

FREE SUBEXPONENTIALITY

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In this article, we introduce the notion of *free subexponentiality*, which extends the notion of subexponentiality in the classical probability setup to the noncommutative probability spaces under freeness. We show that distributions with regularly varying tails belong to the class of free subexponential distributions. This also shows that the partial sums of free random elements having distributions with regularly varying tails are tail equivalent to their maximum in the sense of Ben Arous and Voiculescu [*Ann. Probab.* **34** (2006) 2037–2059]. The analysis is based on the asymptotic relationship between the tail of the distribution and the real and the imaginary parts of the remainder terms in Laurent series expansion of Cauchy transform, as well as the relationship between the remainder terms in Laurent series expansions of Cauchy and Voiculescu transforms, when the distribution has regularly varying tails.

1. Introduction. A noncommutative probability space is a pair (\mathcal{A}, τ) where \mathcal{A} is a unital complex algebra, and τ is a linear functional on \mathcal{A} satisfying $\tau(1) = 1$. A noncommutative analog of independence, based on free products, was introduced by Voiculescu [20]. A family of unital subalgebras $\{\mathcal{A}_i\}_{i \in I} \subset \mathcal{A}$ is called *free* if $\tau(a_1 \cdots a_n) = 0$ whenever $\tau(a_j) = 0$, $a_j \in \mathcal{A}_{i_j}$ and $i_j \neq i_{j+1}$ for all j . The above setup is suitable for dealing with bounded random variables. In order to deal with unbounded random variables, we need to consider a tracial W^* -probability space (\mathcal{A}, τ) with a von Neumann algebra \mathcal{A} and a normal faithful tracial state τ .

A self-adjoint operator X is said to be affiliated to a von Neumann algebra \mathcal{A} , if $f(X) \in \mathcal{A}$ for any bounded Borel function f on the real line \mathbb{R} . A self-adjoint operator affiliated with \mathcal{A} will also be called a random element. For an affiliated random element (i.e., a self-adjoint operator) X , the algebra generated by X is defined as $\mathcal{A}_X = \{f(X) : f \text{ bounded measurable}\}$. The notion of freeness was extended to this context by Bercovici and Voiculescu [10]. A set of random elements $\{X_i\}_{1 \leq i \leq k}$ affiliated with a von Neumann algebra \mathcal{A} , are called freely independent, or simply free, if $\{\mathcal{A}_{X_i}\}_{1 \leq i \leq k}$ are free.

Given a random element X affiliated with \mathcal{A} , the law of X is the unique probability measure μ_X on \mathbb{R} satisfying $\tau(f(X)) = \int_{-\infty}^{\infty} f(t) d\mu_X(t)$ for every bounded Borel function f on \mathbb{R} . If e_A denote the projection valued spectral

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measure associated with X , evaluated at the set A , then it is easy to see that $\mu_X(-\infty, x] = \tau(e_{(-\infty, x]}(X))$. The distribution function of X , denoted by F_X , is given by $F_X(x) = \mu_X(-\infty, x]$.

Let \mathcal{M} be the family of probability measures on \mathbb{R} . On \mathcal{M} , two associative operations $*$ and \boxplus can be defined. The measure $\mu * \nu$ is the classical convolution of μ and ν , which also corresponds to the probability law of a random variable $X + Y$, where X and Y are independent and have laws μ and ν , respectively. Also, given two measures μ and ν , there exists a unique measure $\mu \boxplus \nu$, called the *free convolution* of μ and ν , such that whenever X and Y are two free random elements on a tracial W^* probability space (\mathcal{A}, τ) with laws μ and ν , respectively, $X + Y$ has the law $\mu \boxplus \nu$. The free convolution was first introduced by Voiculescu [20] for compactly supported measures, extended by Maassen [14] to measures with finite variance and by Bercovici and Voiculescu [10] to arbitrary Borel probability measures with unbounded support. The classical and free convolutions of distributions are defined and denoted analogously.

The relationship between $*$ and \boxplus convolution is very striking. They have many similarities like characterizations of infinitely divisible and stable laws [7, 8], weak law of large numbers [6] and central limit theorem [14, 15, 19]. Analogs of many other classical theories have also been derived. In recent times, links with extreme value theory [2, 3] and de Finetti-type theorems [1] have drawn much attention in the literature. However, there are differences too—for example, Cramér’s theorem [11] and Raikov’s theorem [4] fail in the noncommutative setup.

Now we consider an interesting family of distributions in the classical setup called subexponential distributions. The main endeavor of this article is to obtain an analog of this concept in the noncommutative setup under freeness. A probability measure μ on $[0, \infty)$, with $\mu(x, \infty) > 0$ for all $x \geq 0$, is said to be *subexponential*, if for every $n \in \mathbb{N}$,

$$\mu^{*n}(x, \infty) \sim n\mu(x, \infty) \quad \text{as } x \rightarrow \infty.$$

For a random variable X with distribution F and subexponential law μ , X and F are also called subexponential. The above definition can be rephrased in terms of the complementary distribution functions. For a distribution function F , we define its complementary distribution function as $\overline{F} = 1 - F$. Then a subexponential distribution function satisfies, for each natural number n , $\overline{F}^{*n}(x) \sim n\overline{F}(x)$ as $x \rightarrow \infty$. The definition can be extended to probability measures μ and equivalently distribution functions F defined on the entire real line. A distribution function F on the real line is called subexponential if the distribution function F_+ , defined as $F_+(x) = F(x)$, for $x \geq 0$ and $F_+(x) = 0$, for $x < 0$, is subexponential. Thus to discuss the subexponential property of the probability measures, it is enough to consider the ones concentrated on $[0, \infty)$. The subexponential random variables satisfy *the principle of one large jump* as well. If $\{X_i\}$ are i.i.d. subexponential random variables, then for all $n \in \mathbb{N}$,

$$P[X_1 + \dots + X_n > x] \sim nP[X_1 > x] = P\left[\max_{1 \leq i \leq n} X_i > x\right] \quad \text{as } x \rightarrow \infty.$$

Such a property makes subexponential distributions an ideal choice for modeling ruin and insurance problems and has caused wide interest in the classical probability literature; cf. [13, 17].

The classical definition of subexponential distributions can be easily extended to the noncommutative setup by replacing the classical convolution powers by free convolution powers. We shall define a free subexponential measure on $[0, \infty)$ alone, but the definition can be extended to probability measures on the entire real line, as in the classical case. Formally, we define a free subexponential measure as follows:

DEFINITION 1.1. A probability measure μ on $[0, \infty)$, with $\mu(x, \infty) > 0$ for all $x \geq 0$, is said to be free subexponential if for all n ,

$$\mu^{\boxplus n}(x, \infty) = \underbrace{(\mu \boxplus \cdots \boxplus \mu)}_{n \text{ times}}(x, \infty) \sim n\mu(x, \infty) \quad \text{as } x \rightarrow \infty.$$

The above definition can be rewritten in terms of distribution functions as well. A distribution function F is called free subexponential if for all $n \in \mathbb{N}$, $\overline{F^{\boxplus n}}(x) \sim n\overline{F}(x)$ as $x \rightarrow \infty$. A random variable X affiliated to a tracial W^* -probability space is called free subexponential if its distribution is so. One immediate consequence of the definition of free subexponentiality is the principle of one large jump.

Ben Arous and Voiculescu [3] showed that for two distribution functions F and G , there exists a unique measure $F \boxtimes G$, such that whenever X and Y are two free random elements on a tracial W^* -probability space, $F \boxtimes G$ will become the distribution of $X \vee Y$. Here $X \vee Y$ is the maximum of two self-adjoint operators defined using the spectral calculus via the projection-valued operators; see [3] for details. Ben Arous and Voiculescu [3] showed that $F \boxtimes G(x) = \max((F(x) + G(x) - 1), 0)$, and hence $F^{\boxtimes n}(x) = \max((nF(x) - (n - 1)), 0)$. Then we have for each n , $\overline{F^{\boxtimes n}}(x) \sim n\overline{F}(x)$ as $x \rightarrow \infty$. Thus, by the definition of free subexponentiality, we have

PROPOSITION 1.1 (Free one large jump principle). *Free subexponential distributions satisfy the principle of one large jump, namely, if F is freely subexponential, then, for every n ,*

$$\overline{F^{\boxplus n}}(x) \sim \overline{F^{\boxtimes n}}(x) \quad \text{as } x \rightarrow \infty.$$

While the class of free subexponential distributions possess the above important property, it remains to be checked whether the class is nonempty. The answer to this question, which is the main result of this article, is given in Theorem 1.1. The distributions with regularly varying (right) tails of index $-\alpha$, with $\alpha \geq 0$, form an important class of examples of subexponential distributions in the classical setup. (In further discussions, we shall suppress the qualifier “right.”) A (real valued)

measurable function f defined on nonnegative real line is called *regularly varying* (at infinity) with index α if for every $t > 0$, $f(tx)/f(x) \rightarrow t^\alpha$ as $x \rightarrow \infty$. If $\alpha = 0$, then f is said to be slowly varying (at infinity). Regular variation with index α at zero is defined analogously. In fact, f is regularly varying at zero of index α , if the function $x \mapsto f(1/x)$ is regularly varying at infinity of index $-\alpha$. Unless otherwise mentioned, the regular variation of a function will be considered at infinity. For regular variation at zero, we shall explicitly mention so. A distribution function F on $[0, \infty)$ has regularly varying tail of index $-\alpha$ if $\overline{F}(x)$ is regularly varying of index $-\alpha$. Since $\overline{F}(x) \rightarrow 0$ as $x \rightarrow \infty$, we must necessarily have $\alpha \geq 0$. As in the case of subexponential distributions, a distribution F on the entire real line is said to have regularly varying tail if F_+ has so. Note that for $x > 0$, we have $\overline{F_+}(x) = \overline{F}(x)$. A probability measure with regularly varying tail is defined through its distribution function. Equivalently, a measure μ is said to have a regularly varying tail if $\mu(x, \infty)$ is regularly varying.

Other than distributions with regularly varying tails, Weibull distributions with shape parameter less than 1 and lognormal distribution are some other well-known examples of subexponential distributions in the classical setup. The last two distributions have all moments finite unlike the distributions with regularly varying tails of index $-\alpha$, which have all moments higher than α infinite.

The distributions with regularly varying tails have already attracted attention in the noncommutative probability theory. They play a very crucial role in determining the domains of attraction of stable laws [7, 8]. In this article, we shall show that the distributions with regularly varying tails form a subclass of the free subexponential distributions.

THEOREM 1.1. *If a distribution function F has regularly varying tail of index $-\alpha$ with $\alpha \geq 0$, then F is free subexponential.*

The class of distribution functions with regularly varying tails is a significantly large class containing stable distributions, Pareto and Fréchet distributions. For all $\alpha \geq 0$, there are distribution functions, which have regularly varying tail of index $-\alpha$. The class of distribution function with regularly varying tail of index $-\alpha$ have found significant application in finance, insurance, weather, Internet traffic modeling and many other fields.

