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LAPLACE APPROXIMATION FOR ROUGH DIFFERENTIAL EQUATION DRIVEN BY FRACTIONAL BROWNIAN MOTION

BY YUZURU INAHAMA

Nagoya University

We consider a rough differential equation indexed by a small parameter $\varepsilon > 0$. When the rough differential equation is driven by fractional Brownian motion with Hurst parameter H $(1/4 < H < 1/2)$, we prove the Laplace-type asymptotics for the solution as the parameter ε tends to zero.

1. Introduction. The rough path theory was invented by T. Lyons in [30] and summarized in a book [29] with Z. Qian. See also [16, 27, 31]. Roughly speaking, a rough path is a path coupled with its iterated integrals. T. Lyons generalized the line integral of one-form along a path to the one along a rough path. This is a pathwise integral theory and no probability measure is involved. In a natural way, an ordinary differential equation (ODE) is generalized. This is called a rough differential equation (RDE) in this paper. The corresponding Itô map is not only everywhere defined, but is also locally Lipschitz continuous with respect to the topology of geometric rough path space (Lyons' continuity theorem). If a Wiener-like measure is given on the geometric rough path space or, in other words, if a Brownian rough path is mapped by the Itô map, then the solution of the corresponding stochastic differential equation (SDE) of Stratonovich-type is recovered via rough paths. In order to investigate the Brownian motion, one only needs the double integral (i.e., the second level path) as well as the path itself (i.e., the first level path). In short, we can obtain the solution of an SDE as the image of a continuous map. This is basically impossible in the framework of the usual stochastic calculus. Recall that, in the usual stochastic calculus, stochastic integrals and SDEs are defined by the martingale integration theory, which is quite probabilistic by definition. Therefore, those objects have no pathwise meaning.

Brownian motion and Brownian rough path are most important and were studied extensively. There may be other stochastic processes (i.e., probability measures on the usual path space), however, which can be lifted to probability measures on the geometric rough path space. The most typical example is the d-dimensional fractional Brownian motion (fBM) $(w_t^H)_{0\le t\le 1} = (w_t^{H,1}, \dots, w_t^{H,d})_{0\le t\le 1}$ with Hurst parameter $H \in (1/4, 1/2]$ (see Coutin–Qian [10]). Recall that, when $H = 1/2$, it is the Brownian motion. It is worth noting that, if $H \in (1/4, 1/3]$, the third level

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path plays a role, unlike the Brownian motion case. The Schilder-type large deviation for the lift of scaled fBM was proved by Millet and Sanz-Sole [32]. Combined with Lyons' continuity theorem and the contraction principle, this fact implies that the solution of an RDE driven by the lift of scaled fBM also satisfies large deviation.

According to [8, 9], there are several types of path integrals along fBM, namely, (1) deterministic or pathwise integral, (2) integral with generalized covariation, (3) the divergence operator in the sense of the Malliavin calculus, and (4) White noise approach. Clearly, the rough path approach belongs to the first category.

More precisely, we consider the following RDE: for $\varepsilon > 0$,

$$dY_t^\varepsilon = \sigma(Y_t^\varepsilon)\varepsilon\, dW_t^H + \beta(\varepsilon, Y_t^\varepsilon)\, dt, \qquad Y_0^\varepsilon = 0. \tag{1.1}$$

Here, W^H is the fractional Brownian rough path (fBRP), that is, the lift of fBM w^H and $\sigma \in C_b^\infty(\mathbf{R}^n, \mathrm{Mat}(n,d))$ and $\beta \in C_b^\infty([0,1] \times \mathbf{R}^n, \mathbf{R}^n)$. Note that C_b^∞ denotes the set of bounded smooth functions with bounded derivatives.

The main purpose of this paper is to prove the Laplace approximation for (the first level path of) Y^ε as $\varepsilon \searrow 0$. The precise statement is in Theorem 2.1 below. Apparently, in none of the integrals (1)–(4) has the Laplace approximation been proved for the solution of SDE (or RDE) driven by the scaled fBM. Note that it is the precise asymptotics of the large deviation. In this paper, we will prove it in the framework of the rough path theory for $H \in (1/4, 1/2)$. [The case $H > 1/2$ is not so interesting from a viewpoint of rough path analysis. Our method does not work for the case $H \leq 1/4$ since the relation for the Young integral $1/p + 1/q > 1$ in equation (4.2) below fails to hold.]

The history of this kind of problem is long. A partial list could be as follows. First, Azencott [4] showed this kind of asymptotics for finite dimensional SDEs, which is followed by Ben Arous [7]. There are similar results for infinite dimensional SDEs (e.g., Albeverio–Röckle–Steblovskaya [3]) as well as SPDEs (e.g., Rovira–Tindel [35]). In the framework of the Malliavin calculus, there are deep results on the asymptotics of the generalized expectation of generalized Wiener functionals (Takanobu–Watanabe [36], Kusuoka–Stroock [25, 26], Kusuoka–Osajima [24]) which have applications to the asymptotics for the heat kernels on Riemannian manifolds.

In the framework of the rough path theory, Aida studied this problem for finite dimensional Brownian rough paths and gave a new proof for the results in [4, 7]. The same problem for infinite dimensional Brownian rough paths was studied in [18, 20], which has an application to Brownian motion over loop groups.

The organization of this paper is as follows: In Section 2 we give a precise statement of our main result. In Section 3 we review the rough path theory and fractional Brownian rough path. In Section 4 we prove the Hilbert–Schmidt property of the Hessian of the Itô map restricted on the Cameron–Martin space $\mathcal{H}^H$ of fBM. For those who understand the proof of Laplace approximation for Brownian rough path as in [2, 18, 20], this is the most difficult part, because the Cameron–Martin space of fBM is not understood very well. However, thanks to Friz–Victoir's result (Proposition 3.4), such Cameron–Martin paths are Young integrable and, therefore, the Hessian is computable. In Section 5 we give a probabilistic representation of (the stochastic extension of) the Hessian. In Section 6 we give a proof of the main theorem. In Section 7 we consider the Laplace approximation for an RDE, which involves a fractional order term of $\varepsilon > 0$. This has an application to the short time asymptotics of integral quantities of the solution of a fixed RDE driven by fBM. (Similar problems were studied in [5, 34]).

REMARK 1.1. All the results in this paper hold for the case $H = 1/2$, too, with trivial modifications. The only reason we do not treat the case $H = 1/2$ (i.e., the usual Brownian case) is because those results are already well known in that case.

2. Statement of main result.

2.1. *Assumption and main result.* In this section we state our main results in this paper. Throughout this paper, the time interval is [0, 1] except otherwise stated. Let $1/4 < H < 1/2$ and let $\mathcal{H}^H$ be the Cameron–Martin subspace of the d-dimensional fBM $(w_t^H)_{0 \le t \le 1}$. By Friz–Victoir's result, which will be explained in Proposition 3.4 below, $k \in \mathcal{H}^H$ is of finite q-variation for any $(H + 1/2)^{-1} < q < 2$. Hence, the following ODE makes sense in the q-variational setting in the sense of the Young integration:

$$dy_t = \sigma(y_t)\, dk_t + \beta(0, y_t)\, dt, \qquad y_0 = 0.$$

Note that y is again of finite q-variation and we will write $y = \Psi(k)$.

Now we set the following assumptions. In short, we assume that there is only one point that attains the minimum of F_Λ and the Hessian at the point is nondegenerate. These are typical assumptions for Laplace's method of this kind. The space of continuous paths in $\mathbf{R}^n$ with finite p'-variation starting at 0 is denoted by $C_0^{p'\text{-var}}(\mathbf{R}^n)$. Note that the self-adjoint operator A in the fourth assumption turns out to be Hilbert–Schmidt in Theorem 4.1 below.

(H1): F and G are real-valued bounded continuous functions on $C_0^{p'\text{-var}}(\mathbf{R}^n)$ for some $p' > 1/H$.

(H2): The function $F_\Lambda := F \circ \Psi + \| \cdot \|^2_{\mathcal{H}^H}/2$ attains its minimum at a unique point $\gamma \in \mathcal{H}^H$. We will write $\phi^0 = \Psi(\gamma)$.

(H3): F and G are $m+3$ and $m+1$ times Fréchet differentiable on a neighborhood $U(\phi^0)$ of $\phi^0 \in C_0^{p'\text{-var}}(\mathbf{R}^n)$, respectively. Moreover, there are positive constants $M_1, M_2, \ldots$ such that

$$|\nabla^j F(\eta)\langle z, \ldots, z\rangle| \le M_j \|z\|^j_{p'\text{-var}} \qquad (j = 1, \ldots, m+3),$$

$$|\nabla^j G(\eta)\langle z, \ldots, z\rangle| \le M_j \|z\|^j_{p'\text{-var}} \qquad (j = 1, \ldots, m+1)$$

hold for any $\eta \in U(\phi^0)$ and $z \in C_0^{p'\text{-var}}(\mathbf{R}^n)$.

(H4): At the point $\gamma \in \mathcal{H}^H$, the bounded self-adjoint operator A on $\mathcal{H}^H$, which corresponds to the Hessian $\nabla^2(F \circ \Psi)(\gamma)|_{\mathcal{H}^H \times \mathcal{H}^H}$, is strictly larger than $-\mathrm{Id}_{\mathcal{H}^H}$ (in the form sense).

Under these assumptions, the following Laplace-type asymptotics hold. Explicitly, the constant $c = \nabla F(\phi^0)\langle \theta^1 \rangle$, where θ^1 will be given in (6.4) below. [Below, $Y^{\varepsilon,1} = (Y^\varepsilon)^1$ denotes the first level path of Y^ε.]

THEOREM 2.1. *Let the coefficients* $\sigma : \mathbf{R}^n \to \mathrm{Mat}(n, d)$ *and* $\beta : [0, 1] \times \mathbf{R}^n \to \mathbf{R}^n$ *be* C_b^∞. *Then, under Assumptions* (H1)–(H4), *we have the following asymptotic expansion as* $\varepsilon \searrow 0$: *there are real constants* c *and* $\alpha_0, \alpha_1, \dots$ *such that*

$$\begin{aligned}&\mathbb{E}\big[G(Y^{\varepsilon,1}) \exp\big(-F(Y^{\varepsilon,1})/\varepsilon^2\big)\big] \\ &\qquad = \exp\big(-F_\Lambda(\gamma)/\varepsilon^2\big) \exp(-c/\varepsilon) \cdot \big(\alpha_0 + \alpha_1\varepsilon + \cdots + \alpha_m \varepsilon^m + O(\varepsilon^{m+1})\big)\end{aligned}$$

for any $m \geq 0$.

REMARK 2.2. The only reason for the boundedness assumption for σ and b is for safety. It is an important and difficult problem whether Lyons' continuity theorem holds for unbounded coefficients under a mild growth condition. (One of such attempts can be found in [17]). If we have such an extension of the continuity theorem, then Theorem 2.1 could easily be generalized because localization around γ is crucially used in the proof (see Section 6 below).

2.2. *A heuristic "proof"*. Some readers who are not familiar with Laplace-type asymptotics may find the argument in this paper too complicated. So, in this subsection, we try to help them get a bird's eye view of the proof of the main theorem. The argument in this subsection is very heuristic and has no rigorous meaning.

In this subsection, we denote a generic (rough) path by w (instead of w^H or W^H) and assume for simplicity that $G \equiv 1$ and β is independent of ε so that $Y^\varepsilon = \Psi(\varepsilon w)$ holds at least formally. As physicists often do, we will write the law of fBM μ^H heuristically as $\mu^H(dw) = Z^{-1} \exp(-|w|^2_{\mathcal{H}^H}/2)\mathcal{D}w$, where $\mathcal{D}w$ is the nonexistent "Lebesgue measure" and Z is a "normalizing constant."

In this case, we study the following quantity:

$$\begin{aligned}&\int \exp\Big(-\frac{F(\Psi(\varepsilon w))}{\varepsilon^2}\Big)\mu^H(dw) \\ &\qquad = \frac{1}{Z}\int \exp\Big(-\frac{F \circ \Psi(\varepsilon w) + |\varepsilon w|^2_{\mathcal{H}^H}/2}{\varepsilon^2}\Big)\mathcal{D}w.\end{aligned} \tag{2.1}$$

Note that the functional F_Λ in (H2) appears on the right-hand side above. It achieves the minimum at γ. So, as in the calculus for freshmen, one can easily imagine that $(1/2) \times$ Hessian at $w = \gamma$ plays a very important role.

Let us continue. By shifting $w \mapsto w + (\gamma/\varepsilon)$, the right-hand side of (2.1) is equal to

$$
\begin{aligned}
&\frac{1}{Z}\int \exp\Big(-\frac{F\circ\Psi(\varepsilon w+\gamma)+|\varepsilon w+\gamma|^2_{\mathcal{H}^H}/2}{\varepsilon^2}\Big)\mathcal{D}w\\
&\quad=\frac{1}{Z}\int \exp\Big[-\frac{1}{\varepsilon^2}\Big\{F_\Lambda(\gamma)+\varepsilon\cdot 0+\frac{\varepsilon^2}{2}(\langle Aw,w\rangle+|w|^2_{\mathcal{H}^H})+O(\varepsilon^3)\Big\}\Big]\mathcal{D}w\\
&\quad=e^{-F_\Lambda(\Lambda)/\varepsilon^2}\int \exp\Big[-\frac{\langle Aw,w\rangle}{2}+O(\varepsilon)\Big]\mu^H(dw)\\
&\quad\sim e^{-F_\Lambda(\Lambda)/\varepsilon^2}\int \exp\Big[-\frac{\langle Aw,w\rangle}{2}\Big]\mu^H(dw)\qquad \text{as } \varepsilon\searrow 0.
\end{aligned}
\tag{2.2}
$$

Note that what we did is the Taylor expansion of $F\circ\Psi$ at $w=\gamma$. Here, the first order term vanishes, because γ is a stationary point. When the Taylor expansion as above is done, some kind of localization is usually necessary. In this case, however, we can localize around γ, thanks to the large deviation principle.

As we have seen, the Taylor expansion of $F\circ\Psi$ plays a central role. Since we assumed Fréchet differentiability of F around $\phi^0=\Psi(\gamma)$, the key point is the Taylor expansion of the Itô map $w\mapsto\Psi(w)$ around γ. This part is rather hard, but was already proved in the author's previous paper [19]. See Theorem 3.2 below, which is a special case of the result in [19].

To make this argument rigorous, we also need integrability of $\exp(-\langle A\bullet,\bullet\rangle/2)$ with respect to μ^H. If A is Hilbert–Schmidt and $A>-\mathrm{Id}$ in the form sense, this is integrable and its expectation is written in terms of the Carleman–Fredholm determinant $\det_2(\mathrm{Id}+A)$. See Lemma 5.4 and Remark 5.5. Therefore, it is important to prove the Hilbert–Schmidt property of A. (Precisely, the quadratic form $\langle A\bullet,\bullet\rangle$ must be replaced by its stochastic extension, i.e., the element of the second order Wiener chaos corresponding to the Hilbert–Schmidt operator A.)

3. A review of fractional Brownian rough paths. In this section we recall that d-dimensional fBM $(w_t^H)_{0\le t\le 1}$ with Hurst parameter $H\in(1/4,1/2)$ can be lifted as a random variable on the geometric rough path space $G\Omega_p(\mathbf{R}^d)$ for $1/H<p<[1/H]+1$ (Coutin–Qian [10] or Section 4.5 of Lyons–Qian [29]). When $H\in(1/4,1/3]$, not only the first and the second level paths, but also the third level paths play a role.

3.1. *Geometric rough paths, Lyons' continuity theorem and Taylor expansion of Itô maps.* In this subsection we recall definitions of geometric rough paths, and a rough differential equation (RDE) and Lyons' continuity theorem for the Itô map. We also review (stochastic) Taylor expansion for Itô maps around a "nice" path, which was shown in [19]. It plays a crucial role in the proof of the Laplace asymptotic expansion. Note that no probability measure is involved in this subsection. No new results are presented in this subsection.

Before introducing the rough path space, let us first introduce some path spaces in the usual sense and the norms on them. Let $\mathcal{V}$ be a real Banach space. Throughout this paper, we assume $\dim \mathcal{V} < \infty$ and the time interval is $[0, 1]$. In almost all applications in later sections, either $\mathcal{V} = \mathbf{R}^d$ or $\mathcal{V} = \mathrm{Mat}(n, d)$ (the space of $n \times d$ matrices). Let

$$C = C([0, 1], \mathcal{V}) = \{k : [0, 1] \to \mathcal{V} | \text{continuous}\}$$

be the space of $\mathcal{V}$-valued continuous functions with the usual sup-norm. For $p \geq 1$, $C^{p\text{-var}}$ is the set of $k \in C$ such that

$$\|k\|_{p\text{-var}}; = |k_0| + \left(\sup_{\mathcal{P}} \sum_{i=1}^{n} |k_{t_i} - k_{t_{i-1}}|^p \right)^{1/p} < \infty,$$

where $\mathcal{P}$ runs over all the finite partition of $[0, 1]$. If $p, q \geq 1$ with $1/p + 1/q > 1$ and $k \in C^{q\text{-var}}(L(\mathcal{V}, \mathcal{W}))$ and $l \in C^{p\text{-var}}(\mathcal{V})$ with $l_0 = 0$, then the Young integral

$$\int_s^t k_u \, dl_u := \lim_{|\mathcal{P}| \searrow 0} \sum_{i=1}^{N} k_{t_{i-1}} (l_{t_i} - l_{t_{i-1}})$$

is well-defined. Here, $L(\mathcal{V}, \mathcal{W})$ is the set of linear maps from $\mathcal{V}$ to $\mathcal{W}$ and $\mathcal{P} = \{s = t_0 < t_1 < \cdots < t_N = t\}$ is a partition of $[s, t]$. Moreover, $t \mapsto \int_0^t k_u \, dl_u \in \mathbf{R}^n$ is of finite p-variation and $\| \int_0^\cdot k_u \, dl_u \|_{p\text{-var}} \leq \mathrm{const} \cdot \|k\|_{q\text{-var}} \|l\|_{p\text{-var}}$. More precisely, if there is a control function ω such that $|k_t - k_s| \leq \omega(s, t)^{1/q}$, $|l_t - l_s| \leq \omega(s, t)^{1/p}$, then

$$\left| \int_s^t k_u dl_u - k_s(l_t - l_s) \right| \leq \mathrm{const} \cdot \omega(s, t)^{1/p + 1/q}.$$

In particular, if $\tilde{l} \in C^{p\text{-var}}(\mathcal{V})$ and $\tilde{k} \in C^{q\text{-var}}(\mathcal{W})$ with $1/p + 1/q > 1$, then $\int_s^t \tilde{k}_u \otimes d\tilde{l}_u$ is well-defined.

