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On the discrete approximation of occupation time of diffusion processes

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Abstract: Let X be a 1-dimensional diffusion process. We study a simple class of estimators, which rely only on one sample data $\{X_{\frac{i}{n}}, 0 \le i \le nt\}$, for the occupation time $\int_0^t I_A(X_s)ds$ of process X in some set A. The main concern of this paper is the rates of convergence of the estimators. First, we consider the case that A is a finite union of some intervals in $\mathbb{R}$, then we show that the estimator converges at rate $n^{-3/4}$. Second, we consider the so-called stochastic corridor in mathematical finance. More precisely, we let A be a stochastic interval, say $[X_{t_0}, \infty)$ for some $t_0 \in (0,t)$, then we show that the estimator converges at rate $n^{-1/2}$. Some discussions about the exactness of the rates are also presented.

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1. Introduction

Let X be a 1-dimensional diffusion process defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge 0}, \mathbb{P})$ by

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t, \quad X_0 = x_0 \in \mathbb{R} \ a.s, \tag{1.1}$$

where W is a standard Brownian motion. Suppose that process X is observed at each time $t_i = i/n, i = 0, 1, \ldots$, we consider an approximation of the occupation time $\int_0^t I_A(X_s)ds$ of X in some set A by an average of the values $I_A(X_{\frac{i}{n}}), 0 \le i \le nt$, as $n \to \infty$.

The occupation time of a Brownian motion, and more generally of continuous diffusion processes, has been playing an important role in pricing some type of path dependent options like corridor option and eddoko option (see [3, 10] and references therein). Generally speaking, price of such options depends on the amount of time that the continuous time price process, say X, stays in some designated intervals. However, in real market, we can only get the values of process X at a finite set of observation points. It thus naturally raises a question of how to approximate the occupation time of X using the data $\{X_{\frac{i}{n}}, i = 0, \ldots, nt\}$.

Although the approximation of the local time of X by finite data of observation, discretized either in time or in space, has been well studied by many authors (see [1, 6]), there is a few works on the approximation of occupation time (see [9] and references therein). It is quite natural to use a Riemann sum to approximate the occupation time, however the rate of convergence seems to be unknown. The main aim of this paper is to find the rates of convergence by showing the tightness of the estimators.

In this paper, we consider two cases which are of special importance in theory as well as in application. Firstly, we study the deterministic corridor case where A is a finite union of some intervals in $\mathbb{R}$. Secondly, we study the stochastic corridor case where A is a stochastic interval, say $[X_{t_0}, \infty)$ for some $t_0 \in (0, t)$. It is interesting to compare the rates of convergence of the estimators in the former and latter cases, which are $n^{-3/4}$ and $n^{-1/2}$, respectively.

The present paper is organized in the following way. The framework and main results will be stated in Section 2. Proofs and further comments of the results are presented in Section 3.

2. Main results

For each set $A \in \mathcal{B}(\mathbb{R})$, we introduce the following estimator for the occupation time of X in A

$$\Gamma(A)^n_t = \frac{1}{n} \sum_{i=0}^{[nt]} I_A(X_{\frac{i}{n}}),$$

where $[x]$ denotes the integer part of x. Let us denote by ∂A the boundary of A and $\lambda(A)$ the Lebesgue measure of A.

The following proposition shows the convergence of estimator $\Gamma(A)^n_t$ to the occupation time as $n \to \infty$.

Proposition 2.1. *Assume that $\sigma(x_0) \neq 0$, and b and σ satisfy the Lipschitz condition. Let A be a Borel set satisfying $\lambda(\partial A) = 0$, then*

$$\Gamma(A)^n_t \xrightarrow{a.s.} \int_0^t I_A(X_s) ds, \tag{2.1}$$

as $n \to \infty$ for any $t > 0$.

In order to study the rate of convergence, we recall the definition of $C-$ *tightness* (see [8] for more details). First, we denote by $\mathbb{D}(\mathbb{R})$ the Polish space of all càdlàg functions: $\mathbb{R}_+ \to \mathbb{R}$ with Skorokhod topology. A sequence of $\mathbb{D}(\mathbb{R})$-valued random vectors (ξ^n) defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called *tight* if

$$\inf_K \sup_n \mathbb{P}(\xi^n \notin K) = 0,$$

where the infimum is taken over all compact sets K in $\mathbb{D}(\mathbb{R})$. The sequence (ξ^n) is called C-tight if it is tight and all cumulative points of the sequence $\{\mathcal{L}(\xi^n)\}$ are laws of continuous processes.

We introduce $\mathcal{A}$, a collection of all sets A of the following form

$$A = B \cup \bigcup_{i=1}^{n} (a_{2i}, a_{2i+1}),$$

where $-\infty \leq a_0 < a_1 < \cdots < a_{2n+1} \leq +\infty$ and B is a Borel set satisfying $\lambda(B) = 0$. We need the following assumption:

Assumption (H): *σ is a continuously differentiable and strictly positive function with bounded derivative on $\mathbb{R}$. Furthermore, b is a function of linear growth,*

$$|b(x)| \leq K(1 + |x|), \quad \forall x \in \mathbb{R},$$

for some constant $K > 0$.

Assumption (H) is sufficient for equation (1.1) to have a non-exploding, unique strong solution (Proposition V.5.17 [5]).

Now we are in a position to state the first main result of this paper.

Theorem 2.2. *Assume (H) and $A \in \mathcal{A}$. Then the sequence of processes*

$$\Big\{ n^{3/4} \Big(\Gamma(A)_t^n - \int_0^t I_A(X_s) ds \Big) \Big\}_{t \geq 0} \tag{2.2}$$

is C-tight.

The following simple proposition tells that the rate of convergence $n^{-3/4}$ is exact in L^2-sense when X is a standard Brownian motion.

Proposition 2.3. *Let X be a standard Brownian motion. Then there exist constants $K_1, K_2 \in (0, \infty)$ such that for all n, t satisfying $nt > 1$, we have*

$$K_1 \sqrt{t} \leq n^{3/2} \mathbb{E} \Big(\frac{1}{n} \sum_{i=1}^{[nt]} I_{\{X_{\frac{i}{n}} \geq 0\}} - \int_0^t I_{\{X_s \geq 0\}} ds \Big)^2 \leq K_2 \sqrt{t}.$$

Moreover, when X is a standard Brownian motion, we can modify Γ to get an unbiased estimator for the occupation time of X in some interval $[K, \infty)$, $K \in \mathbb{R}$, and then establishing a central limit theorem for the new estimator (see Corollary 3.4). Nevertheless, it is the bias of the general estimator Γ_t^n that makes the problem of showing a central limit theorem become very hard.

Next, we consider the case of so-called stochastic corridor in option pricing theory (see [3, 10]). We fix the horizontal time, say $t = 1$, and for a fixed constant $t_0 \in (0, 1)$, the *stochastic corridor at time* t_0 is defined by

$$R(t_0) = \int_0^1 I_{\{X_u \geq X_{t_0}\}} du.$$

Since X_{t_0} may not be observable, a natural estimator for $R(t_0)$ is

$$\Lambda_n(t_0) = \frac{1}{n} \sum_{i=1}^{n} I_{\{X_{\frac{i}{n}} \geq X_{\frac{[nt_0]}{n}}\}}.$$

We state the second main result of this paper.

Theorem 2.4. *Assume (H). Then the sequence of random variables* $n^{1/2}\big(\Lambda_n(t_0) - R(t_0)\big)$ *is tight.*

The rate of convergence drops from $n^{-3/4}$ to $n^{-1/2}$ due to the discreteness of the estimator which compares $X_{i/n}$ with $X_{[nt_0]/n}$ instead of the unobservable X_{t_0}. In fact, if we take $t_0 = 1$, then we have the following result.

