Admissibility of the usual confidence interval in linear regression

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Abstract: Consider a linear regression model with independent and identically normally distributed random errors. Suppose that the parameter of interest is a specified linear combination of the regression parameters. We prove that the usual confidence interval for this parameter is admissible within a broad class of confidence intervals.

Contents

1. Introduction

Consider the linear regression model $Y = X\beta + \varepsilon$, where Y is a random n-vector of responses, X is a known $n \times p$ matrix with linearly independent columns, β is an unknown parameter p-vector and $\varepsilon \sim N(0, \sigma^2 I_n)$ where σ^2 is an unknown positive parameter. Let $\hat{\beta}$ denote the least squares estimator of β. Also, define $\hat{\sigma}^2 = (Y - X\hat{\beta})^T (Y - X\hat{\beta})/(n - p)$.

Suppose that the parameter of interest is $\theta = a^T\beta$ where a is a given p-vector ($a \neq 0$). We seek a $1 - \alpha$ confidence interval for θ. Define the quantile $t(m)$ by the requirement that $P\left(-t(m) \leq T \leq t(m)\right) = 1 - \alpha$ for $T \sim t_m$. Let $\hat{\Theta}$ denote $a^T\hat{\beta}$, i.e. the least squares estimator of θ. Also let v_{11} denote the variance of $\hat{\Theta}$ divided by σ^2. The usual $1 - \alpha$ confidence interval for θ is

$$I = \left[\hat{\Theta} - t(m)\sqrt{v_{11}}\hat{\sigma},\ \hat{\Theta} + t(m)\sqrt{v_{11}}\hat{\sigma}\right]$$

where $m = n - p$. Is this confidence interval admissible? The admissibility of a confidence interval is a much more difficult concept than the admissibility of a point estimator, since confidence intervals must satisfy a coverage probability constraint. Also, admissibility of confidence intervals can be defined in either weak or strong forms (Joshi, [3, 4]).

Kabaila and Giri [5], Section 3, describe a broad class $\mathcal{D}$ of confidence intervals that includes I. The main result of the present paper, presented in Section 3, is that I is strongly admissible within the class $\mathcal{D}$. An attractive feature of the proof of this result is that, although lengthy, this proof is quite straightforward and elementary. Section 2 provides a brief description of this class $\mathcal{D}$. For completeness, in Section 4 we describe a strong admissibility result, that follows from the results of Joshi [3], for the usual $1 - \alpha$ confidence interval for θ in the somewhat artificial situation that the error variance σ^2 is assumed to be known.

2. Description of the class $\mathcal{D}$

Define the parameter $\tau = c^T\beta - t$ where the vector c and the number t are given and a and c are linearly independent. Let $\hat{\tau}$ denote $c^T\hat{\beta} - t$ i.e. the least squares estimator of τ. Define the matrix V to be the covariance matrix of $(\hat{\Theta}, \hat{\tau})$ divided by σ^2. Let v_{ij} denote the (i, j) th element of V. We use the notation $[a \pm b]$ for the interval $[a - b, a + b]$ ($b > 0$). Define the following confidence interval for θ

$$J(b, s) = \left[\hat{\Theta} - \sqrt{v_{11}}\hat{\sigma}\, b\left(\frac{\hat{\tau}}{\hat{\sigma}\sqrt{v_{22}}}\right) \pm \sqrt{v_{11}}\hat{\sigma}\, s\left(\frac{|\hat{\tau}|}{\hat{\sigma}\sqrt{v_{22}}}\right)\right] \tag{1}$$

where the functions b and s are required to satisfy the following restrictions. The function $b : \mathbb{R} \to \mathbb{R}$ is an odd function and $s : [0, \infty) \to (0, \infty)$. Both b and s are bounded. These functions are also continuous except, possibly, at a finite number of values. Also, $b(x) = 0$ for all $|x| \geq d$ and $s(x) = t(m)$ for all $x \geq d$ where d is a given positive number. Let $\mathcal{F}(d)$ denote the class of pairs of functions (b, s) that satisfy these restrictions, for given d ($d > 0$).

Define $\mathcal{D}$ to be the class of all confidence intervals for θ of the form (1), where c, t, d, b and s satisfy the stated restrictions. Each member of this class is specified by (c, t, d, b, s). Apart from the usual $1 - \alpha$ confidence interval I for θ, the class $\mathcal{D}$ of confidence intervals for θ includes the following:

(a) Suppose that we carry out a preliminary hypothesis test of the null hypothesis $\tau = 0$ against the alternative hypothesis $\tau \neq 0$. Also suppose that we construct a confidence interval for θ with nominal coverage $1 - \alpha$ based on the assumption that the selected model had been given to us *a priori* (as the true model). The resulting confidence interval, called the naive $1 - \alpha$ confidence interval, belongs to the class $\mathcal{D}$ (Kabaila and Giri [5], Section 2).