Next we introduce the Besov space $W^{\delta,p}$ for $p > 1$ and $0 < \delta < 1$. For a measurable function $k : [0, 1] \to \mathcal{V}$, set

$$\|k\|_{W^{\delta,p}} = \|k\|_{L^p} + \left(\iint_{[0,1]^2} \frac{|k_t - k_s|^p}{|t - s|^{1+\delta p}} \, ds \, dt \right)^{1/p}. \tag{3.1}$$

The Besov space $W^{\delta,p}$ is the totality of k's such that $\|k\|_{W^{\delta,p}} < \infty$. When $1/p < \delta$, this Banach space is continuously imbedded in C and basically we only consider such a case. The subspace of functions which start at 0 (i.e., $k_0 = 0$) is denoted by C_0, $C_0^{\alpha\text{-hldr}}$, etc. When we need to specify the range of functions, we write $C^{p\text{-var}}(\mathcal{V})$, $W_0^{\delta,p}(\mathcal{V})$, etc. (The domain is always $[0, 1]$ and, hence, is usually omitted.)

Now we introduce the geometric rough path space. Let $p \geq 1$ for a while. (In later sections, however, only the case $2 < p < 4$ will be considered.) Set $\triangle =$

$\{(s,t)|0 \le s \le t \le 1\}$. The p-variation norm of a continuous map A form $\triangle$ to a real finite dimensional Banach space $\mathcal{V}$ is defined by

$$\|A\|_{p\text{-var}} = \left(\sup_{\mathcal{P}} \sum_{i=1}^{n} |A_{t_{i-1},t_i}|_{\mathcal{V}}^{p}\right)^{1/p},$$

where $\mathcal{P}$ runs over all the finite partition of [0, 1]. A continuous map

$$X = \left(1, X^1, X^2, \dots, X^{[p]}\right) : \triangle \to T^{[p]}(\mathcal{V}) = \mathbf{R} \oplus \mathcal{V} \oplus \mathcal{V}^{\otimes 2} \cdots \oplus \mathcal{V}^{\otimes [p]}$$

is said to be a $\mathcal{V}$-valued rough path of roughness p if it satisfies the following conditions:

(a): For any $s \le u \le t$, $X_{s,t} = X_{s,u} \otimes X_{u,t}$, where $\otimes$ denotes the tensor operation in the truncated tensor algebra $T^{[p]}(\mathcal{V})$. In other words, $X_{s,t}^j = \sum_{i=0}^{j} X_{s,u}^i \otimes X_{u,t}^{j-i}$ for all $1 \le j \le [p]$. This is called Chen's identity.

(b): For all $1 \le j \le [p]$, $\|X^j\|_{p/j\text{-var}} < \infty$.

We usually omit the 0th component 1 and simply write $X = (X^1, \dots, X^{[p]})$. The first level path of X is naturally regarded as an element in $C_0^{p\text{-var}}(\mathcal{V})$ by $t \mapsto X_{0,t}^1$. [We will abuse the notation to write $X^1 \in C_0^{p\text{-var}}(\mathcal{V})$, e.g.] The set of all the $\mathcal{V}$-valued rough paths of roughness p is denoted by $\Omega_p(\mathcal{V})$. With the distance $d_p(X,Y) = \sum_{i=1}^{[p]} \|X^j - Y^j\|_{p/j\text{-var}}$, it becomes a complete metric space.

A $\mathcal{V}$-valued finite variational path $x \in C_0^{1\text{-var}}(\mathcal{V})$ is naturally lifted as an element of $\Omega_p(\mathcal{V})$ by the following iterated Stieltjes integral:

$$X_{s,t}^j = \int_{s \le t_1 \le \cdots \le t_j \le t} dx_{t_1} \otimes dx_{t_2} \otimes \cdots \otimes dx_{t_j}. \tag{3.2}$$

We say X is the smooth rough path lying above x. It is well known that the injection $x \mapsto X \in \Omega_p(\mathcal{V})$ is continuous with respect to the 1-variation norm. The space of geometric rough path $G\Omega_p(\mathcal{V})$ is the closure of $C_0^{1\text{-var}}(\mathcal{V})$ with respect to d_p. Since $\mathcal{V}$ is separable, $G\Omega_p(\mathcal{V})$ is a complete separable metric space.

Let us recall some properties of the q-variational path for $1 \le q < 2$. For the facts presented below, see Section 3.3.2 in Lyons–Qian [29] or Inahama [19], for example. Since $k \in C_0^{q\text{-var}}(\mathcal{V})$ is Young integrable with respect to itself, the iterated integral in (3.2) is still well-defined and k can be lifted to an element $K \in G\Omega_p(\mathcal{V})$ if $p \ge 2$. This injection $C_0^{q\text{-var}}(\mathcal{V}) \hookrightarrow G\Omega_p(\mathcal{V})$ is continuous.

For any $m = 1, 2, \dots$ and any $k \in C_0(\mathcal{V})$, the mth dyadic piecewise linear approximation $k(m)$ is defined by

$$k(m)_t = k_{(l-1)/2^m} + 2^m\left(k_{l/2^m} - k_{(l-1)/2^m}\right)\left(t - (l-1)/2^m\right) \qquad \text{if } t \in \left[\frac{l-1}{2^m}, \frac{l}{2^m}\right].$$

If k is of q-variation ($q \ge 1$), then $k(m)$ converges to k in $(q+\varepsilon)$-variation norm for any $\varepsilon > 0$. It implies that, if $p \ge 2$ and $k \in C_0^{q\text{-var}}(\mathcal{V})$ for $1 \le q < 2$, then $K(m)$ converges to K in $G\Omega_p(\mathcal{V})$.

Suppose that if $p \geq 2$, $1 \leq q < 2$, and $1/p + 1/q > 1$, then the shift

$$(X, k) \in G\Omega_p(\mathcal{V}) \times C_0^{q\text{-var}}(\mathcal{V}) \mapsto X + K \in G\Omega_p(\mathcal{V})$$

is well-defined by the Young integral and this map is continuous. Similarly,

$$(X, k) \in G\Omega_p(\mathcal{V}) \times C_0^{q\text{-var}}(\mathcal{W}) \mapsto (X, K) \in G\Omega_p(\mathcal{V} \oplus \mathcal{W})$$

is well-defined and continuous. These facts are well known. (See Section 9-4 in Friz and Victoir's book [16], e.g.) Note that the notation "$X + K$" above may be somewhat misleading since the geometric rough path space is not an additive group.

Let $\mathcal{V}$ and $\mathcal{W}$ be two finite dimensional real Banach spaces and let $\sigma : \mathcal{W} \to L(\mathcal{V}, \mathcal{W})$ with some regularity condition, which will be specified later. We consider the following differential equation in the rough path sense (rough differential equation or RDE):

$$dY_t = \sigma(Y_t)\, dX_t, \qquad Y_0 = y_0 \in \mathcal{W}. \tag{3.3}$$

When there is a unique solution Y for given X, it is denoted by $Y = \Phi(X)$ and the map $\Phi : G\Omega_p(\mathcal{V}) \to G\Omega_p(\mathcal{W})$ is called the Itô map.

The following is called Lyons' continuity theorem (or universal limit theorem) and is most important in the rough path theory. (See Section 6.3, Lyons–Qian [29]. For a proof of continuity when the coefficient σ also varies, see Inahama [19], e.g.)

THEOREM 3.1. (i) *Let $p \geq 2$ and assume that $\sigma \in C_b^{[p]+1}(\mathcal{V}, \mathcal{W})$. Then, for given $X \in G\Omega_p(\mathcal{V})$ and an initial value $y_0 \in \mathcal{W}$, there is a unique solution $Y \in G\Omega_p(\mathcal{W})$ of RDE* (3.3). *Moreover, there is a constant $C_M > 0$ for $M > 0$ such that, if*

$$|y_0| \leq M, \qquad \sum_{j=1}^{[p]} \|X^j\|_{p/j\text{-var}} \leq M, \qquad \sum_{j=0}^{[p]+1} \sup_{y \in \mathcal{W}} \|\nabla^j \sigma(y)\| \leq M,$$

then $\sum_{j=1}^{[p]} \|Y^j\|_{p/j\text{-var}} \leq C_M$.

(ii) *Keep the same assumption as above. Assume that $X_l \to X$ in $G\Omega_p(\mathcal{V})$ and $y_0^l \to y_0$ in $\mathcal{W}$ as $l \to \infty$. Assume further that $\sigma_l, \sigma \in C_b^{[p]+1}(\mathcal{V}, \mathcal{W})$ satisfy that*

$$\sup_{l \geq 1} \sum_{j=0}^{[p]+1} \sup_{y \in \mathcal{W}} \|\nabla^j \sigma_l(y)\| \leq M$$

for some constant $M > 0$ and

$$\lim_{l \to \infty} \sum_{j=0}^{[p]+1} \sup_{|y|_{\mathcal{W}} \leq N} \|\nabla^j \sigma_l(y) - \nabla^j \sigma(y)\| = 0$$

for each fixed $N > 0$. Then, $Y_l \to Y$ in $G\Omega_p(\mathcal{W})$, where Y_l is the solution of RDE (3.3) *corresponding to (X_l, y_0^l, σ_l).*

In this paper we consider the following RDE indexed by small parameter $\varepsilon > 0$. Let $\sigma \in C_b^\infty(\mathbf{R}^n, \mathrm{Mat}(n,d))$ and $\beta \in C_b^\infty([0,1] \times \mathbf{R}^n, \mathbf{R}^n)$. For fixed $\varepsilon \in [0,1]$, consider

$$dY_t^\varepsilon = \sigma(Y_t^\varepsilon)\varepsilon\, dX_t + \beta(\varepsilon, Y_t^\varepsilon)\, dt, \qquad Y_0^\varepsilon = 0. \tag{3.4}$$

[This is the same RDE as in (1.1)]. If we define $\hat{\sigma}_\varepsilon : \mathbf{R}^n, \mathrm{Mat}(n, d+1)$ by

$$\hat{\sigma}_\varepsilon(y)x' = \sigma(y)x + \beta(\varepsilon, y)x_{d+1}, \qquad x' = (x, x_{d+1}) \in \mathbf{R}^d \oplus \mathbf{R},$$

then $Y^\varepsilon = \hat{\Phi}_\varepsilon(\varepsilon X, \lambda)$. Here, $\lambda_t = t$ and $\hat{\Phi}_\varepsilon : G\Omega_p(\mathbf{R}^{d+1}) \to G\Omega_p(\mathbf{R}^n)$ is the Itô map which corresponds to $\hat{\sigma}_\varepsilon$. Note that $\hat{\sigma}_\varepsilon$ converges to $\hat{\sigma}_{\varepsilon'}$ in the sense of Theorem 3.1(ii) as $\varepsilon \to \varepsilon'$.

Now we consider the (stochastic) Taylor expansion around $\gamma \in C_0^{q\text{-var}}(\mathbf{R}^d)$ with $1/p + 1/q > 1$. Consider $\hat{\Phi}_\varepsilon(\varepsilon X + \gamma, \lambda)$ or, equivalently, the solution of the following RDE:

$$d\tilde{Y}_t^\varepsilon = \sigma(\tilde{Y}_t^\varepsilon)(\varepsilon\, dX_t + d\gamma_t) + \beta(\varepsilon, \tilde{Y}_t^\varepsilon)\, dt, \qquad \tilde{Y}_0^\varepsilon = 0. \tag{3.5}$$

We will write $\phi^{(\varepsilon)} = (\tilde{Y}^\varepsilon)^1$ (the first level path). Note that $\hat{\Phi}_0(\gamma, \lambda)$ is lying above $\phi^0 = \Psi(\gamma) \in C_0^{q\text{-var}}(\mathbf{R}^n)$ which is defined by

$$d\phi_t^0 = \sigma(\phi_t^0)\, d\gamma_t + \beta(0, \phi_t^0)\, dt, \qquad \phi_0^0 = 0. \tag{3.6}$$

In the following theorem, we consider the asymptotic expansion of $\phi^{(\varepsilon)} - \phi^0$. By formally operating $(m!)^{-1}(d/d\varepsilon)^m|_{\varepsilon=0}$ on both sides of (3.5), we get an RDE for the mth term ϕ^m (see [19] for detail). Note that ϕ^m depends on X, γ (although γ is basically fixed in this paper), but independent of ε. (The superscript m does not denote the level of the path ϕ^m. Here we only consider the usual paths or the first level paths.)

In what follows, we will use the following notation; for a geometric rough path X of roughness p,

$$\xi(X) = \|X^1\|_{p\text{-var}} + \|X^2\|_{p/2\text{-var}}^{1/2} + \cdots + \|X^{[p]}\|_{p/[p]\text{-var}}^{1/[p]}. \tag{3.7}$$

THEOREM 3.2. *Let $p \ge 2$, $1 \le q < 2$ with $1/p + 1/q > 1$ and let the notation be as above. Then, for any $m = 1, 2, \ldots,$ we have the following expansion*:

$$\phi^{(\varepsilon)} = \phi^0 + \varepsilon\phi^1 + \cdots + \varepsilon^m\phi^m + R_\varepsilon^{m+1}.$$

The maps $(X, \gamma) \in G\Omega_p(\mathbf{R}^d) \times C_0^{q\text{-var}}(\mathbf{R}^d) \mapsto \phi^k, R_\varepsilon^{m+1} \in C_0^{p\text{-var}}(\mathbf{R}^n)$ are continuous $(0 \le k \le m)$. *Moreover, the following estimates* (a), (b) *hold*:

(a) *For any $r_1 > 0$, there exists $C_1 > 0$ which depends only on r_1 such that, if $\|\gamma\|_{q\text{-var}} \le r_1$, then $\|\phi^k\|_{p\text{-var}} \le C_1(1 + \xi(X))^k$ holds.*

(b) *For any $r_2, r_3 > 0$, there exists $C_2 > 0$ which depends only on r_2, r_3 such that, if $\|\gamma\|_{q\text{-var}} \le r_2$ and $\xi(\varepsilon X) \le r_3$, then $\|R_\varepsilon^{m+1}\|_{p\text{-var}} \le C_2(\varepsilon + \xi(\varepsilon X))^{m+1}$ holds.*

3.2. *Fractional Brownian rough paths*. First we introduce fractional Brownian motion (fBM for short) of Hurst parameter H. There are several books and surveys on fBM (see [8, 9, 33], e.g.). In this paper we only consider the case $1/4 < H < 1/2$. A real-valued continuous stochastic process $(w_t^H)_{t\geq 0}$ starting at 0 is said to a fBM of Hurst parameter H if it is a centered Gaussian process with

$$\mathbb{E}[w_t^H w_s^H] = \tfrac{1}{2}[t^{2H} + s^{2H} - |t-s|^{2H}] \qquad (s,t \geq 0).$$

This process has stationary increments $\mathbb{E}[(w_t^H - w_s^H)^2] = |t-s|^{2H}$ $(s,t\geq 0)$, and self-similarity, that is, for any $c>0$, $(c^{-H} w_{ct}^H)_{t\geq 0}$ and $(w_t^H)_{t\geq 0}$ have the same law. Note that $(w_t^{1/2})_{t\geq 0}$ is the standard Brownian motion. For $d \geq 1$, a d-dimensional fBM is defined by $(w_t^{H,1}, \ldots, w_t^{H,d})_{t\geq 0}$, where $w^{H,i}$ $(i=1,\ldots,d)$ are independent one-dimensional fBM's. Its law μ^H is a probability measure on $C_0(\mathbf{R}^d)$. [Actually, it is a probability measure on $C_0^{p\text{-var}}(\mathbf{R}^d)$ for $p > 1/H$.]

Let $H \in (1/4, 1/2)$. We denote by $w^H(m)$ the mth dyadic piecewise linear approximation of w^H, that is, piecewise linear approximation associated with the partition $\{j2^{-m} | 0 \leq j \leq 2^m\}$. The existence of a fractional Brownian rough path (fBRP for short) was shown by Coutin–Qian [10] as an almost sure limit of $W^H(m)$ as $m \to \infty$, where $W^H(m)$ is the smooth rough path lying above $w^H(m) \in C_0^{1\text{-var}}(\mathbf{R}^d)$. More precisely, they proved

$$\mathbb{E}\left[\sum_{m=1}^{\infty} \|W^H(m+1)^j - W^H(m)^j\|_{p/j\text{-var}}\right] < \infty \qquad (1 \leq j \leq [p]).$$

In particular, $W^H(m)$ converges to W^H in the L^1-sense, too. When $1/3 < H < 1/2$, $[p]=2$ and when $1/4 < H \leq 1/3$, $[p]=3$.