Proposition 2.5. *Assume (H). Then for any* $t > 0$, *the sequence of random variables*

$$n^{3/4}\Big(\frac{1}{n}\sum_{i=1}^{[nt]} I_{\{X_{\frac{i}{n}} \geq X_1\}} - \int_0^t I_{\{X_u \geq X_1\}} du\Big)$$

is tight.

The proofs of all above results and some further comments will be presented in the next section.

3. Proofs

3.1. Preliminary

First we recall some facts about stable convergence. Let ξ^n be a sequence of random vectors with values in a Polish space $\mathcal{X}$, all defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let $\mathcal{G}$ be a sub-σ-algebra of $\mathcal{F}$. We say that ξ^n *converges* $\mathcal{G}$*-stably in law to* ξ, if ξ is an $\mathcal{X}$-valued random vector defined on an extension $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ of the original probability space and if

$$\lim_n \mathbb{E}(Uf(Y_n)) = \tilde{\mathbb{E}}(Uf(Y)),$$

for every bounded continuous functions $f : \mathcal{X} \to \mathbb{R}$ and all bounded $\mathcal{G}$-measurable random variables U. Stable convergence in law is obviously stronger than the convergence in law.

We will make use of the following lemmas.

Lemma 3.1. *Let (ξ^n) be a C-tight sequence of càdlàg processes defined on a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ and Q another probability measure which is equivalent to $\mathbb{P}$. Then the sequence (ξ^n) is also C-tight on probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, Q)$.*

Proof. The result follows easily from Proposition VI.3.26 [8]. □

Lemma 3.2 (Corollary VI.3.33 [8])**.** *Let (ξ^n) and (η^n) be C-tight sequences of càdlàg processes. Then the sequence $(\xi^n + \eta^n)$ is also C-tight.*

3.2. Proof of Proposition 2.1

It follows from Theorem 7.3 [11] that, under the given condition for coefficients b and σ, the law of X_t is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$. We denote A° and $\bar{A}$ the interior and closed bound of A, respectively. Since $\lambda(\partial A) = 0$, applying Fubini's theorem, we get

$$\int_0^t I_A(X_s)ds = \int_0^t I_{\bar{A}}(X_s)ds = \int_0^t I_{A^\circ}(X_s)ds \quad a.s.$$

Applying Fatou's lemma, we get

$$\begin{aligned}\limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{[nt]} I_A(X_{\frac{i}{n}}) &= \limsup_{n\to\infty} \int_0^t I_A(X_{\frac{[ns]}{n}})ds \\ \leq \int_0^t \limsup_{n\to\infty} I_A(X_{\frac{[ns]}{n}})ds &\leq \int_0^t \limsup_{n\to\infty} I_{\bar{A}}(X_{\frac{[ns]}{n}})ds \\ \leq \int_0^t I_{\bar{A}}(X_s)ds &= \int_0^t I_A(X_s)ds \quad a.s.\end{aligned}$$

On the other hand, a similar argument as above yields

$$\liminf_{n\to\infty} \frac{1}{n}\sum_{i=0}^{[nt]} I_A(X_{\frac{i}{n}}) \geq \int_0^t I_A(X_s)ds \quad a.s.$$

Hence we get (2.1).

Remark. By looking again at the above proof, we see that the conclusion of Proposition 2.1 holds for any process X which is continuous and has continuous marginal distribution. We refer to [4] and references therein for other classes of conditions on b and σ, which guarantee the continuity of sample path as well as of marginal distribution of X.

3.3. Proof of Theorem 2.2

It is enough to prove Theorem 2.2 with an additional assumption that b and $1/\sigma$ are bounded functions. Then the proof for general case can be obtained via a well-known localization procedure (see [6, 7] for instance). Furthermore, by applying the argument in Section 2 of [6], it is sufficient to prove Theorem 2.2 when $(\Omega, \mathcal{F})$ is the canonical space $(C[0,\infty), \mathcal{B}(C[0,\infty)))$ with W is the coordinate mapping process and $\mathbb{P}$ is the Wiener measure. Hence, from now on, we suppose that b and $1/\sigma$ are bounded and our probability space is canonical.

We denote $S(x) = \int_{x_0}^{x} \frac{1}{\sigma(u)} du, Y_t = S(X_t)$. It follows from Itô's formula that

$$dY_t = g(X_t)dt + dW_t, \tag{3.1}$$

where $g(x) = \frac{b(x)}{\sigma(x)} - \frac{1}{2}\sigma'(x)$. Since σ is strictly positive, S is strictly increasing and continuous, hence the inverse S^{-1} of S is well-defined. We denote $h(x) = g(S^{-1}(x))$, then equation (3.1) becomes

$$dY(t) = h(Y_t)dt + dW_t.$$

Denote

$$Z_t = \exp\Big(-\int_0^t h(Y_s)dW_s - \frac{1}{2}\int_0^t h(Y_s)^2 ds\Big).$$

Since g is bounded, h is also bounded and therefore Z is a martingale. Applying Girsanov's theorem (Corollary III.5.2 [5]), there is a unique probability measure $\tilde{\mathbb{P}}$ which is absolutely continuous with respect to $\mathbb{P}$, and under $\tilde{\mathbb{P}}$,

$$B_t := W_t + \int_0^t h(Y_s)ds$$

is a Brownian motion. Denote by $(\mathcal{G}_t)$ the filtration generated by the process B, and $\mathcal{G} = \bigvee \mathcal{G}_t$. Denote $\mathcal{G}_i^n = \mathcal{G}_{\frac{i}{n}}, B_i^n = B_{\frac{i}{n}}$ and $\tilde{\mathbb{E}}$ is the expectation with respect to $\tilde{\mathbb{P}}$.

For some $K \in \mathbb{R}$, we set

$$\begin{aligned} Z_{i,n} &= n^{3/4}\Big(\frac{1}{n} I_{[K,\infty)}(B_{i-1}^n) - \int_{\frac{i-1}{n}}^{\frac{i}{n}} I_{[K,\infty)}(B_s)ds\Big),\\ Y_{i,n} &= Z_{i,n} - \tilde{\mathbb{E}}(Z_{i,n}|\mathcal{G}_{i-1}^n),\ i = 1, 2, \ldots \end{aligned}$$

and

$$Z_t^n = \sum_{i=1}^{[nt]} Z_{i,n}, \quad Y_t^n = \sum_{i=1}^{[nt]} Y_{i,n}, \quad T_t^n = \sum_{i=1}^{[nt]} \tilde{\mathbb{E}}(Z_{i,n}|\mathcal{G}_{i-1}^n).$$

Now, the proof is divided into a series of lemmas.

