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Bayesian improvements of a MRE estimator of a bounded location parameter

Éric Marchand[*,†]

Département de mathématiques, Sherbrooke, Canada J1K 2R1
e-mail: eric.marchand@usherbrooke.ca

and

Amir T. Payandeh Najafabadi[†]

Department of Mathematical Sciences, Shahid Beheshti University,
G.C. Evin, 1983963113, Tehran, Iran
e-mail: amirtpayandeh@sbu.ac.ir

Abstract: We study the frequentist risk performance of Bayesian estimators of a bounded location parameter, and focus on conditions placed on the shape of the prior density guaranteeing dominance over the minimum risk equivariant (MRE) estimator. For a large class of even and logconcave densities, even convex loss functions, we demonstrate in a unified manner that symmetric priors which are bowled shaped and logconcave lead to Bayesian dominating estimators. The results generalize similar results obtained by Marchand and Strawderman for the fully uniform prior, as well as those obtained by Kubokawa for squared error loss. Finally, we present a detailed example and several remarks.

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1. The problem

Consider the restricted parameter space estimation problem:

$$X \sim f_0(x-\theta), \text{ loss is } \rho(d-\theta), \theta \in [a,b], \tag{1}$$

where f_0 is a positive Lebesgue density and ρ is convex. Without the restriction $\theta \in [a,b]$, a benchmark estimator is the minimum risk equivariant estimator δ_{mre}, which is also minimax and Bayes with respect to the flat (or noninformative) prior on $(-\infty,\infty)$. In view of the compact interval restriction, the frequentist risk performance of Bayesian alternatives δ_π with respect to prior densities π supported on $[a,b]$, or a subset of $[a,b]$, is of interest. Characterizing Bayesian estimators δ_π, or the prior densities π themselves, that guarantee that δ_π dominate δ_{mre} is of particular interest. In this regard, Marchand and Strawderman [11] (see also [4]) showed that indeed that the fully uniform prior Bayes estimator δ_U dominates δ_{mre} quite generally with respect to f_0 and convex ρ, (or again logconcave f_0 and strict bowl-shaped ρ). The result is achieved, in an unified way with respect to f_0 and ρ and by using Kubokawa's [3] IERD method, via general conditions for an estimator δ to dominate δ_{mre} and showing then that δ_U satisfies these conditions. By making use of such conditions, Kubokawa ([4], Proposition 3.1) provides quite elegant, simple and useful conditions on f_0 and π for δ_π to dominate δ_{mre} for squared error ρ, reproduced here in a slightly weaker version.

Lemma 1 (Kubokawa, [4])**.** *For problem (1) with f_0 even, unimodal and logconcave*[1]*, and squared error ρ, δ_π dominates δ_{mre} whenever the density π is symmetric about $\frac{a+b}{2}$, logconcave on (a,b), and nondecreasing on $(\frac{a+b}{2}, b)$.*

We see indeed that the conditions for dominance are qualitatively appealing. Moreover, we argue that they capture the essential features of priors which lead to dominance. Indeed, these prior densities are bowled shaped in contrast to unimodal priors π which may lead to large frequentist risk $R(\theta,\delta_\pi)$ for θ on or near the boundary $\{a,b\}$. They also do not rise too sharply from the center, as controlled by the logconcavity condition, as otherwise, such as the case of too sharply increasing densities moving away from the center, the frequentist risk may be too large in the center $\frac{a+b}{2}$ of the parameter space.

However, while Marchand and Strawderman's dominance conditions and proof of the superiority of δ_U on δ_{mre} applies for a large class of losses ρ, Lemma 1 is limited to squared error ρ and Kubokawa's proof does indeed exploit analytical properties of δ_π specific to squared error loss. The main motivation and finding below thus consists of a generalization (Theorem 1) of Lemma 1 to a larger class of losses, where convex ρ and logconcave ρ' (the latter is a weak condition) are required. Moreover, the proof is unified with respect to (f_0,ρ), and we believe that it even simplifies some of the intricate analysis of Kubokawa's proof for the squared error case. It also brings into focus precise analytical properties (Lemmas 4 and 5) of Bayes estimators in restricted parameter spaces which will be

[1] It is true that the symmetry and unimodality would suffice as these imply unimodality.

of use in similar problems. An illustration of our results (Example 1) completes the presentation.

