-1} \# X_i)) \\ &= 1_{i=j} U_n - \sum_{k=0}^{n-1} (-1)^k (\mathcal{E}_q - 1 \otimes 1)^k \# (\partial_j \otimes 1(\mathcal{E}_q) \#_2 (U_{n-k-1} \# X_i)) \\ &\quad + 1 \otimes \partial_j(\mathcal{E}_q) \#_1 (U_{n-k-1} \# X_i), \\ \partial_j(m \circ (1 \otimes \tau \otimes 1)(\partial_i^{(q)} \otimes 1(U_n))) &= -\partial_j m \circ 1 \otimes \tau \otimes 1 \left[\sum_{k=0}^{n-1} (-1)^k (\mathcal{E}_q - 1 \otimes 1)^k \# (\partial_i^{(q)} \otimes 1(\mathcal{E}_q) \#_2 (U_{n-k-1})) \right] \\ &= -\sum_{k=0}^{n-1} (-1)^k \times \{ (1 \otimes (m \circ 1 \otimes \tau \otimes 1)) [(\partial_j \otimes 1(\mathcal{E}_q - 1 \otimes 1)^k) \#_2 (\partial_i^{(q)} \otimes 1(\mathcal{E}_q) \#_2 (U_{n-k-1}))] \\ &\quad + ((m \circ 1 \otimes \tau \otimes 1) \otimes 1) [(1 \otimes \partial_j(\mathcal{E}_q - 1 \otimes 1)^k) \#_1 (\partial_i^{(q)} \otimes 1(\mathcal{E}_q) \#_2 (U_{n-k-1}))] \\ &\quad + ((m \circ 1 \otimes \tau \otimes 1) \otimes 1) (\mathcal{E}_q - 1 \otimes 1)^k \# (\partial_i^{(q)} \otimes 1(\mathcal{E}_q) \#_2 (1 \otimes \partial_j(U_{n-k-1}))) \\ &\quad + (\mathcal{E}_q - 1 \otimes 1)^k \# [(1 \otimes (m \circ 1 \otimes \tau \otimes 1)) (\partial_j \otimes 1 \otimes 1 \partial_i^{(q)} \otimes 1(\mathcal{E}_q) \#_3 U_{n-k-1})] \\ &\quad + (\mathcal{E}_q - 1 \otimes 1)^k \# [(m \circ 1 \otimes \tau \otimes 1) \otimes 1) (\partial_i^{(q)} \otimes \partial_j(\mathcal{E}_q) \#_2 U_{n-k-1}) \}. \end{aligned}$$

Preparing for the reintroduction of $\partial_i^{(q)*}(U_{n-k-1})$ we rewrite (a part of) the first line in our last right hand side:

$$\begin{aligned} & \sum_{k=0}^{n-1} (-1)^k [(\partial_j \otimes 1(\mathcal{E}_q - 1 \otimes 1)^k) \#_2 (\partial_i^{(q)} \otimes 1(\mathcal{E}_q) \#_2 (U_{n-k-1}))] \\ &= \sum_{k=0}^{n-1} (-1)^k \times \left[\sum_{l=0}^{k-1} (\mathcal{E}_q - 1 \otimes 1)^l \# (\partial_j \otimes 1(\mathcal{E}_q)) \#_2 (\mathcal{E}_q - 1 \otimes 1)^{k-l-1} \right] \#_2 (\partial_i^{(q)} \otimes 1(\mathcal{E}_q) \#_2 (U_{n-k-1})) \\ &= \sum_{l=0}^{n-2} (-1)^l (\mathcal{E}_q - 1 \otimes 1)^l \# (\partial_j \otimes 1(\mathcal{E}_q)) \\ & \quad \#_2 \left[\sum_{k=l+1}^{n-1} (-1)^{k-l} (\mathcal{E}_q - 1 \otimes 1)^{k-l-1} \# (\partial_i^{(q)} \otimes 1(\mathcal{E}_q) \#_2 (U_{n-k-1})) \right] \\ &= \sum_{l=1}^{n-1} (-1)^l (\mathcal{E}_q - 1 \otimes 1)^l \# (\partial_j \otimes 1(\mathcal{E}_q)) \#_2 (\partial_i^{(q)} \otimes 1(U_{n-l-1})) \end{aligned}$$

(in the last line, note that the term with $l = n - 1$ is zero since $\partial_i^{(q)} \otimes 1(U_0) = 0$).

We will now write $\tilde{\tau} = m \circ 1 \otimes \tau \circ 1$. Putting everything together and reintroducing in the last line $\partial_i^{(q)*}(U_{n-k-1})$ when useful in the right hand side:

$$\begin{aligned} & \partial_j \partial_i^{(q)*}(U_n) \\ &= 1_{i=j} U_n + \sum_{k=0}^{n-1} (-1)^k \{ (\mathcal{E}_q - 1 \otimes 1)^k \# (\tilde{\tau} \otimes 1) (\partial_i^{(q)} \otimes 1(\mathcal{E}_q) \#_2 (1 \otimes \partial_j(U_{n-k-1}))) \\ & \quad + (\mathcal{E}_q - 1 \otimes 1)^k \# (1 \otimes \tilde{\tau}) (1 \otimes \partial_i^{(q)}(\mathcal{E}_q) \#_1 (\partial_j \otimes 1(U_{n-k-1}))) \\ & \quad + (\mathcal{E}_q - 1 \otimes 1)^k \# (1 \otimes \tilde{\tau}) (\partial_j \otimes 1 \otimes 1 \partial_i^{(q)} \otimes 1(\mathcal{E}_q) \#_3 U_{n-k-1} + \partial_j \otimes \partial_i^{(q)}(\mathcal{E}_q) \#_2 U_{n-k-1}) \\ & \quad + (\mathcal{E}_q - 1 \otimes 1)^k \# (\tilde{\tau} \otimes 1) (1 \otimes 1 \otimes \partial_j 1 \otimes \partial_i^{(q)}(\mathcal{E}_q) \#_1 U_{n-k-1} + \partial_i^{(q)} \otimes \partial_j(\mathcal{E}_q) \#_2 U_{n-k-1}) \\ & \quad - (\mathcal{E}_q - 1 \otimes 1)^k \# (\partial_j \otimes 1(\mathcal{E}_q) \#_2 \partial_i^{(q)*}(U_{n-k-1}) + 1 \otimes \partial_j(\mathcal{E}_q) \#_1 \partial_i^{(q)*}(U_{n-k-1})) \}. \end{aligned}$$

We can now deduce from this a bound for $p \in [2, \infty]$ in $L^p(M \otimes M^{\text{op}})$ if we know a bound on $\|\partial_i^{(q)*}(U_k)\|_p$. Under the assumption $|q|N < 1$ we know this is finite for $p = 2$, we will use it later in the case $p = \infty$ under a stronger assumption (the second line bellow corresponds to the last line of our last equation, the first and third to the first and second, the fourth and fifth to the third and fourth).

$$\begin{aligned} & \|\partial_j \partial_i^{(q)*}(U_n)\|_p \\ & \leq 1_{i=j} \|U_n\|_p \\ & \quad + \left(\sup_{k \leq n-1} \|\partial_i^{(q)*}(U_k)\|_p \right) (\|\partial_j \otimes 1(\mathcal{E}_q)\| + \|1 \otimes \partial_j(\mathcal{E}_q)\|) \sum_{k=0}^{n-1} \|\mathcal{E}_q - 1 \otimes 1\|_{M \otimes M^{\text{op}}}^k \\ & \quad + (\|1 \otimes \partial_i^{(q)}(\mathcal{E}_q)\| \|\partial_j \otimes 1(\mathcal{E}_q)\| + \|\partial_i^{(q)} \otimes 1(\mathcal{E}_q)\| \|1 \otimes \partial_j(\mathcal{E}_q)\|) \\ & \quad \times \sum_{k=2}^n \frac{k(k-1)}{2} \|(\mathcal{E}_q - 1 \otimes 1)\|_{M \otimes M^{\text{op}}}^{k-2} \end{aligned}$$

$$\begin{aligned}
 & + (\|\partial_j \otimes \partial_i^{(q)}(\mathcal{E}_q)\| + \|\partial_j \otimes 1 \otimes 1 \partial_i^{(q)} \otimes 1(\mathcal{E}_q)\|) \sum_{k=0}^{n-1} (k+1) \|(\mathcal{E}_q - 1 \otimes 1)\|_{M \overline{\otimes} M^{\text{op}}}^k \\
 & + (\|\partial_i^{(q)} \otimes \partial_j(\mathcal{E}_q)\| + \|1 \otimes 1 \otimes \partial_j 1 \otimes \partial_i^{(q)}(\mathcal{E}_q)\|) \sum_{k=0}^{n-1} (k+1) \|(\mathcal{E}_q - 1 \otimes 1)\|_{M \overline{\otimes} M^{\text{op}}}^k.
 \end{aligned}$$

Since $\|(\mathcal{E}_q - 1 \otimes 1)\|_{M \overline{\otimes} M^{\text{op}}} < 1$ all the sums of the right hand side extended to infinity converge so that we get constants C, D $\|\partial_j \partial_i^{(q)*}(U_n)\|_2 \leq C + D(\sup_{k \leq n-1} \|\partial_i^{(q)*}(U_k)\|_2)$ and thus $\|\partial_i^{(q)*}(U_n)\|_2^2 \leq \|\mathcal{E}_q\| \|U_n\|_2 (C + D(\sup_{k \leq n-1} \|\partial_i^{(q)*}(U_k)\|_2))$, and a standard bound concludes to finiteness of $\sup_k \|\partial_i^{(q)*}(U_k)\|_2$.

Step 2: Bounded conjugate variable under $\nu(q, N) < 1$.

From the previous step, we know:

$$\begin{aligned}
 \partial_i^{(q)*}(U_n) &= U_n \# X_i + \sum_{k=0}^{n-1} (-1)^k (\mathcal{E}_q - 1 \otimes 1)^k \\
 &\quad \# m \circ (1 \otimes \tau \otimes 1) (\partial_i^{(q)} \otimes 1(\mathcal{E}_q) \#_2(U_{n-k-1}) + 1 \otimes \partial_i^{(q)}(\mathcal{E}_q) \#_1(U_{n-k-1})).
 \end{aligned}$$