Lemma 3.3. *The sequence Y^n converges $\mathcal{G}$-stably to a continuous process defined on an extension of the original probability space. In particular, the sequence (Y^n) is C-tight under probability measure $\tilde{\mathbb{P}}$.*

Proof. Denote by Φ the standard normal distribution function. The proof will proceed in several steps.
1) We have

$$\tilde{\mathbb{E}}(Z_{i,n}|\mathcal{G}^n_{i-1}) = n^{3/4}\Big(\frac{1}{n}\tilde{\mathbb{E}}(I_{\{B^n_{i-1}\geq K\}}|\mathcal{G}^n_{i-1}) - \tilde{\mathbb{E}}\Big(\int_{\frac{i-1}{n}}^{\frac{i}{n}} I_{\{B_s\geq K\}}ds|\mathcal{G}^n_{i-1}\Big)\Big).$$

By Markov property,

$$\begin{aligned}
&\tilde{\mathbb{E}}(Z_{i,n}|\mathcal{G}^n_{i-1})\\
&= n^{3/4}\Big(\frac{1}{n}I_{\{B^n_{i-1}\geq K\}} - \int_0^{\frac{1}{n}}\Phi\Big(\frac{B^n_{i-1}-K}{\sqrt{u}}\Big)du\Big)\\
&= \frac{1}{\sqrt{2\pi}}n^{-1/4}\int_{\sqrt{n}(B^n_{i-1}-K)}^{\infty}\Big(1-\frac{n(B^n_{i-1}-K)^2}{t^2}\Big)e^{-t^2/2}dtI_{\{B^n_{i-1}\geq K\}}\\
&\quad - \frac{1}{\sqrt{2\pi}}n^{-1/4}\int_{-\infty}^{\sqrt{n}(B^n_{i-1}-K)}\Big(1-\frac{n(B^n_{i-1}-K)^2}{t^2}\Big)e^{-t^2/2}dtI_{\{B^n_{i-1}< K\}}.
\end{aligned}$$

We denote

$$\begin{aligned}
g_1(x) &= \Big(\int_x^{\infty}\Big(1-\frac{x^2}{t^2}\Big)e^{-t^2/2}dtI_{\{x\geq 0\}} - \int_{-\infty}^{x}\Big(1-\frac{x^2}{t^2}\Big)e^{-t^2/2}dtI_{\{x<0\}}\Big)^2.\\
&= \Big(\int_{|x|}^{\infty}\Big(1-\frac{x^2}{t^2}\Big)e^{-t^2/2}dt\Big)^2,
\end{aligned}$$

and

$$\lambda_1 = \int_{\mathbb{R}} g_1(x)dx = 2\int_0^{\infty}\Big(\int_x^{\infty}\Big(1-\frac{x^2}{t^2}\Big)e^{-t^2/2}dt\Big)^2dx.$$

For any $x\neq 0$, one has

$$0\leq \int_{|x|}^{\infty}\Big(1-\frac{x^2}{t^2}\Big)e^{-t^2/2}dt \leq \int_0^{\infty}e^{-t^2/2}dt = \sqrt{\frac{\pi}{2}},$$

and,

$$\int_{|x|}^{\infty}\Big(1-\frac{x^2}{t^2}\Big)e^{-t^2/2}dt \leq \int_{|x|}^{\infty}e^{-t^2/2}dt \leq \int_{|x|}^{\infty}\frac{t}{|x|}e^{-t^2/2}dt = |x|^{-1}e^{-x^2/2}.$$

Therefore

$$g_1(x)\leq \min\Big\{\frac{\pi}{2}, x^{-2}e^{-x^2}\Big\}.$$

Furthermore, one has

$$\lambda_1 = 2\int_0^{\infty}\int_x^{\infty}\Big(1-\frac{x^2}{t^2}\Big)e^{-t^2/2}dt\int_x^{\infty}\Big(1-\frac{x^2}{s^2}\Big)e^{-s^2/2}dsdx.$$

Since the integrand is non-negative, it follows from Fubini theorem that

$$\begin{aligned}\lambda_1 &= 2\int_1^\infty du \int_1^\infty dv\Big(1-\frac{1}{u^2}\Big)\Big(1-\frac{1}{v^2}\Big)\int_0^\infty x^2\exp\Big(-\frac{(u^2+v^2)x^2}{2}\Big)dx \\ &= \sqrt{2\pi}\int_1^\infty du\int_1^\infty dv\Big(1-\frac{1}{u^2}\Big)\Big(1-\frac{1}{v^2}\Big)(u^2+v^2)^{-3/2}.\end{aligned}$$

And then after a change of variables,

$$\begin{cases} u = \frac{1}{r\sin\alpha} \\ v = \frac{1}{r\cos\alpha}\end{cases} \quad 0 < r \leq 1,\ 0 \leq \alpha \leq \frac{\pi}{2},$$

one gets $\lambda_1 = \frac{7\sqrt{2\pi}}{20}$. According to Theorem 4.1 [6], we have

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{[nt]} g_1(\sqrt{n}(B^n_{i-1}-K)) \xrightarrow{\tilde{\mathbb{P}}} \frac{7\sqrt{2\pi}}{20}L_t(K), \tag{3.2}$$

where $L_t(K)$ is the local time of B defined by

$$L_t(K) = |B_t - K| - |K| - \int_0^t sgn(B_s - K)dB_s. \tag{3.3}$$

Hence

$$\sum_{i=1}^{[nt]}\tilde{\mathbb{E}}(Z_{i,n}|\mathcal{G}^n_{i-1})^2 \xrightarrow{\tilde{\mathbb{P}}} \frac{7}{20\sqrt{2\pi}}L_t(K). \tag{3.4}$$

2) Next, we have

$$\begin{aligned}\tilde{\mathbb{E}}\Big(\Big(\int_{\frac{i-1}{n}}^{\frac{i}{n}} I_{\{B_s\geq K\}}ds\Big)^2\Big|\mathcal{G}^n_{i-1}\Big) &= \tilde{\mathbb{E}}\Big(\Big(\int_{\frac{i-1}{n}}^{\frac{i}{n}} I_{\{B_s - B^n_{i-1}\geq K - B^n_{i-1}\}}ds\Big)^2\Big|\mathcal{G}^n_{i-1}\Big) \\ &= \tilde{\mathbb{E}}\Big(\int_{\frac{i-1}{n}}^{\frac{i}{n}} I_{\{B_s - B^n_{i-1}\geq -r\}}ds\Big)^2\Big|_{r=B^n_{i-1}-K},\end{aligned}$$

the Markov property yields,

$$\tilde{\mathbb{E}}\Big(\Big(\int_{\frac{i-1}{n}}^{\frac{i}{n}} I_{\{B_s\geq K\}}ds\Big)^2\Big|\mathcal{G}^n_{i-1}\Big) = \tilde{\mathbb{E}}\Big(\int_0^{\frac{1}{n}} I_{\{B_s\geq -r\}}ds\Big)^2\Big|_{r=B^n_{i-1}-K}. \tag{3.5}$$

Using formula 1.1.4.4 Borodin *et al.* [2], we get, if $r \leq 0$,

$$\begin{aligned}\tilde{\mathbb{E}}\Big(\int_0^{\frac{1}{n}} I_{\{B_s\geq -r\}}ds\Big)^2 &= \int_0^{1/n}\frac{v^2}{\pi\sqrt{v(\frac{1}{n}-v)}}\exp\Big(-\frac{r^2}{2(\frac{1}{n}-v)}\Big)dv \\ &= \frac{1}{n^2\pi}\int_0^1\frac{z^{3/2}}{\sqrt{1-z}}\exp\Big(-\frac{nr^2}{2(1-z)}\Big)dz, \end{aligned}\tag{3.6}$$

and if $r > 0$,

$$\begin{aligned}
&\tilde{\mathbb{E}}\Big(\int_0^{\frac{1}{n}} I_{\{B_s \geq -r\}} ds\Big)^2 \\
=&\frac{1}{n^2}\Big(1 - \int_0^{\frac{1}{n}} \frac{1}{\pi\sqrt{v(\frac{1}{n}-v)}} \exp\Big(-\frac{r^2}{2v}\Big) dv\Big) + \int_0^{\frac{1}{n}} \frac{v^2}{\pi\sqrt{v(\frac{1}{n}-v)}} \exp\Big(-\frac{r^2}{2v}\Big) dv \\
=&\frac{1}{n^2}\Big(1 - \int_0^{1} \frac{1}{\pi\sqrt{z(1-z)}} \exp\Big(-\frac{nr^2}{2z}\Big) dv\Big) + \frac{1}{n^2\pi} \int_0^{1} \frac{z^{3/2}}{\sqrt{1-z}} \exp\Big(-\frac{nr^2}{2z}\Big) dz.
\end{aligned} \tag{3.7}$$