While it need not be assumed that the measure is concentrated on $[0, \infty)$, both the notions of free subexponentiality and regular variation are defined in terms of the measure restricted to $[0, \infty)$. Thus we shall assume the measure to be supported on $[0, \infty)$, except for the definitions of the relevant transforms in the initial part of Section 2.2 and in the statement and the proof of Theorem 1.1. Due to the lack of coordinate systems and expressions for joint distributions of noncommutative random elements in terms of probability measures, the proofs of the above results deviate from the classical ones. In absence of the higher moments

of the distributions with regularly varying tails, we cannot use the usual moment-based approach used in free probability theory. Instead, Cauchy and Voiculescu transforms become the natural tools to deal with the free convolution of measures. We recall the notions of these transforms in Section 2. We then discuss the relationship between the remainder terms of Laurent expansions of Cauchy and Voiculescu transforms of measures with regularly varying tail of index $-\alpha$. We need to consider four cases separately depending on the maximum number p of integer moments that the measure μ may have. For a nonnegative integer p , let us denote the class of all probability measures μ on $[0, \infty)$ with $\int_0^\infty t^p d\mu(t) < \infty$, but $\int_0^\infty t^{p+1} d\mu(t) = \infty$, by \mathcal{M}_p . We shall also denote the class of all probability measures μ in \mathcal{M}_p with regularly varying tail of index $-\alpha$ by $\mathcal{M}_{p,\alpha}$. Note that we necessarily have $\alpha \in [p, p + 1]$; cf. [13, Proposition A3.8(d)]. Theorems 2.1–2.4 summarize the relationships among the remainder terms for various choices of α and p . These theorems are the key tools of this article. Section 2 is concluded with two Abel–Tauber-type results for Stieltjes transform of measures with regularly varying tail. We then prove Theorem 1.1 in Section 3 using Theorems 2.1–2.4. We use the final two sections to prove Theorems 2.1–2.4. In Section 4, we collect some results about the remainder term in Laurent series expansion of Cauchy transform of measures with regularly varying tails. In Section 5, we study the relationship between the remainder terms in Laurent expansions of Cauchy and Voiculescu transforms through a general analysis of the remainder terms of Taylor expansions of a suitable class of functions and their inverses or reciprocals. Combining the results of Sections 4 and 5, we prove Theorems 2.1–2.4.

2. Some transforms and their related properties. In this section, we collect some notation, definitions and results to be used later in the article. In Section 2.1, we define the concept of nontangential limits. Various transforms in noncommutative probability theory, like Cauchy, Voiculescu and R transforms are introduced in Section 2.2. Theorems 2.1–2.4 regarding the relationship between the remainder terms of Laurent expansions of Cauchy and Voiculescu transforms are given in this subsection as well. Finally, in Section 2.3, two results about measures with regularly varying tails are given.

2.1. Nontangential limits and notation. The complex plane will be denoted by \mathbb{C} and for a complex number z , $\Re z$ and $\Im z$ will denote its real and imaginary parts respectively. We say z goes to infinity (zero, resp.) *nontangentially* to \mathbb{R} (n.t.), if z goes to infinity (zero, resp.), while $\Re z / \Im z$ stays bounded. We can then define that a function f converges or stays bounded as z goes to infinity (or zero) n.t. To elaborate upon the notion, given positive numbers η , δ and M , let us define the following cones:

- (1) $\Gamma_\eta = \{z \in \mathbb{C}^+ : |\Re z| < \eta \Im z\}$ and $\Gamma_{\eta,M} = \{z \in \Gamma_\eta : |z| > M\}$;
- (2) $\Delta_\eta = \{z \in \mathbb{C}^- : |\Re z| < -\eta \Im z\}$ and $\Delta_{\eta,\delta} = \{z \in \Delta_\eta : |z| < \delta\}$,

where \mathbb{C}^+ and \mathbb{C}^- are the upper and the lower halves of the complex plane respectively, namely, $\mathbb{C}^+ = \{z \in \mathbb{C} : \Im z > 0\}$ and $\mathbb{C}^- = -\mathbb{C}^+$. Then we shall say that $f(z) \rightarrow l$ as z goes to ∞ n.t., if for any $\varepsilon > 0$ and $\eta > 0$, there exists $M \equiv M(\eta, \varepsilon) > 0$, such that $|f(z) - l| < \varepsilon$, whenever $z \in \Gamma_{\eta, M}$. The boundedness can be defined analogously.

We shall write $f(z) \approx g(z)$, $f(z) = o(g(z))$ and $f(z) = O(g(z))$ as $z \rightarrow \infty$ n.t. to mean that $f(z)/g(z)$ converges to a nonzero limit, $f(z)/g(z) \rightarrow 0$ and $f(z)/g(z)$ stays bounded as $z \rightarrow \infty$ n.t., respectively. If the nonzero limit is 1 in the first case, we write $f(z) \sim g(z)$ as $z \rightarrow \infty$ n.t. For $f(z) = o(g(z))$ as $z \rightarrow \infty$ n.t., we shall also use the notation $f(z) \ll g(z)$ and $g(z) \gg f(z)$ as $z \rightarrow \infty$ n.t.

The map $z \mapsto 1/z$ maps the set $\Gamma_{\eta, 1/\delta}$ onto $\Delta_{\eta, \delta}$ for each positive η and δ . Thus the analogous concepts can be defined for $z \rightarrow 0$ n.t. using $\Delta_{\eta, \delta}$.

2.2. *Cauchy and Voiculescu transforms.* For a probability measure $\mu \in \mathcal{M}$, its Cauchy transform is defined as

$$G_\mu(z) = \int_{-\infty}^{\infty} \frac{1}{z - t} d\mu(t), \quad z \in \mathbb{C}^+.$$

Note that G_μ maps \mathbb{C}^+ to \mathbb{C}^- . Set $F_\mu = 1/G_\mu$, which maps \mathbb{C}^+ to \mathbb{C}^+ . We shall be also interested in the function $H_\mu(z) = G_\mu(1/z)$ which maps \mathbb{C}^- to \mathbb{C}^- .

By Proposition 5.4 and Corollary 5.5 of [10], for all $\eta > 0$ and for all $\varepsilon \in (0, \eta \wedge 1)$, there exists $\delta \equiv \delta(\eta)$ small enough, such that H_μ is a conformal bijection from $\Delta_{\eta, \delta}$ onto an open set $\mathcal{D}_{\eta, \delta}$, where the range sets satisfy

$$\Delta_{\eta - \varepsilon, (1 - \varepsilon)\delta} \subset \mathcal{D}_{\eta, \delta} \subset \Delta_{\eta + \varepsilon, (1 + \varepsilon)\delta}.$$

If we define $\mathcal{D} = \bigcup_{\eta > 0} \mathcal{D}_{\eta, \delta(\eta)}$, then we can obtain an analytic function L_μ with domain \mathcal{D} by patching up the inverses of H_μ on $\mathcal{D}_{\eta, \delta(\eta)}$ for each $\eta > 0$. In this case L_μ becomes the right inverse of H_μ on \mathcal{D} . Also it was shown that the sets of type $\Delta_{\eta, \delta}$ were contained in the unique connected component of the set $H_\mu^{-1}(\mathcal{D})$. It follows that H_μ is the right inverse of L_μ on $\Delta_{\eta, \delta}$ and hence on the whole connected component by analytic continuation.

We then define R and Voiculescu transforms of the probability measure μ respectively as

$$(2.1) \quad R_\mu(z) = \frac{1}{L_\mu(z)} - \frac{1}{z} \quad \text{and} \quad \phi_\mu(z) = R_\mu(1/z).$$

Arguing as in the case of $G_\mu(1/z)$, it can be shown that F_μ has a left inverse, denoted by F_μ^{-1} on a suitable domain and, in that case, we have

$$\phi_\mu(z) = F_\mu^{-1}(z) - z.$$

Bercovici and Voiculescu [10] established the following relation between free convolution and Voiculescu and R transforms. For probability measures μ and ν ,

$$\phi_{\mu \boxplus \nu} = \phi_\mu + \phi_\nu \quad \text{and} \quad R_{\mu \boxplus \nu} = R_\mu + R_\nu,$$

wherever all the functions involved are defined.

We shall also need to analyze the power and Taylor series expansions of the above transforms. For Taylor series expansion of a function, we need to define the remainder term appropriately, so that it becomes amenable to the later calculations. In fact, for a function A with Taylor series expansion of order p , we define the remainder term as

$$(2.2) \quad r_A(z) = z^{-p} \left(A(z) - \sum_{i=0}^p a_i z^i \right).$$

Note that we divide by z^p after subtracting the polynomial part.

For compactly supported measure μ , Speicher [18] showed that, in an appropriate neighborhood of zero, $R_\mu(z) = \sum_{j=0}^\infty \kappa_{j+1}(\mu)z^j$, where $\{\kappa_j(\mu)\}$ denotes the free cumulant sequence of the probability measure μ . For probability measures μ with finite p moments, Taylor expansions of R_μ and H_μ are given by Theorems 1.3 and 1.5 of [5],

$$(2.3) \quad \begin{aligned} R_\mu(z) &= \sum_{j=0}^{p-1} \kappa_{j+1}(\mu)z^j + z^{p-1}r_{R_\mu}(z) \quad \text{and} \\ H_\mu(z) &= \sum_{j=1}^{p+1} m_{j-1}(\mu)z^j + z^{p+1}r_{H_\mu}(z), \end{aligned}$$

where the remainder terms $r_{R_\mu}(z) \equiv r_R(z) = o(1)$ and $r_{H_\mu}(z) \equiv r_H(z) = o(1)$ as $z \rightarrow 0$ n.t. are defined along the lines of (2.2), $\{\kappa_j(\mu) : j \leq p\}$ denotes the free cumulant sequence of μ as before and $\{m_j(\mu) : j \leq p\}$ denotes the moment sequence of the probability measure μ . When there is no possibility of confusion, we shall sometimes suppress the measure involved in the notation for the moment and the cumulant sequences, as well as the remainder terms. In the study of stable laws and the infinitely divisible laws, the following relationship between Cauchy and Voiculescu transforms of a probability measure μ , obtained in Proposition 2.5 of Bercovici and Pata [7], played a crucial role:

$$(2.4) \quad \phi_\mu(z) \sim z^2 \left[G_\mu(z) - \frac{1}{z} \right] \quad \text{as } z \rightarrow \infty \text{ n.t.}$$

Depending on the number of moments that the probability measure μ may have, its Cauchy and Voiculescu transforms can have Laurent series expansions of higher order. Motivated by this fact, for probability measures $\mu \in \mathcal{M}_p$ (i.e., when μ has only p integral moments), we introduce the remainder terms in Laurent series expansion of Cauchy and Voiculescu transforms (in analogy to the remainder terms in Taylor series expansion),

$$(2.5) \quad r_{G_\mu}(z) \equiv r_G(z) = z^{p+1} \left(G_\mu(z) - \sum_{j=1}^{p+1} m_{j-1}(\mu)z^{-j} \right)$$

and

$$(2.6) \quad r_{\phi_\mu}(z) \equiv r_\phi(z) = z^{p-1} \left(\phi_\mu(z) - \sum_{j=0}^{p-1} \kappa_{j+1}(\mu) z^{-j} \right),$$

where we shall again suppress the measure μ in the notation if there is no possibility of confusion. In (2.6), we interpret the sum on the right-hand side as zero, when $p = 0$. Using the remainder terms defined in (2.5) and (2.6) we provide extensions of (2.4) in Theorems 2.1–2.4 for different choices of α and p . We split the statements into four cases as follows: (i) p is a positive integer, and $\alpha \in (p, p + 1)$; (ii) p is a positive integer, and $\alpha = p$; (iii) $p = 0$, and $\alpha \in [0, 1)$; (iv) p is a non-negative integer and, $\alpha = p + 1$, giving rise to Theorems 2.1–2.4, respectively.