(b) Confidence intervals for θ that are constructed to utilize (in the particular manner described by Kabaila and Giri [5]) uncertain prior information that $\tau = 0$.

Let K denote the usual $1 - \alpha$ confidence interval for θ based on the assumption that $\tau = 0$. The naive $1 - \alpha$ confidence interval, described in (a), may be expressed in the following form:

$$h\left(\frac{|\hat{\tau}|}{\hat{\sigma}\sqrt{v_{22}}}\right) I + \left(1 - h\left(\frac{|\hat{\tau}|}{\hat{\sigma}\sqrt{v_{22}}}\right)\right) K \tag{2}$$

where $h : [0, \infty) \to [0, 1]$ is the unit step function defined by $h(x) = 0$ for all $x \in [0, q]$ and $h(x) = 1$ for all $x > q$. Now suppose that we replace h by a continuous increasing function satisfying $h(0) = 0$ and $h(x) \to 1$ as $x \to \infty$ (a similar construction is extensively used in the context of point estimation by Saleh [10]). The confidence interval (2) is also a member of the class $\mathcal{D}$.

3. Main result

As noted in Section 2, each member of the class $\mathcal{D}$ is specified by (c, t, d, b, s). The following result states that the usual $1 - \alpha$ confidence interval for θ is strongly admissible within the class $\mathcal{D}$.

Theorem 1. *There does not exist $(c, t, d, b, s) \in \mathcal{D}$ such that the following three conditions hold:*

$$(a) \qquad E_{\beta,\sigma^2}\big(\text{length of } J(b, s)\big) \leq E_{\beta,\sigma^2}\big(\text{length of } I\big) \quad \text{for all } (\beta, \sigma^2). \tag{3}$$

$$(b) \qquad P_{\beta,\sigma^2}\big(\theta \in J(b, s)\big) \geq P_{\beta,\sigma^2}\big(\theta \in I\big) \quad \text{for all } (\beta, \sigma^2). \tag{4}$$

$$(c) \qquad \text{Strict inequality holds in either (3) or (4) for at least one } (\beta, \sigma^2).$$

The proof of this result is presented in Appendix A.

An illustration of this result is provided by Figure 3 of Kabaila and Giri [5]. Define $\gamma = \tau/(\sigma\sqrt{v_{22}})$. Also define

$$e(\gamma; s) = \frac{\text{expected length of } J(b, s)}{\text{expected length of } I}.$$

We call this the scaled expected length of $J(b, s)$. Theorem 1 tells us that for any confidence interval $J(b, s)$, with minimum coverage probability $1 - \alpha$, it cannot be the case that $e(\gamma; s) \leq 1$ for all γ, with strict inequality for at least one γ. This fact is illustrated by the bottom panel of Figure 3 of Kabaila and Giri [5].

Define the class $\widetilde{\mathcal{D}}$ to be the subset of $\mathcal{D}$ in which both b and s are continuous functions. Strong admissibility of the confidence interval I within the class $\mathcal{D}$ implies weak admissibility of this confidence interval within the class $\widetilde{\mathcal{D}}$, as the following result shows. Since $(\hat{\beta}, \hat{\sigma}^2)$ is a sufficient statistic for (β, σ), we reduce the data to $(\hat{\beta}, \hat{\sigma}^2)$.

Corollary 1. *There does not exist* $(c, t, d, b, s) \in \widetilde{\mathcal{D}}$ *such that the following three conditions hold:*

$$(a') \qquad \big(\text{length of } J(b, s)\big) \leq \big(\text{length of } I\big) \quad \text{for all } (\hat{\beta}, \hat{\sigma}^2). \tag{5}$$

$$(b') \qquad P_{\beta, \sigma^2}\big(\theta \in J(b, s)\big) \geq P_{\beta, \sigma^2}\big(\theta \in I\big) \quad \text{for all } (\beta, \sigma^2). \tag{6}$$

(c') *Strict inequality holds in either* (5) *or* (14) *for at least one* (β, σ^2).

This corollary is proved in Appendix B.