We write

$$\begin{aligned}
\tilde{\mathbb{E}}(Z_{i,n}^2 | \mathcal{G}_{i-1}^n) = \frac{1}{\sqrt{n}} I_{\{B_{i-1}^n \geq K\}} - 2n^{1/2} I_{\{B_{i-1}^n \geq K\}} \tilde{\mathbb{E}}\Big(\int_{\frac{i-1}{n}}^{\frac{i}{n}} I_{\{B_s \geq K\}} ds \Big| \mathcal{G}_{i-1}^n\Big) \\
+ n^{3/2} \tilde{\mathbb{E}}\Big(\Big(\int_{\frac{i-1}{n}}^{\frac{i}{n}} I_{\{B_s \geq K\}} ds\Big)^2 \Big| \mathcal{G}_{i-1}^n\Big),
\end{aligned}$$

then it follows from (3.5)-(3.7) and an elementary calculation that

$$\begin{aligned}
\tilde{\mathbb{E}}(Z_{i,n}^2 | \mathcal{G}_{i-1}^n) =& \frac{1}{\sqrt{n}} \Big\{ \frac{1}{\pi} \int_0^1 \frac{z^{3/2}}{\sqrt{1-z}} \exp\Big(-\frac{n}{2(1-z)} (B_{i-1}^n - K)^2\Big) dz I_{\{B_{i-1}^n < K\}} \\
&+ 2\int_0^1 \Big(1 - \Phi\Big(\frac{\sqrt{n}}{\sqrt{u}}(B_{i-1}^n - K)\Big)\Big) du I_{\{B_{i-1}^n \geq K\}} \\
&+ \frac{1}{\pi} \int_0^1 \frac{z^{3/2}}{\sqrt{1-z}} \exp\Big(-\frac{n}{2z}(B_{i-1}^n - K)^2\Big) dz I_{\{B_{i-1}^n \geq K\}} \\
&- \frac{1}{\pi} \int_0^1 \frac{1}{\sqrt{z(1-z)}} \exp\Big(-\frac{n}{2z}(B_{i-1}^n - K)^2\Big) dz I_{\{B_{i-1}^n \geq K\}} \Big\}.
\end{aligned}$$

Denote

$$\begin{aligned}
g_2(x) =& \frac{1}{\pi} \int_0^1 \frac{z^{3/2}}{\sqrt{1-z}} \exp\Big(-\frac{x^2}{2(1-z)}\Big) dz I_{\{x<0\}} \\
&+ \Big\{ 2\int_0^1 \Big(1 - \Phi(\frac{x}{\sqrt{u}})\Big) du + \frac{1}{\pi} \int_0^1 \frac{z^{3/2}}{\sqrt{1-z}} \exp\Big(-\frac{x^2}{2z}\Big) dz \\
&- \frac{1}{\pi} \int_0^1 \frac{1}{\sqrt{z(1-z)}} \exp\Big(-\frac{x^2}{2z}\Big) dz \Big\} I_{\{x \geq 0\}}.
\end{aligned}$$

Using the fact that $e^{-u} < 1/u$ for all $u > 0$, once can show that

$$|g_2(x)| \leq K \min\{1, |x|^{-1} e^{-x^2/2} + x^{-2}\}, \quad \forall x \in \mathbb{R},$$

for some positive constant K. Furthermore, by applying Fubini theorem, one gets

$$\int_{-\infty}^{+\infty} g_2(x) dx = \frac{2\sqrt{2}}{5\sqrt{\pi}}.$$

Apply Theorem 4.1 [6] again, we have

$$\sum_{i=1}^{[nt]} \tilde{\mathbb{E}}(Z_{i,n}^2|\mathcal{G}_{i-1}^n) = \frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} g_2\big(\sqrt{n}(B_{i-1}^n - K)\big) \xrightarrow{\tilde{\mathbb{P}}} \frac{2\sqrt{2}}{5\sqrt{\pi}} L_t(K). \tag{3.8}$$

3) It follows from (3.4) and (3.8) that

$$\sum_{i=1}^{[nt]} \tilde{\mathbb{E}}(Y_{i,n}^2|\mathcal{G}_{i-1}^n) \xrightarrow{\tilde{\mathbb{P}}} \frac{9}{20\sqrt{2\pi}} L_t(K). \tag{3.9}$$

4) Next, we have

$$\begin{aligned}
&\sum_{i=1}^{[nt]} \tilde{\mathbb{E}}\big(Y_{i,n}(B_i^n - B_{i-1}^n)|\mathcal{G}_{i-1}^n\big) = \sum_{i=1}^{[nt]} \tilde{\mathbb{E}}\big(Z_{i,n}(B_i^n - B_{i-1}^n)|\mathcal{G}_{i-1}^n\big)\\
=& - n^{3/4} \sum_{i=1}^{[nt]} \tilde{\mathbb{E}}\Big((B_i^n - B_{i-1}^n) \int_{\frac{i-1}{n}}^{\frac{i}{n}} I_{\{B_s \geq K\}} ds|\mathcal{G}_{i-1}^n\Big)\\
=& - n^{3/4} \sum_{i=1}^{[nt]} \int_{\frac{i-1}{n}}^{\frac{i}{n}} \tilde{\mathbb{E}}\Big((B_s - B_{i-1}^n) I_{\{B_s \geq K\}}|\mathcal{G}_{i-1}^n\Big) ds.
\end{aligned}$$

The Markov property yields

$$\begin{aligned}
&\sum_{i=1}^{[nt]} \tilde{\mathbb{E}}\big(Y_{i,n}(B_i^n - B_{i-1}^n)|\mathcal{G}_{i-1}^n\big)\\
=& - n^{3/4} \sum_{i=1}^{[nt]} \int_{\frac{i-1}{n}}^{\frac{i}{n}} \tilde{\mathbb{E}}\Big((B_s - B_{i-1}^n) I_{\{B_s - B_{i-1}^n \geq r\}}\Big) ds\Big|_{r=K-B_{i-1}^n}\\
=& - n^{3/4} \sum_{i=1}^{[nt]} \int_0^{\frac{1}{n}} \tilde{\mathbb{E}}\Big(B_s I_{\{B_s \geq r\}}\Big) ds\Big|_{r=K-B_{i-1}^n}\\
=& - n^{3/4} \sum_{i=1}^{[nt]} \frac{1}{\sqrt{2\pi}} \int_0^{\frac{1}{n}} \sqrt{z} \exp\Big(-\frac{(B_{i-1}^n - K)^2}{2z}\Big) dz.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\tilde{\mathbb{E}}\Big|\sum_{i=1}^{[nt]} \tilde{\mathbb{E}}\big(Y_{i,n}(B_i^n - B_{i-1}^n)|\mathcal{G}_{i-1}^n\big)\Big|\\
\leq& n^{3/4} \sum_{i=1}^{[nt]} \frac{1}{\sqrt{2\pi}} \int_0^{\frac{1}{n}} \sqrt{z} \tilde{\mathbb{E}} \exp\Big(-\frac{(B_{i-1}^n - K)^2}{2z}\Big) dz
\end{aligned}$$