We first consider the case where p is a positive integer and $\alpha \in (p, p + 1)$.

THEOREM 2.1. *Let μ be a probability measure in the class \mathcal{M}_p and $\alpha \in (p, p + 1)$. The following statements are equivalent:*

- (i) $\mu(y, \infty)$ is regularly varying of index $-\alpha$.
- (ii) $\Im r_G(iy)$ is regularly varying of index $-(\alpha - p)$.
- (iii) $\Im r_\phi(iy)$ is regularly varying of index $-(\alpha - p)$, $\Re r_\phi(iy) \gg y^{-1}$ as $y \rightarrow \infty$ and $r_\phi(z) \gg z^{-1}$ as $z \rightarrow \infty$ n.t.

If any of the above statements holds, we also have, as $z \rightarrow \infty$ n.t.,

$$(2.7) \quad r_G(z) \sim r_\phi(z) \gg z^{-1};$$

as $y \rightarrow \infty$,

$$(2.8) \quad \Im r_\phi(iy) \sim \Im r_G(iy) \sim -\frac{\pi(p + 1 - \alpha)/2}{\cos(\pi(\alpha - p)/2)} y^p \mu(y, \infty) \gg \frac{1}{y}$$

and

$$(2.9) \quad \Re r_\phi(iy) \sim \Re r_G(iy) \sim -\frac{\pi(p + 2 - \alpha)/2}{\sin(\pi(\alpha - p)/2)} y^p \mu(y, \infty) \gg \frac{1}{y}.$$

Next we consider the case where p is a positive integer and $\alpha = p$.

THEOREM 2.2. *Let μ be a probability measure in the class \mathcal{M}_p . The following statements are equivalent:*

- (i) $\mu(y, \infty)$ is regularly varying of index $-p$.
- (ii) $\Im r_G(iy)$ is slowly varying.
- (iii) $\Im r_\phi(iy)$ is slowly varying, $\Re r_\phi(iy) \gg y^{-1}$ as $y \rightarrow \infty$ and $r_\phi(z) \gg z^{-1}$ as $z \rightarrow \infty$ n.t.

If any of the above statements holds, we also have, as $z \rightarrow \infty$ n.t.,

$$(2.10) \quad r_G(z) \sim r_\phi(z) \gg z^{-1};$$

as $y \rightarrow \infty$,

$$(2.11) \quad \Im r_\phi(iy) \sim \Im r_G(iy) \sim -\frac{\pi}{2}y^p\mu(y, \infty) \gg \frac{1}{y}$$

and

$$(2.12) \quad \Re r_\phi(iy) \sim \Re r_G(iy) \gg \frac{1}{y}.$$

In the third case, we consider $\alpha \in [0, 1)$.

THEOREM 2.3. *Let μ be a probability measure in the class \mathcal{M}_0 and $\alpha \in [0, 1)$. The following statements are equivalent:*

- (i) $\mu(y, \infty)$ is regularly varying of index $-\alpha$.
- (ii) $\Im r_G(iy)$ is regularly varying of index $-\alpha$.
- (iii) $\Im r_\phi(iy)$ is regularly varying of index $-\alpha$, $\Re r_\phi(iy) \approx \Im r_\phi(iy)$ as $y \rightarrow \infty$ and $r_\phi(z) \gg z^{-1}$ as $z \rightarrow \infty$ n.t.

If any of the above statements holds, we also have, as $z \rightarrow \infty$ n.t.,

$$(2.13) \quad r_G(z) \sim r_\phi(z) \gg z^{-1};$$

as $y \rightarrow \infty$,

$$(2.14) \quad \Im r_\phi(iy) \sim \Im r_G(iy) \sim -\frac{\pi(1-\alpha)/2}{\cos(\pi\alpha/2)}\mu(y, \infty) \gg \frac{1}{y}$$

and

$$(2.15) \quad \Re r_\phi(iy) \sim \Re r_G(iy) \sim -d_\alpha\mu(y, \infty) \gg \frac{1}{y},$$

where

$$d_\alpha = \begin{cases} \frac{\pi(2-\alpha)/2}{\sin(\pi\alpha/2)}, & \text{when } \alpha > 0, \\ 1, & \text{when } \alpha = 0. \end{cases}$$

Finally, we consider the case where p is a nonnegative integer, and $\alpha = p + 1$.

THEOREM 2.4. *Let μ be a probability measure in the class \mathcal{M}_p and $\beta \in (0, 1/2)$. The following statements are equivalent:*

- (i) $\mu(y, \infty)$ is regularly varying of index $-(p + 1)$.
- (ii) $\Re r_G(iy)$ is regularly varying of index -1 .

(iii) $\Re r_\phi(iy)$ is regularly varying of index -1 , $y^{-1} \ll \Im r_\phi(iy) \ll y^{-(1-\beta/2)}$ as $y \rightarrow \infty$ and $z^{-1} \ll r_\phi(z) \ll z^{-\beta}$ as $z \rightarrow \infty$ n.t.

If any of the above statements holds, we also have, as $z \rightarrow \infty$ n.t.,

$$(2.16) \quad z^{-1} \ll r_G(z) \sim r_\phi(z) \ll z^{-\beta};$$

as $y \rightarrow \infty$,

$$(2.17) \quad y^{-(1+\beta/2)} \ll \Re r_\phi(iy) \sim \Re r_G(iy) \sim -\frac{\pi}{2} y^p \mu(y, \infty) \ll y^{-(1-\beta/2)}$$

and

$$(2.18) \quad y^{-1} \ll \Im r_\phi(iy) \sim \Im r_G(iy) \ll y^{-(1-\beta/2)}.$$

It is easy to obtain the equivalent statements for H_μ and R_μ through the simple observation that $G_\mu(z) = H_\mu(1/z)$ and $\phi_\mu(z) = R_\mu(1/z)$. For $p = 0$, Theorems 2.3 and 2.4 together give a special case of (2.4) for the probability measures with regularly varying tail and infinite mean. However, Theorems 2.1–2.4 give more detailed asymptotic behavior of the real and imaginary parts separately, which is required for our analysis.

2.3. *Karamata-type results.* We provide here two results for regularly varying functions, which we shall be using in the proofs of our results. They are variants of Karamata’s Abel–Tauber theorem for Stieltjes transform (cf. [12], Section 1.7.5) and explain the regular variation of Cauchy transform of measures with regularly varying tails.

The first result is quoted from [7].

PROPOSITION 2.1 ([7], Corollary 5.4). *Let ρ be a positive Borel measure on $[0, \infty)$ and fix $\alpha \in [0, 2)$. Then the following statements are equivalent:*

- (i) $y \mapsto \rho[0, y]$ is regularly varying of index α .
- (ii) $y \mapsto \int_0^\infty \frac{1}{t^2+y^2} d\rho(t)$ is regularly varying of index $-(2-\alpha)$.

If either of the above conditions is satisfied, then

$$\int_0^\infty \frac{1}{t^2+y^2} d\rho(t) \sim \frac{\pi\alpha/2}{\sin(\pi\alpha/2)} \frac{\rho[0, y]}{y^2} \quad \text{as } y \rightarrow \infty.$$

The constant pre-factor on the right-hand side is interpreted as 1 when $\alpha = 0$.

The second result uses a different integrand.

PROPOSITION 2.2. *Let ρ be a finite positive Borel measure on $[0, \infty)$ and fix $\alpha \in [0, 2)$. Then the following statements are equivalent:*

- (i) $y \mapsto \rho(y, \infty)$ is regularly varying of index $-\alpha$.

(ii) $y \mapsto \int_0^\infty \frac{t^2}{t^2+y^2} d\rho(t)$ is regularly varying of index $-\alpha$.

If either of the above conditions is satisfied, then

$$\int_0^\infty \frac{t^2}{t^2+y^2} d\rho(t) \sim \frac{\pi\alpha/2}{\sin(\pi\alpha/2)} \rho(y, \infty) \quad \text{as } y \rightarrow \infty.$$

The constant pre-factor on the right-hand side is interpreted as 1 when $\alpha = 0$.

PROOF. Define $d\tilde{\rho}(s) = \rho(\sqrt{s}, \infty) ds$. By a variant of Karamata’s theorem, given in Theorem 0.6(a) of [16], as $\alpha < 2$, we have

$$(2.19) \quad \tilde{\rho}[0, y] \sim \frac{1}{1-\alpha/2} y \rho(\sqrt{y}, \infty)$$

is regularly varying of index $1-\alpha/2$. Then we have

$$\begin{aligned} \int_0^\infty \frac{t^2}{t^2+y^2} d\rho(t) &= y^2 \int_0^\infty \int_0^t \frac{2s ds}{(s^2+y^2)^2} d\rho(t) \\ &= y^2 \int_0^\infty \frac{2s\rho(s, \infty)}{(s^2+y^2)^2} ds = y^2 \int_0^\infty \frac{d\tilde{\rho}(s)}{(s+y^2)^2}. \end{aligned}$$

Now, first applying Theorem 1.7.4 of [12] as $\tilde{\rho}[0, y]$ is regularly varying of index $1-\alpha/2 \in (0, 2]$ and then (2.19), we have

$$\int_0^\infty \frac{t^2}{t^2+y^2} d\rho(t) \sim \frac{(1-\alpha/2)\pi\alpha/2}{\sin(\pi\alpha/2)} y^2 \frac{\tilde{\rho}[0, y^2]}{y^4} \sim \frac{\pi\alpha/2}{\sin(\pi\alpha/2)} \rho(y, \infty). \quad \square$$

3. Free subexponentiality of measures with regularly varying tails. We now use Theorems 2.1–2.4 to prove Theorem 1.1. We shall first look at the tail behavior of the free convolution of two probability measures with regularly varying tails and which are tail balanced. Theorem 1.1 will be proved by suitable choices of the two measures.