4. Admissibility result for known error variance

In this section, we suppose that σ^2 is known. Without loss of generality, we assume that $\sigma^2 = 1$. As before, let $\hat{\beta}$ denote the least squares estimator of β. Since $\hat{\beta}$ is a sufficient statistic for β, we reduce the data to $\hat{\beta}$. Assume that the parameter of interest is $\theta = \beta_1 / \sqrt{\text{Var}(\hat{\beta}_1)}$. Thus the least squares estimator of θ is $\hat{\Theta} = \hat{\beta}_1 / \sqrt{\text{Var}(\hat{\beta}_1)}$. Define

$$\hat{\Delta} = \begin{bmatrix} \hat{\beta}_2 - \ell_2 \hat{\beta}_1 \\ \vdots \\ \hat{\beta}_p - \ell_p \hat{\beta}_1 \end{bmatrix}$$

where $\ell_2, \ldots, \ell_p$ have been chosen such that $\text{Cov}(\hat{\beta}_j - \ell_j \hat{\beta}_1, \hat{\beta}_1) = 0$ for $j = 2, \ldots, p$. Now define

$$\delta = \begin{bmatrix} \beta_2 - \ell_2 \beta_1 \\ \vdots \\ \beta_p - \ell_p \beta_1 \end{bmatrix}.$$

Note that $(\hat{\Theta}, \hat{\Delta})$ is obtained by a one-to-one transformation from $\hat{\beta}$. So, we reduce the data to $(\hat{\Theta}, \hat{\Delta})$. Note that $\hat{\Theta}$ and $\hat{\Delta}$ are independent, with $\hat{\Theta} \sim N(\theta, 1)$ and $\hat{\Delta}$ with a multivariate normal distribution with mean δ and known covariance matrix. Define the number z by the requirement that $P(-z \leq Z \leq z) = 1 - \alpha$ for $Z \sim N(0, 1)$. Let $I = \big[\hat{\Theta} - z, \hat{\Theta} + z\big]$. Define

$$\varphi(\hat{\theta}, \theta) = \begin{cases} 1 & \text{if } \theta \in \big[\hat{\theta} - z, \hat{\theta} + z\big] \\ 0 & \text{otherwise} \end{cases}$$

This is the probability that θ is included in the confidence interval I, when $\hat{\theta}$ is the observed value of $\hat{\Theta}$. The length of the confidence interval I is $\int_{-\infty}^{\infty} \varphi(\hat{\theta}, \theta) \, d\theta = 2z$. Let $p_\theta(\cdot)$ denote the probability density function of $\hat{\Theta}$ for given θ. The coverage probability of I is $\int_{-\infty}^{\infty} \varphi(\hat{\theta}, \theta) \, p_\theta(\hat{\theta}) \, d\hat{\theta} = 1 - \alpha$.

Now let $\mathcal{C}(\hat{\Theta}, \hat{\Delta})$ denote a confidence set for θ. Define

$$\varphi_\delta(\hat{\theta}, \theta) = P_{\theta,\delta}\big(\theta \in \mathcal{C}(\hat{\theta}, \hat{\Delta})\big),$$

where $\hat{\theta}$ denotes the observed value of $\hat{\Theta}$. For each given $\delta \in \mathbb{R}^{p-1}$, the expected Lebesgue measure of $\mathcal{C}(\hat{\Theta}, \hat{\Delta})$ is $E_{\theta,\delta}\Big(\int_{-\infty}^{\infty} \varphi_\delta(\hat{\Theta}, \theta) \, d\theta\Big)$. For each given $\delta \in \mathbb{R}^{p-1}$, the coverage probability of $\mathcal{C}(\hat{\Theta}, \hat{\Delta})$ is $\int_{-\infty}^{\infty} \varphi_\delta(\hat{\theta}, \theta) \, p_\theta(\hat{\theta}) \, d\hat{\theta}$. Theorem 5.1 of Joshi [3] implies the following strong admissibility result. Suppose that $\varphi_\delta(\hat{\theta}, \theta)$ satisfies the following conditions

(i) $E_{\theta,\delta}\Big(\int_{-\infty}^{\infty} \varphi_\delta(\hat{\theta}, \theta) \, d\theta\Big) \leq E_{\theta,\delta}\Big(\int_{-\infty}^{\infty} \varphi(\hat{\theta}, \theta) \, d\theta\Big)$ for all $\theta \in \mathbb{R}$.

(ii) $\int_{-\infty}^{\infty} \varphi_\delta(\hat{\theta}, \theta) \, p_\theta(\hat{\theta}) \, d\hat{\theta} \geq \int_{-\infty}^{\infty} \varphi(\hat{\theta}, \theta) \, p_\theta(\hat{\theta}) \, d\hat{\theta}$ for all $\theta \in \mathbb{R}$.

