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ON AZÉMA–YOR PROCESSES, THEIR OPTIMAL PROPERTIES AND THE BACHELIER–DRAWDOWN EQUATION

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We study the class of Azéma–Yor processes defined from a general semimartingale with a continuous running maximum process. We show that they arise as unique strong solutions of the Bachelier stochastic differential equation which we prove is equivalent to the drawdown equation. Solutions of the latter have the drawdown property: they always stay above a given function of their past maximum. We then show that any process which satisfies the drawdown property is in fact an Azéma–Yor process. The proofs exploit group structure of the set of Azéma–Yor processes, indexed by functions, which we introduce.

We investigate in detail Azéma–Yor martingales defined from a nonnegative local martingale converging to zero at infinity. We establish relations between average value at risk, drawdown function, Hardy–Littlewood transform and its inverse. In particular, we construct Azéma–Yor martingales with a given terminal law and this allows us to rediscover the Azéma–Yor solution to the Skorokhod embedding problem. Finally, we characterize Azéma–Yor martingales showing they are optimal relative to the concave ordering of terminal variables among martingales whose maximum dominates stochastically a given benchmark.

1. Introduction. In [3] Azéma and Yor introduced a family of simple local martingales, associated with Brownian motion or more generally with a continuous martingale, which they exploited to solve the Skorokhod embedding problem. These processes, called Azéma–Yor processes, are simply functions of the underlying process X and its running maximum $\overline{X}_t = \sup_{s\le t} X_s$. They proved to be very useful especially in describing laws of the maximum or of the last passage times of a martingale and were applied in problems ranging from Skorokhod embeddings, through optimal inequalities, to Brownian penalizations (cf. Azéma and Yor [3, 4], Obłój and Yor [28], Roynette, Vallois and Yor [31]). The appearance

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of Azéma–Yor martingales in all these problems was partially explained with a characterization in Obłój [27] as the only local martingales which can be written as a function of the couple $(X, \overline{X}_t)$.

Recently these processes have seen a revived interest with applications in mathematical finance including re-interpretation of classical pricing formulae (see Madan, Roynette and Yor [24]) and portfolio optimization under pathwise constraints (see El Karoui and Meziou [11, 12]). In this paper we uncover a more general structure of these processes and present new characterizations. We explore in depth their properties and present some further applications of Azéma–Yor processes. We work in a general setup and extend the concept of Azéma–Yor processes $M^U(X)$, as defined in (2) below, to the context of an arbitrary semimartingale (X_t) with a continuous running maximum process $\overline{X}_t$.

We start by studying the set of Azéma–Yor processes $M^U(X)$, indexed by increasing absolutely continuous functions U, and show that it has a simple group structure. This allows us to see any semimartingale with continuous running maximum as an Azéma–Yor process. The main contribution of Section 3 is to study how such representations arise naturally. We show that Azéma–Yor processes allow us to solve explicitly the Bachelier equation, which we also identify with the drawdown equation. The solutions to the latter satisfy the drawdown constraint $Y_t \geq w(\overline{Y}_t)$. Conversely, if (Y_t) satisfies the drawdown constraint up to time ζ, then it can be written as $M^U_{t\wedge\zeta}(X)$ for some nonnegative X. Further, if $Y_\zeta = w(\overline{Y}_\zeta)$ a.s., then the inverse process $(X_{t\wedge\zeta})$ is stopped upon existing an interval $(0, b)$. We provide explicit relation between function U which generates Azéma–Yor process and functions w and φ which feature in the drawdown constraint and in the SDEs. This characterizes the processes both in a pathwise manner and differential manner.

Then in Section 4 we specialize further and investigate Azéma–Yor processes defined from $X = N$ a nonnegative local martingale with continuous maximum process and with $N_t \to 0$ as $t \to \infty$. We show how one can identify explicitly Azéma–Yor processes from their terminal values. In Section 4.3 we discuss the average value at risk and the Hardy–Littlewood transform in a unified manner using tail quantiles of probability measures. Then we construct Azéma–Yor martingales with a prescribed terminal law. This allows us to rediscover, in Section 4.4, the Azéma–Yor [3] solution to the Skorokhod embedding problem and give it a new interpretation.