And thus,

$$\begin{aligned}
 \|\partial_i^{(q)*}(U_n)\| &\leq \|U_n\|_{M \hat{\otimes} M^{\text{op}}} \|X_i\| + \sum_{k=0}^{n-1} (k+1) \|\mathcal{E}_q - 1 \otimes 1\|_{M \hat{\otimes} M^{\text{op}}}^k \\
 &\quad \times (\|\partial_i^{(q)} \otimes 1(\mathcal{E}_q)\|_{(M \overline{\otimes} M^{\text{op}}) \hat{\otimes} M^{\text{op}}} + \|1 \otimes \partial_i^{(q)}(\mathcal{E}_q)\|_{M \hat{\otimes} (M \overline{\otimes} M^{\text{op}})}) \\
 &\leq \|X_i\| \sum_{k=0}^{\infty} \|\mathcal{E}_q - 1 \otimes 1\|_{M \hat{\otimes} M^{\text{op}}}^k + \sum_{k=0}^{\infty} (k+1) \|\mathcal{E}_q - 1 \otimes 1\|_{M \hat{\otimes} M^{\text{op}}}^k \\
 &\quad \times (\|\partial_i^{(q)} \otimes 1(\mathcal{E}_q)\|_{(M \overline{\otimes} M^{\text{op}}) \hat{\otimes} M^{\text{op}}} + \|1 \otimes \partial_i^{(q)}(\mathcal{E}_q)\|_{M \hat{\otimes} (M \overline{\otimes} M^{\text{op}})}).
 \end{aligned}$$

The last inequality gives a finite bound for $\nu(q, N) < 1$ as, then, by Corollary 29, we have $\|\mathcal{E}_q - 1 \otimes 1\|_{M \hat{\otimes} M^{\text{op}}} < 1$. Since we showed in step 1 $\partial_i^{(q)*}(U_n) \rightarrow \partial_i^{(q)*}(\mathcal{E}_q^{-1})$ weakly in L^2 up to extraction, this means we have ultraweak convergence of the same extraction. Thus especially $\partial_i^{(q)*}(\mathcal{E}_q^{-1}) \in M$.

Step 3: Lipschitz conjugate variable under $\nu(q, N) < 1$.

Since we now know $\sup_k \|\partial_i^{(q)*}(U_k)\|_M < \infty$ from the second step, the end of the first step gives:

$$\|\partial_j \partial_i^{(q)*}(U_n)\|_{M \overline{\otimes} M^{\text{op}}} \leq C + D \left(\sup_k \|\partial_i^{(q)*}(U_k)\|_M \right).$$

Again since we saw in step one: $\partial_j \partial_i^{(q)*}(U_n) \rightarrow \partial_j \partial_i^{(q)*}(\mathcal{E}_q^{-1})$ weakly in L^2 up to extraction, we got $\partial_j \partial_i^{(q)*}(\mathcal{E}_q^{-1}) \in M \overline{\otimes} M^{\text{op}}$.

Putting everything together, this concludes the proof of the second part of our theorem (the statement on microstate free entropy dimension uses the R^ω embeddability result of [40] and Corollary 25).

Step 4: Analytic conjugate variable coming from a potential under $\pi(q, N) < 1$.

Since by Corollary 29, we have $\|\mathcal{E}_q - 1 \otimes 1\|_R < 1$, we have a non-commutative power series \mathcal{E}_q^{-1} . If we define

$$\xi_i(Y_1, \dots, Y_N) = \mathcal{E}_q^{-1}(Y_1, \dots, Y_N) \# Y_i - m \circ ((1 \otimes \tau) \partial_j^{(q)\text{an}} \otimes 1 + 1 \otimes (\tau \otimes 1) \partial_j^{(q)\text{an}})(\mathcal{E}_q^{-1}(Y_1, \dots, Y_N)),$$

where $\partial_j^{(q)\text{an}}(P(Y)) = \partial_j P(Y) \# \mathcal{E}_q(Y_1, \dots, Y_N)$ is the analytic version of $\partial_j^{(q)}$. This is now obviously a power series with radius of convergence R (τ here is the tracial state of q -Gaussians), we have by the claim in step 1 $\xi_i(X_1, \dots, X_N)$ is the conjugate variable of q -Gaussian variables X_1, \dots, X_N .

Let us define

$$V(Y_1, \dots, Y_N) = N^{-1} \left(\frac{1}{2} \sum_{i=1}^N \xi_i(Y_1, \dots, Y_N) Y_i + Y_i \xi_i(Y_1, \dots, Y_N) \right),$$

where N is the operator defined on non-commutative power series having each monomial of degree n as eigenvector of eigenvalue n . Obviously, V is a selfadjoint potential since Y_i and $\xi_i(Y)$ are self-adjoint. We have to check in the spirit of [46] that $D_i V = \xi_i$. Of course this is equivalent to $D_i(NV) = (1+N)(D_i V) = (1+N)(\xi_i) = \xi_i + \sum_{j=1}^N \partial_j(\xi_i) \# Y_j$. In order to prove this, using Lemma 37 bellow, it suffices to show we have $D_i(NV)(X_1, \dots, X_N) = \xi_i(X_1, \dots, X_N) + \sum_{j=1}^N \partial_j(\xi_i(X_1, \dots, X_N)) \# X_j$.

But by Corollary 5.12 in [44], we have $NV(X_1, \dots, X_N) = \sum_{i=1}^N \xi_i(X_1, \dots, X_N) X_i$.

Note that the computation at the end of step 1, we know e.g. $\partial_i \xi_i(X_1, \dots, X_N) \in L^2(M) \hat{\otimes} L^2(M)$. Note that $D_i \xi_i(X_1, \dots, X_N) = m \circ \text{flip}(\partial_i \xi_i(X_1, \dots, X_N))$ is then defined in L^1 with $\text{flip}(a \otimes b) = b \otimes a$.