Now we prove a theorem of Fernique-type for fBRP for later use. We give a direct proof here for readers' convenience by using a useful estimate in Millet and Sanz-Sole [32]. (The case $H = 1/2$ is shown in [18], e.g.) It should be noted, however, that (i) this proposition is included in Theorems 15.22 and 15.42, [16] and (ii) Friz and Oberhauser [12] recently showed this kind of integrability for a wider class of Gaussian rough paths, by using isoperimetric inequality.

PROPOSITION 3.3. *Let $1/4 < H < 1/2$ and W^H be a d-dimensional fBRP as above.*

(1) *Then, there exists a positive constant c such that*

$$\mathbb{E}[\exp(c\xi(W^H)^2)] = \int_{G\Omega_p(\mathbf{R}^d)} \exp(c\xi(X)^2)\mathbb{P}^H(dX) < \infty,$$

where ξ is given in (3.7) and $\mathbb{P}^H$ denotes the law of W^H.

(2) *For any $r>0$ and $1 \leq j \leq [p]$, $\lim_{m\to\infty} \mathbb{E}[\|W^H(m)^j - W^{H,j}\|_{p/j\text{-var}}^r] = 0$.*

PROOF. In this proof, $c_1, c_2, \dots$ are positive constants which may change from line to line. For a rough path X of roughness p and $\gamma > p-1$, set

$$D_{j,p}(X,Y) = \left(\sum_{n=1}^{\infty} n^{\gamma} \sum_{l=1}^{2^n} \left| X^j_{(l-1)/2^n, l/2^n} - Y^j_{(l-1)/2^n, l/2^n} \right|^{p/j} \right)^{j/p} \qquad (1 \le j \le [p]).$$

When $Y=0$, we write $D_{j,p}(X) = D_{j,p}(X,Y)$ for simplicity. From Section 4.1 in Lyons–Qian [29], the following estimates hold:

$$\begin{aligned}
&\|X^1 - Y^1\|^p_{p\text{-var}} \le c_1 D_{1,p}(X,Y)^p \\
&\|X^2 - Y^2\|^{p/2}_{p/2\text{-var}} \\
&\qquad \le c_1 \bigl[D_{2,p}(X,Y)^{p/2} + D_{1,p}(X,Y)^{p/2} \bigl(D_{1,p}(X)^p + D_{1,p}(Y)^p \bigr)^{1/2} \bigr] \\
&\|X^3 - Y^3\|^{p/3}_{p/3\text{-var}} \\
&\qquad \le c_1 \bigl[D_{3,p}(X,Y)^{p/3} + D_{2,p}(X,Y)^{p/3} \bigl(D_{1,p}(X)^p + D_{1,p}(Y)^p \bigr)^{1/3} \\
&\qquad\qquad + D_{1,p}(X,Y)^{p/3} \bigl(D_{2,p}(X)^{p/2} + D_{2,p}(Y)^{p/2} \bigr)^{2/3} \\
&\qquad\qquad\qquad + D_{1,p}(X,Y)^{p/3} \bigl(D_{1,p}(X)^p + D_{1,p}(Y)^p \bigr)^{2/3} \bigr].
\end{aligned} \tag{3.8}$$

Proposition 2 in [32] states that there is a sequence $\{a_m\}$ of positive numbers converging to 0 such that, for any $r > p$,

$$\mathbb{E}[D_{j,p}(W^H(m), W^H)^r]^{1/r} \le a_m r^{j/2}$$

holds. For simplicity, set $F_m = D_{j,p}(W^H(m), W^H)^{2/j}$. Then, from the above inequality,

$$\mathbb{P}(N < F_m) \le N^{-N} \mathbb{E}[F_m^N] \le c_2^N a_m^N$$

for $N = 4, 5, \dots$. Therefore,

$$\begin{aligned}
\mathbb{E}[e^{cF_m}] &\le \sum_{N=0}^{\infty} e^{c(N+1)} \mathbb{P}(N < F_m \le N+1) \\
&\le (e^c + \cdots + c^{4c}) + e^c \sum_{N=4}^{\infty} e^{cN} \mathbb{P}(N < F_m) \\
&\le (e^c + \cdots + c^{4c}) + e^c \sum_{N=4}^{\infty} \exp[N(c + \log c_2 - \log a_m)].
\end{aligned}$$

For given $c > 0$, there exists m_0 such that $m \ge m_0$ implies $c + \log c_2 - \log a_m < 0$. Thus, we obtain

$$\sup_{m \ge m_0} \mathbb{E}[e^{cF_m}] \le \sup_{m \ge m_0} \mathbb{E}[\exp(cD_{j,p}(W^H(m), W^H)^{2/j})] < \infty.$$

On the other hand, it is easy to see that, for each fixed m_0, there is a constant $c'(m_0) > 0$ such that $D_{j,p}(W^H(m_0))^{1/j} \le c'(m_0)\|w^H\|_\infty$. Hence, the usual Fernique theorem for Gaussian measures applies and $D_{j,p}(W^H(m_0))^{1/j}$ is square exponentially integrable. Using (3.8) and the triangle inequality for $D_{j,p}$, we prove (1). In a similar way, we see that

$$\sup_{m\ge 1}\mathbb{E}[D_{j,p}(W^H(m))^r] < \infty, \qquad \sup_{m\ge 1}\mathbb{E}[\|W^H(m)^j\|^r_{p/j\text{-var}}] < \infty.$$

This implies (2). □

Let $\mathcal{H}^H$ be the Cameron–Martin subspace of fBM [i.e., $k \in C_0(\mathbf{R}^d)$ is an element of $\mathcal{H}^H$ if and only if μ^H and $\mu^H(\cdot + k)$ are mutually absolutely continuous]. When $H = 1/2$, it is easy to see $k \in \mathcal{H}^{1/2}$ is of finite 1-variation. But, when $H \in (1/4, 1, 2)$, does $k \in \mathcal{H}^H$ have a similar nice property in terms of variation norm? The following theorem answers this question. As a result, $\mathcal{H}^H$ is continuously (and compactly) embedded in $G\Omega_p(\mathbf{R}^d)$ for $p \ge 2$.

PROPOSITION 3.4 (Friz–Victoir [13]).

(i) *Let* $0 < \delta < 1$ *and* $p \ge 1$ *such that* $\alpha = \delta - 1/p > 0$ *and set* $q = 1/\delta$. *Then, we have a continuous embedding*

$$W^{\delta,p} \subset C^{q\text{-var}}, \qquad W^{\delta,p} \subset C^{\alpha\text{-hldr}}.$$

More precisely, for $h \in W^{\delta,p}$,

$$\omega(s,t) = \|h\|^q_{W^{\delta,p};[s,t]}(t-s)^{\alpha q}, \qquad 0 \le s \le t \le 1$$

becomes a control function in the sense of Lyons–Qian [29], *page* 16, *and* h *is controlled by a constant multiple of* ω [*i.e.*, $|h_t - h_s| \le \text{const} \times \omega(s,t)^{1/q}$].

(ii) *Let the Hurst parameter* $H \in (0, 1/2)$. *If* $1/2 < \delta < H + 1/2$, *then* $\mathcal{H}^H \Subset W_0^{\delta,2}$ (*compact embedding*). *Therefore, for any* $\alpha \in (0, H)$ *and* $q \in ((H + 1/2)^{-1}, 2)$,

$$\mathcal{H}^H \Subset C_0^{\alpha\text{-hldr}}, \qquad \mathcal{H}^H \Subset C_0^{q\text{-var}}.$$

We give a theorem of a Cameron–Martin type for fBRP W^H. (For BRP, see [18], e.g.) Let $1/4 < H < 1/2$ and $1/H < p < [1/H] + 1$. Then, fBRP W^H exists on $G\Omega_p(\mathbf{R}^d)$ and its law is a probability measure on $G\Omega_p(\mathbf{R}^d)$. By Proposition 3.4, there exists $1 \le q < 2$ such that $\mathcal{H}^H \Subset C_0^{q\text{-var}} \subset G\Omega_p(\mathbf{R}^d)$ and $1/p + 1/q > 1$. Hence, the shift $X \mapsto X + K$ for $k \in \mathcal{H}^H$ is well-defined in $G\Omega_p(\mathbf{R}^d)$, where K is the lift of k as usual.

PROPOSITION 3.5. *Let $\varepsilon > 0$ and let $\mathbb{P}^H_\varepsilon$ be the law of εW^H. Then, for any $k \in \mathcal{H}^H$, $\mathbb{P}^H_\varepsilon$ and $\mathbb{P}^H_\varepsilon(\cdot + K)$ are mutually absolutely continuous and, for any bounded Borel function f on $G\Omega_p(\mathbf{R}^d)$,*

$$\int_{G\Omega_p(\mathbf{R}^d)} f(X+K)\mathbb{P}^H_\varepsilon(dX)$$
$$= \int_{G\Omega_p(\mathbf{R}^d)} f(X)\exp\left(\frac{1}{\varepsilon^2}\langle k, X^1\rangle - \frac{1}{2\varepsilon^2}\|k\|^2_{\mathcal{H}^H}\right)\mathbb{P}^H_\varepsilon(dX).$$

Here, $\langle k, X^1\rangle$ is the measurable linear functional associated with $k \in \mathcal{H}^H = (\mathcal{H}^H)^*$ for the fBM $t \mapsto X^1_{0,t}$ (i.e., the element of the first Wiener chaos of the fBM X^1 associated with k).

PROOF. Since $W^H(m) \to W^H$ in $G\Omega_p(\mathbf{R}^d)$ and $k(m) \to k$ in q-variation norm as $m \to \infty$, respectively, $W^H + K = \lim_{m\to\infty}[W^H(m) + K(m)]$. On the other hand, $W^H(m) + K(m)$ is the lift of $w^H(m) + k(m) = (w^H + k)(m)$. Hence, the problem reduces to the usual Cameron–Martin theorem for fBM w^H. □

In the end of this subsection we give a Schilder-type large deviation principle for the law of εW^H as $\varepsilon \searrow 0$. This was shown by Millet and Sanz-Sole [32] (and by Friz–Victoir [13, 14]).

PROPOSITION 3.6. *Let $\mathbb{P}^H_\varepsilon$ be the law of εW^H as above* ($1/4 < H < 1/2$). *As before, $1/H < p < [1/H] + 1$ and $G\Omega_p(\mathbf{R}^d)$ is equipped with the p-variation metric. Then, as $\varepsilon \searrow 0$, $\{\mathbb{P}^H_\varepsilon\}_{\varepsilon>0}$ satisfies a large deviation principle with a good rate function I, which is given by*

$$I(X) = \begin{cases} \frac{1}{2}\|k\|^2_{\mathcal{H}^H} & (\textit{if } X \textit{ is lying above } k \in \mathcal{H}^H), \\ \infty & (\textit{otherwise}). \end{cases}$$

4. Hilbert–Schmidt property of Hessian. In this section we consider the Itô map restricted on the Cameron–Martin space $\mathcal{H}^H$ of the fBM with Hurst parameter $H \in (1/4, 1/2)$ and prove that its Hessian is a symmetric Hilbert–Schmidt bilinear form.

Throughout this section we set $\beta_0(y) = \beta(0, y)$ for simplicity. Consider the following RDE:

$$dY_t = \sigma(Y_t)\,dX_t + \beta_0(Y_t)\,dt, \qquad Y_0 = 0. \tag{4.1}$$

The Itô map $X \in G\Omega_p(\mathbf{R}^d) \mapsto \hat{\Phi}_0(X, \lambda) = Y \in G\Omega_p(\mathbf{R}^n)$ restricted on the Cameron–Martin space $\mathcal{H}^H$ of fBM is denoted by Ψ, that is, $\Psi(k) = \hat{\Phi}_0(K, \lambda)$ for $k \in \mathcal{H}^H$. Here, K is a geometric rough path lying above k and $\lambda_t = t$. (Since k is of finite q-variation for some $q < 2$, as we will see below, this is well-defined.

Regularity of $k \in \mathcal{H}^H$ in a p-variational setting is studied by Friz–Victoir [13]. Fortunately, h is of finite q-variation for some $q < 2$ and, hence, the Young integral is possible.)

The aim of this section is to prove the following theorem. Let F and p' be as in Assumption (H1).

THEOREM 4.1. *$\nabla^2(F \circ \Psi)(\gamma)\langle \cdot, \cdot \rangle$ is a symmetric Hilbert–Schmidt bilinear form on $\mathcal{H}^H$ for any $\gamma \in \mathcal{H}^H$.*

REMARK 4.2. The reader may find arguments in this section a little bit messy. So, we give a brief summary here. The most difficult part in proving the above theorem is to show that the bilinear functional

$$(f, k) \in \mathcal{H}^H \times \mathcal{H}^H \mapsto \int_0^{\cdot} f_u \otimes dk_u \in C_0^{p\text{-var}}(\mathbf{R}^d \otimes \mathbf{R}^d)$$

is "Hilbert–Schmidt." To compute the Hilbert–Schmidt norm, we need a simple orthonormal basis. However, we do not know a good basis of $\mathcal{H}^H$. Therefore, we first imbed $\mathcal{H}^H$ into a larger Hilbert space $L^{\delta,2}_{\mathrm{real}}$, since it has a very simple orthonormal basis of cosine functions, and then prove the Hilbert–Schmidt property for the norm of $L^{\delta,2}_{\mathrm{real}}$. [See (4.2) below for the definition of p and $\delta = 1/q$.]

Note that

$$\begin{aligned}\nabla^2(F \circ \Psi)(\gamma)\langle f, k\rangle &= \nabla F(\Psi(\gamma))\langle \nabla^2 \Psi(\gamma)\langle f, k\rangle\rangle \\ &\quad + \nabla^2 F(\Psi(\gamma))\langle \nabla \Psi(\gamma)\langle f\rangle, \nabla \Psi(\gamma)\langle k\rangle\rangle.\end{aligned}$$

ODEs for $\nabla\Psi(\gamma)\langle k\rangle$ and $\nabla^2\Psi(\gamma)\langle f, k\rangle$ will be given in (4.5)–(4.7) below.

Now we set conditions on parameters. First we have the Hurst parameter $H \in (1/4, 1/2)$. Then, we can choose p and $q = \delta^{-1}$ such that

$$\begin{gathered}\frac{1}{p'} \vee \frac{1}{[1/H]+1} < \frac{1}{p} < H, \qquad \frac{3}{4} < \frac{1}{q} < H + \frac{1}{2}, \\ \frac{1}{p} + \frac{1}{q} > 1, \qquad \frac{1}{q} - \frac{1}{p} > \frac{1}{2}.\end{gathered} \tag{4.2}$$

For example, $1/p = H - 2\varepsilon$ and $1/q = H + 1/2 - \varepsilon$ for sufficiently small $\varepsilon > 0$ satisfy (4.2). Indeed,

$$\frac{1}{p} + \frac{1}{q} = 1 + 2\left(H - \frac{1}{4}\right) - 3\varepsilon, \qquad \frac{1}{q} - \frac{1}{p} = \frac{1}{2} + \varepsilon.$$

For this p and $q = \delta^{-1}$, the fBM with the Hurst parameter H can be lifted to $G\Omega_p(\mathbf{R}^d)$ and its Cameron–Martin space $\mathcal{H}^H$ satisfies Proposition 3.4 above. In particular, the Young integral of $k \in \mathcal{H}^H$ with respect to itself is possible since

$q < 2$. The shift and the pairing of $X \in G\Omega_p(\mathbf{R}^d)$ by $k \in \mathcal{H}^H$ can be defined since $1/p + 1/q > 1$. In what follows we always assume (4.2).

The Banach space $W^{\delta,p}$ is defined by (3.1). In Adams [1], its original definition is given by a kind of real interpolation (precisely, the trace space of J. L. Lions, see paragraph 7.35, [1]) of $W^{1,p}$ and $W^{0,p} = L^p$. Those are equivalent Banach spaces (paragraph 7.48, [1]). On the other hand, $L^{\delta,p}$ is defined by the complex interpolation of (the complexification of) $W^{1,p}$ and $W^{0,p} = L^p$, that is, $L^{\delta,p} = [W^{1,p}, L^p]_{1-\delta}$. If $p = 2$, $L^{\delta,2}$ and (the complexification of) $W^{\delta,2}$ are equivalent Hilbert spaces (not unitarily equivalent, see paragraph 7.59, [1]). As a result , $L^{\delta,2}_{\mathrm{real}}$ and $W^{\delta,2}$ are equivalent real Hilbert spaces, where $L^{\delta,2}_{\mathrm{real}}$ is the subspace of $\mathbf{R}^d$-valued functions in $L^{\delta,2}$.

THEOREM 4.3. *The following functions of $t \in [0, 1]$ form an orthonormal basis of $L^{\delta,2}_{\mathrm{real}}$ and of $L^{\delta,2} = L^{\delta,2}_{\mathrm{real}} \otimes \mathbf{C}$:*

$$\{1 \cdot \mathbf{e}_i \,|\, 1 \le i \le d\} \cup \left\{ \frac{\sqrt{2}}{(1+n^2)^{\delta/2}} \cos(n\pi t)\mathbf{e}_i \,\middle|\, n \ge 1, 1 \le i \le d \right\}.$$

Here, $\{\mathbf{e}_1, \dots, \mathbf{e}_d\}$ is the canonical orthonormal basis of $\mathbf{R}^d$.