In contrast to the problem of dominating the (more plausible) truncation of δ_{mre}, where conditions applicable to δ_π ([6, 7, 8, 9]) are limited to not too large parameter spaces (i.e., $b-a$ not too large), the results of Lemma 1 and those below apply regardless of the given parameter space $[a,b]$. There has been a fair amount of work on decision theoretic approaches to restricted parameter space problems (see for instance [10, 13]), with many remaining challenges such as those solved with Lemma 1 and its generalization below. As an example, Hartigan [2] studies a multivariate version of problem (1) with $X \sim N_p(\theta, I_p)$, loss $\|d-\theta\|^2$, $\theta \in C$ where C is a convex set with a non-empty interior, and shows that the fully uniform Bayes estimator δ_U (although here the prior can be improper) dominates quite generally X regardless of C and p. His results apply to problem (1) but have not been extended to other priors, or to other models or losses for $p > 1$.

2. Main results

For problem (1), a minimum risk equivariant estimator δ_{mre} exists and is, under mild conditons which we assume, uniquely given by $X + c_0$, where c_0 minimizes the constant risk $R(\theta, X+c) = E_0[\rho(X+c)]$ in c. Unless specified otherwise, we take throughout in (1) $a = -m, b = m$ without loss of generality, we assume f_0 to be even, logconcave, and we further assume that ρ is absolutely continuous, symmetric about 0, and strict bowled-shaped such that $\rho \geq 0$, $\rho(0) = 0$, $\rho'(u) < 0$ for $u < 0$ and $\rho'(u) > 0$ for $u > 0$. We point out that the logconcave assumption on f_0 equates to a strict monotone likelihood ratio (mlr) property for the family of densities of X. Under the above assumptions, the MRE estimator is simply X.

Marchand and Strawderman's conditions for an estimator $X + h(X)$ to dominate δ_{mre} specialize as follows for Bayesian estimators δ_π associated with symmetric densities π. The conditions are rather simple, qualitatively appealing, and bring into play the benchmark fully uniform Bayes estimator δ_U.

Lemma 2. *For problem (1) with f_0 even, with ρ symmetric about 0, logconcave and strict bowled-shaped, the Bayes estimator $\delta_\pi(x) = x + h_\pi(x)$ with respect to a symmetric about 0 prior density π, dominates $\delta_{mre}(X) = X$ whenever*

(i) *$h_\pi(x)$ decreases in x, for $x > 0$;*
(ii) *and $|\delta_\pi| \geq |\delta_U|$.*

Proof. From Marchand and Strawderman's ([11], Theorem 5.1(ii) and Remark 5.1) sufficient conditions for dominance, we require for $X + h_\pi(X)$ to dominate $\delta_{\text{mre}}(X)$: **(a)** $h_\pi(\cdot)$ to be nonincreasing on $\Re$, **(b)** there exists x_0 such that $\delta_\pi(x_0) = \delta_U(x_0) = \delta_{\text{mre}}(x_0)$, and **(c)** and $|h| \leq |h_U|$; where $x + h_U(x)$ is the fully uniform Bayes estimator. Now, under the given symmetry assumptions on f_0, ρ, and π, $\delta_\pi(\cdot)$ will be an odd function (i.e., equivariant with respect to a sign change) and hence $\delta_\pi(x) = -\delta_\pi(-x)$, or equivalently $h_\pi(x) = -h_\pi(-x)$. Thus, **(a)** is equivalent to **(i)**, **(b)** is satisfied with $x_0 = 0$, and **(c)** is equivalent to **(ii)**, thus establishing the result. □

Synthesizing the previous lemma, our task is simple to describe: select Bayesian estimators which shrink (towards 0) on the δ_{mre} and increasingly as $|x|$ increases (i.e., $|x-\delta_\pi(x)|$ increases in $|x|$), but at the same time expand (away from 0) on the fully uniform Bayes estimator δ_U. The difficulty, of course, is relating these features to the prior π. We pursue with a useful lemma.