Then $\varphi_\delta(\hat{\theta}, \theta) = \varphi(\hat{\theta}, \theta)$ for almost all $(\hat{\theta}, \theta) \in \mathbb{R}^2$. This result is true for each $\delta \in \mathbb{R}^{p-1}$. Using standard arguemnts, this entails that $I \backslash \mathcal{C}(\hat{\Theta}, \hat{\Delta})$ and $\mathcal{C}(\hat{\Theta}, \hat{\Delta}) \backslash I$ are Lebesgue-null sets, for (Lebesgue-) almost all values of $(\hat{\Theta}, \hat{\Delta})$.

5. Remarks

The purpose of this section is to discuss (a) the breadth of the class of confidence intervals $\mathcal{D}$ and (b) the challenges to showing admissibility within a wider class of confidence intervals. We present an argument in favour of the claim that the class $\mathcal{D}$ is about as broad as possible for the case $p = 2$. We then describe a wider class of confidence intervals $\mathcal{E}$. We note that a similar argument may be mounted in favour of the claim that the class $\mathcal{E}$ is about as broad as possible for the case $p = 3$. We point out, however, that the extension of the arguments of the present paper to this class does not seem feasible.

Provided that one accepts some compelling equivariance arguments and some further reasonable conditions on the confidence interval, the class $\mathcal{D}$ is as broad as possible for the case $p = 2$. To see this, we proceed as follows. Suppose that $p = 2$. By sufficiency, reduce the data to $(\hat{\Theta}, \hat{\tau}, \hat{\sigma})$. Now apply the 3 equivariance arguments described in Appendix A of Kabaila and Giri [5] to obtain a confidence interval of the form (1), where $b : \mathbb{R} \to \mathbb{R}$ is an odd function and $s : [0, \infty) \to (0, \infty)$. These equivariance arguments seem compelling. Requiring that the functions b and s are (a) bounded and (b) continuous except, possibly, at a finite number of values, seems reasonable. The condition that $b(x) = 0$ for all $|x| \geq d$ and $s(x) = t(m)$ for all $x \geq d$ (for d a given positive number) is motivated by the requirement that the confidence interval revert to the usual $1 - \alpha$ confidence interval I with probability approaching 1, as $|\tau/\sigma| \to \infty$. This condition therefore seems reasonable. In this way we are led to the class $\mathcal{D}$.

Farchione [2] considers the following case. Consider the same linear regression model and parameter of interest θ as described in the introduction of the present paper. Define the parameters $\tau_2 = c_2^T \beta - t_2$ and $\tau_3 = c_3^T \beta - t_3$, where a, c_2 and c_3 are given linearly independent vectors and t_2 and t_3 are given numbers. Let $\hat{\tau}_2 = c_2^T \hat{\beta} - t_2$ and $\hat{\tau}_3 = c_3^T \hat{\beta} - t_3$. Define V to be the covariance matrix of $(\hat{\Theta}, \hat{\tau}_2, \hat{\tau}_3)$ divided by σ^2 and let v_{ij} denote that (i, j) th element of V. Farchione [2] considers the class $\mathcal{E}$ of confidence intervals for θ of the form

$$\left[\hat{\Theta} - \sqrt{v_{11}} \hat{\sigma} \, b\left(\frac{\hat{\tau}_2}{\hat{\sigma}\sqrt{v_{22}}}, \frac{\hat{\tau}_3}{\hat{\sigma}\sqrt{v_{33}}} \right) \pm \sqrt{v_{11}} \hat{\sigma} \, s\left(\frac{\hat{\tau}_2}{\hat{\sigma}\sqrt{v_{22}}}, \frac{\hat{\tau}_3}{\hat{\sigma}\sqrt{v_{33}}} \right) \right] \tag{7}$$

where b and s satisfy conditions analogous to those stated in Section 2 of the present paper. The class $\mathcal{E}$ is wider than the class $\mathcal{D}$. One can argue, in a manner similar to that presented in the previous paragraph, that the following is true. Provided that one accepts some compelling equivariance arguments and some further reasonable conditions on the confidence interval, the class $\mathcal{E}$ is as broad as possible for the case $p = 3$. An extension of the arguments of the present paper to the class $\mathcal{E}$ does not seem feasible due to (a) the complicatedness of the conditions that must be satisfied by b and s and (b) the complicatedness of the expressions derived by Farchione [2] for the coverage probability and the expected length of a confidence interval belonging to the class $\mathcal{E}$.

Appendix A: Proof of Theorem 1

Suppose that c is a given vector (such that c and a are linearly independent), t is a given number and d is a given positive number. The proof of Theorem 1 now proceeds as follows. We present a few definitions and a lemma. We then apply this lemma to prove this theorem.