LEMMA 3.1. *Suppose μ and ν are two probability measures on $[0, \infty)$ with regularly varying tails, which are tail balanced; that is, for some $c > 0$, we have $\nu(y, \infty) \sim c\mu(y, \infty)$. Then*

$$\mu \boxplus \nu(y, \infty) \sim (1+c)\mu(y, \infty).$$

PROOF. We shall now indicate the associated probability measures in the remainder terms, moments and the cumulants to avoid any confusion. Since μ and ν are tail balanced and have regularly varying tails, for some nonnegative integer p and $\alpha \in [p, p+1]$, we have both μ and ν in the same class $\mathcal{M}_{p,\alpha}$. When $\alpha \in [p, p+1)$, depending on the choice of p and α , we apply one of Theorems 2.1, 2.2 and 2.3 on the imaginary parts of the remainder terms in Laurent

expansion of Voiculescu transforms. On the other hand, for $\alpha = p + 1$, we apply Theorem 2.4 on the real parts of the corresponding objects. We work out only the case $\alpha \in [p, p + 1)$ in details, while the other case $\alpha = p + 1$ is similar.

For $\alpha \in [p, p + 1)$, by Theorems 2.1–2.3, we have

$$\begin{aligned}
 (3.1) \quad & r_{\phi_\mu}(z) \gg z^{-1} \quad \text{and} \quad r_{\phi_\nu}(z) \gg z^{-1}, \\
 (3.2) \quad & \Re r_{\phi_\mu}(-iy) \gg y^{-1} \quad \text{and} \quad \Re r_{\phi_\nu}(-iy) \gg y^{-1}, \\
 (3.3) \quad & \Im r_{\phi_\mu}(iy) \sim -\frac{\pi(p+1-\alpha)/2}{\cos(\pi(\alpha-p)/2)} y^p \mu(y, \infty) \quad \text{and} \\
 & \Im r_{\phi_\nu}(iy) \sim -\frac{\pi(p+1-\alpha)/2}{\cos(\pi(\alpha-p)/2)} y^p \nu(y, \infty).
 \end{aligned}$$

For $p = 0$ and $\alpha \in [0, 1)$, we further have

$$\begin{aligned}
 (3.4) \quad & \Im r_{\phi_\mu}(iy) \approx \Re r_{\phi_\mu}(iy) \approx \mu(y, \infty) \quad \text{and} \\
 & \Im r_{\phi_\nu}(iy) \approx \Re r_{\phi_\nu}(iy) \approx \nu(y, \infty).
 \end{aligned}$$

We also know that both Voiculescu transforms and cumulants add up in case of free convolution. Hence,

$$(3.5) \quad r_{\phi_{\mu \boxplus \nu}}(z) = r_{\phi_\mu}(z) + r_{\phi_\nu}(z).$$

Further, we shall have $\kappa_p(\mu \boxplus \nu) < \infty$, but $\kappa_{p+1}(\mu \boxplus \nu) = \infty$ and similar results hold for the moments of $\mu \boxplus \nu$ as well. Then Theorems 2.1–2.3 will also apply for $\mu \boxplus \nu$. Thus, applying (3.5) and its real and imaginary parts evaluated at $z = iy$, together with (3.1)–(3.4), respectively, we get

$$\begin{aligned}
 r_{\phi_{\mu \boxplus \nu}}(z) &\gg z^{-1} && \text{as } z \rightarrow \infty \text{ n.t.}, \\
 \Re r_{\phi_{\mu \boxplus \nu}}(iy) &\gg y^{-1} && \text{as } y \rightarrow \infty
 \end{aligned}$$

and

$$(3.6) \quad \Im r_{\phi_{\mu \boxplus \nu}}(iy) \sim -(1+c) \frac{\pi(p+1-\alpha)/2}{\cos(\pi(\alpha-p)/2)} y^p \mu(y, \infty) \quad \text{as } y \rightarrow \infty,$$

which is regularly varying of index $-(\alpha - p)$. Further, for $p = 0$ and $\alpha \in [0, 1)$, we have

$$\Im r_{\phi_{\mu \boxplus \nu}}(iy) \approx \Re r_{\phi_{\mu \boxplus \nu}}(iy).$$

In the last two steps, we also use the hypothesis that $\nu(y, \infty) \sim c\mu(y, \infty)$ as $y \rightarrow \infty$. Thus, again using Theorems 2.1–2.3, we have

$$(3.7) \quad -\frac{\pi(p+1-\alpha)/2}{\cos(\pi(\alpha-p)/2)} y^p \mu \boxplus \nu(y, \infty) \sim \Im r_{\phi_{\mu \boxplus \nu}}(iy).$$

Combining (3.6) and (3.7), the result follows. \square

We are now ready to prove the subexponentiality of a distribution with regularly varying tail.

PROOF OF THEOREM 1.1. Let μ be the probability measure on $[0, \infty)$ associated with the distribution function F_+ . Then μ also has regularly varying tail of index $-\alpha$. We prove that

$$(3.8) \quad \mu^{\boxplus n}(y, \infty) \sim n\mu(y, \infty) \quad \text{as } y \rightarrow \infty$$

by induction on n . To prove (3.8) for $n = 2$, apply Lemma 3.1 with both the probability measures as μ and the constant $c = 1$. Next assume (3.8) holds for $n = m$. To prove (3.8) for $n = m + 1$, apply Lemma 3.1 again with the probability measures μ and $\mu^{\boxplus m}$ and the constant $c = m$. \square

4. Cauchy transform of measures with regularly varying tail. As a first step toward proving Theorems 2.1–2.4, we now collect some results about $r_G(z)$, when the probability measure μ has regularly varying tails. These results will be useful in showing equivalence between the tail of μ and $r_G(iy)$. It is easy to see by induction that

$$\frac{1}{z-t} - \sum_{j=0}^p \frac{t^j}{z^{j+1}} = \left(\frac{t}{z}\right)^{p+1} \frac{1}{z-t}.$$

Integrating and multiplying by z^{p+1} , we get

$$(4.1) \quad r_G(z) = \int_0^\infty \frac{t^{p+1}}{z-t} d\mu(t).$$

We use (4.1) to obtain asymptotic upper and lower bounds for $r_G(z)$ as $z \rightarrow \infty$ n.t. Similar results about r_H can be obtained easily from the fact that $r_G(z) = r_H(1/z)$, but will not be stated separately. We consider the lower bound first.

PROPOSITION 4.1. *Suppose $\mu \in \mathcal{M}_p$ for some nonnegative integer p , then*

$$z^{-1} \ll r_G(z) \quad \text{as } z \rightarrow \infty \text{ n.t.}$$

PROOF. We need to show that for any $\eta > 0$, as $|z| \rightarrow \infty$ with z in the cone Γ_η , we have $|zr_G(z)| \rightarrow \infty$. Note that for $z = x + iy \in \Gamma_\eta$, we have $|x| < \eta y$. Now, as $|z-t|^2 = (z-t)(\bar{z}-t)$ and $z(\bar{z}-t) = |z|^2 - zt$, using (4.1), we have

$$zr_G(z) = z \int_0^\infty \frac{t^{p+1}}{z-t} d\mu(t) = |z|^2 \int_0^\infty \frac{t^{p+1}}{|z-t|^2} d\mu(t) - z \int_0^\infty \frac{t^{p+2}}{|z-t|^2} d\mu(t),$$

which gives

$$(4.2) \quad \Re(zr_G(z)) = |z|^2 \int_0^\infty \frac{t^{p+1}}{|z-t|^2} d\mu(t) - \Re z \int_0^\infty \frac{t^{p+2}}{|z-t|^2} d\mu(t)$$

and

$$(4.3) \quad \Im(zr_G(z)) = -\Im z \int_0^\infty \frac{t^{p+2}}{|z-t|^2} d\mu(t).$$

On Γ_η and for $t \in [0, \eta y]$, $|t-x| \leq t+|x| \leq 2\eta y$. Thus, we have

$$(4.4) \quad \begin{aligned} \int_0^\infty \frac{|z|^2 t^{p+1}}{|z-t|^2} d\mu(t) &\geq \int_0^{\eta y} \frac{y^2 t^{p+1}}{(t-x)^2 + y^2} d\mu(t) \\ &\geq \frac{1}{1+4\eta^2} \int_0^{\eta y} t^{p+1} d\mu(t) \rightarrow \infty, \end{aligned}$$

as $y \rightarrow \infty$, since $\mu \in \mathcal{M}_p$.

Now fix $\eta > 0$, and consider a sequence $\{z_n = x_n + iy_n\}$ in Γ_η , such that $|z_n| \rightarrow \infty$, that is, $|x_n| \leq \eta y_n$ and $y_n \rightarrow \infty$. Assume toward contradiction that $\{|z_n r_G(z_n)|\}$ is a bounded sequence. Then both the real and the imaginary parts of the sequence will be bounded. However, then the boundedness of the real part and (4.2) and (4.4) give

$$\left| \Re z_n \int_0^\infty \frac{t^{p+2}}{|z_n-t|^2} d\mu(t) \right| \rightarrow \infty.$$

Then, using (4.3) and the fact that $|\Re z| \leq \eta \Im z$ on Γ_η , we have

$$\Im(z_n r_G(z_n)) \geq \frac{1}{\eta} \left| \Re z_n \int_0^\infty \frac{t^{p+2}}{|z_n-t|^2} d\mu(t) \right| \rightarrow \infty,$$

which contradicts the fact that the imaginary part of $\{z_n r_G(z_n)\}$ is bounded and completes the proof. \square

We now consider the upper bound for $r_G(z)$. The result and the proof of the following proposition are inspired by Lemma 5.2(iii) of [9].

PROPOSITION 4.2. *Let μ be a probability measure in the class $\mathcal{M}_{p,\alpha}$ for some nonnegative integer p and $\alpha \in (p, p+1]$. Then, for any $\beta \in [0, (\alpha-p)/(\alpha-p+1))$, we have*

$$(4.5) \quad r_G(z) = o(z^{-\beta}) \quad \text{as } z \rightarrow \infty \text{ n.t.}$$

REMARK 4.1. We consider the principal branch of logarithm of a complex number with positive imaginary part, while defining the fractional powers in (4.5) above and elsewhere.

REMARK 4.2. Note that (4.5) holds also for $p = \alpha$ with $\beta = 0$, which can be readily seen from Theorem 1.5 of [5].

PROOF OF PROPOSITION 4.2. Define a measure ρ_0 as $d\rho_0(t) = t^p d\mu(t)$. Since $\mu \in \mathcal{M}_p$, ρ_0 is a finite measure. Further, since $p < \alpha$, using Theorem 1.6.5 of [12], we have $\rho_0(y, \infty) \sim \frac{\alpha}{\alpha-p} y^p \mu(y, \infty)$, which is regularly varying of index $-(\alpha - p)$.