Lemma 3. *Suppose g_1 and g_0 are two distinct, positive densities on (a,b) with respect to a $\sigma-$ finite measure μ, such that $\frac{g_1}{g_0}$ increases on (a,b). Consider a function $k(\cdot)$ that changes signs exactly once from $-$ to $+$ on (a,b), in the sense that there exists $y_0 \in (a,b)$ such that $k(y) < 0$ for $y < y_0$, and $k(y) > 0$ for $y > y_0$. Then, we have $E_{g_0}[k(Y)] = 0 \Rightarrow E_{g_1}[k(Y)] > 0$, and $E_{g_1}[k(Y)] = 0 \Rightarrow E_{g_0}[k(Y)] < 0$.*

Proof. The result is known. Here is a short proof for completeness. We have

$$\begin{aligned} E_{g_0}[k(Y)] &= \int_a^{y_0} k(y)\frac{g_0(y)}{g_1(y)}\, g_1(y)\, d\mu(y) + \int_{y_0}^b k(y)\frac{g_0(y)}{g_1(y)}\, g_1(y)\, d\mu(y) \\ &< \frac{g_0(y_0)}{g_1(y_0)} \int_a^b k(y)\, g_1(y) d\mu(y) \;=\; \frac{g_0(y_0)}{g_1(y_0)} E_{g_1}[k(Y)], \end{aligned}$$

and the result follows directly. □

Here below is a key lemma which addresses condition **(i)** of Lemma 2, showing that priors with logconcave densities on $(-m,m)$ lead to Bayes estimates $\delta_\pi(x)$ such that $\delta_\pi(x) - x$ decreases in x. The result, which applies for location models and invariant losses with strict mlr densities and strict bowl-shaped losses, or positive densities and convex losses; generalizes Marchand and Strawderman's ([11], Lemma 5.1) uniform prior result, as well as Kubokawa's ([4], Proposition 3.1.) squared error loss result. We do not assume for this lemma that f_0 is symmetric, nor that ρ or π is symmetric so that the phenomenon is actually much more general than required here.

Lemma 4. *For problem (1) with b=-a=m, logconcave f_0, and strict bowled-shaped ρ, the Bayes estimator $\delta_\pi(x) = x + h_\pi(x)$ possesses the property that $h_\pi(x) = \delta_\pi(x) - x$ decreases in x, for all $x \in \Re$, as long as the prior density π is logconcave on $(-m,m)$.*

Proof. The Bayes estimate $\delta_\pi(x)$ minimizes the expected posterior loss $E(\rho(d-\theta)|x)$ in d, and hence satisfies the equation $\int_{(-m,m)} \rho'(\delta_\pi(x)-\theta)\, f_0(x-\theta)\, \pi(\theta) d\theta = 0$, for all x, or equivalently

$$E_x[\rho'(h_\pi(x) + c_0 + U)] = 0; x \in \Re; \tag{2}$$

where $U =^d X - \theta|x$ has density $f_{U,x}(u)$ supported on $(x-m, x+m)$ and proportional to $f_0(u)\, \pi(-u+x)$. Now observe that, for $x_1 > x_0$, the ratio $\frac{f_{U,x_1}(u)}{f_{U,x_0}(u)} \propto \frac{\pi(-u+x_1)\, 1_{(x_1-m,x_1+m)}(u)}{\pi(-u+x_0)\, 1_{(x_0-m,x_0+m)}(u)}$ is increasing in u given the logconcavity of π. Hence, the family of densities $f_{U,x}$ possesses an increasing monotone likelihood ratio (in U) with parameter x. Now, take any $x_1 > x_0$ and suppose in order to arrive at a contradiction that $a' = h_\pi(x_1) - h_\pi(x_0) > 0$. Then, on one hand, with equation (2) and the mlr property of the densities $f_{U,x}$, Lemma 3 would tell us that

$$0 = E_{x_1}[\rho'(h_\pi(x_1) + c_0 + U)] > E_{x_0}[\rho'(h_\pi(x_1) + c_0 + U)]\,.$$