Finally, in the last section, we apply the previous results to uncover optimal properties of Azéma–Yor martingales. More precisely, we show that all uniformly integrable martingales whose maximum dominates stochastically a given floor distribution are dominated by an Azéma–Yor martingale in the concave ordering of terminal values. This problem is an extension of the more intuitive problem, motivated by finance, to find an optimal martingale for the concave order dominating (pathwise) a given floor process. It is rather surprising to find that the two problems have the same solution. We recover in this way the Δ operator of Kertz and Rösler

[22] and give a direct way to compute it. These dual results are compared with the classical primal result stating that among all uniformly integrable martingales with a fixed terminal law the Azéma–Yor martingale has the largest maximum (relative to the stochastic order). Furthermore, in both problems we can show that any optimal martingale is necessarily an appropriate Azéma–Yor martingale.

2. The set of Azéma–Yor processes. Throughout, all processes are defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ a filtered probability space satisfying the usual hypothesis and assumed to be taken right-continuous with left limits (càdlàg), up to ∞ included if needed. All functions are assumed to be Borel measurable. Given a process (X_t) we denote its running maximum $\overline{X}_t = \sup_{s\le t} X_s$. In what follows, we are essentially concerned with semimartingales with continuous running maximum, that we call *max-continuous semimartingales*. Observe that under this assumption, the process $\overline{X}_t = \sup_{s\le t} X_s$ only increases when $\overline{X}_t = X_t$ or equivalently

$$\int_0^T (\overline{X}_t - X_t)\, d\overline{X}_t = 0 \qquad \text{for any } T > 0. \tag{1}$$

We let $\tau^b(X) = \tau^b_X = \inf\{t \ge 0 : X_t \ge b\}$ be the first up-crossing time of the level b by process X, with the standard convention that $\inf\{\varnothing\} = \infty$. Note that by max-continuity $X_{\tau^b(X)} = b$, if $0 < \tau^b(X) < \infty$. With a slight abuse of notation, τ^∞_X denotes the explosion time of X.

2.1. *Definition and properties.* There are two different ways to introduce Azéma–Yor processes, and their equivalence has been proven by several authors (see the comments below).

DEFINITION 2.1. Let (X_t) be a max-continuous semimartingale starting from $X_0 = a$, and $\overline{X}_t$ its (continuous) running maximum.

With any locally bounded Borel function u we associate the primitive U, defined on $[a, +\infty)$, with initial condition a^*, that is, $U(x) = a^* + \int_a^x u(s)\, ds$. The Azéma–Yor process associated with U and X is defined by one of these two equations:

$$M^U_t(X) := U(\overline{X}_t) - u(\overline{X}_t)(\overline{X}_t - X_t) \quad \text{or} \tag{2}$$

$$= a^* + \int_0^t u(\overline{X}_s)\, dX_s. \tag{3}$$

In consequence, $M^U(X)$ is a semimartingale and it is a local martingale when X is a local martingale.

Observe that the process $M^U(X)$ is càdlàg, since $U(\overline{X})$ is continuous and $u(\overline{X}_t)(\overline{X}_t - X_t)$ is nonzero only on the intervals of constancy of $\overline{X}_t$, where the nonregular process $u(\overline{X}_t)$ is constant. Moreover the jumps of $M_t^U(X)$ are given explicitly by $-u(\overline{X}_t)(X_t - X_{t-})$.

We note also that, when u is defined only on some interval $[a, b)$ but $U(b)$ is well defined (finite or infinite), then we can still define $M_t^U(X)$ for $t \le \tau^b(X)$ and $M^U_{\tau^b(X)}(X) = U(b)$. Further, using regularity of paths of (X_t) we have that (2)–(3) hold with $t \wedge \tau^b(X)$ instead of t and $u(b) := 0$.