Since $e^{-u} < 1$ if $u > 0$, we have

$$
\begin{aligned}
&\tilde{\mathbb{E}}\Big|\sum_{i=1}^{[nt]}\tilde{\mathbb{E}}\big(Y_{i,n}(B_i^n - B_{i-1}^n)|\mathcal{G}_{i-1}^n\big)\Big| \\
\leq & \frac{1}{n^{3/4}} + n^{3/4}\sum_{i=2}^{[nt]}\frac{1}{\sqrt{2\pi}}\int_0^{\frac{1}{n}}dz\int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi\frac{i-1}{n}}}\sqrt{z}\exp\Big(-\frac{(x-K)^2}{2z}\Big)dx \\
\leq & \frac{1}{n^{3/4}} + n^{3/4}\sum_{i=2}^{[nt]}\frac{\sqrt{n}}{n^2\sqrt{i-1}} \\
\leq & 2n^{-1/4}(1+\sqrt{t}).
\end{aligned}
$$

Therefore,

$$
\sum_{i=1}^{[nt]}\tilde{\mathbb{E}}\big(Y_{i,n}(B_i^n - B_{i-1}^n)|\mathcal{G}_{i-1}^n\big) \xrightarrow{\tilde{\mathbb{P}}} 0. \tag{3.10}
$$

5) We have

$$
\sum_{i=1}^{[nt]}\tilde{\mathbb{E}}\big(Y_{i,n}^4|\mathcal{G}_{i-1}^n\big) \leq 16\sum_{i=1}^{[nt]}\tilde{\mathbb{E}}\big(Z_{i,n}^4|\mathcal{G}_{i-1}^n\big)
$$

Because of Markov property, we get

$$
\begin{aligned}
\sum_{i=1}^{[nt]}\tilde{\mathbb{E}}\big(Y_{i,n}^4|\mathcal{G}_{i-1}^n\big) &\leq 16n^3\sum_{i=1}^{[nt]}\tilde{\mathbb{E}}\Big(\frac{1}{n}I_{\{r\geq 0\}} - \int_0^{\frac{1}{n}}I_{\{B_s\geq -r\}}ds\Big)^4\Big|_{r=B_{i-1}^n-K} \\
&\leq 16n^3\sum_{i=1}^{[nt]}\tilde{\mathbb{E}}\Big(\int_0^{\frac{1}{n}}I_{\{B_s\geq r\}}ds\Big)^4\Big|_{r=|B_{i-1}^n-K|} \\
&\leq 16\sum_{i=1}^{[nt]}\int_0^{\frac{1}{n}}\Big(1-\Phi\Big(\frac{|B_{i-1}^n-K|}{\sqrt{s}}\Big)\Big)ds.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\tilde{\mathbb{E}}\Big|\sum_{i=1}^{[nt]}\tilde{\mathbb{E}}\big(Y_{i,n}^4|\mathcal{G}_{i-1}^n\big)\Big| \leq 16\sum_{i=1}^{[nt]}\int_0^{\frac{1}{n}}\tilde{\mathbb{E}}\Big(1-\Phi\Big(\frac{|B_{i-1}^n-K|}{\sqrt{s}}\Big)\Big)ds \\
\leq & 16n^{-1} + 16\sum_{i=2}^{[nt]}\int_0^{\frac{1}{n}}ds\int_{-\infty}^{+\infty}dx\int_0^{+\infty}\frac{\sqrt{n}}{\sqrt{i-1}}I_{\{t\geq\frac{|x-K|}{\sqrt{s}}\}}e^{-t^2/2}dt
\end{aligned}
$$

Applying Fubini theorem, one gets

$$\begin{aligned}\tilde{\mathbb{E}}\Big|\sum_{i=1}^{[nt]}\tilde{\mathbb{E}}\big(Y_{i,n}^4|\mathcal{G}_{i-1}^n\big)\Big| \leq & 16n^{-1}+16\sum_{i=2}^{[nt]}\int_0^{\frac{1}{n}}ds\int_0^{+\infty}\frac{\sqrt{n}}{\sqrt{i-1}}2t\sqrt{s}e^{-t^2/2}dt\\ =&16n^{-1}+\frac{64}{3}\sum_{i=2}^{[nt]}\frac{1}{n\sqrt{i-1}}\\ \leq&\frac{64}{3}\Big(\frac{1}{n}+\frac{\sqrt{t}}{\sqrt{n}}\Big).\end{aligned}$$

Therefore

$$\sum_{i=1}^{[nt]}\tilde{\mathbb{E}}\big(Y_{i,n}^4|\mathcal{G}_{i-1}^n\big)\xrightarrow{\tilde{\mathbb{P}}}0. \tag{3.11}$$

6) We have that $Y^n=(Y_{i,n},\mathcal{G}_i^n)$ is a martingale. Further, under $\tilde{\mathbb{P}}$, any martingale with respect to $(\mathcal{G}_t)$ orthogonal to B is constant. Hence it follows from (3.9)-(3.11) and Theorem IX.7.28 [8] that Y^n converges $\mathcal{G}$-stably to a continuous process defined on an extension of the original probability space. In particular, the sequence (Y^n) is C-tight under probability measure $\tilde{\mathbb{P}}$. □

It should be noted here that although the Riemann estimator $\Gamma(.)$ of the occupation time is consistent, it is biased in general. However, Lemma 3.3 tells us that we can construct an unbiased and consistent estimator for the occupation time of Brownian motion. More precisely, we have the following central limit theorem in which we refer to [8] for notions we have not defined so far.

Corollary 3.4. *Suppose that B is a Brownian motion defined on a filtered probability space $\mathcal{B}=(\Omega,\mathcal{G},(\mathcal{G})_t,\tilde{\mathbb{P}})$. For each $n\geq 1$, $t>0$ and $K\in\mathbb{R}$, we denote*

$$\tilde{\Gamma}(K)_t^n=\sum_{i=1}^{[nt]}\int_{\frac{i-1}{n}}^{\frac{i}{n}}\Phi\Big(\frac{K-B_{(i-1)/n}}{\sqrt{s-\frac{i-1}{n}}}\Big)ds+\int_{\frac{[nt]}{n}}^{t}\Phi\Big(\frac{K-B_{[nt]/n}}{\sqrt{s-\frac{[nt]}{n}}}\Big)ds.$$

Then $\tilde{\Gamma}(K)_t^n$ is an unbiased estimator for the occupation time $\tilde{\Gamma}(K)_t=\int_0^t I_{\{B_s\geq K\}}ds$. Moreover, there is a very good extension $\tilde{\mathcal{B}}$ of $\mathcal{B}$ and a continuous B-biased $\mathcal{G}$-progressive conditional martingale with independent increment X' on this extension with

$$\langle X',X'\rangle_t=\frac{9}{20\sqrt{2\pi}}L_t(K),\quad \langle X',B\rangle=0,$$

such that $n^{3/4}\big(\tilde{\Gamma}(K)^n-\tilde{\Gamma}(K)\big)$ converges $\mathcal{G}$-stably to X', where $L_t(K)$ is defined in (3.3).