Applying cyclic gradients and using the relation

$$D_i(PQ) = \text{flip}(\partial_i(P)) \# Q + \text{flip}(\partial_i(Q)) \# P,$$

we thus deduce:

$$D_i \sum_{j=1}^N \xi_j(X_1, \dots, X_N) X_j = \sum_{j=1}^N \text{flip}(\partial_i(\xi_j(X_1, \dots, X_N))) \# X_j + \xi_i(X_1, \dots, X_N).$$

To conclude we just have to recall $\text{flip}(\partial_i(\xi_j(X_1, \dots, X_N))) = \partial_j(\xi_i(X_1, \dots, X_N))$, a priori in $L^2(M \otimes M)$ thus also in the subspace $L^2(M) \hat{\otimes} L^2(M)$. This follows by a duality argument in Lemma 36. \square

Lemma 36. *If X_1, \dots, X_N have conjugate variables $\xi_1, \dots, \xi_N \in L^2$, then for any $a, b \in D(\partial_j) \cap M$:*

$$\langle \xi_i, \partial_j^*(a \otimes b) \rangle = \langle \xi_j, \partial_i^*(b \otimes a) \rangle.$$

Proof. The result is shown, by density, for a, b non-commutative polynomials in X_1, \dots, X_N , using Lemma 4 and coassociativity of the free difference quotient:

$$\begin{aligned} & \langle \xi_i, \partial_j^*(a \otimes b) \rangle \\ &= \tau(\xi_i^* a \xi_j b) - \tau(\xi_i^* [(1 \otimes \tau \partial_j)(a)b + a(\tau \otimes 1 \partial_j)(b)]) \\ &= \langle \xi_j, \partial_i^*(b \otimes a) \rangle + \tau(\xi_j^* [(1 \otimes \tau \partial_i)(b)a + b(\tau \otimes 1 \partial_i)(a)]) \\ &\quad - \tau \otimes \tau((\partial_i \otimes \tau)(\partial_j)(a)b) - \tau \otimes \tau(a((\tau \otimes \partial_i) \partial_j)(b)) \\ &\quad - \tau \otimes \tau((1 \otimes \tau \partial_j)(a) \partial_i(b)) - \tau \otimes \tau(\partial_i(a)((\tau \otimes 1) \partial_j)(b)) \\ &= \langle \xi_j, \partial_i^*(b \otimes a) \rangle + \tau \otimes \tau([(\partial_j \otimes \tau \partial_i)(b)a + b((\tau \otimes \partial_j) \partial_i)(a)]) \\ &\quad + \tau \otimes \tau([(1 \otimes \tau \partial_i)(b) \partial_j(a) + \partial_j(b)((\tau \otimes 1) \partial_i)(a)]) \\ &\quad - \tau((\tau \otimes ((1 \otimes \tau) \partial_j))(\partial_i)(a)b) - \tau(a((\tau \otimes 1) \partial_j) \otimes \tau) \partial_i(b) \\ &\quad - \tau((1 \otimes \tau) \partial_j(a)(1 \otimes \tau) \partial_i(b)) - \tau((\tau \otimes 1) \partial_i(a)((\tau \otimes 1) \partial_j)(b)) \\ &= \langle \xi_j, \partial_i^*(b \otimes a) \rangle. \end{aligned} \quad \square$$

Lemma 37. *Assume $\Phi^*(X_1, \dots, X_n) < \infty$ then there is no non-zero non-commutative power series $P(X_1, \dots, X_n)$ of radius of convergence $R > \|X_i\|$ such that $P(X_1, \dots, X_n) = 0$.*

Proof. Since $\Phi^*(X_1, \dots, X_n) < \infty$, the free difference quotient is closable. As a consequence, taking a sequence of polynomials $P_n \rightarrow P$ in analytic norm, we have $P_n(X_1, \dots, X_n) \rightarrow 0$ in L^2 norm. $\partial_i P(X_1, \dots, X_n)$ converges to

$\partial_i P(X_1, \dots, X_n)$ (since $\|\partial_i P_n - \partial_i P\|_S \rightarrow 0$ for any $S < R$), thus by closability $\partial_i P(X_1, \dots, X_n) = 0$. We also get vanishing of any higher order derivatives by induction.

Taking successive non-commutative derivatives and multiplying, if we assume for contradiction $P \neq 0$ one can assume $P(0, \dots, 0) \neq 0$. Now for any non-commutative polynomial Q , one has

$$Q(X_1 + Y_1, \dots, X_n + Y_n) = \sum_{k=0}^{\infty} \sum_{i_1, \dots, i_k \in [1, n]} [(\partial_{i_1} \otimes 1^{\otimes k-1}) \circ \dots \circ \partial_{i_k} (Q)](X_1, \dots, X_n) \#(Y_{i_1}, \dots, Y_{i_k}), \tag{17}$$

where $(a_0 \otimes \dots \otimes a_k) \#(Y_{i_1}, \dots, Y_{i_k}) = a_0 Y_{i_1} a_1 \dots Y_{i_k} a_k$ and the sum over k is finite here.

Let $T = \max_i (\|X_i\|) < R$. Consider $c_n(P)$ the sum of absolute values of coefficients of degree n of P . Then $\phi_P(x) = \sum_{n=0}^{\infty} c_n(P) x^n$ is a commutative power series of radius of convergence at least R and $\sum_{i_1, \dots, i_k \in [1, n]} \|[(\partial_{i_1} \otimes 1^{\otimes k-1}) \circ \dots \circ \partial_{i_k} (P)]\|_T \leq \frac{1}{k!} \phi_P^{(k)}(T)$.