PROOF. It is sufficient to prove the case $d = 1$. Note that

$$W^{1,2} = \left\{ f = c_0 + \sum_{n=1}^{\infty} c_n \sqrt{2}\cos(n\pi t) \,\middle|\, c_n \in \mathbf{C}, \sum_{n=0}^{\infty} (1+n^2)|c_n|^2 < \infty \right\}$$

and $\|f\|^2_{W^{1,2}} = \sum_{n=0}^{\infty} (1+n^2)|c_n|^2$. Similarly,

$$L^2 = \left\{ f = c_0 + \sum_{n=1}^{\infty} c_n \sqrt{2}\cos(n\pi t) \,\middle|\, c_n \in \mathbf{C}, \sum_{n=0}^{\infty} |c_n|^2 < \infty \right\}$$

and $\|f\|^2_{L^2} = \sum_{n=0}^{\infty} |c_n|^2$. Therefore, $W^{1,2}$ and L^2 are unitarily isometric to $l_2^{(1)}$ and $l_2^{(0)} = l_2$, respectively, where

$$l_2^{(\delta)} = \left\{ \mathbf{c} = (c_n)_{n=0,1,2,\dots} \in \mathbf{C}^\infty \,\middle|\, \|\mathbf{c}\|^2_{l_2^{(\delta)}} = \sum_{n=0}^{\infty} (1+n^2)^\delta |c_n|^2 \right\} \qquad (\delta \in \mathbf{R}).$$

Thus, the problem is reduced to the complex interpolation of two Hilbert spaces of sequences. A simple calculation shows that $[l_2^{(1)}, l_2]_{1-\delta} = l_2^{(\delta)}$. This implies

$$L^{\delta,2} = \left\{ f = c_0 + \sum_{n=1}^{\infty} c_n \sqrt{2}\cos(n\pi x) \,\middle|\, c_n \in \mathbf{C}, \sum_{n=0}^{\infty} (1+n^2)^\delta |c_n|^2 < \infty \right\}$$

with $\|f\|^2_{L^{\delta,2}} = \sum_{n=0}^{\infty} (1+n^2)^\delta |c_n|^2$, which ends the proof. $\square$

We compute the p-variation norm of cosine functions. The following lemma is taken from Nate Eldredge's unpublished manuscripts [11]. Before stating it, we introduce some definitions. Let x be a one-dimensional continuous path with $x_0 = 0$. We say that $s \in [0, 1]$ is a forward maximum (or forward minimum) if $x_s = \max x|_{[s,1]}$ (or $x_s = \min x|_{[s,1]}$, resp.). Suppose x is piecewise monotone with local extrema $\{0 = s_0 < s_1 < s_2 < \cdots < s_n = 1\}$. (For simplicity, we assume $s_0, s_2, \ldots$ are local minima and $s_1, s_3, \ldots$ are local maxima. The reverse case is easily dealt with by just replacing x with $-x$.) If $s_2, s_4, \ldots$ are not only local minima but also forward minima, and $s_1, s_3, \ldots$ are not only local maxima but also forward maxima, then we say x is jog-free. (Note that x_0 is not required to be a forward extremum.)

PROPOSITION 4.4. *Let $p \geq 1$.* (i) *If a one-dimensional continuous path x with $x_0 = 0$ is jog-free with extrema $\{0 = s_0 < s_1 < s_2 < \cdots < s_n = 1\}$, then*

$$\|x\|_{p\text{-var}} = \left(\sum_{i=1}^{n} |x_{s_i} - x_{s_{i-1}}|^p \right)^{1/p}.$$

(ii) *In particular, the p-variation norm of $c_n(t) = \cos(n\pi t) - 1$ is given by $\|c_n\|_{p\text{-var}} = 2n^{1/p}$.*

PROOF. (ii) is immediate from (i). We show (i). For a continuous path y and a partition $\mathcal{P} = \{0 = t_0 < t_1 < t_2 < \cdots < t_n = 1\}$, we set $V_{p,\mathcal{P}}(y) = (\sum_{i=1}^{n} |y_{t_i} - y_{t_{i-1}}|^p)^{1/p}$. Then, $\|y\|_{p\text{-var}} = \sup_{\mathcal{P}} V_{p,\mathcal{P}}(y)$. First, note that if y is monotone increasing (or decreasing) on $[t_{i-1}, t_{i+1}]$, then it is easy to see that $V_{p,\mathcal{P}\setminus\{t_i\}}(y) \geq V_{p,\mathcal{P}}(y)$. In other words, intermediate points in monotone intervals should not be included.

Let x be jog-free with extrema $\mathcal{Q} = \{0 = s_0 < s_1 < s_2 < \cdots < s_n = 1\}$ as in the statement of (i) and let $\mathcal{P} = \{0 = t_0 < t_1 < t_2 < \cdots < t_n = 1\}$ be a partition which does not include all the s_j's. We will show below that there exists an s_j such that $V_{p,\mathcal{P}\cup\{s_j\}}(x) \geq V_{p,\mathcal{P}}(x)$.

Let s_j be the first extremum not contained in $\mathcal{P}$. (For simplicity, we assume it is local and forward maximum.) Let t_i be the last element of $\mathcal{P}$ less than s_j. Then, $s_{j-1} \leq t_i \leq s_j \leq t_{i+1}$. Since x is increasing on $[s_{j-1}, s_j]$ and x_{s_j} is forward maximum,

$$x_{s_j} - x_{t_i} \geq x_{t_{i+1}} - x_{t_i}, \qquad x_{s_j} - x_{t_{i+1}} \geq x_{t_i} - x_{t_{i+1}},$$

which yields that $|x_{s_j} - x_{t_i}|^p + |x_{s_j} - x_{t_{i+1}}|^p \geq |x_{t_{i+1}} - x_{t_i}|^p$. Therefore, $V_{p,\mathcal{P}\cup\{s_j\}}(x) \geq V_{p,\mathcal{P}}(x)$.

For any $\varepsilon > 0$, there exists $\mathcal{P}$ such that $V_{p,\mathcal{P}}(x) \geq \|x\|_{p\text{-var}} - \varepsilon$. First by adding all the s_j's, then by removing all the intermediate points (i.e., t_i's which are not one of s_j's), we get $V_{p,\mathcal{Q}}(x) \geq \|x\|_{p\text{-var}} - \varepsilon$. Letting $\varepsilon \searrow 0$, we complete the proof of (i). $\square$

Now we calculate the Hessian of Ψ, which is defined in (4.1). For $q < 2$, ODE like (4.1) is well-defined in the q-variation sense, thanks to the Young integral. The continuity of Ψ is well known. Smoothness of the Itô map in the $q(< 2)$-variation setting is studied in Li–Lyons [28]. The explicit form of the derivatives are obtained in a similar way to the case of (stochastic) Taylor expansion.

Let $q \in [1, 2)$ for a while and fix $\gamma \in C_0^{q\text{-var}}$. Then $\phi^0 = \Psi(\gamma)$ is also of finite q-variation, which takes values in $\mathbf{R}^n$. Set

$$d\Omega_t = \nabla\sigma(\phi_t^0)\langle\cdot, d\gamma_t\rangle + \nabla\beta_0(\phi_t^0)\langle\cdot\rangle\, dt.$$

Then, Ω is an $\mathrm{End}(\mathbf{R}^n)$-valued path of finite q-variation. Next, consider the following $\mathrm{End}(\mathbf{R}^n)$-valued ODE in the q-variation sense:

$$dM_t = d\Omega_t \cdot M_t, \qquad M_0 = \mathrm{Id}_n. \tag{4.3}$$

Its inverse satisfies a similar ODE:

$$dM_t^{-1} = -M_t^{-1} \cdot d\Omega_t, \qquad M_0^{-1} = \mathrm{Id}_n, \tag{4.4}$$

although the coefficients of these ODEs are not bounded, thanks to their special forms, to a unique solution [For this kind of equation with unbounded coefficients, existence of a local solution and uniqueness are easier. The problem is existence of a global solution. If M is a local solution of (4.3) and $a \in \mathrm{GL}(n, \mathbf{R})$, then Ma is a local solution to (4.3) with a initial condition $M_0 = a$. This fact, combined with existence of a local solution, implies existence of a global solution.] The map $\gamma \mapsto M$ in the q-variational setting is locally Lipschitz continuous. (In this paper, however, γ is always fixed and, hence, so are Ω and M.) If γ is controlled by a control function ω, then M and M^{-1} are controlled by $\hat{\omega}(s,t) = C(\omega(s,t) + (t - s))$, where $C > 0$ is a constant which depends on q and $\omega(0, 1)$. A rigorous proof for this paragraph can be found in [19], for instance.

Set $\chi(k) = (\nabla\Psi)(\gamma)\langle k\rangle$ for simplicity. This is a continuous path of finite q-variation, if k is of finite q-variation. Then, it satisfies an $\mathbf{R}^n$-valued ODE:

$$d\chi_t - \nabla\sigma(\phi_t^0)\langle\chi_t, d\gamma_t\rangle - \nabla\beta_0(\phi_t^0)\langle\chi_t\rangle\, dt = \sigma(\phi_t^0)\, dk_t, \qquad \chi_0 = 0. \tag{4.5}$$

From this, we can obtain an explicit expression as follows:

$$\chi(k)_t = (\nabla\Psi)(\gamma)\langle k\rangle_t = M_t \int_0^t M_s^{-1}\sigma(\phi_s^0)\, dk_s. \tag{4.6}$$

Note that the right-hand side is a Young integral and $k \mapsto \chi(k)$ extends to a continuous map from $C_0^{p\text{-var}}(\mathbf{R}^d)$ to $C_0^{p\text{-var}}(\mathbf{R}^n)$.

In a similar way, $\psi_t = \nabla^2\Psi(\gamma)\langle k, k\rangle_t$ satisfies the following ODE:

$$\begin{aligned} &d\psi_t - \nabla\sigma(\phi_t^0)\langle\psi_t, d\gamma_t\rangle - \nabla\beta_0(\phi_t^0)\langle\psi_t\rangle\, dt \\ &\quad = 2\nabla\sigma(\phi_t^0)\langle\chi(k)_t, dk_t\rangle + \nabla^2\sigma(\phi_t^0)\langle\chi(k)_t, \chi(k)_t, d\gamma_t\rangle \\ &\qquad + \nabla^2\beta_0(\phi_t^0)\langle\chi(k)_t, \chi(k)_t\rangle\, dt, \qquad \psi_0 = 0. \end{aligned} \tag{4.7}$$

From this and by polarization, we see that

$$
\begin{aligned}
&\nabla^2\Psi(\gamma)\langle f,k\rangle_t \\
&\quad = M_t\int_0^t M_s^{-1}\{\nabla\sigma(\phi_s^0)\langle\chi(f)_s, dk_s\rangle + \nabla\sigma(\phi_s^0)\langle\chi(k)_s, df_s\rangle\} \\
&\qquad + M_t\int_0^t M_s^{-1}\{\nabla^2\sigma(\phi_t^0)\langle\chi(f)_s,\chi(k)_s, d\gamma_s\rangle \\
&\qquad\qquad + \nabla^2\beta_0(\phi_s^0)\langle\chi(f)_s,\chi(k)_s\rangle\, ds\} \\
&\quad =: V_1(f,k)_t + V_2(f,k)_t.
\end{aligned}
\tag{4.8}
$$

It is obvious that

$$
(f,k)\in C_0^{q\text{-var}}(\mathbf{R}^d)\times C_0^{q\text{-var}}(\mathbf{R}^d)\mapsto \nabla^2\Psi(\gamma)\langle f,k\rangle\in C_0^{q\text{-var}}(\mathbf{R}^n)
$$

is a symmetric bounded bilinear functional.

Note that $\chi(k)$ and $\psi(k,k)/2$ are similar to ϕ^1 and ϕ^2, respectively, when $X=k$. Indeed, they are the first and the second term in the Taylor expansion for $\hat{\Phi}_0(\varepsilon X+\gamma,\lambda)$. [See (6.3) and (6.5) below and compare.] Therefore, $k\mapsto\chi(k)$, $\psi(k,k)$ extend to continuous maps from $G\Omega_p(\mathbf{R}^d)$ to $C_0^{p\text{-var}}(\mathbf{R}^n)$. We will write $\chi(X)$, $\psi(X,X)$ for $X\in G\Omega_p(\mathbf{R}^d)$.

LEMMA 4.5. *Let $1/4<H<1/2$ and choose p and q as in (4.2) Then, for any bounded linear functional $\alpha\in C_0^{p\text{-var}}(\mathbf{R}^n)^*$, the symmetric bounded bilinear form $\alpha\circ V_2\langle\cdot,\cdot\rangle$ on the Cameron–Martin space $\mathcal{H}^H$ is of trace class. In particular, if $p'\ge p$, $\nabla F(\phi^0)\circ V_2$ is of trace class for a Fréchet differentiable function $F: C_0^{p'\text{-var}}(\mathbf{R}^n)\to\mathbf{R}$. Moreover, $\alpha\circ V_2$ extends to a bounded bilinear form on $C_0^{p\text{-var}}(\mathbf{R}^d)$. A similar fact holds for $\nabla^2F(\phi^0)\langle\chi(\cdot),\chi(\cdot)\rangle$, too.*

PROOF. Since $t\mapsto M_t$ and $t\mapsto M_t^{-1}\sigma(\phi_t^0)$ are of finite q-variation, the map $h\mapsto\chi(h)$ extends to a bounded linear map from $C_0^{p\text{-var}}(\mathbf{R}^d)$ to $C_0^{p\text{-var}}(\mathbf{R}^n)$, thanks to the Young integral. By using the Young integral again, we see that $(h,k)\mapsto V_2(h,k)$ extends to a bounded bilinear map from $C_0^{p\text{-var}}(\mathbf{R}^d)\times C_0^{p\text{-var}}(\mathbf{R}^d)$ to $C_0^{q\text{-var}}(\mathbf{R}^n)\subset C_0(\mathbf{R}^d)$.

On the other hand, μ^H (the law of the fBM with the Hurst parameter H) is supported in $C_0^{p\text{-var}}(\mathbf{R}^d)$. In other words, $(\mathcal{X},\mathcal{H}^H,\mu^H)$ is an abstract Wiener space, where $\mathcal{X}$ is the closure of $\mathcal{H}^H$ with respect to the p-variation norm. [According to Jain and Monrad [21], pages 47–48, $C_0^{p\text{-var}}(\mathbf{R}^d)$ is not separable and, consequently, $\mathcal{H}^H$ cannot be dense in $C_0^{p\text{-var}}(\mathbf{R}^d)$. So, we use $\mathcal{X}$ instead of $C_0^{p\text{-var}}(\mathbf{R}^d)$, because an abstract Wiener space must be separable by definition.]

Therefore, $\alpha\circ V_2$ is a bounded bilinear form on an abstract Wiener space. By Goodman's theorem (Theorem 4.6, Kuo [23]), its restriction on the Cameron–Martin space is of trace class. $\square$

Now we compute V_1.

LEMMA 4.6. *Let $1/4 < H < 1/2$ and choose p and q as in (4.2). Then, for any bounded linear functional $\alpha \in C_0^{p\text{-var}}(\mathbf{R}^n)^*$, the symmetric bounded bilinear form $\alpha \circ V_1\langle \cdot, \cdot \rangle$ on the Cameron–Martin space $\mathcal{H}^H$ is Hilbert–Schmidt. In particular, if $p' \geq p$, $\nabla F(\phi^0) \circ V_1$ is Hilbert–Schmidt for a Fréchet differentiable function $F : C_0^{p'\text{-var}}(\mathbf{R}^n) \to \mathbf{R}$. Moreover, if α_l is weak* convergent to α as $l \to \infty$ in $C_0^{p\text{-var}}(\mathbf{R}^n)^*$, then $\alpha_l \circ V_1$ converges to $\alpha \circ V_1$ as $l \to \infty$ in the Hilbert–Schmidt norm.*

The rest of this section is devoted to proving this lemma. An integration by parts yields that

$$V_1\langle f, k \rangle = R_1\langle f, k \rangle + R_1\langle k, f \rangle - (R_2\langle f, k \rangle + R_2\langle k, f \rangle),$$

where, from (4.6),

$$R_1\langle f, k \rangle_t = M_t \int_0^t M_s^{-1} \nabla \sigma(\phi_s^0) \langle \sigma(\phi_s^0) f_s, dk_s \rangle,$$

$$R_2\langle f, k \rangle_t = M_t \int_0^t M_s^{-1} \nabla \sigma(\phi_s^0) \left\langle M_s \int_0^s d[M_u^{-1} \sigma(\phi_u^0)] f_u, dk_s \right\rangle.$$

LEMMA 4.7. *Let R_2 be as above and $\alpha \in C_0^{p\text{-var}}(\mathbf{R}^n)^*$. Then, as a bilinear form on $\mathcal{H}^H$, $\alpha \circ R_2$ is of trace class. Moreover, if α_l is weak* convergent to α as $l \to \infty$ in $C_0^{p\text{-var}}(\mathbf{R}^n)^*$, then $\alpha_l \circ R_2$ converges to $\alpha \circ R_2$ as $l \to \infty$ in the Hilbert–Schmidt norm.*

PROOF. We use the Young integral. Since $u \mapsto M_u^{-1} \sigma(\phi_u^0)$ is of finite q-variation, we see that

$$\left\| \int_0^{\cdot} d[M_u^{-1} \sigma(\phi_u^0)] f_u \right\|_{q\text{-var}} \leq c_1 \| M_\cdot^{-1} \sigma(\phi_\cdot^0) \|_{q\text{-var}} \| f \|_{p\text{-var}} \leq c_2 \| f \|_{p\text{-var}}.$$

Similarly, since $s \mapsto M_s^{-1} \nabla \sigma(\phi_s^0)$, M_s are of finite q-variation,

$$\left\| M_\cdot \int_0^{\cdot} M_s^{-1} \nabla \sigma(\phi_s^0) \left\langle M_s \int_0^s d[M_u^{-1} \sigma(\phi_u^0)] f_u, dk_s \right\rangle \right\|_{p\text{-var}} \leq c_3 \| f \|_{p\text{-var}} \| k \|_{p\text{-var}}. \tag{4.9}$$

Thus, $(f, k) \mapsto R_2\langle f, k \rangle$ is a bounded bilinear map from $C_0^{p\text{-var}}(\mathbf{R}^d) \times C_0^{p\text{-var}}(\mathbf{R}^d)$ to $C_0^{p\text{-var}}(\mathbf{R}^n) \subset C_0(\mathbf{R}^n)$. In particular, $\alpha \circ R_2$ is a bounded bilinear form on $C_0^{p\text{-var}}(\mathbf{R}^n)$. Again, by Goodman's theorem (Theorem 4.6, [23]), its restriction on the Cameron–Martin space is of trace class.