Proof. $\tilde{\Gamma}(K)_t^n$ is an unbiased estimator for $\tilde{\Gamma}(K)_t$ because

$$\tilde{\Gamma}(K)_t^n=\sum_{i=1}^{[nt]}\mathbb{E}\Big(\int_{\frac{i-1}{n}}^{\frac{i}{n}}I_{[K,\infty)}(B_s)ds\big|B_{\frac{i-1}{n}}\Big)+\mathbb{E}\Big(\int_{\frac{[nt]}{n}}^{t}I_{[K,\infty)}(B_s)ds\big|B_{\frac{[nt]}{n}}\Big).$$

Furthermore, we have

$$n^{3/4}\int_{\frac{[nt]}{n}}^{t}\Phi\Big(\frac{K-B_{[nt]/n}}{\sqrt{s-\frac{[nt]}{n}}}\Big)ds\le n^{-1/4}.$$

Then by following the same argument as in the proof of Lemma 3.3 we get the desired result. □

We continue the proof of Theorem 2.2.

Lemma 3.5. *The sequence (T^n) is C-tight under probability measure $\tilde{\mathbb{P}}$.*

Proof. We write

$$\begin{aligned}\sum_{i=1}^{[nt]}\tilde{\mathbb{E}}(Z_{i,n}|\mathcal{G}_{i-1}^n)&=n^{3/4}\sum_{i=1}^{[nt]}\Big(\frac{1}{n}I_{\{B_{i-1}^n\ge K\}}-\frac{1}{n}\int_0^1\Phi\Big(\frac{\sqrt{n}(B_{i-1}^n-K)}{\sqrt{u}}\Big)du\Big)\\&=n^{-1/4}\sum_{i=1}^{[nt]}g_3(\sqrt{n}(B_{i-1}^n-K)),\end{aligned}$$

where

$$g_3(x)=I_{\{x\ge 0\}}-\int_0^1\Phi(xu^{-1/2})du=sgn(x)\int_0^1\big(1-\Phi(|x|u^{-1/2})\big)du.$$

Since $\int_{-\infty}^{+\infty}g_3(x)dx=0$ and $|g_3(x)|\le\min\{1,|x|^{-1}e^{-x^2/2}\}$, applying Theorem 1.2 [6], we have T^n converges $\mathcal{G}$-stably to a continuous process defined on an extension of the original probability space. In particular, the sequence (T^n) is C-tight under probability measure $\tilde{\mathbb{P}}$. □

We are now in a position to finish the proof of Theorem 2.2. Indeed, according to Lemmas 3.3, 3.5 and Lemma 3.2, the sequence of processes

$$\Big\{n^{3/4}\big(\Gamma([K,\infty))_t^n-\int_0^t I_{[K,\infty)}(X_s)ds\big)\Big\}_{t\ge 0}$$

is C-tight under $\tilde{\mathbb{P}}$. Moreover, for each $t>0$, we define a signed measure μ_t on $(\mathbb{R},\mathcal{B}(\mathbb{R}))$ by

$$\mu_t(A)=\Gamma(A)-\int_0^t I_A(X_s)ds.$$

It follows from Fubini theorem that $\mu_t(A)=0$ for any Borel set A satisfying $\lambda(A)=0$. Furthermore, for any $a<b$, $\mu_t([a,b))=\mu_t([a,\infty))-\mu_t([b,\infty))$, hence it follows from Lemma 3.2 that the sequence of process $\big\{n^{3/4}\mu_t([a,b))\big\}$ is C-tight under $\tilde{\mathbb{P}}$. By repeating the same argument, one can show that the sequence $\big\{n^{3/4}\big(\Gamma(A)_t^n-\int_0^t I_A(X_s)ds\big)\big\}_{t\ge 0}$ is C-tight under $\tilde{\mathbb{P}}$ for any $A\in\mathcal{A}$. Applying Lemma 3.1, we get the desired result.

3.4. Proof of Proposition 2.3

Suppose that X is a standard Brownian motion. We need the following equality, which is more or less known but simple to prove.

Lemma 3.6. *For any $t > s > 0$,*

$$\mathbb{P}(X_s \geq 0, X_t \geq 0) = \frac{1}{4} + \frac{1}{2\pi} \arctan \sqrt{\frac{s}{t-s}}. \tag{3.12}$$

Proof. Since X is a Brownian motion, one has

$$\begin{aligned}
\mathbb{P}(X_s \geq 0, X_t \geq 0) &= \mathbb{E}\Big(\mathbb{E}\big(I_{\{X_{t-s} \geq -x\}} I_{\{x \geq 0\}}\big)|_{X_s = x}\Big) \\
&= \int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \Phi\Big(\frac{\sqrt{s}x}{\sqrt{t-s}}\Big) dx \\
&= \int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \Big(\frac{1}{2} + \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-y^2/2} I_{\{\sqrt{t-s}y \leq \sqrt{s}x\}} dy\Big) dx.
\end{aligned}$$

An elementary calculation by using change of variables formula yields the desired result. $\square$

We also make use of the following estimate for the error due to the Riemann approximation of function $f(x) = \sqrt{x - x^2}$ defined on $[0, 1]$. The difficulty comes from the fact that the derivative of f is not bounded on $(0, 1)$.

Lemma 3.7. *There exist positive constants κ_1, κ_2 such that for any positive integer m,*

$$\kappa_1 m^{-3/2} \leq \frac{\pi}{8} - \frac{1}{m} \sum_{i=1}^m \frac{i}{m} \sqrt{\frac{m}{i} - 1} \leq \kappa_2 m^{-3/2}. \tag{3.13}$$

Proof. On the Cartesian plane Oxy, let $(\mathcal{C})$ be a semicircle with equation

$$\begin{cases} y^2 = x - x^2 \\ y \geq 0. \end{cases}$$

We denote $I(1/2, 0)$ the central point of the circle. Set $M_i = (\frac{i}{m}, f(\frac{i}{m})) \in (\mathcal{C})$, $i = 0, \ldots, m$, and $\alpha_i = \angle M_i I M_{i+1}$, $i = 0, \ldots, m-1$. It is clear that

$$\frac{\pi}{8} - \frac{1}{m} \sum_{i=1}^m \frac{i}{m} \sqrt{\frac{m}{i} - 1} = \frac{1}{8} \sum_{i=0}^{m-1} (\alpha_i - \sin \alpha_i) > 0.$$

since $\sin \alpha < \alpha$ for all $\alpha > 0$. Hence it is sufficient to verify inequality (3.13) for m large enough.

Since $\lim_{\alpha \to 0} \frac{\alpha - \sin \alpha}{\alpha^3} = \frac{1}{6}$, there exists $\delta > 0$ such that for all $\alpha < \delta$, $\alpha - \sin \alpha > \frac{\alpha^3}{12}$. Hence, for m large enough, we have

$$\frac{\pi}{8} - \frac{1}{m} \sum_{i=1}^m \frac{i}{m} \sqrt{\frac{m}{i} - 1} \geq \frac{1}{8} (\alpha_0 - \sin \alpha_0) \geq \frac{1}{96} \alpha_0^3.$$

Since $\cos\alpha_0 = 1 - \frac{2}{m}$, one has,

$$\sin\alpha_0 = \sqrt{1-(1-\frac{2}{m})^2} \geq \frac{1}{\sqrt{m}}.$$

Therefore

$$\frac{\pi}{8} - \frac{1}{m}\sum_{i=1}^{m}\frac{i}{m}\sqrt{\frac{m}{i}-1} \geq \frac{1}{96}m^{-3/2},$$

for all m large enough.