Since ϕ_P is analytic in the ball of center 0 and radius R , it admits a Taylor power series expansion around T and as a consequence, the right hand side of (17) makes sense if $\|Y_i\| < R - T$. As a consequence approximating P by polynomials, one gets (17) for P and such Y_i 's. Applying this for $Y_i = (t - 1)X_i$ one gets $P(tX_1, \dots, tX_n) = 0$ for t close to 1 and then after iterating for $t \in [0, 1]$, this contradicts $P(0, \dots, 0) \neq 0$. \square

4.4. Group cocycles

Since Assumption 1 is hard to verify in practice, it is interesting to work only under assumption 0, and prove directly that the ultramild solution of Theorem 11(i) satisfy $\|X_t\|_2 = \|X_0\|_2$ a.e. to get a stationary solution. In this part, we find a necessary and sufficient condition for derivations coming from group cocycles to get results in the spirit of Corollary 3 in [39].

Let Γ be a discrete group. To a(n additive left) cocycle c with value in the regular representation $c \in C^1(\Gamma, \ell^2(\Gamma))$ we associate a derivation $\delta_c : \mathbb{C}\Gamma \rightarrow \ell^2(\Gamma) \otimes \ell^2(\Gamma) = L^2(M_0 \otimes M_0)$ (M_0 the group von Neumann algebra of Γ) given by $\delta_c(\gamma) = B(c(\gamma))\gamma$ where $B : \ell^2(\Gamma) \rightarrow \ell^2(\Gamma) \otimes \ell^2(\Gamma)$ the isometric map given by $B(\gamma) = \gamma \otimes \gamma^{-1}$. Indeed, $\delta(\gamma_1 \gamma_2) = B(\gamma_1 c(\gamma_2) + c(\gamma_1))\gamma_1 \gamma_2 = \gamma_1 B(c(\gamma_2))\gamma_1^{-1} \gamma_1 \gamma_2 + \delta_c(\gamma_1)\gamma_2 = \gamma_1 \delta_c(\gamma_2) + \delta_c(\gamma_1)\gamma_2$ so that δ_c is a derivation with the same bimodule structure used earlier on $L^2(M_0 \otimes M_0)$. Moreover δ_c is easily seen to be a real derivation if c takes values in $i\mathbb{R}$ (we will consider only such cocycles). Let us note that $\langle \delta_c(\gamma), 1 \otimes 1 \rangle = 0$ for any γ so that we easily deduce that $\delta_c^*(1 \otimes 1) = 0$ so that δ_c is always closable. Any $\delta_{c_1}, \dots, \delta_{c_n}$ therefore satisfy assumption 0. Moreover, as noted e.g. in the proof of Corollary 19 in [39], $\langle \delta_c \gamma, \delta_c \gamma' \rangle = \delta_{\gamma'}' \|c(\gamma)\|_2^2$ so that $\delta_c^* \delta_c(\gamma) = \|c(\gamma)\|_2^2 \gamma$. We now fix c_1, \dots, c_n such cocycles and write δ_i the extension to M of δ_{c_i} described at the beginning of Section 2. We write X_t, X_t^ε the ultramild (resp mild) solution given by Theorem 11 when the initial condition is X_0 .

We now want to describe a first equivalent formulation of the isometry $\|X_t\|_2 = \|X_0\|_2$. To this end, we want to give an equation on certain components of the free product $L^2(M)$. Let us call N the von Neumann algebra generated by free Brownian motions, it is well known that M is the orthogonal direct sum of $L^2(N)$ and $L^2(N)\gamma_1(L^2(N) \ominus \mathbb{C})\gamma_2 \dots (L^2(N) \ominus \mathbb{C})\gamma_n L^2(N)$ where γ_i 's run over $\Gamma - \{1\}$. Since X_t and X_t^ε are orthogonal to $L^2(N)$ (since $\delta_i^* 1 \otimes 1 = 0$) we may consider only $X_{t; \gamma_1, \dots, \gamma_n}^\varepsilon \in L^2(N) \otimes (L^2(N) \ominus \mathbb{C})^{n-1} \otimes L^2(N)$ such that $X_{t; \gamma_1, \dots, \gamma_n}^\varepsilon \#(\gamma_1 \otimes \dots \otimes \gamma_n)$ are the orthogonal projections on those spaces. We wrote here $U\#(\gamma_1 \otimes \dots \otimes \gamma_n)$ the extension given by freeness of $(a_1 \otimes \dots \otimes a_{n+1})\#(\gamma_1 \otimes \dots \otimes \gamma_n) = a_1 \gamma_1 a_2 \dots a_n \gamma_n a_{n+1}$. We will also write $(a_1 \otimes \dots \otimes a_i \otimes a_{i+1} \dots \otimes a_n)\#_i(1 \otimes (S_t - S_s) \otimes 1) = a_1 \otimes \dots \otimes a_i \otimes (S_t - S_s) \otimes a_{i+1} \dots \otimes a_n$, $(a_1 \otimes \dots \otimes a_i \otimes a_{i+1} \dots \otimes a_n)\#_i((S_t - S_s) \otimes 1) = a_1 \otimes \dots \otimes a_i (S_t - S_s) \otimes a_{i+1} \dots \otimes a_n$, $(a_1 \otimes \dots \otimes a_i \otimes a_{i+1} \dots \otimes a_n)\#_i(1 \otimes (S_t - S_s)) = a_1 \otimes \dots \otimes a_i \otimes (S_t - S_s) a_{i+1} \dots \otimes a_n$ and the obvious corresponding adapted stochastic integrals. We now have the following:

Proposition 38. *Assume $X_0 = \gamma$, then:*

$$X_{t; \gamma_1, \dots, \gamma_n}^\varepsilon = \delta_{n=1} \delta_{\gamma_1 = \gamma} e^{-(t/2)(\sum_{j=1}^n \|c_j(\gamma)\|_2^2)} 1 \otimes 1 + (1 - \varepsilon) \sum_{i=1}^n \sum_{j=1}^n \int_0^t e^{((s-t)/2)(\sum_{i=1}^n \sum_{j=1}^n \|c_j(\gamma)\|_2^2)} X_{s; \gamma_1, \dots, \gamma_n}^\varepsilon$$

$$\begin{aligned} & \#_i(\langle \gamma_i, c_j(\gamma_i) \rangle \mathbf{1} \otimes dS_s^{(j)} + \langle \mathbf{1}, c_j(\gamma_i) \rangle dS_s^{(j)} \otimes \mathbf{1}) \\ & + (1 - \varepsilon) \delta_{n \neq 1} \sum_{i=1}^{n-1} \sum_{j=1}^N \langle \gamma_i, c_j(\gamma_i \gamma_{i+1}) \rangle \\ & \times \int_0^t e^{((s-t)/2)(\sum_{i=1}^n \sum_{j=1}^N \|c_j(\gamma_i)\|_2^2)} X_{s; \gamma_1, \dots, \gamma_i \gamma_{i+1}, \dots, \gamma_n}^\varepsilon \#_i \mathbf{1} \otimes dS_s^{(j)} \otimes \mathbf{1}, \end{aligned}$$

which is non-zero only if $\gamma_1 \cdots \gamma_n = \gamma$. Moreover this relation with $\varepsilon = 0$ is thus also valid for X_t (by the weak convergence defining it).

As a consequence, using freeness and the definition of the space where $X_{t; \gamma_1, \dots, \gamma_n}^\varepsilon$ lives (especially the orthogonal complements to \mathbb{C}) we get:

$$\begin{aligned} \|X_{t; \gamma_1, \dots, \gamma_n}\|_2^2 &= \delta_{n=1} \delta_{\gamma_1 = \gamma} e^{-t(\sum_{j=1}^N \|c_j(\gamma)\|_2^2)} \\ &+ \sum_{i=1}^n \sum_{j=1}^N (|\langle \gamma_i, c_j(\gamma_i) \rangle|^2 + |\langle \mathbf{1}, c_j(\gamma_i) \rangle|^2) \int_0^t ds e^{(s-t)(\sum_{i=1}^n \sum_{j=1}^N \|c_j(\gamma_i)\|_2^2)} \|X_{s; \gamma_1, \dots, \gamma_n}\|_2^2 \\ &+ \delta_{n \neq 1} \sum_{i=1}^{n-1} \sum_{j=1}^N |\langle \gamma_i, c_j(\gamma_i \gamma_{i+1}) \rangle|^2 \int_0^t ds e^{(s-t)(\sum_{i=1}^n \sum_{j=1}^N \|c_j(\gamma_i)\|_2^2)} \|X_{s; \gamma_1, \dots, \gamma_i \gamma_{i+1}, \dots, \gamma_n}\|_2^2. \end{aligned}$$

As a consequence, solving the equation by variation of constants, and using the following convenient notation $\|\hat{c}_j(\gamma_i)\|_2^2 = \|c_j(\gamma_i)\|_2^2 - (|\langle \gamma_i, c_j(\gamma_i) \rangle|^2 + |\langle \mathbf{1}, c_j(\gamma_i) \rangle|^2)$, we obtain the following:

Proposition 39. *Assume $X_0 = \gamma$, then*

$$\begin{aligned} \|X_{t; \gamma_1, \dots, \gamma_n}\|_2^2 &= \delta_{n=1} \delta_{\gamma_1 = \gamma} e^{-t(\sum_{j=1}^N \|\hat{c}_j(\gamma)\|_2^2)} \\ &+ \delta_{n \neq 1} \sum_{i=1}^{n-1} \sum_{j=1}^N |\langle \gamma_i, c_j(\gamma_i \gamma_{i+1}) \rangle|^2 \int_0^t ds e^{(s-t)(\sum_{i=1}^n \sum_{j=1}^N \|\hat{c}_j(\gamma_i)\|_2^2)} \|X_{s; \gamma_1, \dots, \gamma_i \gamma_{i+1}, \dots, \gamma_n}\|_2^2. \end{aligned}$$

This equation is nothing but a forward Kolmogorov equation, and the question we ask is whether $1 = \|\gamma\|_2^2 = \|X_0\|_2^2 = \sum_n \sum_{\gamma_1, \dots, \gamma_n} \|X_{t; \gamma_1, \dots, \gamma_n}\|_2^2$, i.e. nothing but if the solution of the Kolmogorov equation is conservative. In order to state a result, let us define a corresponding continuous time Markov chain to give a probabilistic counterpart to the stationarity of X_t , using usual results on Kolmogorov equations (cf. e.g. [24]).