The symbol $M^U(X)$ is a slight abuse of notation since this process depends explicitly on the derivative u rather than the function U. The equivalence between (2) and (3) is easy to establish when u is smooth enough to apply Itô's formula, since the continuity of the running maximum implies from (1) that $\int_0^t (\overline{X}_t - X_t)\, du(\overline{X}_t) \equiv 0$. These results may be extended to all bounded functions u via monotone class theorem and to all locally bounded functions u via a localization argument. Alternatively, the equivalence can be argued using the general balayage formula (see Nikeghbali and Yor [25]). The case of locally integrable function u can be attained for a continuous local martingale X, as shown in Obłój and Yor [28].

The family of processes in (2)–(3), or their analog with the local time in zero replacing the running maximum, were first exhibited when (X_t) is a continuous local martingale. Azéma [1] obtained them as an application of formulae for dual predictable projections resulting from supermartingale representations, and Azéma and Yor [2] as direct application of their balayage formula. Azéma and Yor [3, 4] described these processes in more detail and used them to solve the Skorokhod embedding problem.

The importance of the family of Azéma–Yor martingales is well exhibited by Obłój [27] who proves that in the case of a continuous local martingale (X_t) all local martingales which are functions of the couple $(X_t, \overline{X}_t)$, $M_t = H(X_t, \overline{X}_t)$ can be represented as a $M = M^U(X)$ local martingale associated with a locally integrable function u. We note that such processes are sometimes called *max-martingales*.

2.2. *Monotonic transformations and Azéma–Yor processes.* We want to investigate further the structure of the set of Azéma–Yor processes associated with a max-continuous semimartingale (X_t). One of the most remarkable properties of these processes is that their running maximum can be easily computed, when the function U is nondecreasing ($u \ge 0$).

We denote by $\mathcal{U}_m$ the set of such functions, that is absolutely continuous functions defined on an appropriate interval with a locally bounded and nonnegative derivative. This set is stable by composition, that is, if U and F are in $\mathcal{U}_m$, and defined on appropriate intervals, then $U \circ F(x) = U(F(x))$ is in $\mathcal{U}_m$. We let $\mathcal{U}_m^+$ be the set of increasing functions $U \in \mathcal{U}_m$, with inverse function $V \in \mathcal{U}_m$, or equivalently of functions U such that $u = U' > 0$, and both u and $1/u$ are locally bounded. Throughout, when we consider an inverse function V of $U \in \mathcal{U}_m^+$ then we choose $v(y) = V'(y) = 1/u(V(y))$.

In light of (2), we then have:

PROPOSITION 2.2. (a) *Let* $U \in \mathcal{U}_m$, X *be a max-continuous semimartingale and* $M^U(X)$ *be the Azéma–Yor process in* (2). *Then*

$$\overline{M_t^U(X)} = U(\overline{X}_t), \tag{4}$$

and $M^U(X)$ *is a max-continuous semimartingale*.

(b) *Let* $F \in \mathcal{U}_m$ *defined on an appropriate interval, so that* $U \circ F$ *is well defined. Then*

$$M_t^U(M^F(X)) = M_t^{U \circ F}(X).$$

REMARK 2.3. It follows from point (b) above that the set of Azéma–Yor processes indexed by $U \in \mathcal{U}_m^+$ defined on whole $\mathbb{R}$ with $U(\mathbb{R}) = \mathbb{R}$, is a group under the operation $\otimes$ defined by

$$M^U \otimes M^F := M^{U \circ F}.$$

Note that $M_t^{\mathrm{Id}}(X) = X_t$, where $\mathrm{Id}(x) = x$ is the identity mapping.

PROOF OF PROPOSITION 2.2. (a) In light of (2), when u is nonnegative, the Azéma–Yor process $M_t^U(X)$ is dominated by $U(\overline{X}_t)$, with equality if t is a point of increase of $\overline{X}_t$. Since U is nondecreasing we obtain (4). Moreover, since $U(\overline{X})$ is a continuous process, $M^U(X)$ is a max-continuous semimartingale and we may take an Azéma–Yor process of it.