Now fix $\eta > 0$. It is easy to check that for $t \geq 0$ and $z \in \Gamma_\eta$, $t/|z - t| < \sqrt{1 + \eta^2}$. For $z = x + iy$, we have $|z - t| > y$ and hence for $t \in [0, y^{1/(\alpha-p+1)}]$, we have $t/|z - t| < y^{-(\alpha-p)/(\alpha-p+1)}$. Then, using (4.1) and the definition of ρ_0 ,

$$\begin{aligned} |r_G(z)| &\leq \int_0^{y^{1/(\alpha-p+1)}} \left| \frac{t}{z-t} \right| d\rho_0(t) + \sqrt{1 + \eta^2} \rho_0(y^{1/(\alpha-p+1)}, \infty) \\ &\leq y^{-(\alpha-p)/(\alpha-p+1)} \int_0^\infty t^p d\mu(t) + \sqrt{1 + \eta^2} \rho_0(y^{1/(\alpha-p+1)}, \infty) = o(y^{-\beta}) \end{aligned}$$

for any $\beta \in [0, (\alpha - p)/(\alpha - p + 1))$, as the second term is regularly varying of index $-(\alpha - p)/(\alpha - p + 1)$. Further, for $z = x + iy \in \Gamma_\eta$, we have $|z| = \sqrt{x^2 + y^2} \leq y\sqrt{1 + \eta^2}$, and hence we have the required result. \square

Next we specialize to the asymptotic behavior of $r_G(iy)$, as $y \rightarrow \infty$. Observe that

$$\begin{aligned} \Re r_G(iy) &= - \int_0^\infty \frac{t^{p+2}}{t^2 + y^2} d\mu(t) \quad \text{and} \\ \Im r_G(iy) &= -y \int_0^\infty \frac{t^{p+1}}{t^2 + y^2} d\mu(t). \end{aligned} \tag{4.6}$$

PROPOSITION 4.3. *Let μ be a probability measure in the class \mathcal{M}_p . If $\alpha \in (p, p + 1)$, then the following statements are equivalent:*

- (i) μ has regularly varying tail of index $-\alpha$.
- (ii) $\Re r_G(iy)$ is regularly varying of index $-(\alpha - p)$.
- (iii) $\Im r_G(iy)$ is regularly varying of index $-(\alpha - p)$.

If any of the above statements holds, then

$$\begin{aligned} \frac{\sin(\pi(\alpha - p)/2)}{\pi(p + 2 - \alpha)/2} \Re r_G(iy) &\sim \frac{\cos(\pi(\alpha - p)/2)}{\pi(p + 1 - \alpha)/2} \Im r_G(iy) \\ &\sim -y^p \mu(y, \infty) \quad \text{as } y \rightarrow \infty. \end{aligned}$$

Further, $\Re r_G(iy) \gg y^{-1}$ and $\Im r_G(iy) \gg y^{-1}$ as $y \rightarrow \infty$.

If $\alpha = p$, then statements (i) and (iii) above are equivalent. Also, if either of the statements holds, then

$$\Im r_G(iy) \sim -\frac{\pi}{2} y^p \mu(y, \infty) \quad \text{as } y \rightarrow \infty. \tag{4.7}$$

Further, $\Im r_G(iy) \gg y^{-1}$ as $y \rightarrow \infty$.

If $\alpha = p + 1$, then statements (i) and (ii) above are equivalent. Also, if either of the statements holds, then

$$(4.8) \quad \Re r_G(iy) \sim -\frac{\pi}{2} y^p \mu(y, \infty) \quad \text{as } y \rightarrow \infty.$$

Further, for any $\varepsilon > 0$, $\Re r_G(iy) \gg y^{-(1+\varepsilon)}$ as $y \rightarrow \infty$.

REMARK 4.3. Note that for $\alpha = p + 1$, $\Re r_G(iy)$ is regularly varying of index -1 , and the asymptotic lower bound $\Re r_G(iy) \gg y^{-1}$ need not hold. This causes some difficulty in the proofs of Propositions 5.1 and 5.2. The lack of the asymptotic lower bound has to be compensated for by the stronger upper bound obtained in Proposition 4.2, which holds for $\alpha = p + 1$. This is reflected in condition (R4'') for the class $\mathcal{R}_{p,\beta}$ with $\beta > 0$, defined in Section 5. Further note that the situation reverses for $\alpha = p$, as Proposition 4.2 need not hold. The case, where $\alpha \in (p, p + 1)$ is not an integer, is simple, as the asymptotic lower bounds hold for both the real and imaginary parts of $r_G(iy)$ (Proposition 4.3), as well as the stronger asymptotic upper bound works (Proposition 4.2). However, the case of noninteger $\alpha \in (p, p + 1)$ is treated simultaneously with the case $\alpha = p$ as the class $\mathcal{R}_{p,0}$; cf. Section 5 in Propositions 5.1 and 5.2.

PROOF OF PROPOSITION 4.3. The asymptotic lower bounds for the real and the imaginary parts of $r_G(iy)$ are immediate from (ii) and (iii), respectively. So, we only need to show (4.8) and the equivalence between (i) and (ii) when $\alpha \in (p, p + 1]$ and (4.7) and the equivalence between (i) and (iii) when $\alpha \in [p, p + 1)$.

Let $d\rho_j(t) = t^{p+j} d\mu(t)$, for $j = 1, 2$. Then, by Theorem 1.6.4 of [12], we have for $\alpha \in [p, p + 1)$, $\rho_1[0, y] \sim \alpha/(p + 1 - \alpha)y^{p+1}\mu(y, \infty)$, which is regularly varying of index $p + 1 - \alpha \in (0, 1]$, and for $\alpha \in (p, p + 1]$, $\rho_2[0, y] \sim \alpha/(p + 2 - \alpha)y^{p+2}\mu(y, \infty)$, which is regularly varying of index $p + 2 - \alpha \in [1, 2)$. Further, from (4.6), we get

$$\Re r_G(iy) = -\int_0^\infty \frac{1}{t^2 + y^2} d\rho_2(t) \quad \text{and} \quad \Im r_G(iy) = -y \int_0^\infty \frac{1}{t^2 + y^2} d\rho_1(t).$$

Then the results follow immediately from Proposition 2.1. \square

While asymptotic equivalences between $\Re r_G(iy)$ and tail of μ for $\alpha = p$ and $\Im r_G(iy)$ and tail of μ for $\alpha = p + 1$ are not true in general, we obtain the relevant asymptotic bounds in these cases. We also obtain the exact asymptotic orders when $p = 0$.

PROPOSITION 4.4. Consider a probability measure μ in the class \mathcal{M}_p .

If μ has regularly varying tail of index $-p$, then for any $\varepsilon > 0$, $\Re r_G(iy) \gg y^{-\varepsilon}$ as $y \rightarrow \infty$. Further, if $p = 0$, then $\Re r_G(iy) \sim -\mu(y, \infty)$ as $y \rightarrow \infty$.

If μ has regularly varying tail of index $-(p + 1)$, then $\Im r_G(iy)$ is regularly varying of index -1 and $y^{-1} \ll \Im r_G(iy) \ll y^{-(1-\varepsilon)}$ as $y \rightarrow \infty$, for any $\varepsilon > 0$.

REMARK 4.4. Note that $\mathfrak{S}r_G(iy)$ is regularly varying for the case $\alpha = p + 1$ in contrast to $\mathfrak{R}r_G(iy)$ for the case $\alpha = p > 0$. Further, for the case $\alpha = p + 1$, the lower bound for $\mathfrak{S}r_G(iy)$ is sharper than that for $\mathfrak{R}r_G(iy)$ and coincides with that of $\mathfrak{S}r_G(iy)$ for the case $\alpha \in [p, p + 1)$ discussed in Proposition 4.3.

PROOF OF PROPOSITION 4.4. First consider the case where μ has regularly varying tail of index $-p$. We use the notation $d\rho_0(t) = t^p d\mu(t)$ introduced in the proof of Proposition 4.2. However, in the current situation Theorem 1.6.4 of [12] will not apply. If $p = 0$, then $\rho_0 = \mu$ and $\rho_0(y, \infty)$ is slowly varying. If $p > 0$, observe that, as $\int t^p d\mu(t) < \infty$, we have

$$\rho_0(y, \infty) = y^p \mu(y, \infty) + p \int_0^y s^{p-1} \mu(s, \infty) ds \sim p \int_0^y s^{p-1} \mu(s, \infty) ds,$$

which is again slowly varying, where we use Theorem 0.6(a) of [16]. Thus, in either case, $\rho_0(y, \infty)$ is slowly varying and converges to zero as $y \rightarrow \infty$. Now, from (4.6) and Proposition 2.2, we also have

$$\mathfrak{R}r_G(iy) = - \int_0^\infty \frac{t^2}{t^2 + y^2} d\rho_0(t) \sim -\rho_0(y, \infty)$$

as $y \rightarrow \infty$. Since $\rho_0(y, \infty)$ is slowly varying, for any $\varepsilon > 0$, we have $|y^\varepsilon \times \mathfrak{R}r_G(iy)| \rightarrow \infty$ as $y \rightarrow \infty$. Also, for $p = 0$, we have

$$\mathfrak{R}r_G(iy) \sim -\rho_0(y, \infty) = -\mu(y, \infty).$$

Next consider the case where $\mu \in \mathcal{M}_p$ has regularly varying tail of index $-(p + 1)$. Define again $d\rho_1(t) = t^{p+1} d\mu(t)$. Then,

$$\rho_1[0, y] = (p + 1) \int_0^y s^p \mu(s, \infty) ds - y^{p+1} \mu(y, \infty) \sim (p + 1) \int_0^y s^p \mu(s, \infty) ds$$

is slowly varying, again by Theorem 0.6(a) of [16]. Then, by (4.6) and Proposition 2.1, we have

$$\mathfrak{S}r_G(iy) = y \int_0^\infty \frac{d\rho_1(t)}{t^2 + y^2} \sim \frac{1}{y} \rho_1[0, y]$$

is regularly varying of index -1 . Further, $\rho_1[0, y] \rightarrow \int_0^\infty t^{p+1} d\mu(t) = \infty$ as $y \rightarrow \infty$. Then the asymptotic upper and lower bounds follow immediately. \square

5. Relationship between Cauchy and Voiculescu transforms. The results of the previous section relate the tail of a regularly varying probability measure and the behavior of the remainder term in Laurent series expansion of its Cauchy transform. In this section, we shall relate the remainder terms in Laurent series expansion of Cauchy and Voiculescu transforms. Finally, we collect the results from Sections 4 and 5 to prove Theorems 2.1–2.4.

To study the relation between the remainder terms in Laurent series expansion of Cauchy and Voiculescu transforms, we consider a class of functions, which include the functions H_μ for the probability measures μ with regularly varying tails. We then show that the class is closed under appropriate operations. See Propositions 5.1 and 5.2.

Let \mathcal{H} denote the set of analytic functions A having a domain \mathcal{D}_A such that for all positive η , there exists $\delta > 0$ with $\Delta_{\eta,\delta} \subset \mathcal{D}_A$.