Now we prove the convergence. Note that (4.9) still holds even when f or k do not start at 0. Consider the following continuous inclusions (see Proposition 3.4. Below, all the function space is $\mathbf{R}^d$-valued):

$$\mathcal{H}^H \hookrightarrow W_0^{\delta,2} \cong L_{0,\mathrm{real}}^{\delta,2} \hookrightarrow L_{\mathrm{real}}^{\delta,2} \hookrightarrow C^{q\text{-var}} \hookrightarrow C^{p\text{-var}},$$

where $\delta = 1/q$ and $\cong$ denotes isomorphism (but not unitary) of Hilbert spaces. Let us first consider $R_2|_{L_{\mathrm{real}}^{\delta,2} \times L_{\mathrm{real}}^{\delta,2}}$. We will show that, for an ONB $\{f_k\}_{k=1,2,\dots}$ of $L_{\mathrm{real}}^{\delta,2}$, it holds that $\sum_{k,j=1}^{\infty} \|R_2\langle f_k, f_j\rangle\|_{p\text{-var}}^2 < \infty$. As in Theorem 4.3, we set $f_{0,i}(t) = 1 \cdot \mathbf{e}_i$ and $f_{m,i}(t) = (1+m^2)^{-\delta/2}\sqrt{2}\cos(m\pi t)\mathbf{e}_i$ $(m = 1, 2, \dots)$. By Proposition 4.4,

$$\|f_{m,i}\|_{p\text{-var}} \le (1+m^2)^{-\delta/2}\sqrt{2}(1+2m^{1/p}) \le c\left(\frac{1}{1+m}\right)^{1/q-1/p}$$

for some constant $c > 0$. From this and (4.9),

$$\begin{aligned}
&\sum_{i,i'=1}^{d} \sum_{m,m'=0}^{\infty} \|R_2\langle f_{m,i}, f_{m',i'}\rangle\|_{p\text{-var}}^2 \\
&\quad \le c \sum_{i,i'=1}^{d} \sum_{m,m'=0}^{\infty} \|f_{m,i}\|_{p\text{-var}}^2 \|f_{m',i'}\|_{p\text{-var}}^2 \\
&\quad \le c \sum_{m=0}^{\infty} \left(\frac{1}{1+m}\right)^{2(1/q-1/p)} \sum_{m'=0}^{\infty} \left(\frac{1}{1+m'}\right)^{2(1/q-1/p)} < \infty,
\end{aligned}$$

because $1/q - 1/p > 1/2$. Here, the constant $c > 0$ may change from line to line.

By the Banach–Steinhaus theorem, $\|\alpha_l - \alpha\|_{C^{p\text{-var},*}} \le c$ for some constant $c > 0$. Hence,

$$|(\alpha_l - \alpha) \circ R_2\langle f_{m,i}, f_{m',i'}\rangle|^2 \le c^2 \|R_2\langle f_{m,i}, f_{m',i'}\rangle\|_{p\text{-var}}^2.$$

By the dominated convergence theorem, $\|\alpha_k \circ R_2 - \alpha \circ R_2\|_{\mathrm{HS}-L_{\mathrm{real}}^{\delta,2}} \to 0$ as $k \to \infty$. (The norm denotes the Hilbert–Schmidt norm.) This implies that

$$\|\alpha_l \circ R_2 - \alpha \circ R_2\|_{\mathrm{HS}-\mathcal{H}^H} \le \|\iota\|_{\mathrm{op}} \|\iota^*\|_{\mathrm{op}} \|\alpha_l \circ R_2 - \alpha \circ R_2\|_{\mathrm{HS}-L_{\mathrm{real}}^{\delta,2}} \to 0$$

as $l \to \infty$, where $\iota : \mathcal{H}^H \hookrightarrow L_{\mathrm{real}}^{\delta,2}$ denotes the inclusion. □

LEMMA 4.8. *Let R_1 be as above and $\alpha \in C_0^{p\text{-var}}(\mathbf{R}^n)^*$. Then, as a bilinear form on $\mathcal{H}^H$, $\alpha \circ R_1$ is Hilbert–Schmidt. Moreover, if α_l is weak* convergent to α as $l \to \infty$ in $C_0^{p\text{-var}}(\mathbf{R}^n)^*$, then $\alpha_l \circ R_1$ converges to $\alpha \circ R_1$ as $l \to \infty$ in the Hilbert–Schmidt norm.*

PROOF. The proof is similar to the one for Lemma 4.7. It is sufficient to show that

$$\sum_{i,i'=1}^{d}\sum_{m,m'=0}^{\infty}\|R_1\langle f_{m,i}, f_{m',i'}\rangle\|_{p\text{-var}}^2<\infty. \tag{4.10}$$

In this proof, $c>0$ is a constant which may change from line to line.

It is easy to see that, if $m\neq m'$,

$$\begin{aligned}\sqrt{2}\cos(m\pi t)d\big[\sqrt{2}\cos(m'\pi t)\big]&=-2m'\pi\cos(m\pi t)\sin(m'\pi t)\,dt\\&=-m'\pi\big\{\sin\big((m'+m)\pi t\big)+\sin\big((m'-m)\pi t\big)\big\}\,dt\\&=m'd\left[\frac{\cos((m'+m)\pi t)}{m'+m}+\frac{\cos((m'-m)\pi t)}{m'-m}\right],\end{aligned}$$

and that, if $m=m'$, $\sqrt{2}\cos(m\pi t)d[\sqrt{2}\cos(m\pi t)]=d[\cos(2m\pi t)]/2$.

In the following, fix i, i'. First, we consider the case $m=m'$:

$$\begin{aligned}&R_1\langle f_{m,i}, f_{m,i'}\rangle_t\\&\quad=M_t\int_0^t M_s^{-1}\nabla\sigma(\phi_s^0)\langle\sigma(\phi_s^0)\mathbf{e}_i,\mathbf{e}_{i'}\rangle\sqrt{2}\frac{\cos(m\pi s)}{(1+m^2)^{1/2q}}\,d\left[\frac{\sqrt{2}\cos(m\pi s)}{(1+m^2)^{1/2q}}\right]\\&\quad=\frac{1/2}{(1+m^2)^{1/q}}M_t\int_0^t M_s^{-1}\nabla\sigma(\phi_s^0)\langle\sigma(\phi_s^0)\mathbf{e}_i,\mathbf{e}_{i'}\rangle\,d[\cos(2m\pi s)].\end{aligned}$$

By the Young integral and Proposition 4.4, we see that

$$\begin{aligned}\|R_1\langle f_{m,i}, f_{m,i'}\rangle\|_{p\text{-var}}^2&\le\frac{c}{(1+m^2)^{2/q}}\|\cos(2m\pi\cdot)-1\|_{p\text{-var}}^2\\&\le\frac{cm^{2/p}}{(1+m^2)^{2/q}}\le\frac{c}{(1+m)^{4/q-2/p}}.\end{aligned}$$

Since $4/q-2/p>1$,

$$\sum_{m=0}^{\infty}\|R_1\langle f_{m,i}, f_{m,i'}\rangle\|_{p\text{-var}}^2<\infty. \tag{4.11}$$

Next we consider the case $m\neq m'$:

$$\begin{aligned}&R_1\langle f_{m,i}, f_{m',i'}\rangle_t\\&\quad=M_t\int_0^t M_s^{-1}\nabla\sigma(\phi_s^0)\langle\sigma(\phi_s^0)\mathbf{e}_i,\mathbf{e}_{i'}\rangle\sqrt{2}\frac{\cos(m\pi s)}{(1+m^2)^{1/2q}}\,d\left[\frac{\sqrt{2}\cos(m'\pi s)}{(1+m'^2)^{1/2q}}\right]\\&\quad=\frac{m'}{(1+m^2)^{1/2q}(1+m'^2)^{1/2q}(m'+m)}\\&\qquad\times M_t\int_0^t M_s^{-1}\nabla\sigma(\phi_s^0)\langle\sigma(\phi_s^0)\mathbf{e}_i,\mathbf{e}_{i'}\rangle\,d\big[\cos\big((m'+m)\pi s\big)\big]\end{aligned}$$

$$
\begin{aligned}
&+ \frac{m'}{(1+m^2)^{1/2q}(1+m'^2)^{1/2q}(m'-m)} \\
&\qquad \times M_t \int_0^t M_s^{-1} \nabla\sigma(\phi_s^0)\langle \sigma(\phi_s^0)\mathbf{e}_i, \mathbf{e}_{i'}\rangle \, d\bigl[\cos\bigl((m'-m)\pi s\bigr)\bigr] \\
&=: \hat{R}_1^{i,i'}(m,m')_t + \hat{R}_2^{i,i'}(m,m')_t.
\end{aligned}
$$

By using the estimate for the Young integral again, we see that

$$
\begin{aligned}
&\|\hat{R}_1^{i,i'}\langle f_{m,i}, f_{m,i'}\rangle\|_{p\text{-var}}^2 \\
&\quad \le \frac{cm'^2}{(1+m^2)^{1/q}(1+m'^2)^{1/q}|m'+m|^2} \bigl\|\cos\bigl((m'+m)\pi\cdot\bigr) - 1\bigr\|_{p\text{-var}}^2 \\
&\quad \le \frac{cm'^2|m'+m|^{2/p}}{(1+m^2)^{1/q}(1+m'^2)^{1/q}|m'+m|^2} \\
&\quad \le \frac{c|(m'+m)-m|^{2(1-1/q)}}{(1+|m|)^{2/q}|m'+m|^{2(1-1/p)}} \\
&\quad \le \frac{c}{(1+|m|)^{2/q}(1+|m'+m|)^{2(1/q-1/p)}} \\
&\qquad + \frac{c}{(1+|m|)^{4(1/q-1/2)}(1+|m'+m|)^{2(1-1/p)}}.
\end{aligned}
$$

It is easy to see that $2/q > 1$ and $2(1-1/p) > 1$ hold. From (4.2), $2(1/q-1/p) > 1$ and $4(1/q-1/2) > 1$. (The condition $1/q > 3/4$ is used here.) Therefore,

$$
\begin{aligned}
&\sum_{0\le m,m'<\infty, m\ne m'} \|\hat{R}_1^{i,i'}\langle f_{m,i}, f_{m,i'}\rangle\|_{p\text{-var}}^2 \\
&\quad \le c \sum_{m,m'\in\mathbf{Z}} \biggl(\frac{1}{(1+|m|)^{2/q}(1+|m'+m|)^{2(1/q-1/p)}} \\
&\qquad\qquad + \frac{1}{(1+|m|)^{4(1/q-1/2)}(1+|m'+m|)^{2(1-1/p)}} \biggr) \qquad (4.12) \\
&\quad = c \sum_{m\in\mathbf{Z}} \frac{1}{(1+|m|)^{2/q}} \biggl(\sum_{m'\in\mathbf{Z}} \frac{1}{(1+|m'+m|)^{2(1/q-1/p)}} \biggr) \\
&\qquad + c \sum_{m\in\mathbf{Z}} \frac{1}{(1+|m|)^{4(1/q-1/2)}} \biggl(\sum_{m'\in\mathbf{Z}} \frac{1}{(1+|m'+m|)^{2(1-1/p)}} \biggr) < \infty.
\end{aligned}
$$

In the same way as above,

$$
\sum_{0\le m,m'<\infty, m\ne m'} \|\hat{R}_2^{i,i'}\langle f_{m,i}, f_{m,i'}\rangle\|_{p\text{-var}}^2 < \infty. \tag{4.13}
$$

From (4.11), (4.12) and (4.13), we have (4.10), which completes the proof. □

5. A probabilistic representation of Hessian. Throughout this section we assume (4.2). Let $(\mathcal{X}, \mathcal{H}^H, \mu^H)$ be the abstract Wiener space for the fBM as in the previous section. Here, $\mathcal{X}$ is the closure of the Cameron–Martin space $\mathcal{H}^H$ in $C_0^{p\text{-var}}(\mathbf{R}^d)$. A generic element of $\mathcal{X}$ is denoted by w^H. Under μ^H, $(w_t^H)_{0\le t\le 1}$ is the canonical realization of d-dimensional fBM.

Any $\langle k, \cdot\rangle \in (\mathcal{H}^H)^*$ extends to a measurable linear functional on $\mathcal{X}$, which is denoted by $\langle k, w^H\rangle$ with a slight abuse of notation. It satisfies

$$\int_{\mathcal{X}} e^{\sqrt{-1}\langle k, w^H\rangle} \mu^H(dw^H) = e^{\|k\|^2_{\mathcal{H}^H}/2}.$$

For a cylinder function $F(w^H) = f(\langle k_1, w^H\rangle, \dots, \langle k_m, w^H\rangle)$, where $f : \mathbf{R}^m \to \mathbf{R}$ is a bounded smooth function with bounded derivatives, we set

$$D_k F(w^H) = \sum_{j=1}^m \partial_j f(\langle k_1, b\rangle, \dots, \langle k_m, b\rangle)(k_j, k)_{\mathcal{H}^H}, \qquad k \in \mathcal{H}^H,$$

and

$$DF(w^H) = \sum_{j=1}^m \partial_j f(\langle k_1, w^H\rangle, \dots, \langle k_m, w^H\rangle) k_j.$$

Note that DF is an $\mathcal{H}^H$-valued function.

Let $\mathcal{C}_n = \mathcal{C}_n(\mu^H)(n = 0, 1, 2, \dots)$ be the nth Wiener chaos of w^H. It is well known that $\mathcal{C}_n$ are mutually orthogonal and $L^2(\mu^H) = \bigoplus_{n=0}^\infty \mathcal{C}_n$. For example, $\mathcal{C}_0 = \{\text{constants}\}$ and $\mathcal{C}_1 = \{\langle k, \cdot\rangle | k \in \mathcal{H}^H\}$. The second Wiener chaos $\mathcal{C}_2$ is unitarily isometric with the space of symmetric Hilbert–Schmidt operators (or symmetric Hilbert–Schmidt bilinear forms) $\mathcal{H}^H \otimes_{sym} \mathcal{H}^H$ in a natural way.

LEMMA 5.1. *Let V_1 be as in (4.8) and consider $V_1(w^H(m), w^H(m))_t$, where $w^H(m)$ denotes the mth dyadic polygonal approximation of w^H. Then, for $k, \hat{k} \in \mathcal{H}^H$,*

$$\tfrac{1}{2} D_k V_1(w^H(m), w^H(m))_t = V_1(k(m), w^H(m))_t,$$

$$\tfrac{1}{2} D_{\hat{k}} D_k V_1(w^H(m), w^H(m))_t = V_1(k(m), \hat{k}(m))_t.$$

Moreover, as $m \to \infty$, the right-hand sides of the above equations converge to

$$V_1(k, w^H)_t \quad \textit{and} \quad V_1(k, \hat{k})_t$$

almost surely and in $L^2(\mu^H)$. [Note that the above quantities are well-defined since w^H is of finite p-variation and $k, \hat{k}$ is of finite q-variation with $1/p + 1/q > 1$. Since $k, \hat{k}$ are of finite $(q - \varepsilon)$-variation for sufficiently small $\varepsilon > 0$, $k(m), \hat{k}(m)$ converge to $k, \hat{k}$ in q-variation norm, resp.]