Define $W = \hat{\sigma}/\sigma$. Note that W has the same distribution as $\sqrt{Q/m}$ where $Q \sim \chi_m^2$. Let f_W denote the probability density function of W. Also let ϕ denote the $N(0, 1)$ probability density function. Now define

$$R_1(b, s; \gamma) = \frac{\text{expected length of } J(b, s)}{\text{expected length of } I} - 1.$$

It follows from (7) of Kabaila and Giri [5] that

$$R_1(b, s; \gamma) = \frac{1}{t(m) \, E(W)} \int_0^\infty \int_{-d}^d \left(s(|x|) - t(m) \right) \phi(wx - \gamma) \, dx \, w^2 \, f_W(w) \, dw. \tag{8}$$

Thus, for each $(b, s) \in \mathcal{F}(d)$, $R_1(b, s; \gamma)$ is a continuous function of γ.

Also define $R_2(b, s; \gamma) = P\left(\theta \notin J(b, s) \right) - \alpha$. We make the following definitions, also used by Kabaila and Giri [5]. Define $\rho = v_{12}/\sqrt{v_{11}v_{22}}$ and $\Psi(x, y; \mu, v) = P(x \leq Z \leq y)$, for $Z \sim N(\mu, v)$. Now define the functions

$$k^\dagger(h, w, \gamma, \rho) = \Psi\left(-t(m)w, t(m)w; \rho(h - \gamma), 1 - \rho^2 \right)$$
$$k(h, w, \gamma, \rho) = \Psi\left(b(h/w)w - s(|h|/w)w, b(h/w)w + s(|h|/w)w; \rho(h - \gamma), 1 - \rho^2 \right).$$

It follows from (6) of Kabaila and Giri [5], that

$$R_2(b, s; \gamma) = -\int_0^\infty \int_{-d}^d \big(k(wx, w, \gamma, \rho)$$
$$- k^\dagger(wx, w, \gamma, \rho)\big)\, \phi(wx - \gamma)\, dx\, w\, f_W(w)\, dw. \qquad (9)$$

Thus, for each $(b, s) \in \mathcal{F}(d)$, $R_2(b, s; \gamma)$ is a continuous function of γ.

Now $E(W^2) = 1$ and so

$$\int_0^\infty w^2\, f_W(w)\, dw = 1.$$

It follows from (8) that

$$\int_{-\infty}^\infty R_1(b, s; \gamma)\, d\gamma = \frac{2}{t(m)\, E(W)} \int_0^d \big(s(x) - t(m)\big)\, dx. \qquad (10)$$

Thus $\int_{-\infty}^\infty R_1(b, s; \gamma)\, d\gamma$ exists for all $(b, s) \in \mathcal{F}(d)$.

Since $k(wx, w, \gamma, \rho)$ and $k^\dagger(wx, w, \gamma, \rho)$ are probabilities,

$$|R_2(b, s; \gamma)| \leq \int_0^\infty \int_{-d}^d \phi(wx - \gamma)dx\, w f_W(w)\, dw,$$

so that

$$\int_{-\infty}^\infty |R_2(b, s; \gamma)|\, d\gamma \leq 2d \int_0^\infty w f_W(w)\, dw = 2dE(W) < \infty.$$

Thus $\int_{-\infty}^\infty R_2(b, s; \gamma)\, d\gamma$ exists for all $(b, s) \in \mathcal{F}(d)$.

Thus, we may define

$$g(b, s; \lambda) = \lambda \int_{-\infty}^\infty R_1(b, s; \gamma)\, d\gamma + (1 - \lambda) \int_{-\infty}^\infty R_2(b, s; \gamma)\, d\gamma,$$

for each $(b, s) \in \mathcal{F}(d)$, where $0 < \lambda < 1$. Kempthorne [7–9] presents results on what he calls compromise decision theory. Initially, these results were applied only to the solution of some problems of point estimation. Kabaila and Tuck [6] develop new results in compromise decision theory and apply these to a problem of interval estimation. The following lemma, which will be used in the proof of Theorem 1, is in the style of these compromise decision theory results.

Lemma 1. *Suppose that c is a given vector (such that c and a are linearly independent), t is a given number and d is a given positive number. Also suppose that λ is given and that (b^*, s^*) minimizes $g(b, s; \lambda)$ with respect to $(b, s) \in \mathcal{F}(d)$. Then there does not exist $(b, s) \in \mathcal{F}(d)$ such that*

(a) $R_1(b, s; \gamma) \leq R_1(b^*, s^*; \gamma)$ for all γ.
(b) $R_2(b, s; \gamma) \leq R_2(b^*, s^*; \gamma)$ for all γ.
(c) Strict inequality holds in either (a) or (b) for at least one γ.