For a nonnegative integer p and $\beta \in [0, 1/2)$, let $\mathcal{R}_{p,\beta}$ denote the set of all functions $A \in \mathcal{H}$ which satisfy the following conditions:

(R1) A has Taylor series expansion with real coefficients of the form

$$A(z) = z + \sum_{j=1}^p a_j z^{j+1} + z^{p+1} r_A(z),$$

where a_1, \dots, a_p are real numbers. For $p = 0$, we interpret the sum in the middle term as absent.

(R2) $z \ll r_A(z) \ll z^\beta$ as $z \rightarrow 0$ n.t.

(R3) $\Re r_A(-iy) \gg y^{1+\beta/2}$ and $\Im r_A(-iy) \gg y$ as $y \rightarrow 0+$.

For $p = 0 = \beta$, we further require that

(R4') $\Re r_A(-iy) \approx \Im r_A(-iy)$ as $y \rightarrow 0+$.

For $\beta \in (0, 1/2)$, we further require that,

(R4'') $\Re r_A(-iy) \ll y^{1-\beta/2}$ and $\Im r_A(-iy) \ll y^{1-\beta/2}$ as $y \rightarrow 0+$.

Note that the functions in $\mathcal{R}_{p,\beta}$ satisfy (R1)–(R3) for $p \geq 1$. For $p = 0 = \beta$, the functions in $\mathcal{R}_{p,\beta}$ satisfy (R1)–(R3) as well as (R4'). Finally, for nonnegative integers p and $\beta \in (0, 1/2)$, the functions in $\mathcal{R}_{p,\beta}$ satisfy (R1)–(R3) and (R4'').

The classes $\mathcal{R}_{p,\beta}$ as p varies over the set of nonnegative integers and β varies over $[0, 1/2]$, include the functions H_μ where $\mu \in \mathcal{M}_{p,\alpha}$ with p varying over nonnegative integers and α varying over $[p, p + 1]$:

Case I: p positive integer and $\alpha \in [p, p + 1)$: By Proposition 4.1 and 4.3, we have $H_\mu \in \mathcal{R}_{p,0}$.

Case II: $p = 0, \alpha \in [0, 1)$: By Proposition 4.1, 4.3 and 4.4, $H_\mu \in \mathcal{R}_{0,0}$.

Proposition 4.4 is required to prove (R4') for $p = \alpha = 0$ only.

Case III: p nonnegative integer, $\alpha = p + 1$: By Proposition 4.2 and 4.4, H_μ will be in $\mathcal{R}_{p,\beta}$ for any $\beta \in (0, 1/2)$.

We do not impose the condition $\Re r_A(-iy) \approx \Im r_A(-iy)$ for $p > 0$, as it may fail for some measures in $\mathcal{M}_{p,p}$.

The first result deals with the reciprocals. Note that $U(z)$ and $zU(z)$ have the same remainder functions, and if one belongs to the class \mathcal{H} , so does the other.

PROPOSITION 5.1. *Suppose $zU(z) \in \mathcal{H}$ be a function belonging to $\mathcal{R}_{p,\beta}$ for some nonnegative integer p and $0 \leq \beta < 1/2$, such that U does not vanish in a*

neighborhood of zero. Then the reciprocal $V = 1/U$ is defined and $zV(z)$ is also in $\mathcal{R}_{p,\beta}$. Furthermore, we have:

- (F1) $r_V(z) \sim -r_U(z)$, as $z \rightarrow 0$ n.t.;
- (F2) $\Re r_V(-iy) \sim -\Re r_U(-iy)$, as $y \rightarrow 0+$;
- (F3) $\Im r_V(-iy) \sim -\Im r_U(-iy)$, as $y \rightarrow 0+$.

The second result shows that for each of the above classes, when we consider a bijective function from the class, its inverse is also in the same class.

PROPOSITION 5.2. *Suppose $U \in \mathcal{H}$ be a bijective function with the inverse in \mathcal{H} as well and $U \in \mathcal{R}_{p,\beta}$ for some nonnegative integer p and $0 \leq \beta < 1/2$. Then the inverse V is defined and is also in $\mathcal{R}_{p,\beta}$. Furthermore, we have:*

- (I1) $r_V(z) \sim -r_U(z)$, as $z \rightarrow 0$ n.t.;
- (I2) $\Re r_V(-iy) \sim -\Re r_U(-iy)$, as $y \rightarrow 0+$;
- (I3) $\Im r_V(-iy) \sim -\Im r_U(-iy)$, as $y \rightarrow 0+$.

Next we prove Propositions 5.1 and 5.2. In both the proofs, all the limits will be taken as $z \rightarrow 0$ n.t. or $y \rightarrow 0+$, unless otherwise mentioned, and these conventions will not be stated repeatedly. We shall also use that for any nonnegative integer p and $\beta \in [0, 1/2)$, with $U \in \mathcal{R}_{p,\beta}$, we have

$$(5.1) \quad |\Re r_U(-iy)| \leq |r_U(-iy)| \ll 1 \quad \text{and} \quad |\Im r_U(-iy)| \leq |r_U(-iy)| \ll 1.$$

The proofs of Propositions 5.1 and 5.2 will be broken down into cases $p = 0$ and $p \geq 1$. Each of these cases will be further split into subcases $\beta = 0$ and $\beta \in (0, 1/2)$. The $p \geq 1$ is more involved compared to the case $p = 0$. However, the proofs, specially that of Proposition 5.2, have substantial parts in common for different cases.

We first prove the result regarding the reciprocal.

PROOF OF PROPOSITION 5.1. Note that since $zU(z)$ belongs to \mathcal{H} , given $\eta > 0$, there exists $\delta > 0$, such that $\Delta_{\eta,\delta}$ is contained in the domain of $zU(z)$, and U does not vanish on $\Delta_{\eta,\delta}$. Thus, $V(z)$ and hence $zV(z)$ will also be defined on $\Delta_{\eta,\delta}$. So $zV(z)$ also belongs to \mathcal{H} .

Observe that if we verify (F1)–(F3), then $zV(z)$ is automatically in $\mathcal{R}_{p,\beta}$ as well, since $V(z)$ and $zV(z)$ have same remainder functions. We shall prove (F1)–(F3) using the fact that $V(z) = 1/U(z)$ and the properties of $zU(z)$ as an element of $\mathcal{R}_{p,\beta}$.

Case I: $p = 0$. Let $zU(z) = z + zr_U(z)$ be a function in this class. Then $V(z) = 1 - r_U(z) + O(|r_U(z)|^2)$. By uniqueness of Taylor’s expansion from Lemma A.1 of [5], we have

$$(5.2) \quad r_V(z) = -r_U(z) + O(|r_U(z)|^2).$$

Since, by (R2), $r_U(z) \ll 1$, we have $r_V(z) \sim -r_U(z)$, which checks (F1).

Further, evaluating (5.2) at $z = -iy$ and equating the real and the imaginary parts, we have

$$\Re r_V(-iy) = -\Re r_U(-iy) + O(|r_U(-iy)|^2)$$

and

$$\Im r_V(-iy) = -\Im r_U(-iy) + O(|r_U(-iy)|^2).$$

Thus, to obtain the equivalences (F2) and (F3), it is enough to show that $|r_U(-iy)|^2 = |\Re r_U(-iy)|^2 + |\Im r_U(-iy)|^2$ is negligible with respect to both the real and the imaginary parts of $r_U(-iy)$. We prove the negligibility separately for two subcases $\beta = 0$ and $\beta \in (0, 1/2)$.

Subcase Ia: $p = 0, \beta = 0$. Using (5.1) and $\Re r_U(-iy) \approx \Im r_U(-iy)$ from (R4'), we have the required negligibility condition.

Subcase Ib: $p = 0, \beta \in (0, 1/2)$. Using (R3) and (R4''), we have

$$\frac{|\Im r_U(-iy)|^2}{|\Re r_U(-iy)|} = \frac{y^{1+\beta/2}}{|\Re r_U(-iy)|} \left(\frac{|\Im r_U(-iy)|}{y^{1-\beta/2}} \right)^2 y^{1-3\beta/2} \rightarrow 0$$

and

$$\frac{|\Re r_U(-iy)|^2}{|\Im r_U(-iy)|} = \frac{y}{|\Im r_U(-iy)|} \left(\frac{|\Re r_U(-iy)|}{y^{1-\beta/2}} \right)^2 y^{1-\beta} \rightarrow 0.$$

They, together with (5.1), give the required negligibility condition, thus proving (F2) and (F3).

Case II: $p \geq 1$. Let $zU(z) = z + \sum_{j=1}^p u_j z^{j+1} + z^{p+1} r_U(z)$ be a function in this class. Note that, as $p \geq 1$ and by (R2), as $z \ll r_U(z)$, we have $\sum_{j=1}^p u_j z^j + z^p r_U(z) = u_1 z + O(z r_U(z))$. Thus, using (R2), we have

$$V(z) = 1 + \sum_{j=1}^p (-1)^j \left(\sum_{m=1}^p u_m z^m + z^p r_U(z) \right)^j + (-1)^{p+1} u_1^{p+1} z^{p+1} + O(z^{p+1} r_U(z)).$$

Now we expand the second term on the right-hand side. As $z \ll r_U(z)$ from (R2), all powers of z with indices greater than $(p + 1)$ can be absorbed in the last term on the right-hand side. Then collect the $(p + 1)$ th powers of z in the second and third terms to get $c_1 z^{p+1}$ for some real number c_1 . The remaining powers of z form a polynomial $P(z)$ of degree at most p with real coefficients. Finally we consider the terms containing some power of $r_U(z)$. It will contain terms of the form $z^{l_1} (z^p r_U(z))^{l_2}$ for integers $l_1 \geq 0$ and $l_2 \geq 1$, with the leading term being $-z^p r_U(z)$. Since $p \geq 1$ and from (R2) we have $r_U(z) \ll 1$, the remaining terms can be absorbed in the last term on the right-hand side. Thus we get

$$V(z) = 1 + P(z) - z^p r_U(z) + c_1 z^{p+1} + O(z^{p+1} r_U(z)).$$

By uniqueness of Taylor series expansion from Lemma A.1 of [5], we have

$$r_V(z) = -r_U(z) + c_1z + O(zr_U(z)).$$

The form of r_V immediately gives $r_V(z) \sim -r_U(z)$, since $z \ll r_U(z)$, by (R2). This proves (F1).

Also, using (5.1), $\Im r_V(-iy) = -\Im r_U(-iy) + O(y)$ and as $y \ll \Im r_U(-iy)$ from (R3), we have $\Im r_V(-iy) \sim -\Im r_U(-iy)$. This shows (F3). Further, as c_1 is real, $\Re r_V(-iy) = -\Re r_U(-iy) + O(y|r_U(-iy)|)$. Thus, to conclude (F2), it is enough to show that $y|r_U(-iy)| \ll \Re r_U(-iy)$, for which it is enough to show that $y\Im r_U(-iy) \ll \Re r_U(-iy)$. We show this separately for two subcases.