On the other hand, observe that the family of densities of $T = h_\pi(x_0)+c_0+a+U$, with $a > 0$ possesses an increasing mlr property as well with parameter a, so that a further application of (2) and Lemma 3 would tell us that

$$\begin{aligned} E_{x_0}[\rho'(h_\pi(x_1) + c_0 + U)] &= E_{a'}[\rho'(T)] > E_{a=0}[\rho'(T)] \\ &= E_{x_0}[\rho'(h_\pi(x_0) + c_0 + U)] = 0\,, \end{aligned}$$

and would thus lead to a contradiction. Hence, we must have $h_\pi(x_1) \leq h_\pi(x_0)$. □

Remark 1. The result also holds for positive densities and convex losses as the key property of a decreasing monotone likelihood ratio of the posterior densities of $U = \theta - x$, with parameter x, follows from the logconcavity of the prior, and the sign change argument of Lemma 3 is not required when ρ' is increasing. In both this case, and the situation in the lemma, the Bayes estimators δ_π are (essentially) unique (see [11], page 133, for a similar situation).

There remains to address condition **(ii)** of Lemma 2. We prove in what follows a more general result ordering the absolute value of Bayes estimates δ_{π_1} and δ_{π_0} in cases where the ratio of densities $\frac{\pi_1(\theta)}{\pi_0(\theta)}$ is monotone in $|\theta|$. This quite plausible property, which we will exploit for $\delta_{\pi_0} \equiv \delta_U$, is very useful as it involves simple conditions for which δ_{π_1} expands, or shrinks, on δ_{π_0}. A squared error loss version of the following lemma was given by [9], while a multivariate normal and squared error loss version was previously established by [7], so that the novel feature below lies with the departure from squared error loss. We assume below that ρ is convex, even, and that ρ' is logconcave on $(0,\infty)$. The latter assumption is weak, includes L^p losses $|d-\theta|^p$, with $p \geq 1$, Linex loss $\rho_\alpha(t) = e^{\alpha t} - \alpha t - 1$ and the symmetrized version $\rho_\alpha(t) + \rho_\alpha(-t) = e^{\alpha t} + e^{-\alpha t} - 2$, but discounts very sharp penalizing losses such as $e^{|d-\theta|^p}$ with $p > 1$.

Lemma 5. *Consider problem (1) with b=-a=m, f_0 logconcave and even, ρ convex and even, and such that ρ' is logconcave on $(0,\infty)$. Suppose that π_0 and π_1 are symmetric about 0 prior densities with respect to a σ-finite measure μ such that $\frac{\pi_1(\theta)}{\pi_0(\theta)}$ increases in $\theta \in [0,m]$. Then, we have $|\delta_{\pi_1}(x)| \geq |\delta_{\pi_0}(x)|$ for all $x \in \Re$.*

Proof. As in Lemma 2, $\delta_{\pi_1}(\cdot)$ and $\delta_{\pi_0}(\cdot)$ will be odd given the assumptions on $(f_0, \rho, \pi_1, \pi_0)$, so that we only need consider $x > 0$. Now, for an even density π, the Bayes estimator δ_π satisfies, for all $x > 0$:

$$\begin{aligned} & \int_{[-m,m]} f_0(x-\theta)\,\pi(\theta)\,\rho'(\delta_\pi(x)-\theta)\,d\mu(\theta) = 0, \\ \Longleftrightarrow \quad & \int_{[0,m]} k_\pi(x,\lambda)\,\pi(\lambda)\,d\mu(\lambda) \;= 0\,, \qquad (3) \end{aligned}$$