Proof. Suppose that c is a given vector (such that c and a are linearly independent), t is a given number and d is a given positive number. The proof is by contradiction. Suppose that there exist $(b, s) \in \mathcal{F}(d)$ such that (a), (b) and (c) hold. Now,

$$g(b^*, s^*; \lambda) - g(b, s; \lambda) = \lambda \int_{-\infty}^{\infty} \left(R_1(b^*, s^*; \gamma) - R_1(b, s; \gamma) \right) d\gamma$$

$$+ (1 - \lambda) \int_{-\infty}^{\infty} \left(R_2(b^*, s^*; \gamma) - R_2(b, s; \gamma) \right) d\gamma$$

By hypothesis, one of the following 2 cases holds.

<u>Case 1</u> (a) and (b) hold and $R_1(b^*, s^*; \gamma) - R_1(b, s; \gamma) > 0$ for at least one γ. Since $R_1(b^*, s^*; \gamma) - R_1(b, s; \gamma)$ is a continuous function of γ,

$$\int_{-\infty}^{\infty} \left(R_1(b^*, s^*; \gamma) - R_1(b, s; \gamma) \right) d\gamma > 0.$$

Thus $g(b^*, s^*; \lambda) > g(b, s; \lambda)$ and we have established a contradiction.

<u>Case 2</u> (a) and (b) hold and $R_2(b^*, s^*; \gamma) - R_2(b, s; \gamma) > 0$ for at least one γ. Since $R_2(b^*, s^*; \gamma) - R_2(b, s; \gamma)$ is a continuous function of γ,

$$\int_{-\infty}^{\infty} \left(R_2(b^*, s^*; \gamma) - R_2(b, s; \gamma) \right) d\gamma > 0.$$

Thus $g(b^*, s^*; \lambda) > g(b, s; \lambda)$ and we have established a contradiction.

Lemma 1 follows from the fact that this argument holds for every given vector c (such that c and a are linearly independent), every given number t and every given positive number d.

$\square$

We will first find the (b^*, s^*) that minimizes $g(b, s; \lambda)$ with respect to $(b, s) \in \mathcal{F}(d)$, for given λ. We will then choose λ such that $J(b^*, s^*) = I$, the usual $1 - \alpha$ confidence interval for θ. Theorem 1 is then a consequence of Lemma 1.

By changing the variable of integration in the inner integral in (9), it can be shown that $R_2(b, s; \gamma)$ is equal to

$$- \int_0^{\infty} \int_0^d \left(\left(k(wx, w, \gamma, \rho) - k^\dagger(wx, w, \gamma, \rho) \right) \phi(wx - \gamma) \right.$$

$$\left. + \left(k(-wx, w, \gamma, \rho) - k^\dagger(-wx, w, \gamma, \rho) \right) \phi(wx + \gamma) \right) dx \, w \, f_W(w) \, dw$$

Using this expression and the restriction that b is an odd function, we find that $\int_{-\infty}^{\infty} R_2(b, s; \gamma) \, d\gamma$ is equal to

$$-\int_0^d \int_0^\infty \int_{-\infty}^\infty \Big(\Psi\big(b(x)w - s(x)w, b(x)w + s(x)w; \rho y, 1 - \rho^2\big)$$

$$- \Psi\big(-t(m)w, t(m)w; \rho y, 1 - \rho^2\big)$$

$$+ \Psi\big(-b(x)w - s(x)w, -b(x)w + s(x)w; -\rho y, 1 - \rho^2\big)$$

$$- \Psi\big(-t(m)w, t(m)w; -\rho y, 1 - \rho^2\big)\Big) \phi(y)\, dy\, w\, f_W(w)\, dw\, dx.$$

Hence, to within an additive constant that does not depend on (b, s), $\int_{-\infty}^\infty R_2(b, s; \gamma)\, d\gamma$ is equal to

$$-\int_0^d \int_0^\infty \int_{-\infty}^\infty \Big(\Psi\big(b(x)w - s(x)w, b(x)w + s(x)w; \rho y, 1 - \rho^2\big)$$

$$+ \Psi\big(-b(x)w - s(x)w, -b(x)w + s(x)w; -\rho y, 1 - \rho^2\big)\Big)$$

$$\times \phi(y)\, dy\, w\, f_W(w)\, dw\, dx.$$

Thus, to within an additive constant that does not depend on (b, s),

$$g(b, s; \lambda) = \int_0^d q(b, s; x)\, dx,$$

where $q(b, s; x)$ is equal to

$$\frac{2\lambda}{t(m)\, E(W)} s(x)$$

$$- (1 - \lambda) \int_0^\infty \int_{-\infty}^\infty \Big(\Psi(b(x)w - s(x)w, b(x)w + s(x)w; \rho y, 1 - \rho^2)$$

$$+ \Psi(-b(x)w - s(x)w, -b(x)w + s(x)w; -\rho y, 1 - \rho^2)\Big)\, \phi(y)\, dy\, w\, f_W(w)\, dw.$$