PROOF. On $[(l-1)/2^m, l/2^m]$, $dw^H(m)_t = 2^n(w^H_{l/2^m} - w^H_{(l-1)/2^m})\,dt$. Therefore,

$$D_k dw^H(m)_t = 2^n D_k\big(w^H_{l/2^m} - w^H_{(l-1)/2^m}\big)\,dt = 2^n\big(k_{l/2^m} - k_{(l-1)/2^m}\big)\,dt = dk(m)_t.$$

From this, we see that

$$\begin{aligned} D_k\chi(w^H(m))_t &= M_t \int_0^t M_s^{-1}\sigma(\phi_s^0) D_k\, dw^H(m)_s \\ &= M_t \int_0^t M_s^{-1}\sigma(\phi_s^0)\, dk(m)_s = \chi(k(m))_t. \end{aligned}$$

Since $\|k(m) - k\|_{q\text{-var}}$ as $m \to \infty$ and $k \mapsto \chi(k)$ is bounded linear from $C_0^{q\text{-var}}(\mathbf{R}^d)$ to $C_0^{q\text{-var}}(\mathbf{R}^d)$, $\|\chi(k(m)) - \chi(k)\|_{q\text{-var}}$ as $m \to \infty$. [Note that, for sufficiently small $\varepsilon > 0$, $k \in C_0^{(q-\varepsilon)\text{-var}}$ still holds.] In a similar way,

$$\begin{aligned} &\frac{1}{2} D_k V_1(w^H(m), w^H(m))_t \\ &\quad = M_t \int_0^t M_s^{-1}\{\nabla\sigma(\phi_s^0)\langle D_k\chi(w^H(m))_s, dw^H(m)_s\rangle \\ &\qquad\qquad + \nabla\sigma(\phi_s^0)\langle \chi(w^H(m))_s, D_k\, dw^H(m)_s\rangle\} \\ &\quad = M_t \int_0^t M_s^{-1}\{\nabla\sigma(\phi_s^0)\langle \chi(k(m))_s, dw^H(m)_s\rangle \\ &\qquad\qquad + \nabla\sigma(\phi_s^0)\langle \chi(w^H(m))_s, dk(m)_s\rangle\} \\ &\quad = V_1(k(m), w^H(m))_t. \end{aligned}$$

Since $\|w^H(m) - w^H\|_{p\text{-var}} \to 0$ as $m \to \infty$ almost surely and in L^r for any $r > 0$ (see [32]), $(1/2)D_k V_2(w^H(m), w^H(m))_t \to V_1(k, w^H)_t$ almost surely and in L^2. Finally,

$$(1/2)D_{\hat{k}} D_k V_2(w^H(m), w^H(m))_t = V_1(k(m), \hat{k}(m))_t,$$

which is nonrandom and clearly converges to $V_1(k, \hat{k})_t$ as $m \to \infty$. □

PROPOSITION 5.2. *Let V_1 be as in (4.8) and consider $V_1(w^H(m), w^H(m))_t^i$. Here, i stands for the ith component ($1 \le i \le n$). Then, for each fixed t, $V_1(w^H(m), w^H(m))_t^i$ converges almost surely and in $L^2(\mu^H)$ as $m \to \infty$. More precisely,*

$$\lim_{m\to\infty} V_1(w^H(m), w^H(m))_t^i = \Theta_t^i + \Lambda_t^i.$$

Here, Θ_t^i is an element in $\mathcal{C}_2$ which corresponds to the symmetric Hilbert–Schmidt bilinear form $V_1(\bullet,\bullet)_t^i$ and $t \mapsto \Lambda_t^i := \lim_{m\to\infty} E[V_1(w^H(m), w^H(m))_t^i]$ is of finite p-variation.

PROOF. First note that $V_1(x,x)$ has a rough path representation. Recall the (stochastic) Taylor expansion of the Itô map (4.1) around γ. Then, $V_1(x,x)$ was essentially calculated in computation for the second Taylor term. There is a continuous map $V': G\Omega_p(\mathbf{R}^d) \to G\Omega_p(\mathbf{R}^n)$ such that $V_1(x,x) = V'(X)^1$ for all $x \in C_0^{q\text{-var}}(\mathbf{R}^d)$. Here, the superscript means the first level path and $X \in G\Omega_p(\mathbf{R}^d)$ is the lift of x. Moreover, since the integral that defines V_1 or V' in (4.8) is of second order, V' has the following property: there exists a constant $c > 0$ such that, for all $X, Y \in G\Omega_p(\mathbf{R}^d)$,

$$\|V'(X)^1\|_{p\text{-var}} \le c\bigl(1 + \xi(X)^2\bigr),$$

$$\|V'(X)^1 - V'(Y)^1\|_{p\text{-var}} \le c\bigl(1 + \xi(X)^c\bigr) \sum_{j=1}^{[p]} \|X^j - Y^j\|_{p/j\text{-var}}.$$

Here, $\xi(X) = \sum_{j=1}^{[p]} \|X^j\|_{p/j\text{-var}}^{1/j}$. From this, a.s.-convergence of $V_1(w^H(m), w^H(m)) = V'(w^H(m)))^1$ to $V'(W^H))^1$ is obvious.

It is shown in [10] that $E[\|W^H(m)^j - W^{H,j}\|_{p/j\text{-var}}] \to 0$ as $m \to \infty$. From Proposition 3.3, $\sup_m E[\|W^H(m)^j\|_{p/j\text{-var}}^r] < \infty$ for any $r > 0$ and $1 \le j \le [p]$. Then, we easily see from these and Hölder's inequality that

$$E[\|V'(W^H(m))^1 - V'(W^H)^1\|_{p\text{-var}}^2] \to 0 \qquad \text{as } m \to \infty.$$

This implies the L^2-convergence. Since $V'(W^H) = \lim_{m\to\infty} V_1(w^H(m), w^H(m))$ is a $C_0^{p\text{-var}}(\mathbf{R}^n)$-valued random variable,

$$\|E[V'(W^H)^1]\|_{p\text{-var}} \le E[\|V'(W^H)^1\|_{p\text{-var}}] < \infty,$$

which shows that Λ is of finite p-variation.

By Lemma 5.1 and the closability of the derivative operator D in $L^2(\mu^H)$,

$$\tfrac{1}{2} D_k V'(W^H)_t^{1,i} = V_1(k, w^H)_t^i, \qquad \tfrac{1}{2} D_{\hat{k}} D_k V'(W^H)_t^{1,i} = V_1(k, \hat{k})_t^i,$$

where the superscript i denotes the ith component of $\mathbf{R}^n$. These equality imply that $V'(W^H)_t^{1,i} - E[V'(W^H)_t^{1,i}]$ is in $\mathcal{C}_2$, which corresponds to $V_1(\bullet,\bullet)_t^i$. $\square$

LEMMA 5.3. *Let $p' > p$ and $F: C_0^{p'\text{-var}}(\mathbf{R}^n)$ be a Fréchet differentiable function. Let*

$$\Theta_t = \lim_{m\to\infty} V_1(w^H(m), w^H(m))_t - E\Bigl[\lim_{m\to\infty} V_1(w^H(m), w^H(m))_t\Bigr]$$

be as in Proposition 5.2. Then, $\nabla F(\phi^0)\langle\Theta\rangle \in \mathcal{C}_2(\mu^H)$ which corresponds to the symmetric Hilbert–Schmidt bilinear form $\nabla F(\phi^0) \circ V_1 = \nabla F(\phi^0)\langle V_1(\bullet,\bullet)\rangle$ on $\mathcal{H}^H$.

PROOF. Denote by g_K the element of $\mathcal{C}_2(\mu^H)$ which corresponds to a symmetric Hilbert–Schmidt bilinear form (or, equivalently, operator) K and set

$$M := \{\alpha \in C_0^{p\text{-var}}(\mathbf{R}^n)^* | \alpha\langle\Theta(w)\rangle = g_{\alpha\circ V_1}(w) \text{ a.a. } w\ (\mu^H)\}.$$

Obviously, M is a linear subspace. Moreover, from Lemma 4.6, M is closed under weak*-limit. By Lemma 5.2, the evaluation map ev_t^i ($t \in [0,1]$, $1 \le i \le n$) defined by $\mathrm{ev}_t^i\langle y\rangle = y_t^i$ is in M. Denote by $\pi_m : C_0^{p\text{-var}}(\mathbf{R}^n) \to C_0^{p\text{-var}}(\mathbf{R}^n)$ the projection defined by $\pi(m)y = y(m)$, where $y(m)$ is the mth dyadic piecewise linear approximation of $y \in C_0^{p\text{-var}}(\mathbf{R}^n)$. Note that $\nabla F(\phi^0)\langle \pi(m)y\rangle$ can be written as a linear combination of $y^i_{k/2^m}$ ($1 \le k \le 2^m$, $1 \le i \le n$). Hence, $\nabla F(\phi^0) \circ \pi(m) \in M$. Since $p' > p$, $y(m) \to y$ in p'-variation norm. This implies that $\nabla F(\phi^0) \circ \pi(m) \to \nabla F(\phi^0)$ in the weak*-topology. Hence, $\nabla F(\phi^0) \in M$. $\square$

Let A_1 be a self-adjoint Hilbert–Schmidt operator on $\mathcal{H}^H$ which corresponds to

$$\nabla F(\phi^0)\langle V_1(\bullet, \bullet)\rangle.$$

Then, $A - A_1$ is a self-adjoint Hilbert–Schmidt operator on $\mathcal{H}^H$ which corresponds to

$$\nabla F(\phi^0)\langle V_2(\bullet, \bullet)\rangle + \nabla^2 F(\phi^0)\langle \chi(\bullet), \chi(\bullet)\rangle.$$

Obviously, this bilinear form extends to one on $C_0^{p\text{-var}}(\mathbf{R}^d)$ and, hence, is of trace class by Goodman's theorem. See (4.8) for the definition of V_1, V_2. Combining these all, we see that

$$k \in \mathcal{H}^H \mapsto \langle Ak, k\rangle_{\mathcal{H}^H} = \nabla F(\phi^0)\langle \psi(k,k)\rangle + \nabla^2 F(\phi^0)\langle \chi(k), \chi(k)\rangle$$

extends to a continuous map on $G\Omega_p(\mathbf{R}^d)$ and we denote it by $\langle AX, X\rangle$ for $X \in G\Omega_p(\mathbf{R}^d)$.

LEMMA 5.4. *Let $\alpha \ge 1$ be such that $\mathrm{Id}_{\mathcal{H}^H} + \alpha A$ is strictly positive in the form sense. Then,*

$$\begin{aligned}\int_{\mathcal{X}} &\exp\left(-\frac{\alpha}{2}\langle AW^H, W^H\rangle\right)\mu^H(dw^H)\\ &= \int_{G\Omega_p(\mathbf{R}^d)} \exp\left(-\frac{\alpha}{2}\langle AX, X\rangle\right)\mathbb{P}^H(dX) < \infty.\end{aligned}$$

In particular, $e^{-\langle A\bullet,\bullet\rangle/2}$ is in $L^r(G\Omega_p(\mathbf{R}^d), \mathbb{P}^H)$ for some $r > 1$.

PROOF. As a functional of w^H, $\langle (A - A_1)W^H, W^H\rangle$ is a sum of $\mathrm{Tr}(A - A_1)$ and the second order Wiener chaos corresponding to $A - A_1$. From Proposition 5.2 and Lemma 5.3, $\langle AW^H, W^H\rangle$ is a sum of a constant $\mathrm{Tr}(A - A_1) + \nabla F(\phi^0)\langle \Lambda\rangle$ and the second order Wiener chaos corresponding to A (which is denoted by Ξ_A below). It is well known (see Remark 5.5 below) that

$$\mathbb{E}[e^{-\alpha \Xi_A/2}] = \det_2(\mathrm{Id}_{\mathcal{H}^H} + \alpha A)^{-1/2},$$

where $\det_2$ stands for the Carleman–Fredholm determinant. $\square$

REMARK 5.5. Let $(\hat{\mathcal{X}}, \hat{H}, \hat{\mu})$ be any abstract Wiener space. For a symmetric Hilbert–Schmidt operator $\hat{A} : \hat{H} \to \hat{H}$, we denote by $\hat{\Xi}$ the corresponding element in the second Wiener chaos $\hat{\mathcal{C}}_2$. If $\hat{A} > -\mathrm{Id}$ in the form sense, then

$$\mathbb{E}[e^{-\hat{\Xi}/2}] = \det_2(\mathrm{Id} + \hat{A})^{-1/2} \qquad \left(:= \prod_{j=1}^{\infty} \{(1+\lambda_j)e^{-\lambda_j}\}^{-1/2}\right). \tag{5.1}$$

Here, $\{\lambda_j\}_{j=1,2,\dots}$ are eigenvalues of $\hat{A}$.

This fact is well known. For the reader's convenience, however, we give a simple (and somewhat heuristic) proof below. Suppose $\hat{\mathcal{X}} = \hat{H} = \mathbf{R}^l$ and $\hat{\mu}$ is the standard normal distribution. Let $\hat{A}$ be such that $\langle \hat{A}\xi, \eta \rangle = \sum_{j=1}^{l} \lambda_j \xi_j \eta_j$, $(\xi, \eta \in \mathbf{R}^l)$. We assume that $\lambda_j > -1$ for all j. In this case, $\hat{\Xi}$ is given by the following Hermite polynomial: $\hat{\Xi}(u) = \sum_{j=1}^{l} \lambda_j (u_j^2 - 1)$ $(u \in \mathbf{R}^l)$. This clearly corresponds to $\hat{A}$ because $(1/2) D_\xi D_\eta \hat{\Xi}(u) = \langle \hat{A}\xi, \eta \rangle$. (Recall that this is the way we identified symmetric Hilbert–Schmidt operators with elements of second Wiener chaos in the previous argument.) A simple calculation shows that

$$\begin{aligned}
\mathbb{E}[e^{-\hat{\Xi}/2}] &= \int_{\mathbf{R}^l} e^{-\hat{\Xi}(u)/2} \prod_{j=1}^{l} \frac{1}{\sqrt{2\pi}} e^{-u_j^2/2}\, du_j \\
&= \prod_{j=1}^{l} \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}^l} \exp\left(\frac{\lambda_j - (1+\lambda_j) u_j^2}{2} \right) du_j = \prod_{j=1}^{l} \{(1+\lambda_j)e^{-\lambda_j}\}^{-1/2}.
\end{aligned}$$

We can easily do a similar computation in the case of $(\mathbf{R}^\infty, l^2, \mu^\infty)$, where μ^∞ denotes the countable product of the one-dimensional standard normal distribution. The general case reduces to the case of $\mathbf{R}^\infty$, after $\hat{A}$ is diagonalized.

Another method to verify (5.1) is to use an explicit formula for the characteristic function of the quadratic Wiener functional, which has been studied extensively. Let $B : \hat{H} \to \hat{H}$ be any symmetric Hilbert–Schmidt operator and let Ξ_B be the corresponding element in the second Wiener chaos $\hat{\mathcal{C}}_2$. Then, we have

$$\int_{\hat{\mathcal{X}}} \exp((\zeta/2)\Xi_B)\, d\hat{\mu} = \det_2(\mathrm{Id} - \zeta B)^{-1/2} \qquad \text{for any } \zeta \in \mathbf{C} \text{ with } |\zeta| < 1/\|B\|_{\mathrm{op}}.$$

For example, see Janson [22], page 78, or Taniguchi [37], page 13. The formula (5.1) immediately follows from this.

6. Proof of Laplace approximation.

6.1. *Large deviation for the law of Y^ε as $\varepsilon \searrow 0$.* In this section we prove the main theorem (Theorem 2.1). Let Y^ε be a solution of RDE (3.4). The law of $(Y^\varepsilon)^1 = Y^{\varepsilon,1}$ is the probability measure on $C_0^{p\text{-var}}(\mathbf{R}^n)$ for any $p > 1/H$. Then, by Theorem 3.1 and Proposition 3.6 we can use the contraction principle to see that the law of $\{Y^{\varepsilon,1}\}_{\varepsilon>0}$ satisfies large deviation as $\varepsilon \searrow 0$. The good rate function is given as follows:

$$I(y) = \begin{cases} \inf\{\|k\|^2_{\mathcal{H}^H}/2 \mid y = \hat{\Phi}_0(k,\lambda)^1\} & (\text{if } y = \hat{\Phi}_0(k,\lambda)^1 \text{ for some } k \in \mathcal{H}^H), \\ \infty & (\text{otherwise}). \end{cases}$$

Here, $\hat{\Phi}_\varepsilon$ is the Itô map corresponding to RDE (3.4) and $\lambda_t = t$. For a bounded continuous function F on $C_0^{p\text{-var}}(\mathbf{R}^n)$, it holds that

$$\lim_{\varepsilon\searrow 0} \varepsilon^2 \log \mathbb{E}\bigl[\exp\bigl(-F(Y^{\varepsilon,1})/\varepsilon^2\bigr)\bigr] = -\inf\{F(y) + I(y) \mid y \in C_0^{p\text{-var}}(\mathbf{R}^n)\}.$$

Now, let us consider Laplace's method, that is, the precise asymptotic behavior of the following integral:

$$\begin{aligned}\mathbb{E}\bigl[\exp\bigl(-F(Y^{\varepsilon,1})/\varepsilon^2\bigr)\bigr] &= \int_{G\Omega_p(\mathbf{R}^d)} \exp\bigl(-F(\hat{\Phi}_\varepsilon(\varepsilon X,\lambda)^1)/\varepsilon^2\bigr)\mathbb{P}^H(dX) \\ &= \int_{G\Omega_p(\mathbf{R}^d)} \exp\bigl(-F(\hat{\Phi}_\varepsilon(X,\lambda)^1)/\varepsilon^2\bigr)\mathbb{P}^H_\varepsilon(dX)\end{aligned}$$

as $\varepsilon \searrow 0$ under Assumptions (H1)–(H4).

Let $\gamma \in \mathcal{H}^H \subset G\Omega_p(\mathbf{R}^d)$ be the unique element at which $F(\hat{\Phi}_0(\cdot,\lambda)) + \|\cdot\|^2_{\mathcal{H}^H}/2$ attains minimum ($F_\Lambda(\gamma) =: a$) as in (H2). By a well-known argument, for any neighborhood of $O \subset G\Omega_p(\mathbf{R}^d)$ of γ, there exist positive constants δ, C such that

$$\int_{O^c} \exp\bigl(-F(\hat{\Phi}_\varepsilon(X,\lambda)^1)/\varepsilon^2\bigr)\mathbb{P}^H_\varepsilon(dX) \le Ce^{-(a+\delta)/\varepsilon^2}, \qquad \varepsilon \in (0,1].$$

This decays very fast and does not contribute to the asymptotic expansion.