where $k_\pi(x,\lambda) = \rho'(\delta_\pi(x)-\lambda)\, f_0(x-\lambda) + \rho'(\delta_\pi(x)+\lambda)\, f_0(x+\lambda)$, $x>0$, $\lambda \in [0,m]$. Turning to the sign of $k_\pi(x,\cdot)$ for a fixed $x>0$, observe that $k_\pi(x,\cdot)$ must change sign at least once on $[0,m]$ given (3), and that $k_\pi(x,\cdot)$ is positive on $[0,\delta_\pi(x)]$ given the properties of ρ. For $\lambda > \delta_\pi(x)$, we write with f_0 even and ρ' odd:

$$k_\pi(x,\lambda) = \{ \frac{\rho'(\lambda+\delta_\pi(x))}{\rho'(\lambda-\delta_\pi(x))} - \frac{f_0(\lambda-x)}{f_0(\lambda+x)} \} f_0(\lambda+x)\, \rho'(\lambda-\delta_\pi(x))\,.$$

We infer from the above that $k_\pi(x,\cdot)$ changes signs once from $+$ to $-$ on $(\delta_\pi(x),m]$ since: **(i)** $\frac{\rho'(\lambda+\delta_\pi(x))}{\rho'(\lambda-\delta_\pi(x))}$ decreases in λ given the logconcavity of ρ', **(ii)** $\frac{f_0(\lambda-x)}{f_0(\lambda+x)}$ increases in λ given the logconcavity of f_0, and **(iii)** $\rho'(\lambda-\delta_\pi(x))>0$ for $\lambda > \delta_\pi(x)$. Now, we fix $x>0$ and we assume in order to arrive at a contradiction that $\delta_{\pi_1}(x) < \delta_{\pi_0}(x)$ which would imply $k_{\pi_1}(x,\lambda) < k_{\pi_0}(x,\lambda)$ given that ρ' is increasing (i.e., ρ convex). We apply Lemma 3 to $-k_\pi(x,\lambda)$ to infer that under this assumption, we would have

$$\begin{aligned}\int_{[0,m]} k_{\pi_0}(x,\lambda)\pi_0(\lambda)d\mu(\lambda) \quad &> \quad \int_{[0,m]} k_{\pi_0}(x,\lambda)\pi_1(\lambda)d\mu(\lambda) \\ &> \quad \int_{[0,m]} k_{\pi_1}(x,\lambda)\pi_1(\lambda)d\mu(\lambda)\,,\end{aligned}$$

which is not possible given (3) applied to π_0 and π_1, and concludes the proof. □

Remark 2. We point out that Lemma 5 holds for non-symmetric π_0 or π_1 by replacing the condition of the monotone increasing ratio $\frac{\pi_1(\cdot)}{\pi_0(\cdot)}$ by the monotone increasing ratio $\frac{\pi_1^*(\cdot)}{\pi_0^*(\cdot)}$ (on $[0,m]$), where π^* is the density of $\lambda = |\theta|$.

Having addressed conditions **(i)** and **(ii)** of Lemma 2, our main result, which generalizes Lemma 1, follows immediately from Lemma 2, Lemma 4 and Lemma 5.

Theorem 1. *For problem (1) with f_0 even and logconcave, with ρ convex and even, and such that ρ' is logconcave on $(0,\infty)$, we have that δ_π dominates δ_{mre} whenever the density π is symmetric about $\frac{a+b}{2}$, logconcave on (a,b), and nondecreasing on $(\frac{a+b}{2},b)$.*

We pursue with an illustration.