Note that x enters into the expression for $q(b, s; x)$ only through $b(x)$ and $s(x)$. To minimize $g(b, s; \lambda)$ with respect to $(b, s) \in \mathcal{F}(d)$, it is therefore sufficient to minimize $q(b, s; x)$ with respect to $(b(x), s(x))$ for each $x \in [0, d]$. The situation here is similar to the computation of Bayes rules, see e.g. Casella and Berger [1], pp. 352–353. Therefore, to minimize $g(b, s; \lambda)$ with respect to $(b, s) \in \mathcal{F}(d)$, we simply minimize

$$\tilde{q}(b, s) = \frac{2\lambda}{t(m)\, E(W)} s$$

$$- (1 - \lambda) \int_0^\infty \int_{-\infty}^\infty \Big(\Psi(bw - sw, bw + sw; \rho y, 1 - \rho^2)$$

$$+ \Psi(-bw - sw, -bw + sw; -\rho y, 1 - \rho^2)\Big)\, \phi(y)\, dy\, w\, f_W(w)\, dw$$

with respect to $(b, s) \in \mathbb{R} \times (0, \infty)$, to obtain (b', s') and then set $b(x) = b'$ and $s(x) = s'$ for all $x \in [0, d]$.

Let the random variables A and B have the following distribution

$$\begin{bmatrix} A \\ B \end{bmatrix} \sim N \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} \right).$$

Note that the distribution of A, conditional on $B = y$, is $N(\rho y, 1 - \rho^2)$. Thus

$$\Psi(bw - sw, bw + sw; \rho y, 1 - \rho^2) = P\big(bw - sw \leq A \leq bw + sw \,\big|\, B = y\big)$$

Hence

$$\int_0^\infty \int_{-\infty}^\infty \Psi(bw - sw, bw + sw; \rho y, 1 - \rho^2) \, \phi(y) \, dy \, w \, f_W(w) \, dw$$

$$= \int_0^\infty P(bw - sw \leq A \leq bw + sw) \, w \, f_W(w) \, dw. \tag{11}$$

Let Φ denote the $N(0, 1)$ cumulative distribution function. For every fixed $w > 0$ and $s > 0$,

$$P(bw - sw \leq A \leq bw + sw) = \Phi(bw + sw) - \Phi(bw - sw)$$

is maximized by setting $b = 0$. Thus, for each fixed $s > 0$, (11) is maximized with respect to $b \in \mathbb{R}$ by setting $b = 0$.

Now let the random variables $\tilde{A}$ and $\tilde{B}$ have the following distribution

$$\begin{bmatrix} \tilde{A} \\ \tilde{B} \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & -\rho \\ -\rho & 1 \end{bmatrix} \right).$$

Note that the distribution of $\tilde{A}$, conditional on $\tilde{B} = y$, is $N(-\rho y, 1 - \rho^2)$. Thus

$$\Psi(-bw - sw, -bw + sw; -\rho y, 1 - \rho^2) = P\big(-bw - sw \leq \tilde{A} \leq -bw + sw \,\big|\, \tilde{B} = y\big)$$

Hence

$$\int_0^\infty \int_{-\infty}^\infty \Psi(-bw - sw, -bw + sw; -\rho y, 1 - \rho^2) \, \phi(y) \, dy \, w \, f_W(w) \, dw$$

$$= \int_0^\infty P(-bw - sw \leq \tilde{A} \leq -bw + sw) \, w \, f_W(w) \, dw. \tag{12}$$

For every fixed $w > 0$ and $s > 0$,

$$P\big(-bw - sw \leq \tilde{A} \leq -bw + sw\big) = \Phi(-bw + sw) - \Phi(-bw - sw)$$

is maximized by setting $b = 0$. Thus, for each fixed $s > 0$, (12) is maximized with respect to $b \in \mathbb{R}$ by setting $b = 0$.