6.2. *Computation of* α_0. In this subsection we compute the first term α_0 in the asymptotic expansion when $G \equiv 1$ (constant) and show $\alpha_0 > 0$. To do so, we need the (stochastic) Taylor expansion (Theorem 3.2) up to order $m = 2$. Once this is done, expansion up to higher order terms can be obtained rather easily.

For $\rho > 0$, set $U_\rho = \{X \in G\Omega_p(\mathbf{R}^d) \mid \xi(X) < \rho\}$, where ξ is given in (3.7). Then, taking $O = \gamma + U_\rho$, we see from the theorem of the Cameron–Martin type (Proposition 3.5) that

$$\begin{aligned}&\int_{\gamma+U_\rho} \exp\bigl(-F(\hat{\Phi}_\varepsilon(X,\lambda)^1)/\varepsilon^2\bigr)\mathbb{P}^H_\varepsilon(dX) \\ &\quad= \int_{U_\rho} \exp\bigl(-F\bigl(\hat{\Phi}_\varepsilon(X+\gamma,\lambda)^1\bigr)/\varepsilon^2\bigr) \\ &\qquad\quad \times \exp\Bigl(-\frac{1}{\varepsilon^2}\langle\gamma, X^1\rangle - \frac{1}{2\varepsilon^2}\|\gamma\|^2_{\mathcal{H}^H}\Bigr)\mathbb{P}^H_\varepsilon(dX) \\ &\quad= \int_{\{\xi(\varepsilon X)<\rho\}} \exp\Bigl(-\frac{F(\phi^{(\varepsilon)})}{\varepsilon^2} - \frac{1}{\varepsilon}\langle\gamma, X^1\rangle - \frac{1}{2\varepsilon^2}\|\gamma\|^2_{\mathcal{H}^H}\Bigr)\mathbb{P}^H(dX).\end{aligned} \tag{6.1}$$

As we will see, $\langle \gamma, \cdot \rangle$ extends to a continuous linear functional on $C_0^{p\text{-var}}(\mathbf{R}^d)$ and, in particular, everywhere defined.

For sufficiently small ρ (i.e., $\rho \le \rho_0$ for some ρ_0), $\phi^{(\varepsilon)}$ is in the neighborhood of ϕ^0 as in Assumption (H3). So, from the Taylor expansion for F,

$$\begin{aligned} F(\phi^{(\varepsilon)}) &= F(\phi^0) + \nabla F(\phi^0)\langle \phi^{(\varepsilon)} - \phi^0 \rangle + \frac{1}{2}\nabla^2 F(\phi^0)\langle \phi^{(\varepsilon)} - \phi^0, \phi^{(\varepsilon)} - \phi^0 \rangle \\ &\quad + \frac{1}{6}\int_0^1 d\theta \nabla^3 F\big(\theta\phi^{(\varepsilon)} + (1-\theta)\phi^0\big)\langle \phi^{(\varepsilon)} - \phi^0, \phi^{(\varepsilon)} - \phi^0, \phi^{(\varepsilon)} - \phi^0 \rangle \\ &= F(\phi^0) + \nabla F(\phi^0)\langle \varepsilon\phi^1 + \varepsilon^2\phi^2 \rangle + \frac{1}{2}\nabla^2 F(\phi^0)\langle \varepsilon\phi^1, \varepsilon\phi^1 \rangle + Q_\varepsilon^3. \end{aligned}$$

Here, the remainder term Q_ε^3 satisfies the following estimates: there exists a positive constant $C = C(\rho_0)$ such that

$$|Q_\varepsilon^3| \le C\big(\varepsilon + \xi(\varepsilon X)\big)^3 \qquad \text{on the set } \{\xi(\varepsilon X) < \rho_0\}. \tag{6.2}$$

Note that C is independent of the choice of ρ ($\rho \le \rho_0$).

Now we compute the shoulder of exp on the right-hand side of (6.1). Terms of order -2 are computed as follows:

$$-\frac{1}{\varepsilon^2}\left(F(\phi^0) + \frac{1}{2}\|\gamma\|_{\mathcal{H}^H}^2\right) = -\frac{a}{\varepsilon^2}.$$

Since $k \in \mathcal{H}^H \mapsto F(\Phi_0(k,\lambda)) + \|k\|_{\mathcal{H}^H}^2/2$ takes its minimum at $k = \gamma$, we see that

$$\langle k, \gamma \rangle_{\mathcal{H}^H} + \nabla F(\phi^0)\langle \chi(k) \rangle = 0,$$

where $\chi(k)$ is given by (4.5) or (4.6). By (4.6) and the Young integral, $k \mapsto \nabla F(\phi^0)\langle \chi(k) \rangle$ extends to a continuous linear map from $C_0^{p\text{-var}}(\mathbf{R}^d)$ and so does $\langle \gamma, \cdot \rangle_{\mathcal{H}^H}$. Hence, the measurable linear functional (i.e., the first Wiener chaos) associated with γ is this continuous extension.

An ODE for $\phi^1 = \phi^1(k) = \phi^1(k,\gamma)$ is as follows [$k \in C_0^{q\text{-var}}(\mathbf{R}^d)$]:

$$\begin{aligned} &d\phi_t^1 - \nabla\sigma(\phi_t^0)\langle \phi_t^1, d\gamma_t \rangle - \nabla_y \beta(0,\phi_t^0)\langle \phi_t^1 \rangle\, dt \\ &\qquad = \sigma(\phi_t^0)\, dk_t + \nabla_\varepsilon \beta(0,\phi_t^0)\, dt, \qquad \phi_0^1 = 0. \end{aligned} \tag{6.3}$$

Note that both ϕ^1 and χ extend to a continuous map from $G\Omega_p(\mathbf{R}^d)$. The difference $\theta_t^1 := \phi_t^1(X) - \chi_t(X)$ is independent of X (i.e., nonrandom), of finite q-variation, and satisfies

$$\begin{aligned} &d\theta_t^1 - \nabla\sigma(\phi_t^0)\langle \theta_t^1, d\gamma_t \rangle - \nabla_y \beta(0,\phi_t^0)\langle \theta_t^1 \rangle\, dt \\ &\qquad = \nabla_\varepsilon \beta(0,\phi_t^0)\, dt, \qquad \theta_0^1 = 0. \end{aligned} \tag{6.4}$$

Or, equivalently, $\theta_t^1 = M_t \int_0^t M_s^{-1} \nabla_\varepsilon \beta(0, \phi_s^0)\, ds$. Consequently, terms of order -1 are computed as follows:

$$-\frac{1}{\varepsilon}\big(\nabla F(\phi^0)\langle \phi^1\rangle + \langle \gamma, X^1\rangle\big) = -\frac{\nabla F(\phi^0)\langle \theta^1\rangle}{\varepsilon}.$$

Now we compute terms of order 0. The second term $\phi^2 = \phi^2(k) = \phi^2(k, \gamma)$ in the expansion in Theorem 3.2 satisfies the following ODE (see [19], e.g.):

$$\begin{aligned} & d\phi_t^2 - \nabla\sigma(\phi_t^0)\langle \phi_t^2, d\gamma_t\rangle - \nabla_y \beta(0, \phi_t^0)\langle \phi_t^2\rangle\, dt \\ & \quad = \nabla\sigma(\phi_t^0)\langle \phi_t^1, dk_t\rangle + \tfrac{1}{2}\nabla^2\sigma(\phi_t^0)\langle \phi_t^1, \phi_t^1, d\gamma_t\rangle \\ & \qquad + \tfrac{1}{2}\nabla_y^2 \beta(\phi_t^0)\langle \phi_t^1, \phi_t^1\rangle\, dt \\ & \qquad + \nabla_y \nabla_\varepsilon \beta(\phi_t^0)\langle \phi_t^1\rangle\, dt + \tfrac{1}{2}\nabla_\varepsilon^2 \beta(0, \phi_t^0)\, dt, \qquad \phi_0^2 = 0. \end{aligned} \tag{6.5}$$

Let χ and ψ be as in (4.5) and (4.7), respectively. By the same argument for (stochastic) Taylor expansion (Theorem 3.2), those extend to continuous maps from $G\Omega_p(\mathbf{R}^d)$ and we write $\chi(X)$ and $\psi(X) = \psi(X, X)$. If we set $\theta^2(k) := \phi^2(k) - \psi(k)/2$, then θ^2 satisfies the following ODE:

$$\begin{aligned} & d\theta_t^2 - \nabla\sigma(\phi_t^0)\langle \theta_t^2, d\gamma_t\rangle - \nabla_y \beta(0, \phi_t^0)\langle \theta_t^2\rangle\, dt \\ & \quad = \nabla\sigma(\phi_t^0)\langle \theta_t^1, dk_t\rangle + \tfrac{1}{2}\nabla^2\sigma(\phi_t^0)\langle \theta_t^1, \theta_t^1, d\gamma_t\rangle + \nabla^2\sigma(\phi_t^0)\langle \theta_t^1, \chi_t, d\gamma_t\rangle \\ & \qquad + \tfrac{1}{2}\nabla_y^2 \beta(\phi_t^0)\langle \theta_t^1, \theta_t^1\rangle\, dt + \nabla_y^2 \beta(\phi_t^0)\langle \theta_t^1, \chi_t\rangle\, dt \\ & \qquad + \nabla_y \nabla_\varepsilon \beta(\phi_t^0)\langle \theta_t^1 + \chi_t\rangle\, dt + \tfrac{1}{2}\nabla_\varepsilon^2 \beta(0, \phi_t^0)\, dt, \qquad \theta_0^2 = 0. \end{aligned} \tag{6.6}$$

Or, equivalently,

$$\begin{aligned} \theta_t^2 = M_t \int_0^t M_s^{-1} \bigg(& \nabla\sigma(\phi_s^0)\langle \theta_s^1, dk_s\rangle + \frac{1}{2}\nabla^2\sigma(\phi_s^0)\langle \theta_s^1, \theta_t^1, d\gamma_s\rangle \\ & + \nabla^2\sigma(\phi_s^0)\langle \theta_s^1, \chi_s, d\gamma_s\rangle \\ & + \frac{1}{2}\nabla_y^2 \beta(\phi_s^0)\langle \theta_s^1, \theta_s^1\rangle\, ds + \nabla_y^2 \beta(\phi_s^0)\langle \theta_s^1, \chi_t\rangle\, ds \\ & + \nabla_y \nabla_\varepsilon \beta(\phi_s^0)\langle \theta_t^1 + \chi_s\rangle\, ds + \frac{1}{2}\nabla_\varepsilon^2 \beta(0, \phi_s^0)\, ds \bigg). \end{aligned}$$

This is just a Young integral and $k \mapsto \theta^2(k)$ extends to a continuous map from $C_0^{p\text{-var}}(\mathbf{R}^d)$ or from $G\Omega_p(\mathbf{R}^d)$ to $C_0^{p\text{-var}}(\mathbf{R}^n)$. Moreover, θ^2 is of first order, that is, for some constant $C > 0$, $\|\theta^2(X)\|_{p\text{-var}} \le C(1 + \xi(X))$ holds for any $X \in G\Omega_p(\mathbf{R}^d)$. In particular, by the Fernique-type theorem (Proposition 3.3), (a constant multiple of) θ^2 is exponentially integrable.

Hence, terms of order 0 on the shoulder of exp on the right-hand side of (6.1) are as follows:

$$
\begin{aligned}
&\nabla F(\phi^0)\langle\phi^2\rangle + \tfrac{1}{2}\nabla^2 F(\phi^0)\langle\phi^1, \phi^1\rangle \\
&\quad = \tfrac{1}{2}[\nabla F(\phi^0)\langle\psi\rangle + \nabla^2 F(\phi^0)\langle\chi, \chi\rangle] + \nabla F(\phi^0)\langle\theta^2\rangle \\
&\qquad + \tfrac{1}{2}\nabla^2 F(\phi^0)\langle\theta^1, \theta^1\rangle + \nabla^2 F(\phi^0)\langle\theta^1, \chi\rangle.
\end{aligned} \tag{6.7}
$$

Note that the last three terms on the right-hand side are dominated by $C(1+\xi(X))$ and that the first term is $\langle AX, X\rangle/2$ as in Lemma 5.4. By Proposition 3.3 and Lemma 5.4,

$$
\exp\bigl(-\nabla F(\phi^0)\langle\phi^2\rangle - \tfrac{1}{2}\nabla^2 F(\phi^0)\langle\phi^1, \phi^1\rangle\bigr) \in L^r(G\Omega_p(\mathbf{R}^d), \mathbb{P}^H) \quad \text{for some } r > 1.
$$

If $\rho > 0$ is chosen sufficiently small, then $\exp[2C\rho(1+\xi(X))^2] \in L^{r'}(G\Omega_p(\mathbf{R}^d), \mathbb{P}^H)$ for the conjugate exponent r', that is, $1/r + 1/r' = 1$. (We determine ρ, here.) We easily see that, if $\varepsilon \le \rho$,

$$
\begin{aligned}
&\mathbf{1}_{\{\xi(\varepsilon X)<\rho\}} \exp\bigl(-\nabla F(\phi^0)\langle\phi^2\rangle - \tfrac{1}{2}\nabla^2 F(\phi^0)\langle\phi^1, \phi^1\rangle\bigr) \exp(-\varepsilon^{-2} Q_\varepsilon^3) \\
&\quad \le \exp\bigl(-\nabla F(\phi^0)\langle\phi^2\rangle - \tfrac{1}{2}\nabla^2 F(\phi^0)\langle\phi^1, \phi^1\rangle\bigr) \exp\bigl[2C\rho\bigl(1+\xi(X)\bigr)^2\bigr].
\end{aligned} \tag{6.8}
$$

The right-hand side is integrable and independent of ε. So, we may use the dominated convergence theorem to obtain that

$$
\begin{aligned}
&\lim_{\varepsilon \searrow 0} \int_{\{\xi(\varepsilon X)<\rho\}} \exp\Bigl(-\nabla F(\phi^0)\langle\phi^2\rangle - \frac{1}{2}\nabla^2 F(\phi^0)\langle\phi^1, \phi^1\rangle - \frac{1}{\varepsilon^2} Q_\varepsilon^3\Bigr) \mathbb{P}^H(dX) \\
&\quad = \int_{G\Omega_p(\mathbf{R}^d)} \exp\Bigl(-\nabla F(\phi^0)\langle\phi^2\rangle - \frac{1}{2}\nabla^2 F(\phi^0)\langle\phi^1, \phi^1\rangle\Bigr) \mathbb{P}^H(dX).
\end{aligned}
$$

By Lemma 5.4, the right-hand side exists. Thus, we have computed (the asymptotics of) (6.1) up to α_0.

6.3. *Asymptotic expansion up to any order.* In this subsection we obtain the Laplace asymptotic expansion up to any order. Since this is routine once α_0 is obtained, we only give a sketch of the proof.

By combining the (stochastic) Taylor expansions for F, G and $\phi^{(\varepsilon)}$, we get

$$
\begin{aligned}
F\bigl(\phi^{(\varepsilon)}\bigr) - F(\phi^0) &\sim \varepsilon\eta^1 + \cdots + \varepsilon^n \eta^n + Q_\varepsilon^{n+1} \qquad \text{as } \varepsilon \searrow 0, \\
G\bigl(\phi^{(\varepsilon)}\bigr) - G(\phi^0) &\sim \varepsilon\hat{\eta}^1 + \cdots + \varepsilon^n \hat{\eta}^n + \hat{Q}_\varepsilon^{n+1} \qquad \text{as } \varepsilon \searrow 0.
\end{aligned}
$$

Here, the remainder terms Q_ε^{n+1}, $\hat{Q}_\varepsilon^{n+1}$ satisfy similar estimates to (6.2).

From this we see that

$$
\begin{aligned}
&\int_{\gamma+U_\rho} G(\hat{\Phi}_\varepsilon(X,\lambda)^1)\exp\bigl(-F(\hat{\Phi}_\varepsilon(X,\lambda)^1)/\varepsilon^2\bigr)\mathbb{P}^H_\varepsilon(dX) \\
&\qquad = e^{-a/\varepsilon^2}e^{-\nabla F(\phi^0)\langle\theta^1\rangle/\varepsilon} \\
&\qquad\quad \times \int_{\{\xi(\varepsilon X)<\rho\}} G(\phi^{(\varepsilon)})\exp\Bigl(-\nabla F(\phi^0)\langle\phi^2\rangle - \frac{1}{2}\nabla^2 F(\phi^0)\langle\phi^1,\phi^1\rangle\Bigr) \\
&\qquad\qquad\qquad \times \exp(-Q^3_\varepsilon/\varepsilon^2)\mathbb{P}^H(dX)
\end{aligned}
\tag{6.9}
$$

can easily be expanded. Note that

$$
\left|e^u - \left(1 + \frac{u}{1!} + \cdots + \frac{u^{n-1}}{(n-1)!}\right)\right| \le \frac{e^{|u|}|u|^n}{n!} \qquad (\text{with } u = -Q^3_\varepsilon/\varepsilon^2)
$$

and that $Q^3_\varepsilon = \varepsilon^3\eta^3 + \cdots + \varepsilon^n\eta^n + Q^{n+1}_\varepsilon$. Thus, we have shown the main theorem (Theorem 2.1).

7. Fractional order case: with an application to short time expansion. In this section we consider an RDE, which involves a fractional order term of ε. As a result, a fractional order term of ε appears in the asymptotic expansion. By time change, this has an application to the short time problems for the solutions of the RDE driven by fBRP.