Next we show the upper bound of (3.13). For each $j < (m-1)/2$, we denote $\beta_j = \sum_{i=0}^{j}\alpha_i$. We have $\cos(\beta_j) = 1 - \frac{2(j+1)}{m}$, hence, for $1 \leq i \leq (m-1)/2$,

$$\begin{aligned}
\sin\alpha_{i-1} &= \sin(\beta_i - \beta_{i-1}) \\
&= \sqrt{1-\Big(1-\frac{2(i+1)}{m}\Big)^2}\Big(1-\frac{2i}{m}\Big) - \sqrt{1-\Big(1-\frac{2i}{m}\Big)^2}\Big(1-\frac{2i+2}{m}\Big) \\
&= \frac{\big(\frac{4}{m}-\frac{4(2i+1)}{m^2}\big)\big(1-\frac{2i}{m}\big)}{\sqrt{1-\big(1-\frac{2(i+1)}{m}\big)^2}+\sqrt{1-\big(1-\frac{2i}{m}\big)^2}} + \frac{2}{m}\sqrt{1-\Big(1-\frac{2i}{m}\Big)^2} \\
&\leq \frac{\frac{4}{m}}{\sqrt{\frac{4i}{m}-\frac{4i^2}{m^2}}} + \frac{2}{m},
\end{aligned}$$

and this implies

$$\sin\alpha_i \leq \frac{6}{\sqrt{mi}}. \tag{3.14}$$

On the other hand, since $\alpha - \sin\alpha < \alpha^3/6$ for all $\alpha > 0$, one has

$$\frac{\pi}{8} - \frac{1}{m}\sum_{i=1}^{m}\frac{i}{m}\sqrt{\frac{m}{i}-1} \leq \frac{1}{48}\sum_{i=0}^{m-1}\alpha_i^3.$$

For m large enough, we have $\alpha_i < 2\sin\alpha_i$, hence,

$$\frac{\pi}{8} - \frac{1}{m}\sum_{i=1}^{m}\frac{i}{m}\sqrt{\frac{m}{i}-1} \leq \frac{1}{6}\sum_{i=0}^{m-1}\sin^3\alpha_i.$$

Because of the symmetry, $\alpha_i = \alpha_{m-i-1}$ for all $i = 0,\ldots,m-1$. This fact together with inequality (3.14) yields

$$\frac{\pi}{8} - \frac{1}{m}\sum_{i=1}^{m}\frac{i}{m}\sqrt{\frac{m}{i}-1} \leq C\sum_{i=1}^{m}\frac{1}{(mi)^{3/2}} \leq C'\frac{1}{m^{3/2}},$$

for some positive constants C and C'. $\square$

Next, we denote $X^n_j = X_{\frac{j}{n}}$ and

$$S_n = \frac{1}{n}\sum_{i=1}^{[nt]} I_{\{X^n_i \geq 0\}} - \int_0^{\frac{[nt]}{n}} I_{\{X_s \geq 0\}} ds.$$

We write

$$\mathbb{E}S_n^2 = T^n_1 + T^n_2 + T^n_3, \tag{3.15}$$

where

$$\begin{aligned}
T^n_1 &= \frac{1}{n^2}\mathbb{E}\Big(\sum_{i=1}^{[nt]} I_{\{X^n_i \geq 0\}}\Big)^2, \\
T^n_2 &= -\frac{2}{n}\sum_{i=1}^{[nt]}\mathbb{E}\Big(I_{\{X^n_i \geq 0\}}\int_0^{\frac{[nt]}{n}} I_{\{X_s \geq 0\}} ds\Big), \\
T^n_3 &= \mathbb{E}\Big(\int_0^{\frac{[nt]}{n}} I_{\{X_s \geq 0\}} ds\Big)^2.
\end{aligned}$$

It follows from (3.12) that

$$\begin{aligned}
n^2 T^n_1 &= \sum_{i=1}^{[nt]}\mathbb{E}(I_{\{X^n_i \geq 0\}}) + 2\sum_{1\leq i<j\leq [nt]}\mathbb{E}\big(I_{\{X^n_i \geq 0\}} I_{\{X^n_j \geq 0\}}\big) \\
&= \frac{[nt]}{2} + 2\sum_{1\leq i<j\leq [nt]}\Big(\frac{1}{4} + \frac{1}{2\pi}\arctan\sqrt{\frac{i}{j-i}}\Big) \\
&= \frac{[nt]([nt]+1)}{4} + \frac{1}{2\pi}\sum_{\substack{i,j\geq 1\\ i+j\leq [nt]}}\Big(\arctan\sqrt{\frac{i}{j}} + \arctan\sqrt{\frac{j}{i}}\Big), \\
&= \frac{[nt](3[nt]+1)}{8},
\end{aligned}$$

here, and in the following, we use the fact that $\arctan(x) + \arctan(1/x) = \pi/2$. Moreover,

$$\begin{aligned}
-\frac{n}{2}T^n_2 &= \sum_{i=1}^{[nt]}\int_0^{\frac{[nt]}{n}}\mathbb{E}\big(I_{\{X^n_i \geq 0\}} I_{\{X_s \geq 0\}}\big) ds \\
&= \sum_{i=1}^{[nt]}\int_0^{\frac{i}{n}}\Big(\frac{1}{4} + \frac{1}{2\pi}\arctan\sqrt{\frac{s}{\frac{i}{n} - s}}\Big) ds \\
&\quad + \sum_{i=1}^{[nt]}\int_{\frac{i}{n}}^{\frac{[nt]}{n}}\Big(\frac{1}{4} + \frac{1}{2\pi}\arctan\sqrt{\frac{\frac{i}{n}}{s - \frac{i}{n}}}\Big) ds.
\end{aligned}$$

An elementary calculation shows that

$$-\frac{n}{2}T_2^n = \frac{[nt]}{4} + \frac{1}{2\pi}\sum_{i=1}^{[nt]}\Big(\frac{i\pi}{4n} + \frac{[nt]}{n}\arctan\sqrt{\frac{i}{[nt]-i}} + \frac{i}{n}\sqrt{\frac{[nt]}{i}-1} - \frac{i\pi}{2n}\Big)$$
$$= \frac{[nt]^2}{4n} + \frac{[nt]([nt]+1)}{16n} + \frac{1}{2\pi}\sum_{i=1}^{[nt]}\frac{i}{n}\sqrt{\frac{[nt]}{i}-1}.$$

By arcsin law, we have

$$T_3^n = \frac{3[nt]^2}{8n^2}.$$

Therefore,

$$\mathbb{E}S_n^2 = \frac{[nt]^2}{8n^2} - \frac{[nt]}{n^2\pi}\sum_{i=1}^{[nt]}\frac{i}{[nt]}\sqrt{\frac{[nt]}{i}-1}. \tag{3.16}$$

Applying Lemma 3.7, there exist positive constants κ_1, κ_2 such that

$$\kappa_1[nt]^{-3/2} \le \frac{\pi}{8} - \frac{1}{[nt]}\sum_{i=1}^{[nt]}\frac{i}{[nt]}\sqrt{\frac{[nt]}{i}-1} \le \kappa_2[nt]^{-3/2}.$$

Hence

$$\frac{\kappa_1}{\pi}n^{-3/2}t^{1/2} \le \mathbb{E}S_n^2 \le \frac{\kappa_2}{\pi}n^{-3/2}t^{1/2}.$$

This relation together with the fact that

$$\mathbb{E}\Big(\int_{\frac{[nt]}{n}}^t I_{\{X_s\ge 0\}}ds\Big)^2 \le n^{-2}$$

yield the desired result.

3.5. Proof of Theorem 2.4

We first state an auxiliary lemma.