Notation 40. *Given a countable group Γ and additive left cocycles with value in the left regular representation c_1, \dots, c_N as above. We write $M(\Gamma; c_1, \dots, c_N)$ the continuous time Markov process defined on the countable state space of finite non-trivial sequences valued in Γ : $F(\Gamma) = (\Gamma - \{1\})^{(<\omega)}$ defined by the following rates $R((\gamma_1, \dots, \gamma_n)) = \sum_{i=1}^n \sum_{j=1}^N \|\hat{c}_j(\gamma_i)\|_2^2$, and with transition probabilities non-zero only from $(\gamma_1, \dots, \gamma_n)$ to $(\gamma_1, \dots, \delta_i, \delta'_i, \dots, \gamma_n)$ with $\delta_i \delta'_i = \gamma_i$ (of course $\delta_i, \delta'_i \neq 1$), given by*

$$P((\gamma_1, \dots, \gamma_n), (\gamma_1, \dots, \delta_i, \delta'_i, \dots, \gamma_n)) = \frac{\sum_{j=1}^N |\langle \delta_i, c_j(\gamma_i) \rangle|^2}{R((\gamma_1, \dots, \gamma_n))}.$$

We can now state the following trivial:

Corollary 41. *Let X_t be the ultramild solution given by Theorem 11 with $\delta = (\delta_1, \dots, \delta_N)$ associated as above to cocycles (c_1, \dots, c_N) . Then $\|X_t\|_2 = \|X_0\|_2$ (for any $X_0 \in \ell^2(\Gamma)$) for all $t \in [0, T)$ (and as a consequence is stationary in $[0, T)$ on $M_0 = L(\Gamma)$) if and only if $M(\Gamma; c_1, \dots, c_N)$ has almost surely no explosion before T .*

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References

- [1] S. Avsec. Strong solidity of the q -Gaussian algebras for all $-1 < q < 1$. Available at [arXiv:1110.4918](https://arxiv.org/abs/1110.4918).
- [2] P. Biane, M. Capitaine and A. Guionnet. Large deviation bounds for matrix Brownian motion. *Invent. Math.* **152** (2003) 433–459. [MR1975007](#)
- [3] P. Biane and R. Speicher. Stochastic calculus with respect to free Brownian motion. *Probab. Theory Related Fields* **112** (1998) 373–409. [MR1660906](#)
- [4] P. Biane and D. Voiculescu. A free probability analogue of the Wasserstein metric on the trace-state space. *Geom. Func. Anal.* **11** (2001) 1125–1138. [MR1878316](#)
- [5] M. Bożejko. Ultracontractivity and strong Sobolev inequality for q -Ornstein–Uhlenbeck semigroup ($-1 < q < 1$). *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **2** (1998) 203–220. [MR1811255](#)
- [6] M. Bożejko, B. Kummerer and R. Speicher. q -Gaussian processes: Non-commutative and classical aspects. *Commun. Math. Phys.* **185** (1997) 129–154. [MR1463036](#)
- [7] M. Bożejko and R. Speicher. An example of a generalized Brownian motion. *Commun. Math. Phys.* **137** (1991) 519–531. [MR1105428](#)
- [8] A. M. Cheboratev and F. Fagnola. Sufficient conditions for conservativity of minimal quantum dynamical semigroups. *J. Funct. Anal.* **153** (1998) 382–404. [MR1614586](#)
- [9] F. Cipriani and J.-L. Sauvageot. Derivations as square roots of Dirichlet forms. *J. Funct. Anal.* **201** (2003) 78–120. [MR1986156](#)
- [10] A. Connes and D. Shlyakhtenko. L^2 -homology for von Neumann algebras. *J. Reine Angew. Math.* **586** (2005) 125–168. [MR2180603](#)
- [11] Y. Dabrowski. A note about proving non- Γ under a finite non-microstates free Fisher information Assumption. *J. Funct. Anal.* **258** (2008) 3662–3674. [MR2606868](#)
- [12] Y. Dabrowski. A non-commutative path space approach to stationary free stochastic differential equations. Preprint, 2010, available at [arXiv:1006.4351](https://arxiv.org/abs/1006.4351).
- [13] G. Da Prato and J. Zabczyk. *Stochastic Equations in Infinite Dimensions*. Cambridge Univ. Press, Cambridge, 1992. [MR1207136](#)
- [14] E. B. Davis and J. M. Lindsay. Non-commutative symmetric Markov semigroups. *Math. Z.* **210** (1992) 379–411. [MR1171180](#)
- [15] C. Donati-Martin. Stochastic integration with respect to q Brownian motion. *Probab. Theory Related Fields* **125** (2003) 77–95. [MR1952458](#)
- [16] K. Dykema and A. Nica. On the Fock representation of the q -commutation relations. *J. Reine Angew. Math.* **440** (1993) 201–212. [MR1225964](#)
- [17] F. Fagnola. H-P quantum stochastic differential equations. In *Non-Commutativity, Infinite-Dimensionality, and Probability at the Crossroads. The Proceedings of the RIMS Workshop on Infinite Dimensional Analysis and Quantum Probability: Kyoto, Japan, 20–22 November, 2001* 51–96. N. Obata, T. Matsui and A. Hora (Eds). World Scientific, River Edge, NJ, 2002. [MR2059857](#)
- [18] F. Fagnola and S. J. Wills. Mild solutions of quantum stochastic differential equations. *Electron. Commun. Probab.* **5** (2000) 158–171. [MR1800118](#)
- [19] A. Guionnet and D. Shlyakhtenko. Free diffusions and matrix models with strictly convex interaction. *Geom. Func. Anal.* **18** (2007) 1875–1916. [MR2491694](#)
- [20] T. Kato. *Perturbation Theory for Linear Operators*, 2nd edition. Springer-Verlag, Berlin, 1980. [MR1335452](#)
- [21] T. Kotelenetz. *Stochastic Ordinary and Stochastic Partial Differential Equations: Transition from Microscopic to Macroscopic Equations*. Springer, Berlin, 2007. [MR2370567](#)
- [22] N. V. Krylov. An analytic approach to SPDEs. In *Stochastic Partial Differential Equations: Six Perspectives*. R. A. Carmona and B. L. Rozovskii (Eds). *Mathematical Surveys and Monographs* **64**. American Mathematical Society, Providence, 1999. [MR1661766](#)
- [23] N. V. Krylov and B. L. Rozovskii. Stochastic evolution equations (in Russian). *Current Problems in Mathematics* **14** (1979) 71–147. Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Informatsii, Moscow. [MR0570795](#)
- [24] T. M. Liggett. *Continuous Time Markov Processes: An Introduction*. American Mathematical Society, Providence, 2010. [MR2574430](#)
- [25] W. Lück. *L^2 -Invariants: Theory and Applications to Geometry and K-Theory*. Springer, Berlin, 2002. [MR1926649](#)
- [26] Z. M. Ma and M. Röckner. *Introduction to the Theory of (Non-Symmetric) Dirichlet Forms*. Universitext. Springer, Berlin, 1992. [MR1214375](#)
- [27] I. Mineyev and D. Shlyakhtenko. Non-microstate free entropy dimension for groups. *Geom. Func. Anal.* **15** (2005) 476–490. [MR2153907](#)
- [28] A. Nou. Non injectivity of the q -deformed von Neumann algebra. *Math. Ann.* **330** (2004) 17–38. [MR2091676](#)
- [29] F. Otto and C. Villani. Generalization of an inequality by Talagrand and links with the logarithmic Sobolev inequality. *J. Funct. Anal.* **173** (2000) 361–400. [MR1760620](#)
- [30] E. Pardoux. Sur des équations aux dérivés partielles stochastiques monotones. *C. R. Acad. Sci. Paris Sér. A-B* **275** (1972) A101–A103. [MR0312572](#)
- [31] E. Pardoux. Équations aux dérivées partielles stochastiques de type monotone. In *Séminaire sur les Équations aux Dérivées Partielles (1974–1975), III* Exp. No. 2 1–10. Collège de France, Paris, 1975. [MR0651582](#)
- [32] J. Peterson. A 1-cohomology characterisation of Property (T) in von Neumann algebras. *Pacific J. Math.* **243** (2009) 181–199. [MR2550142](#)
- [33] J. Peterson. L^2 -rigidity in von Neumann algebras. *Invent. Math.* **175** (2009) 417–433. [MR2470111](#)
- [34] C. Prévôt and M. Röckner. *A Concise Course on Stochastic Partial Differential Equations. Lecture Notes in Mathematics* **1905**. Springer, Berlin, 2007. [MR2329435](#)