Subcase IIa: $p \geq 1, \beta = 0$. We have by (R3),

$$\frac{y\Im r_U(-iy)}{\Re r_U(-iy)} = \frac{y}{\Re r_U(-iy)} \cdot \Im r_U(-iy).$$

Subcase IIb: $p \geq 1, \beta \in (0, 1/2)$. Using the properties (R3) and (R4'') we get

$$\frac{y\Im r_U(-iy)}{\Re r_U(-iy)} = \frac{y^{1+\beta/2}}{\Re r_U(-iy)} \cdot \frac{\Im r_U(-iy)}{y^{1-\beta/2}} \cdot y^{1-\beta}.$$

It is easy to see that the limit is zero in either subcase. \square

Before proving the result regarding the inverse, we provide a result connecting a function in the class \mathcal{H} and its derivative.

LEMMA 5.1. *Let $v \in \mathcal{H}$ satisfy $v(z) = o(z^\beta)$ as $z \rightarrow 0$ n.t., for some real number β . Then $v'(z) = o(z^{\beta-1})$ as $z \rightarrow 0$ n.t.*

PROOF. The result for $\beta = 0$ follows from the calculations in the proof of Proposition A.1(ii) of [5]. For the general case, define $w(z) = z^{-\beta}v(z)$. Then $w \in \mathcal{H}$ and $w(z) = o(1)$. So by the case $\beta = 0$, we have $w'(z) = -\beta z^{-\beta-1}v(z) + z^{-\beta}v'(z) = o(z^{-1})$. Thus, $zw'(z) = -\beta z^{-\beta}v(z) + z^{-(\beta-1)}v'(z)$, where the left-hand side and the first term on the right-hand side are $o(1)$, and hence the second term on the right-hand side is $o(1)$ as well. \square

We are now ready to prove the result regarding the inverse.

PROOF OF PROPOSITION 5.2. We begin with some estimates which work for all values of p and β before breaking into cases and subcases. Since U is of the form

$$U(z) = z + \sum_{j=1}^p u_j z^{j+1} + z^{p+1}r_U(z)$$

and $r_U(z) \ll 1$, by Proposition A.3 of [5], the inverse function V also has the same form with the remainder term r_V satisfying

$$(5.3) \quad r_V(z) \ll 1.$$

Also note that $V(z) \sim z$. Further, Lemma A.1 of [5] shows that the coefficients are determined by the limits of the derivatives of the function at 0. Hence, the real coefficients of U guarantee that the coefficients of V are real. So we only need to check the asymptotic equivalences of the remainder functions given in (I1)–(I3). We shall achieve this by analyzing $I(z) = r_U(V(z)) - r_U(z)$, the fact that $U(V(z)) = z$ and the properties of U as an element in $\mathcal{R}_{p,\beta}$. For that purpose, we define

$$I(z) = r_U(V(z)) - r_U(z) = \int_{\gamma_z} r'_U(\zeta) d\zeta,$$

where γ_z is the closed line segment joining z and $V(z)$. Using the part (a) in the proof of Proposition A.3 of [5], given any $\eta > 0$, we have for all small enough $\delta > 0$,

$$\Delta_{2\eta,2\delta} \subset \mathcal{D}_U \quad \text{and} \quad V(\Delta_{\eta,\delta}) \subset \Delta_{2\eta,2\delta}.$$

Thus, given any $\eta > 0$, there exists $\delta > 0$, such that whenever $z \in \Delta_{\eta,\delta}$, $V(z)$ belongs to $\Delta_{2\eta,2\delta}$. Note that $\Delta_{2\eta,2\delta}$ is a convex set. Hence, whenever $z \in \Delta_{\eta,\delta}$, γ_z is contained in $\Delta_{2\eta,2\delta} \subset \mathcal{D}_U$, and r'_U is defined on the entire line segment γ_z . We shall need the following estimate, that

$$|I(z)| \leq |\gamma_z| \sup_{\zeta \in \gamma_z} |r'_U(\zeta)| = |V(z) - z| \sup_{\zeta \in \gamma_z} |r'_U(\zeta)| = |V(z) - z| |r'_U(\zeta_0(z))|,$$

for some $\zeta_0(z) \in \gamma_z$, since γ_z is compact. Note that $\zeta_0(z) = z + \theta(z)(V(z) - z)$, for some $\theta(z) \in [0, 1]$ and hence $\zeta_0(z) \sim z$. Now, $r_U(z) = o(z^\beta)$ by (R2), and thus, by Lemma 5.1, we have $r'_U(\zeta_0(z)) = o(\zeta_0(z)^{\beta-1}) = o(z^{\beta-1})$. Further estimates for $I(z)$ depend on the functions of $V(z)$ which are separate for the cases $p = 0$ and $p \geq 1$. Using $V(z) = z + zr_V(z)$ for $p = 0$ and $V(z) = z + O(z^2)$ for $p \geq 1$, we have

$$(5.4) \quad |I(z)| = \begin{cases} o(z^\beta r_V(z)), & \text{for } p = 0, \\ o(z^{1+\beta}), & \text{for } p \geq 1. \end{cases}$$

Case I: $p = 0$. Then $U(z) = z + zr_U(z)$ and $V(z) = z + zr_V(z)$. Using $U(V(z)) = z$ and $I(z) = r_U(V(z)) - r_U(z)$, we get $0 = zr_V(z) + (z + zr_V(z)) \times (r_U(z) + I(z))$. Further canceling z and using (5.3), we have

$$(5.5) \quad 0 = r_U(z) + r_V(z) + r_U(z)r_V(z) + O(I(z)).$$

Using (5.4) for $p = 0$ and $r_U(z) \ll 1$ from (R2), we have $r_V(z) \sim -r_U(z)$, which proves (I1). Further, using (R2) and evaluating at $z = -iy$, we have, for $\beta \in [0, 1/2)$,

$$(5.6) \quad |r_V(-iy)| \ll y^\beta.$$

Evaluating (5.5) at $z = iy$ and equating the real and the imaginary parts, we have

$$(5.7) \quad 0 = \Re r_U(-iy) + \Re r_V(-iy) + O(|r_U(-iy)||r_V(-iy)|) + O(|I(-iy)|)$$

and

$$(5.8) \quad 0 = \Im r_U(-iy) + \Im r_V(-iy) + O(|r_U(-iy)||r_V(-iy)|) + O(|I(-iy)|).$$

We split the proofs of (I2) and (I3) for the case $p = 0$ into further subcases $\beta = 0$ and $\beta \in (0, 1/2)$.

Subcase Ia: $p = 0, \beta = 0$. By (I1) for $z = -iy$ and (R4'), we have

$$|I(-iy)| \ll |r_V(-iy)| \sim |r_U(-iy)| \approx |\Re r_U(-iy)| \approx |\Im r_U(-iy)|.$$

Thus, the last term on the right-hand side of (5.7) and (5.8) are negligible with respect to $\Re r_U(-iy)$ and $\Im r_U(-iy)$, respectively. Then, further using $r_U(-iy) \rightarrow 0$ from (R2), the third term on the right-hand side of (5.7) and (5.8) are negligible with respect to $\Re r_U(-iy)$ and $\Im r_U(-iy)$, respectively, and hence we get $\Re r_U(-iy) \sim -\Re r_V(-iy)$ and $\Im r_U(-iy) \sim -\Im r_V(-iy)$, which prove (I2) and (I3).

Subcase Ib: $p = 0, \beta \in (0, 1/2)$. We have, by (R3) and (R4''),

$$y^\beta \frac{|\Im r_U(-iy)|}{|\Re r_U(-iy)|} = \frac{|\Im r_U(-iy)|}{y^{1-\beta/2}} \frac{y^{1+\beta/2}}{|\Re r_U(-iy)|} \rightarrow 0$$

and

$$y^\beta \frac{|\Re r_U(-iy)|}{|\Im r_U(-iy)|} = \frac{|\Re r_U(-iy)|}{y^{1-\beta/2}} \frac{y}{|\Im r_U(-iy)|} y^{\beta/2} \rightarrow 0.$$

They, together with (5.1), give $y^\beta |r_U(-iy)|$ which is negligible with respect to both the real and the imaginary parts of $r_U(-iy)$. Further, using (5.4) and (5.6), respectively, we have

$$|I(-iy)| \ll y^\beta |r_V(-iy)| \sim y^\beta |r_U(-iy)| \quad \text{and} \\ |r_U(-iy)r_V(-iy)| \ll y^\beta |r_U(-iy)|.$$

Thus, both $|I(-iy)|$ and $|r_U(-iy)r_V(-iy)|$ which are the last two terms of (5.7) and (5.8), are negligible with respect to both the real and the imaginary parts of $r_U(-iy)$. Then, from (5.7) and (5.8), we immediately have $\Re r_U(-iy) \sim -\Re r_V(-iy)$ and $\Im r_U(-iy) \sim -\Im r_V(-iy)$, which prove (I2) and (I3).

Case II: $p \geq 1$. In this case $U(z) = z + \sum_{j=1}^p u_j z^{j+1} + z^{p+1} r_U(z)$ and $V(z) = z + \sum_{j=1}^p v_j z^{j+1} + z^{p+1} r_V(z) = z(1 + v_1 z(1 + o(1)))$. Using $z = U(V(z))$ and canceling z on both sides, we have

$$(5.9) \quad 0 = \sum_{j=1}^p v_j z^{j+1} + z^{p+1} r_V(z) + \sum_{m=1}^p u_m \left(z + \sum_{j=1}^p v_j z^{j+1} + z^{p+1} r_V(z) \right)^{m+1} \\ + z^{p+1} (r_U(z) + I(z)) (1 + (p+1)v_1 z(1 + o(1))).$$

Note that all the coefficients on the right-hand side are real. We collect the powers of z up to degree $p + 1$ on the right side in the polynomial $Q(z)$. Let $c' \in \mathbb{R}$ be the coefficient of z^{p+2} on the right side. The remaining powers of z on the right-hand side will be $O(z^{p+3})$. We next consider the terms with $r_V(z)$ as a factor and observe that $z^{p+1}r_V(z)$ is the leading term and the remaining terms contribute $O(z^{p+2}r_V(z))$. Finally, the last term on the right-hand side gives $z^{p+1}r_U(z) + O(z^{p+2}r_U(z)) + O(z^{p+1}I(z))$. Since $z \ll r_U(z)$ by (R2), the term $O(z^{p+3})$ can be absorbed in $O(z^{p+2}r_U(z))$. Combining the above facts and dividing (5.9) by z^{p+1} , we get

$$(5.10) \quad \begin{aligned} 0 = z^{-(p+1)}Q(z) + (r_U(z) + c'z + O(I(z)) + O(zr_U(z))) \\ + (r_V(z) + O(zr_V(z))). \end{aligned}$$

As $I(z) \ll z^{1+\beta} \ll z \ll r_U(z)$ by (5.4) and (R2), we have $r_U(z) + c'z + O(I(z)) + O(zr_U(z)) = r_U(z)(1 + o(1))$. Also $r_V(z) + O(zr_V(z)) = r_V(z)(1 + o(1))$. Thus, the last two terms on the right-hand side of (5.10) goes to zero. However, the first term on the right-hand side of (5.10), Q being a polynomial of degree at most p , becomes unbounded unless $Q \equiv 0$. So we must have $Q \equiv 0$. Thus, (5.10) simplifies to

$$(5.11) \quad r_U(z) + c'z + O(I(z)) + O(zr_U(z)) = -r_V(z) + O(zr_V(z)).$$

As observed earlier, the left-hand side is $r_U(z)(1 + o(1))$, and the right-hand side is $-r_V(z)(1 + o(1))$, giving $r_U(z) \sim -r_V(z)$, which proves (I1).