Example 1. We study applications of our findings for a normal model with $X \sim N(\theta,1)$ and $\theta \in [-m,m]$, $m>0$. Consider prior densities $\pi_a(\theta) \propto e^{a|\theta|} 1_{(-m,m)}(\theta)$ with $a \geq 0$. Such choices satisfy the conditions of Theorem 1 (nondecreasing on $(0,m)$, even, logconcave) and include the uniform on $(-m,m)$ case for $a=0$. Noting ϕ and Φ the pdf and cdf respectively of a standard normal distribution, we obtain that the posterior density is given by

$$\pi_a(\theta|x) = \frac{1}{k}\{\phi(\theta-(x-a))\, 1_{(-m,0)}(\theta) + \phi(\theta-(x+a))\, 1_{(0,m)}(\theta)\}\,,$$

with $k = \Phi(a-x) - \Phi(-m+a-x) + \Phi(m-x-a) - \Phi(-x-a)$. Interestingly, this posterior density is seen to as a superimposition of two truncated normals on $(-m, 0)$ and $(0, m)$ with means $x - a$ and $x + a$ respectively. When $a = 0$, these means coincide and we simply obtain a truncated $N(x, 1)$ on $(-m, m)$. Now, Theorem 1 tells us that the associated Bayes estimators δ_{π_a} dominates necessarily δ_{mre} for all $a \geq 0$ and for losses ρ that satisfy the given conditions. Such losses include the interesting case of absolute value loss. In this case, the posterior median is computable directly from the above posterior density yielding the dominating estimator $\delta_{\pi_a}(x) = x - a + \Phi^{-1}(\frac{k}{2} + \Phi(-m+a-x))$, for $x \leq 0$, and $\delta_{\pi_a}(x) = -\delta_{\pi}(-x)$ for $x > 0$.

Recapitulating the dominance findings of this paper in an historical context which are applicable to the **normal case** and to the π_a's above, we begin by pointing out that the dominance result for the specific case $a = 0$ and $\rho(t) = t^2$ is due to [1]. Extensions for $a = 0$ to other losses ρ are due to [11], while extensions to $a > 0$ with $\rho(t) = t^2$ follow from [4](i.e., Lemma 1). The main dominance findings of this paper unifies and generalizes the above for $a > 0$ and for losses ρ satisfying the conditions of Theorem 1. Finally, we do reemphasize the greater generality of Kubokawa's and our findings to other (than normal) symmetric and logconcave densities (and to other priors); as well as Marchand and Strawderman's results for the uniform prior with respect to many other asymmetric pairs (f_0, ρ).

3. Concluding remarks

We have demonstrated that essential features of the prior density π on $[-m, m]$, namely symmetry, bowl-shapedness and logconcavity are persistent conditions for a Bayes estimator δ_π to dominate δ_{mre} in problem (1), for a wide class of models f_0 and losses ρ. Moreover, the approach is unified and extends or complements previous results of Kubokawa, and Marchand and Strawderman. The main application for a finite i.i.d. sample undoubtedly arises for a normal model f_0 giving sufficiency (but see [4] or [11] for some developments in this regard). Implications for some scale parameter problems $X_1 \sim \frac{1}{\sigma} f_1(\frac{x_1}{\sigma})$ with σ restricted to a interval are available as corollaries with the transformation $X_1 \to \log(X_1)$ to a location problem (see [12]), but are not pursued here. Given the symmetric condition on the location density, the original scale parameter problem will require the equidistributional property $\frac{X_1}{\sigma} \sim (\frac{X_1}{\sigma})^{-1}$ with common examples of such distributions given by lognormal, half-Cauchy and Fisher with equal degrees of freedom on numerator and denominator.

Potential findings for other unsolved problems may well benefit by the methods and results of this paper. These include: **(i)** asymmetric versions where either f_0 or ρ is not symmetric; and **(ii)** extensions to location-scale problems with a bounded coefficient of variation (e.g., [5]). Finally, with Hartigan's result applying for the model $X \sim N_p(\theta, I_p)$ with a constraint to a ball, such as $\|\theta\| \leq m$, and guaranteeing that the fully uniform Bayes estimator dominates X under loss $\|d - \theta\|^2$, extensions to other spherically symmetric priors, or to other losses, or to other spherically symmetric models, such as those obtained here for $p = 1$, seem plausible and quite interesting to pursue.

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