Therefore, $\tilde{q}(b, s)$ is, for each fixed $s > 0$, minimized with respect to b by setting $b = 0$. Thus $b' = 0$ and so $b^*(x) = 0$ for all $x \in \mathbb{R}$. Hence, to find s' we need to minimize

$$\frac{\lambda}{t(m)E(W)} s - (1 - \lambda) \int_0^\infty \big(2\Phi(sw) - 1 \big) \, w f_W(w) \, dw$$

with respect to $s > 0$. Therefore, to find s' we may minimize

$$r(s) = \ell(\lambda) \, s - 2 \int_0^\infty \Phi(sw) \, w f_W(w) \, dw$$

with respect to $s > 0$, where

$$\ell(\lambda) = \frac{\lambda}{(1 - \lambda)t(m)E(W)}.$$

Note that $\ell(\lambda)$ is an increasing function of λ, such that $\ell(\lambda) \downarrow 0$ as $\lambda \downarrow 0$ and $\ell(\lambda) \uparrow \infty$ as $\lambda \uparrow 1$. Choose $\lambda = \lambda^*$, where

$$\ell(\lambda^*) = 2 \int_0^\infty \phi\big(t(m)w\big) \, w^2 \, f_W(w) \, dw.$$

Note that $0 < \ell(\lambda^*) < \sqrt{2/\pi}$. Now

$$\frac{dr(s)}{ds} = \ell(\lambda^*) - 2 \int_0^\infty \phi(sw) \, w^2 f_W(w) \, dw.$$

Since $\int_0^\infty \phi(sw) \, w^2 f_W(w) \, dw$ is a decreasing function of $s > 0$, $dr(s)/ds$ is an increasing function of $s > 0$. Also, for $s = 0$, $\int_0^\infty \phi(sw) \, w^2 f_W(w) \, dw = 1/\sqrt{2\pi}$. Thus, to minimize $r(s)$ with respect to $s > 0$, we need to solve

$$\ell(\lambda^*) - 2 \int_0^\infty \phi(sw) \, w^2 \, f_W(w) \, dw = 0$$

for $s > 0$. Obviously, this solution in $s = t(m)$. Thus $s^*(x) = t(m)$ for all $x \geq 0$. In other words, $J(b^*, s^*) = I$. By Lemma 1, there does not exist $(b, s) \in \mathcal{F}(d)$ such that

(a) $\quad E_{\beta,\sigma^2}\big(\text{length of } J(b, s)\big) \leq E_{\beta,\sigma^2}\big(\text{length of } I\big) \quad \text{for all } (\beta, \sigma^2).$ (13)

(b) $\quad P_{\beta,\sigma^2}\big(\theta \in J(b, s)\big) \geq P_{\beta,\sigma^2}\big(\theta \in I\big) \quad \text{for all } (\beta, \sigma^2).$ (14)

(c) $\quad$ Strict inequality holds in either (13) or (14) for at least one (β, σ^2).

Theorem 1 follows from the fact that this argument holds for every given vector c (such that c and a are linearly independent), every given number t and every given positive number d.

Appendix B: Proof of Corollary 1

The proof of Corollary 1 is by contradiction. Suppose that c is a given vector (such that c and a are linearly independent), t is a given number and d is a given positive number. Also suppose that there exists $(b, s) \in \mathcal{F}(d)$ such that both b and s are continuous and (a'), (b') and (c'), in the statement of Corollary 1, hold. Now (a') implies that

$$E_{\beta,\sigma^2}\big(\text{length of } J(b, s)\big) \leq E_{\beta,\sigma^2}\big(\text{length of } I\big) \quad \text{for all } (\beta, \sigma^2),$$

so that (a) holds. By hypothesis, one of the following two cases holds.

<u>Case 1</u> $\big($length of $J(b,s)\big) < \big($length of $I\big)$ for at least one $(\hat{\beta}, \hat{\sigma}^2)$. Now

$$\big(\text{length of } J(b,s)\big) = 2\sqrt{v_{11}}\,\hat{\sigma}\,s\left(\frac{|\hat{\tau}|}{\hat{\sigma}\sqrt{v_{22}}}\right),$$

which is a continuous function of $(\hat{\beta}, \hat{\sigma}^2)$. Hence $\big($length of $I\big) - \big($length of $J(b,s)\big)$ is a continuous function of $(\hat{\beta}, \hat{\sigma}^2)$. Thus

$$E_{\beta,\sigma^2}\big(\text{length of } J(b,s)\big) < E_{\beta,\sigma^2}\big(\text{length of } I\big) \quad \text{for at least one } (\beta, \sigma^2).$$

Thus there exists $(b,s) \in \mathcal{F}(d)$ such that (a), (b) and (c), in the statement of Theorem 1, hold. We have established a contradiction.

<u>Case 2</u> There is strict inequality in (b') for at least one (β, σ^2). Thus there exists $(b,s) \in \mathcal{F}(d)$ such that (a), (b) and (c), in the statement of Theorem 1, hold. We have established a contradiction.

Corollary 1 follows from the fact that this argument holds for every given vector c (such that c and a are linearly independent), every given number t and every given positive number d.

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