Further, as in the case $p = 0$, we have (5.6) from $r_U(z) \sim -r_V(z)$. Also, (5.11) becomes

$$(5.12) \quad -r_V(z) = r_U(z) + c'z + O(I(z)) + O(zr_U(z)).$$

Evaluating (5.12) at $z = -iy$ and equating the imaginary parts, we have, using (5.4),

$$-\Im r_V(-iy) = \Im r_U(-iy) + O(y).$$

This gives (I3), that is, $-\Im r_V(-iy) \sim \Im r_U(-iy)$, since $y \ll \Im r_U(-iy)$ by (R3).

Evaluating (5.11) at $z = -iy$ again and now equating the real parts, we have, as c' is real,

$$-\Re r_V(-iy) = \Re r_U(-iy) + O(|I(-iy)|) + O(y|r_U(-iy)|).$$

From (5.4) and (R3), we have $|I(-iy)| \ll y^{1+\beta} \ll \Re r_U(-iy)$. Thus, to obtain (I2), that is, $-\Re r_V(-iy) \sim \Re r_U(-iy)$, we only need to show that $y|r_U(-iy)| \ll \Re r_U(-iy)$, which follows using $r_U(-iy) \sim -r_V(-iy)$, (5.6) and (R3), since

$$\frac{y|r_U(-iy)|}{|\Re r_U(-iy)|} = \frac{y^{1+\beta/2}}{|\Re r_U(-iy)|} \frac{|r_U(-iy)|}{y^\beta} y^{\beta/2}. \quad \square$$

We wrap up the article by collecting the results from Sections 4 and 5 and proving Theorems 2.1–2.4.

PROOFS OF THEOREMS 2.1–2.4. We shall prove all the theorems together, as the proofs are very similar.

The statements involving the tail of the probability measure μ and the remainder term in Laurent expansion of Cauchy transform, r_{G_μ} can be obtained from the results in Section 4 as follows: For all the theorems, the equivalence of the statements (i) and (ii) about the tail of the probability measure and Cauchy transform (the imaginary part in Theorems 2.1–2.3 and the real part in Theorem 2.4) are given in Proposition 4.3. The asymptotic equivalences between the tail of the measure and (the real and the imaginary parts of) the remainder term in Laurent series expansion of Cauchy transform, given in (2.8), (2.9), (2.11), (2.14) and (2.17) are also given in Proposition 4.3. The similar asymptotic equivalence in (2.15) follows from Propositions 4.3 and 4.4 for the cases $\alpha \in (0, 1)$ and $\alpha = 1$ respectively. We consider the asymptotic upper and lower bounds next. The asymptotic lower bounds in (2.7), (2.10), (2.13) and (2.16) follow from Proposition 4.1. The asymptotic upper bound in (2.16) follows from Proposition 4.2. The asymptotic lower bounds in (2.8), (2.9), (2.11), (2.14) and (2.17) follow from Proposition 4.3. The asymptotic upper bound in (2.17) follows from the fact that $y^p \mu(y, \infty)$ is a regularly varying function of index -1 . The asymptotic lower bound in (2.12) follows from Proposition 4.4, while the asymptotic lower bound in (2.15) follows as the tail of the measure is regularly varying of index $-\alpha$ with $\alpha \in [0, 1)$. Finally both the asymptotic bounds in (2.18) follow from Proposition 4.4.

To complete the proofs of Theorems 2.1–2.4, we need to check the equivalence of the statements (ii) and (iii) involving the remainder terms in Laurent expansion of Cauchy and Voiculescu transforms for all the theorems and the asymptotic equivalences between the remainder terms in Laurent series expansion of Cauchy and Voiculescu transforms and their real and imaginary parts given in (2.7)–(2.18). Note that all these claims about Cauchy and Voiculescu transforms of μ have analogs about H_μ and R_μ , due to the facts that $r_G(z) = r_H(1/z)$ and $r_\phi(z) = r_R(1/z)$. We shall actually deal with the functions H_μ and R_μ .

For any probability measure $\mu \in \mathcal{M}_p$, the function $H \equiv H_\mu$ is invertible, belongs to the class \mathcal{H} and the leading term of its Taylor expansion is z . Further, by Proposition A.3 of [5], the above statement about H is equivalent to the same statement about its inverse, denoted by $L \equiv L_\mu$. Since the leading term of Taylor expansion of L has leading term z , the leading term of Taylor’s expansion of $L(z)/z$ is 1, and it is also in \mathcal{H} . Define $K(z) = z/L(z)$. Then K is also in \mathcal{H} , and its Taylor expansion has leading term 1. We shall also use the following facts obtained from (2.1):

$$(5.13) \quad zR_\mu(z) = (K(z) - 1) \quad \text{and} \quad zK(z) = z(1 + zR_\mu(z)).$$

Hence a Taylor expansion of K will also lead to a Taylor expansion of R of degree one less than that of K . However, due to the definition of the remainder term of the Taylor expansion given in (2.2), the corresponding remainder terms will be related by $r_K \equiv r_R$. Thus, we can move from the function r_H to r_K ($\equiv r_R$) through inverse and reciprocal and vice versa as follows:

$$\begin{aligned}
 (5.14) \quad H(z) &\xrightarrow[\text{Proposition 5.2}]{L(z)=H^{-1}(z)} L(z) = z \cdot \frac{L(z)}{z} \\
 &\xleftarrow[\text{Proposition 5.1}]{K(z)=z/L(z)} zK(z) \xrightarrow[r_K=r_R]{R(z)=(K(z)-1)/z} R(z).
 \end{aligned}$$

These observations set up the stage for Propositions 5.1 and 5.2. We shall use the class $\mathcal{R}_{p,0}$ for Theorems 2.1–2.3 and the class $\mathcal{R}_{p,\beta}$ with any $\beta \in (0, 1/2)$ for Theorem 2.4.

Suppose $\mu \in \mathcal{M}_p$ with $\alpha \in [p, p + 1)$. This condition holds for Theorems 2.1–2.3, and we prove these three theorems first. In these cases, $H_\mu(z)$ and $zK(z) = z(1 + zR_\mu(z))$ necessarily have Taylor expansions of the form given in the hypothesis (R1) for the class $\mathcal{R}_{p,0}$ with $r_H(z) \ll 1$ and $r_R(z) \ll 1$ as $z \rightarrow \infty$.

For all three theorems, first assume statement (ii) that $\Im r_G(iy)$ is regularly varying of index $-(\alpha - p)$. Then, from the already proven lower bounds in (2.7)–(2.15), we have the asymptotic lower bounds for $r_G(z)$, $\Re r_G(iy)$ and $\Im r_G(iy)$ under the setup of each of the three theorems. They translate to the asymptotic lower bounds for the function H_μ , as required by the hypotheses (R2) and (R3). The asymptotic upper bound in (R2) holds, as the remainder term in Taylor series expansion of H satisfies $r_H \ll 1$. For Theorem 2.3, we have $p = 0$, and we need to check the extra condition (R4'), which follows from the already proven asymptotic equivalences (2.14) and (2.15). Thus, for each of Theorems 2.1–2.3, H_μ belongs to $\mathcal{R}_{p,0}$.

We now refer to the schematic diagram given in (5.14). As H_μ is also invertible with $L = H^{-1} \in \mathcal{H}$, by Proposition 5.2, we also have $L \in \mathcal{R}_{p,0}$ and $r_H(z) \sim -r_L(z)$, $\Re r_H(-iy) \sim -\Re r_L(-iy)$ and $\Im r_H(-iy) \sim -\Im r_L(-iy)$. Clearly, then Proposition 5.1 applies to the function $L(z)/z$, which has reciprocal $K \in \mathcal{H}$. Thus, r_K and r_L satisfy the relevant asymptotic equivalences. Furthermore, since, $r_R \equiv r_K$, combining, we have $r_H(z) \sim r_R(z)$, $\Re r_H(-iy) \sim \Re r_R(-iy)$ and $\Im r_H(-iy) \sim \Im r_R(-iy)$. Further, for Theorem 2.3, we have $p = 0$ and $H \in \mathcal{R}_{p,0}$ satisfies (R4'). Hence, we also have $\Re r_R(-iy) \approx \Im r_R(-iy)$. Then R_μ inherits the appropriate properties from H_μ and passes them on to ϕ_μ , which gives us the statement (iii) about the remainder term in Laurent expansion of Voiculescu transform in each of Theorems 2.1–2.3.

Conversely, assume the statement (iii). Then the assumptions on r_ϕ imply the analogous properties for $r_R \equiv r_K$. Further, as μ is in \mathcal{M}_p , $zK(z) = z(1 + zR_\mu(z))$ satisfies the hypothesis (R1) for the class $\mathcal{R}_{p,0}$. Also, the remainder term of Taylor series expansion of $zK(z)$ is also given by $r_R \equiv r_K \ll 1$. The lower bound

for the imaginary part of the remainder term in the hypothesis (R3) follows from its regular variation and the fact that $\alpha \in [p, p + 1)$. The lower bound in the hypothesis (R2) is part of the statement (iii). The lower bound for the real part of the remainder term in the hypothesis (R3) is also a part of the statement (iii) for Theorems 2.1 and 2.2, while it follows from the statement (iii) for Theorem 2.3, as both the real and imaginary parts become asymptotically equivalent and regularly varying of index α with $\alpha \in [0, 1)$. Finally, the asymptotic equivalence in (R4') for Theorem 2.3 is a part of the statement (iii). Thus, again for each of Theorems 2.1–2.3, $zK(z)$ belongs to $\mathcal{R}_{p,0}$. Then apply Proposition 5.1 on K and then Proposition 5.2 on $z/K(z) = L(z)$ to obtain $H_\mu(z)$. Arguing, by checking the asymptotic equivalences as in the direct case, we obtain the required conclusions about r_H and hence r_G given in the statement (ii) for each of Theorems 2.1–2.3.

The argument is same in the case $\alpha = p + 1$, which applies to Theorem 2.4, with the observation that the stronger bounds required in the hypotheses (R2), (R3) and (R4') with $\beta > 0$ are assumed for r_ϕ and hence for r_R and is proved for r_G and hence for r_H in Propositions 4.2 and 4.4. \square

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