Lemma 3.8. *Let B be a standard Brownian motion. For each $t_0 \in (0,1]$, there exist positive constants $\kappa_1, \kappa_2, \kappa_3$ such that for all $n > 0$,*

$$\mathbb{E}\Big|\frac{1}{n}\sum_{i=1}^n I_{\{B_{\frac{i}{n}} \ge B_{\frac{[nt_0]}{n}}\}} - \int_0^1 I_{\{B_u \ge B_{\frac{[nt_0]}{n}}\}}du\Big| \le \kappa_1 n^{-3/4}, \tag{3.17}$$

and if $t_0 < 1$,

$$\kappa_2\sqrt{t_0 - \frac{[nt_0]}{n}} \le \mathbb{E}\Big|\int_0^1 I_{\{B_u\ge B_{t_0}\}}du - \int_0^1 I_{\{B_u \ge B_{\frac{[nt_0]}{n}}\}}du\Big| \le \kappa_3\sqrt{t_0 - \frac{[nt_0]}{n}}. \tag{3.18}$$

Proof. Let denote $t_n = \frac{[nt_0]}{n}, n = 1, 2, \ldots$ for simplicity. We write

$$\begin{aligned}
&\frac{1}{n}\sum_{i=1}^{n} I_{\{B_{\frac{i}{n}} \geq B_{t_n}\}} - \int_0^1 I_{\{B_u \geq B_{t_n}\}} du \\
=&\frac{1}{n}\sum_{i=1}^{[nt_0]} I_{\{B_{\frac{i}{n}} - B_{t_n} \geq 0\}} - \int_0^{t_n} I_{\{B_u - B_{t_n} \geq 0\}} du \\
&+ \frac{1}{n}\sum_{i=[nt_0]+1}^{n} I_{\{B_{\frac{i}{n}} - B_{t_n} \geq 0\}} - \int_{t_n}^1 I_{\{B_u - B_{t_n} \geq 0\}} du.
\end{aligned}$$

It is well-known that for each n, processes $\hat{B} = \{B_{t+t_n} - B_{t_n}\}_{t \geq 0}$ and $\tilde{B} = \{B_{t_n - t} - B_{t_n}\}_{0 \leq t \leq t_n}$ are Brownian motions. By applying Proposition 2.3, we get (3.17).

Now we consider inequality (3.18). Since the case $t_0 = t_n$ is trivial, we suppose $t_n < t_0$. Moreover, it is sufficient to verify (3.18) when n is big enough, say, n satisfies $nt_0 > 1$ and $t_0 < \frac{n-1}{n}$.

Because of the symmetry of Brownian motion, we get

$$\mathbb{E}\Big| \int_0^1 I_{\{B_u \geq B_{t_0}\}} du - \int_0^1 I_{\{B_u \geq B_{\frac{[nt_0]}{n}}\}} du \Big| = 2\mathbb{E}\Big(\int_0^1 I_{\{B_{t_n} \geq B_u \geq B_{t_0}\}} du \Big). \tag{3.19}$$

We decompose the right hand side of (3.19) in a sum of three terms:

$$\mathbb{E}\Big(\int_0^1 I_{\{B_{t_n} \geq B_u \geq B_{t_0}\}} du \Big) = J_1 + J_2 + J_3,$$

where $J_1 = \mathbb{E}\big(\int_0^{t_n} I_{\{B_{t_n} \geq B_u \geq B_{t_0}\}} du \big)$, $J_2 = \mathbb{E}\big(\int_{t_n}^{t_0} I_{\{B_{t_n} \geq B_u \geq B_{t_0}\}} du \big)$, and $J_3 = \mathbb{E}\big(\int_{t_0}^{1} I_{\{B_{t_n} \geq B_u \geq B_{t_0}\}} du \big)$.

For the first term J_1, we write

$$\begin{aligned}
J_1 &= \int_0^{t_n} \mathbb{E}\big(I_{\{B_{t_n} \leq B_u \leq B_{t_0}\}} \big) du \\
&= \int_0^{t_n} \mathbb{E}\big(I_{\{0 \leq B_u - B_{t_n} \leq B_{t_0} - B_{t_n}\}} \big) du \\
&= \int_0^{t_n} du \int_0^{+\infty} dx \int_x^{+\infty} dy \frac{1}{2\pi\sqrt{(t_n - u)(t_0 - t_n)}} \exp\Big(-\frac{x^2}{2(t_n - u)} - \frac{y^2}{2(t_0 - t_n)} \Big) \\
&= \frac{1}{2\pi} \int_0^{t_n} \arctan\sqrt{\frac{t_0 - t_n}{u}} du.
\end{aligned}$$

It is elementary to show that for any $x \in (0, 1)$,

$$\frac{x}{2} \leq \arctan(x) \leq x.$$

Moreover, the upper bound also holds for arbitrary positive x. Hence, for all $t_0 > 0$,

$$J_1 \leq \frac{\sqrt{t_n}}{\pi}\sqrt{t_0 - t_n}, \tag{3.20}$$

and if $nt_0 \geq 1$,

$$J_1 \geq \frac{\sqrt{t_n} - \sqrt{t_0 - t_n}}{2\pi}\sqrt{t_0 - t_n}. \tag{3.21}$$

Using a similar argument, we have, for all $t_0 > 0$,

$$J_3 \leq \frac{\sqrt{1 - t_0}}{\pi}\sqrt{t_0 - t_n}, \tag{3.22}$$

and if $t_0 \leq \frac{n-1}{n}$,

$$J_3 \geq \frac{\sqrt{1 - t_0} - \sqrt{t_0 - t_n}}{2\pi}\sqrt{t_0 - t_n}. \tag{3.23}$$

Furthermore, it is trivial that

$$0 \leq J_2 \leq t_0 - t_n. \tag{3.24}$$

Putting together inequalities (3.20)-(3.24) we get (3.18). $\square$

Before proving Theorem 2.4, we state a corollary of Lemma 3.8, which has interest on its own.

Corollary 3.9. *Let B be a standard Brownian motion. The sequence of random variables*

$$n^{1/2}\Big(\frac{1}{n}\sum_{i=1}^{n} I_{\{B_{\frac{i}{n}} \geq B_{\frac{[nt_0]}{n}}\}} - \int_0^1 I_{\{B_u \geq B_{t_0}\}} du\Big)$$

is bounded in L^1. Furthermore, for any $\delta > 0$, the sequence of random variables

$$n^{1/2+\delta}\Big(\frac{1}{n}\sum_{i=1}^{n} I_{\{B_{\frac{i}{n}} \geq B_{\frac{[nt_0]}{n}}\}} - \int_0^1 I_{\{B_u \geq B_{t_0}\}} du\Big)$$

is not bounded in L^1.

Corollary 3.9 shows that in the case of Brownian motion, the discrete estimators of stochastic corridor converge at the rate $n^{-1/2}$ and this rate is exact in L^1-sense.

We now return to the proof of Theorem 2.4. Using the same notations as in Section 3.3, we write

$$\sqrt{n}\Big(\frac{1}{n}\sum_{i=1}^{n} I_{\{X_{\frac{i}{n}} \geq X_{\frac{[nt_0]}{n}}\}} - R(t_0)\Big) = \sqrt{n}\Big(\frac{1}{n}\sum_{i=1}^{n} I_{\{Y_{\frac{i}{n}} \geq Y_{\frac{[nt_0]}{n}}\}} - \int_0^1 I_{\{Y_u \geq Y_{t_0}\}} du\Big), \tag{3.25}$$

where Y is a standard Brownian motion under probability measure $\tilde{\mathbb{P}}$ which is equivalent to $\mathbb{P}$. Corollary 3.9 tells that the sequence of random variables in the right hand side of (3.25) is bounded in $L^1(\tilde{\mathbb{P}})$, which yields the desired result.

3.6. Proof of Proposition 2.5

By using the similar argument as above, we can show that Proposition 2.5 is followed from inequality (3.17).

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