(b) Let F be in $\mathcal{U}_m$, $f = F' \geq 0$, such that $U \circ F$ is well defined. We have from (4) and (2)

$$\begin{aligned} M_t^U(M^F(X)) &= U(F(\overline{X}_t)) - u(F(\overline{X}_t)) f(\overline{X}_t)(\overline{X}_t - X_t) \\ &= M_t^{U \circ F}(X), \end{aligned} \tag{5}$$

where we used $(U(F(x)))' = u(F(x)) f(x)$. $\square$

The two properties described in Proposition 2.2 are rather simple but extremely useful. We phrase part (b) above for stopped processes and for $F = V = U^{-1}$ as a separate corollary.

COROLLARY 2.4. *Let* $a < b \leq \infty$, $U \in \mathcal{U}_m^+$ *the primitive function of a locally bounded* $u : [a, b) \to (0, \infty)$ *with* $U(a) = a^*$. *Let* $V : [a^*, U(b)) \to [a, b)$ *be the inverse of* U *with locally bounded derivative* $v(y) = 1/u(V(y))$.

Then for any max-continuous semimartingale (X_t), $X_0 = a$, *stopped at the time* $\tau^b = \tau^b(X) = \inf\{t : X_t \geq b\}$ *we have*

$$X_{t \wedge \tau^b} = M_{t \wedge \tau^b}^V(M^U(X)). \tag{6}$$

From the differential point of view, on $[0, \tau^b)$,

$$dY_t = u(\overline{X}_t)\, dX_t \quad \textit{and} \quad dX_t = v(\overline{Y}_t)\, dY_t \qquad \textit{where } Y_t = M_t^U(X). \tag{7}$$

Consider u as above with $b = U(b) = \infty$. As a consequence of the above, any max-continuous semimartingale (X_t) can be seen as an Azéma–Yor process associated with U. Indeed, $X_t = M_t^U(Y)$ with $Y_t = M_t^V(X)$. In the following section we study how such representations arise in a natural way.

3. The Bachelier–drawdown equation. In his paper, "Théorie des probabilités continues," published in 1906, French mathematician Louis Bachelier [5] was the first to consider and study stochastic differential equations. Obviously, he did not prove in his paper existence and uniqueness results but focused his attention on some particular types of SDEs. In this way, he obtained the general structure of processes with independent increments and continuous paths, the definition of diffusions (in particular, he solved the Langevin equation), and generalized these concepts to higher dimensions.

3.1. *The Bachelier equation.* In particular, Bachelier ([5], pages 287–290) considered and "solved" an SDE depending on the maximum of the solution, $dY_t = \varphi(\overline{Y}_t)\,dX_t$ which we call the Bachelier equation. Let $U \in \mathcal{U}_m^+$ and $V \in \mathcal{U}_m^+$ its inverse function with derivative v. From (7) and (4) we see that the Azéma–Yor process $Y = M^U(X)$ verifies the Bachelier equation for $\varphi(y) = 1/v(y)$. Now, we can solve the Bachelier equation as an inverse problem. We present a rigorous, simple and explicit solution to this equation. We note that a similar approach is developed in Revuz and Yor [30], Exercice VI.4.21.

THEOREM 3.1. *Let $(X_t : t \geq 0)$, $X_0 = a$, be a max-continuous semimartingale. Consider a positive Borel function $\varphi : [a^*, \infty) \to (0, \infty)$ such that φ and $1/\varphi$ are locally bounded. Let $V(y) = a + \int_{a^*}^{y} \frac{ds}{\varphi(s)}$, and U its inverse defined on $(a, V(\infty))$.*

The Bachelier equation,

$$dY_t = \varphi(\overline{Y}_t)\,dX_t, \qquad Y_0 = a^*, \tag{8}$$

has a strong, pathwise unique, max-continuous solution defined up to its explosion time $\tau_Y^\infty = \tau_X^{V(\infty)}$ given by $Y_t = M_t^U(X)$, $t < \tau_X^{V(\infty)}$.

When X is a continuous local martingale it suffices to assume that $1/\varphi$ is a locally integrable function.