- [35] B. L. Rozovskii. *Stochastic Evolution Systems*. Kluwer Academic, Dordrecht, 1990. [MR1135324](#)
- [36] J.-L. Sauvageot. Strong Feller semigroups on C^* -algebras. *J. Operator Theory* **42** (1999) 83–102. [MR1694793](#)
- [37] D. Shlyakhtenko. Some estimates for non-microstate free entropy dimension with applications to q -semicircular families. *Int. Math. Res. Not.* **51** (2004) 2757–2772. [MR2130608](#)
- [38] D. Shlyakhtenko. Remarks on free entropy dimension. In *Operator Algebras Abel Symposia, Volume 1* 249–257. Springer, Berlin, 2006. [MR2265052](#)
- [39] D. Shlyakhtenko. Lower estimates on microstate free entropy dimension. *Anal. PDE* **2** (2009) 119–146. [MR2547131](#)
- [40] P. Śniady. Gaussian random matrix models for q -deformed Gaussian variables. *Comm. Math. Phys.* **216** (2001) 515–537. [MR1815716](#)
- [41] A. S. Ustunel. On the regularity of the solutions of stochastic partial differential equations. In *Stochastic Differential Systems Filtering and Control. Lecture Notes in Control and Information Sciences* **69** 71–75. Springer, Berlin, 1985. [MR0798309](#)
- [42] D. Voiculescu. The analogs of entropy and of Fisher’s information measure in free probability theory, II. *Invent. Math.* **118** (1994) 411–440. [MR1296352](#)
- [43] D. Voiculescu. The analogs of entropy and of Fisher’s information measure in free probability theory, V: Non commutative Hilbert Transforms. *Invent. Math.* **132** (1998) 189–227. [MR1618636](#)
- [44] D. Voiculescu. The analogs of entropy and of Fisher’s information measure in free probability theory, VI: Liberation and mutual free information. *Adv. Math.* **146** (1999) 101–166. [MR1711843](#)
- [45] D. Voiculescu. Free entropy. *Bull. Lond. Math. Soc.* **34** (2002) 257–278. [MR1887698](#)
- [46] D. Voiculescu. A note on cyclic gradients. *Indiana Univ. Math. J.* **49** (2000) 837–841. [MR1803213](#)
- [47] J. B. Walsh. An introduction to stochastic partial differential equations. In *École d’été de probabilités de Saint-Flour, XIV–1984. Lecture Notes in Math.* **1180** 265–439. Springer, Berlin, 1986. [MR0876085](#)
- [48] N. Weaver. Lipschitz algebras and derivations of von Neumann algebras. *J. Funct. Anal.* **139** (1996) 261–300. [MR1402766](#)
- [49] D. Zagier. Realizability of a model in infinite statistics. *Commun. Math. Phys.* **147** (1992) 199–210. [MR1171767](#)