First we see the scale invariance of fBRP. It is well known that, for $0 < c \le 1$, $(c^{-H}w^H_{ct})_{0\le t\le 1}$ and w^H have the same law. A similar fact holds for the law of fBRP $W^H = (W^H_{s,t})_{0\le s\le t\le 1}$. This is not so obvious from the scale invariance of fBM w^H, since fBRP W^H is constructed via the dyadic partition of [0, 1].

PROPOSITION 7.1. *Let $H \in (1/4, 1/2)$ and $0 < c \le 1$. Then, $(c^{-H} \times W^H_{cs,ct})_{0\le s\le t\le 1}$ and W^H have the same law.*

PROOF. (i) Baudoin and Coutin showed this statement in [6].

(ii) Friz and Victoir [15] showed the following: If a sequence of partitions of [0, 1] whose mesh tending to zero satisfies a condition called "nested," then the lift of w^H via this sequence gives the same W^H again. Combining this result with the scaling property of w^H, we can easily see the Proposition holds at least for $c \in \mathbf{Q}$. For $c \notin \mathbf{Q}$, just take a limit. □

Let $H \in (1/4, 1/3) \cup (1/3, 1/2)$. For simplicity, we consider the following RDE:

$$
dY^\varepsilon_t = \sigma(Y^\varepsilon_t)\varepsilon\, dX_t + \varepsilon^{1/H}\hat{\beta}(Y^\varepsilon_t)\, dt, \qquad Y^\varepsilon_0 = 0. \tag{7.1}
$$

Here, σ is as in Theorem 2.1, but we assume that a C_b^∞-function $\hat{\beta}:\mathbf{R}^n \to \mathbf{R}^n$ and the drift term is of this special form in this case. Set $\beta(\varepsilon, y) = \varepsilon^{1/H}\hat{\beta}(y)$. We also consider the following RDE, which is independent of ε:

$$dV_t = \sigma(V_t)\,dX_t + \hat{\beta}(V_t)\,dt, \qquad V_0 = 0. \tag{7.2}$$

Basically, when we introduce randomness, we always set $X = W^H$ in (7.1) and (7.2). Then, by the scale invariance of W^H (see Proposition 7.1 below), $(V_{\varepsilon^{1/H}s,\varepsilon^{1/H}t})_{0\le s\le t\le 1}$ and $(Y^\varepsilon_{s,t})_{0\le s\le t\le 1}$ have the same law. In particular, for each fixed $T \in (0,1]$, the $\mathbf{R}^n$-valued random variables $V^1_{0,T}$ and $(Y^{T^H})^1_{0,1}$ have the same law. Therefore, the short time asymptotics for $V^1_{0,t}$ is related to the small asymptotics of $(Y^\varepsilon)^1$.

Let us fix some notation for fractional order expansions. For

$$M = \left\{ n_1 + \frac{n_2}{H} \middle| n_1, n_2 = 0, 1, 2, \dots \right\},$$

let $0 = \kappa_0 < \kappa_1 < \kappa_2 < \cdots$ be all elements of M in increasing order. More concretely, leading terms are as follows:

$$\begin{aligned} &(\kappa_0, \kappa_1, \kappa_2, \dots) \\ &\quad = \left(0, 1, 2, \frac{1}{H}, 3, 1 + \frac{1}{H}, 4, 2 + \frac{1}{H}, 5 \wedge \frac{2}{H}, \dots\right) \qquad \text{if } H \in (1/3, 1/2), \\ &(\kappa_0, \kappa_1, \kappa_2, \dots) \\ &\quad = \left(0, 1, 2, 3, \frac{1}{H}, 4, 1 + \frac{1}{H}, 5, \dots,\right) \qquad \text{if } H \in (1/4, 1/3). \end{aligned} \tag{7.3}$$

As in the previous sections, we write $Y^\varepsilon = \hat{\Phi}_\varepsilon(\varepsilon X)$, $\tilde{Y}^\varepsilon = \hat{\Phi}_\varepsilon(\varepsilon X + \gamma)$, and $\phi^\varepsilon = (\tilde{Y}^\varepsilon)^1$ for the solution of (7.1). By slightly modifying Theorem 3.2, we can prove the (stochastic) Taylor expansion (around γ) for

$$\phi^{(\varepsilon)} = \phi^0 + \varepsilon^{\kappa_1}\phi^{\kappa_1} + \varepsilon^{\kappa_2}\phi^{\kappa_2} + \cdots + \varepsilon^{\kappa_m}\phi^{\kappa_m} + R_\varepsilon^{\kappa_{m+1}}.$$

In this case, ϕ^0 satisfies the following ODE (in q-variation sense):

$$d\phi^0_t = \sigma(\phi^0_t)\,d\gamma_t, \qquad \phi^0_0 = 0. \tag{7.4}$$

REMARK 7.2. Although $(d/d\varepsilon)^m|_{\varepsilon=0}$ does not operate on the right-hand side of the following (formal) ODE,

$$d\phi^{(\varepsilon)}_t = \sigma(\phi^{(\varepsilon)}_t)d(\varepsilon X_t + \gamma) + \varepsilon^{1/H}\hat{\beta}(\phi^{(\varepsilon)}_t)\,dt, \qquad \tilde{Y}^\varepsilon_0 = 0, \tag{7.5}$$

the proof of expansion in [19], which is similar to Azencott's argument in [4], does not use the ε-derivative and can be easily modified to our case.

Roughly and formally speaking, the proof goes as follows. First, combine

$$\phi^{(\varepsilon)} - \phi^0 = \varepsilon^{\kappa_1}\phi^{\kappa_1} + \cdots + \varepsilon^{\kappa_m}\phi^{\kappa_m} + \cdots$$

and the Taylor expansion of σ and $\hat{\beta}$ around ϕ_t^0. Next, pick up the terms of order $\alpha_m (m = 1, 2, \ldots)$. Then, we obtain a very simple ODE of first order for ϕ^{κ_m} recursively. This, in turn, can be used to rigorously define ϕ^{κ_m}. In the end, we prove growth of the remainder term is of an expected order. (This part is nontrivial and requires much computation.) Note that this method can be used both in integer order and in fractional order cases.

In the same way as in the previous sections, we have the following modification of the main theorem (Theorem 2.1).

THEOREM 7.3. *Let the coefficients $\sigma : \mathbf{R}^n \to \mathrm{Mat}(n, d)$ and $\hat{\beta} : \mathbf{R}^n \to \mathbf{R}^n$ be C_b^∞ and consider the RDE* (7.1) *with $X = W^H$, where $H \in (1/4, 1/3) \cup (1/3, 1/2)$. For simplicity, assume* (H1)–(H4) *for any order m. Then, we have the following asymptotic expansion as $\varepsilon \searrow 0$: there are real constants c and $\alpha_{\kappa_0} (= \alpha_0), \alpha_{\kappa_1}, \alpha_{\kappa_2}, \ldots$ such that*

$$\begin{aligned} &\mathbb{E}\big[G(Y^{\varepsilon,1}) \exp\big(-F(Y^{\varepsilon,1})/\varepsilon^2\big)\big] \\ &\qquad = \exp\big(-F_\Lambda(\gamma)/\varepsilon^2\big) \exp(-c/\varepsilon) \cdot \big(\alpha_{\kappa_0} + \alpha_{\kappa_1} \varepsilon^{\kappa_1} + \cdots + \alpha_{\kappa_m} \varepsilon^{\kappa_m} + O(\varepsilon^{\kappa_{m+1}})\big) \end{aligned}$$

for any $m \geq 0$.

REMARK 7.4. It is important to note that, in (7.3), indices up to degree two (i.e., $\kappa_0, \kappa_1, \kappa_2$) are the same as in the previous sections. The most difficult part of the proof of Theorem 2.1 is obtaining α_0 [or checking that $\alpha_0 \in (0, \infty)$ when $G \equiv 1$], in which the (stochastic) Taylor expansion of $\phi^{(\varepsilon)}$ up to ϕ^2 is used (see Section 6.2). Therefore, the proof in Section 6.2 holds true without modification in this case, too. Higher order terms are different in the fractional order case. But, the argument in Section 6.3 is simple anyway and can easily be modified. Thus, we can prove Theorem 7.3 without much difficulty.

As a corollary, we have the following short time expansion. In the following, ev_1 denotes the evaluation map at time 1, that is, $\mathrm{ev}_1(x) = x_1$ for an $\mathbf{R}^n$-valued path x.

COROLLARY 7.5. *Let the coefficients $\sigma : \mathbf{R}^n \to \mathrm{Mat}(n, d)$ and $\hat{\beta} : \mathbf{R}^n \to \mathbf{R}^n$ be C_b^∞ and consider the RDE* (7.2) *above with $X = W^H$, where $H \in (1/4, 1/3) \cup (1/3, 1/2)$. Let f and g be real-valued C_b^∞-functions on $\mathbf{R}^n$ such that $F := f \circ \mathrm{ev}_1$ and $G := g \circ \mathrm{ev}_1$ satisfy Assumptions* (H1)–(H4). *Then, we have the following asymptotic expansion as $t \searrow 0$: there are real constants c and $\hat{\alpha}_{\kappa_0} (= \hat{\alpha}_0), \hat{\alpha}_{\kappa_1}, \hat{\alpha}_{\kappa_2}, \ldots$ such that*

$$\begin{aligned} &\mathbb{E}\big[g(V_{0,t}^1) \exp\big(-f(V_{0,t}^1)/t^{2H}\big)\big] \\ &\qquad = \exp\big(-F_\Lambda(\gamma)/t^{2H}\big) \exp(-c/t^H) \\ &\qquad\quad \times \big(\hat{\alpha}_{\kappa_0} + \hat{\alpha}_{\kappa_1} t^{\kappa_1 H} + \cdots + \hat{\alpha}_{\kappa_m} t^{\kappa_m H} + O(t^{\kappa_{m+1} H})\big) \end{aligned}$$

for any $m \geq 0$.

REMARK 7.6. Very roughly speaking, in [5, 34], they studied the sort time asymptotics of the following quantity under mild assumptions:

$$\mathbb{E}[g(V_{0,t}^1)].$$

If f is identically zero in Corollary 7.5, then it is the same short time problems studied in [5, 34], at least formally. (It does not seem to the author that either [5, 34] or the Corollary 7.5 implies the other.)

REFERENCES

[1] ADAMS, R. A. (1975). *Sobolev Spaces*. Academic Press, New York–London. MR0450957

[2] AIDA, S. (2007). Semi-classical limit of the bottom of spectrum of a Schrödinger operator on a path space over a compact Riemannian manifold. *J. Funct. Anal.* **251** 59–121. MR2353701

[3] ALBEVERIO, S., RÖCKLE, H. and STEBLOVSKAYA, V. (2000). Asymptotic expansions for Ornstein–Uhlenbeck semigroups perturbed by potentials over Banach spaces. *Stochastics Stochastics Rep.* **69** 195–238. MR1760977

[4] AZENCOTT, R. (1982). Formule de Taylor stochastique et développement asymptotique d'intégrales de Feynman. In *Seminar on Probability, XVI, Supplement. Lecture Notes in Math.* **921** 237–285. Springer, Berlin. MR0658728

[5] BAUDOIN, F. and COUTIN, L. (2007). Operators associated with a stochastic differential equation driven by fractional Brownian motions. *Stochastic Process. Appl.* **117** 550–574. MR2320949

[6] BAUDOIN, F. and COUTIN, L. (2008). Self-similarity and fractional Brownian motions on Lie groups. *Electron. J. Probab.* **13** 1120–1139. MR2424989

[7] BEN AROUS, G. (1988). Methods de Laplace et de la phase stationnaire sur l'espace de Wiener. *Stochastics* **25** 125–153. MR0999365

[8] BIAGINI, F., HU, Y., ØKSENDAL, B. and ZHANG, T. (2008). *Stochastic Calculus for Fractional Brownian Motion and Applications*. Springer, London. MR2387368

[9] COUTIN, L. (2007). An introduction to (stochastic) calculus with respect to fractional Brownian motion. In *Séminaire de Probabilités XL. Lecture Notes in Math.* **1899** 3–65. Springer, Berlin. MR2408998

[10] COUTIN, L. and QIAN, Z. (2002). Stochastic analysis, rough path analysis and fractional Brownian motions. *Probab. Theory Related Fields* **122** 108–140. MR1883719

[11] ELDREDGE, N. (2005). Computing p-variation. Unpublished manuscript. Univ. California, San Diego.

[12] FRIZ, P. and OBERHAUSER, H. (2010). A generalized Fernique theorem and applications. *Proc. Amer. Math. Soc.* **138** 3679–3688. MR2661566

[13] FRIZ, P. and VICTOIR, N. (2006). A variation embedding theorem and applications. *J. Funct. Anal.* **239** 631–637. MR2261341

[14] FRIZ, P. and VICTOIR, N. (2007). Large deviation principle for enhanced Gaussian processes. *Ann. l'Inst. Henri Poincaré Probab. Stat.* **43** 775–785.

[15] FRIZ, P. and VICTOIR, N. (2010). Differential equations driven by Gaussian signals. *Ann. Inst. Henri Poincaré Probab. Stat.* **46** 369–413. MR2667703

[16] FRIZ, P. K. and VICTOIR, N. B. (2010). *Multidimensional Stochastic Processes as Rough Paths: Theory and Applications. Cambridge Studies in Advanced Mathematics* **120**. Cambridge Univ. Press, Cambridge. MR2604669

[17] GUBINELLI, M. and LEJAY, A. (2009). Global existence for rough differential equations under linear growth condition. Preprint.

[18] INAHAMA, Y. (2006). Laplace's method for the laws of heat processes on loop spaces. *J. Funct. Anal.* **232** 148–194. MR2200170

[19] INAHAMA, Y. (2010). A stochastic Taylor-like expansion in the rough path theory. *J. Theoret. Probab.* **23** 671–714. MR2679952

[20] INAHAMA, Y. and KAWABI, H. (2007). Asymptotic expansions for the Laplace approximations for Itô functionals of Brownian rough paths. *J. Funct. Anal.* **243** 270–322. MR2291439

[21] JAIN, N. C. and MONRAD, D. (1983). Gaussian measures in B_p. *Ann. Probab.* **11** 46–57. MR0682800

[22] JANSON, S. (1997). *Gaussian Hilbert Spaces. Cambridge Tracts in Mathematics* **129**. Cambridge Univ. Press, Cambridge. MR1474726

[23] KUO, H. H. (1975). *Gaussian Measures in Banach Spaces. Lecture Notes in Math.* **463**. Springer, Berlin. MR0461643

[24] KUSUOKA, S. and OSAJIMA, Y. (2008). A remark on the asymptotic expansion of density function of Wiener functionals. *J. Funct. Anal.* **255** 2545–2562. MR2473267

[25] KUSUOKA, S. and STROOCK, D. W. (1991). Precise asymptotics of certain Wiener functionals. *J. Funct. Anal.* **99** 1–74. MR1120913

[26] KUSUOKA, S. and STROOCK, D. W. (1994). Asymptotics of certain Wiener functionals with degenerate extrema. *Comm. Pure Appl. Math.* **47** 477–501. MR1272385

[27] LEJAY, A. (2003). An introduction to rough paths. In *Séminaire de Probabilités XXXVII. Lecture Notes in Math.* **1832** 1–59. Springer, Berlin. MR2053040

[28] LI, X.-D. and LYONS, T. J. (2006). Smoothness of Itô maps and diffusion processes on path spaces. I. *Ann. Sci. Éc. Norm. Supér.* (4) **39** 649–677. MR2290140

[29] LYONS, T. and QIAN, Z. (2002). *System Control and Rough Paths*. Oxford Univ. Press, Oxford. MR2036784

[30] LYONS, T. J. (1998). Differential equations driven by rough signals. *Rev. Mat. Iberoam.* **14** 215–310. MR1654527

[31] LYONS, T. J., CARUANA, M. and LÉVY, T. (2007). *Differential Equations Driven by Rough Paths. Lecture Notes in Math.* **1908**. Springer, Berlin. MR2314753

[32] MILLET, A. and SANZ-SOLÉ, M. (2006). Large deviations for rough paths of the fractional Brownian motion. *Ann. Inst. Henri Poincaré Probab. Stat.* **42** 245–271. MR2199801

[33] MISHURA, Y. S. (2008). *Stochastic Calculus for Fractional Brownian Motion and Related Processes. Lecture Notes in Math.* **1929**. Springer, Berlin. MR2378138

[34] NEUENKIRCH, A., NOURDIN, I., RÖSSLER, A. and TINDEL, S. (2009). Trees and asymptotic expansions for fractional stochastic differential equations. *Ann. Inst. Henri Poincaré Probab. Stat.* **45** 157–174. MR2500233

[35] ROVIRA, C. and TINDEL, S. (2000). Sharp Laplace asymptotics for a parabolic SPDE. *Stochastics Stochastics Rep.* **69** 11–30. MR1751716

[36] TAKANOBU, S. and WATANABE, S. (1993). Asymptotic expansion formulas of the Schilder type for a class of conditional Wiener functional integrations. In *Asymptotic Problems in Probability Theory: Wiener Functionals and Asymptotics* (*Sanda/Kyoto*, 1990). *Pitman Res. Notes Math. Ser.* **284** 194–241. Longman Sci. Tech., Harlow. MR1354169

[37] TANIGUCHI, S. (2008). Quadratic Wiener functionals of square norms on measure spaces. *Commun. Stoch. Anal.* **2** 11–26. MR2446908

GRADUATE SCHOOL OF MATHEMATICS
NAGOYA UNIVERSITY
FUROCHO, CHIKUSA-KU, NAGOYA 464-8602
JAPAN
E-MAIL: inahama@math.nagoya-u.ac.jp