PROOF. The assumptions on φ imply that V and therefore U are in $\mathcal{U}_m^+$ with $U(a) = a^*$. With $u = \varphi(V)$, Definition 2.1 and (4) gives that the Azéma–Yor process $M^U(X)$ verifies

$$dM_t^U(X) = u(\overline{X}_t)\,dX_t = \varphi(\overline{M_t^U(X)})\,dX_t, \qquad t < \tau_X^{V(\infty)}.$$

Furthermore, on $\tau^{V(\infty)}(X) < \infty$, $M^U_{\tau^{V(n)}}(X) = U(V(n)) = n$ and we see that if $V(\infty) < \infty$ then $\tau_X^{V(\infty)}$ is the explosion time of $M^U(X)$. So, $M^U(X)$ is a solution of (8).

Now let Y be a max-continuous solution to equation (8). Equation (3) in Definition 2.1 and (8) imply that $dM_t^V(Y) = dX_t$ on $[0, \tau_Y^\infty)$. It follows from Corollary 2.4 that $Y_t = M_t^U(X)$ and $\tau_X^{V(\infty)}$ is the explosion time τ_Y^∞ of Y.

The above result extends to more general φ whenever U, V and $M^U(X)$ are well defined. When X is a continuous local martingale, to define V and U it is sufficient (and necessary) to assume $1/\varphi$ is locally integrable. That $M^U(X)$ is then well defined follows from Obłój and Yor [28]. □

The above extends naturally to the case when a and a^* are some $\mathcal{F}_0$-measurable random variables. It suffices to assume that φ is well defined on $[l, \infty)$ where $-\infty \le l$ is the lower bound of the support of a^*. We could also consider X which is only defined up to its explosion time τ_X^∞ which would induce $\tau_Y^\infty = \tau_X^\infty \wedge \tau_X^{V(\infty)}$.

In Section 4 we will also consider the case when $\varphi \equiv 0$ on (r, ∞) and then (Y_t) is stopped upon hitting r.

Finally note that under a stronger assumption that X has no positive jumps, *any* solution of the Bachelier equation has no positive jumps and hence is a max-continuous semimartingale.

3.2. *Drawdown constraint and drawdown equation.* In various applications, in particular in financial mathematics, one is interested in processes which remain above a (given) function w of their running maximum. The purpose of this section is to show that Azéma–Yor processes provide a direct answer to this problem when the underlying process X is positive. The following notion will be central throughout the rest of the paper.

DEFINITION 3.2. Given a function w, we say that a càdlàg process (M_t) satisfies *w-drawdown* (w-DD) constraint up to the (stopping) time ζ, if $\min\{M_{t-}, M_t\} > w(\overline{M}_t)$ for all $0 \le t < \zeta$ a.s.

We will see in Section 4 that for a local martingale M it suffices to impose $M_t > w(\overline{M}_t)$ in the above definition.

Azéma–Yor processes, $Y = M^U(X)$ defined from a *positive* max-continuous semimartingale X and function $U \in \mathcal{U}_m^+$ provide an example of such processes with DD-constraint function w defined from U and $V = U^{-1}$ by

$$w(y) = h(V(y)) = y - V(y)/v(y) \qquad \text{where } h(x) = U(x) - xu(x). \tag{9}$$

Indeed, thanks to the positivity of X and u we have

$$\begin{aligned} \min\{Y_{t-}, Y_t\} &= U(\overline{X}_t) - u(\overline{X}_t)\overline{X}_t + u(\overline{X}_t) \cdot \min\{X_{t-}, X_t\} \\ &> U(\overline{X}_t) - u(\overline{X}_t)\overline{X}_t = h(\overline{X}_t) = w(\overline{Y}_t). \end{aligned} \tag{10}$$

The converse is more interesting. We show below that if we start with a given w then $M^U(X)$ satisfies the w-DD constraint for $U = V^{-1}$ and V given in (11) below. Furthermore, it turns out that all processes which satisfy a drawdown constraint are of this type. More precisely, given a max-continuous semimartingale Y satisfying the w-DD constraint we can find explicitly X such that Y is the Azéma–Yor process $M^U(X)$. Moreover, the first instant Y violates the drawdown constraint is precisely the first hitting time to zero of X. For a precise statement we need to introduce the set of admissible functions w.

DEFINITION 3.3. We say that $w : [a^*, \infty] \to \mathbb{R}$ is a *drawdown function* if it is nondecreasing and there exists $r_w \le \infty$ such that $y - w(y) > 0$ is locally bounded and locally bounded away from zero on $[a^*, r_w)$ and $w(y) = y$ for $y \ge r_w$.

We impose w nondecreasing as it is intuitive for applications. It will also arise naturally in Section 4. We introduced here r_w as it gives a convenient way to stop the process upon hitting a given level and again it will be used in Section 4. If a drawdown function w is defined on $[a^*, \infty)$, then we put $w(\infty) = \lim_{y\uparrow\infty} w(y)$, and the above definition requires that $w(\infty) = \infty$. In fact for the results in this section it is not necessary to require any monotonicity from w or that $w(\infty) = \infty$, we comment this below.

We let $\tau_0(X) = \tau_0^X = \inf\{t : \min\{X_{t-}, X_t\} \le 0\}$ and note that when X is nonnegative then $X_{\tau_0^X} \ge 0$ on the set $\{\tau_0^X < \infty\}$. Further let $\zeta_w^Y = \inf\{t : \min\{Y_{t-}, Y_t\} \le w(\overline{Y}_t)\}$. As mentioned before, definitions of both τ_0 and ζ_w simplify for local martingales (see Lemma 4.1 and Corollary 4.2 in Section 4).

THEOREM 3.4. *Consider a drawdown function w of Definition 3.3. Then the solution V of the ODE (9) with $V(a^*) = a > 0$ is given as*

$$V(y) = a \exp\left(\int_{a^*}^{y} \frac{1}{s - w(s)}\, ds\right), \qquad y \ge a^*. \tag{11}$$

For a nonnegative max-continuous semimartingale (X_t), $X_0 = a$, and $\zeta := \tau_0(X) \wedge \tau^{V(r_w)}(X)$ the drawdown equation

$$dY_t = \big(Y_{t_-} - w(\overline{Y}_t)\big)\frac{dX_t}{X_{t_-}}, \qquad t < \zeta, \tag{12}$$

has a strong, pathwise unique, max-continuous solution which satisfies w-DD constraint on $[0, \zeta)$ and $Y_0 = a^*$, given by $Y_t = M_t^U(X)$, where U is the inverse of V. We have $\zeta_w^Y = \zeta$ and further $Y_{\zeta_w^Y} = w(\overline{Y}_{\zeta_w^Y})$ on $\{X_\zeta \in \{0, V(r_w)\}\}$.

Conversely, given (Y_t) a max-continuous semimartingale satisfying w-DD constraint up to $\zeta := \zeta_w^Y$, with $Y_0 = a^*$, there exists a pathwise unique max-continuous semimartingale $(X_t : t < \zeta)$, $X_0 = a$, which solves (12). X may be deduced from Y by the Azéma–Yor bijection $X_t = M_t^V(Y)$ and $\zeta = \tau_0(X) \wedge \tau^{V(r_w)}(X)$.

REMARK 3.5. Naturally $V(y) \equiv \infty$ for $y > r_w$. However, $V(r_w)$ could be either finite or infinite and consequently $\tau^{V(r_w)}(X)$ can be either a hitting time of a finite level or the explosion time for X.

Observe that $\{X_\zeta \in \{0, V(r_w)\}\}$ could be larger than $\{\zeta < \infty\}$. This will be the case in Section 4 where $X_t \to 0$ as $t \to \infty$ and in fact $X_\zeta \in \{0, V(r_w)\}$ a.s. Naturally, we also have $Y_{\zeta_w^Y -} = w(\overline{Y}_{\zeta_w^Y})$ on $\{X_{\zeta -} \in \{0, V(r_w)\}\}$. Note also that in the converse part of the theorem we could have $Y_\zeta < w(\overline{Y}_\zeta)$ which would correspond to $X_\zeta < 0$.

REMARK 3.6. It will be clear from the proof that the theorem holds without assuming any monotonicity on w or that $w(\infty)$ is defined and equal to ∞. The only changes are $Y_{\zeta_w^Y} = w(\overline{Y}_{\zeta_w^Y})$ on $\{X_\zeta = V(r_w)\}$ if and only if $w(r_w) = r_w$ and if $V(\infty) < \infty$ then Y explodes at $\tau_X^{V(\infty)}$.

PROOF OF THEOREM 3.4. Expression for V in terms of w follows as $v(y) = V(y)/(y - w(y))$. Note that $V(\infty) = \infty$. Hence, for $t < \zeta$, $Y_t = M_t^U(X)$ is well defined, and recall from Corollary 2.4 that $X_t = M_t^V(Y)$ and $\overline{X}_t = V(\overline{Y}_t)$. Direct computation yields

$$Y_{t-} - w(\overline{Y}_t) = Y_{t-} - U(\overline{X}_t) + u(\overline{X}_t)\overline{X}_t = u(\overline{X}_t)X_{t-}.$$

Thanks to the positivity of u and X and X_- on $t < \zeta$, we have that $Y_{t-}, Y_t > w(\overline{Y}_t)$ and it follows from (3) that $Y = M^U(X)$ solves (12).

Now consider any max-continuous solution Y of (12), $\min\{Y_{t-}, Y_t\} > w(Y_t)$ for $t < \zeta$. Then, using (2) and (3), we have

$$\frac{dY_t}{Y_{t_-} - w(\overline{Y}_{t_-})} = \frac{v(\overline{Y}_t)}{M_{t_-}^V(Y)}\, dY_t = \frac{dM_t^V(Y)}{M_{t_-}^V(Y)}.$$

Since Y is solution of (12), X and $M^V(Y)$ have the same relative stochastic differential and the same initial condition. Then, there are undistinguishable processes and Corollary 2.4 yields $Y_t = M_t^U(X)$.

Finally, when $X_\zeta = 0$ [resp., $X_\zeta = V(r_w)$] we have $Y_\zeta = U(\overline{X}_\zeta) - u(\overline{X}_\zeta)\overline{X}_\zeta = w(V(\overline{X}_\zeta)) = w(\overline{Y}_\zeta)$ [resp., $Y_\zeta = r_w = \overline{Y}_\zeta = w(\overline{Y}_\zeta)$] and $\zeta = \zeta_w^Y$. If $X_\zeta \notin \{0, V(r_w)\}$, then $X_{\zeta -} = 0$ or $\zeta = \infty$, and in both cases $\zeta = \zeta_w^Y$.

Consider now the second part of the theorem. We can rewrite (12) as

$$\frac{dY_t}{Y_{t_-} - w(\overline{Y}_t)} = \frac{dX_t}{X_{t_-}}, \qquad t < \zeta. \tag{13}$$

This equation defines without ambiguity a positive process X starting from $X_0 = a > 0$. By assumption on w, the solution V of (9) is a positive finite increasing function on $[a^*, r_w)$, $V(y)/v(y) = y - w(y)$. Put $\widehat{X}_t = M_t^V(Y)$, and observe that the differential properties of V imply that $\widehat{X}_t = v(\overline{Y}_t)(Y_t - w(\overline{Y}_t)) > 0$, for $t < \zeta$. Then, the stochastic differential of $M_t^V(Y) = \widehat{X}_t$ is

$$d\widehat{X}_t = v(\overline{Y}_t)\, dY_t = \widehat{X}_{t-}\big(Y_{t-} - w(\overline{Y}_t)\big)^{-1}\, dY_t,$$

and hence both $\widehat{X}$ and X are solutions of the same stochastic differential equation and have the same initial conditions. So, they are undistinguishable processes. Identification of ζ follows as previously. $\square$

Naturally, since $Y = M^U(X)$ solves both the Bachelier equation (8) and the drawdown equation (12) these equations are equivalent. We phrase this as a corollary in the case $\zeta = \infty$ a.s. in (12).

COROLLARY 3.7. *Let $(X_t : t \geq 0)$, $X_0 = a$, be a positive max-continuous semimartingale, $\tau_0^X = \infty$ a.s., and φ, V as in Theorem 3.1. Then, the Bachelier equation (8) is equivalent to the drawdown equation (12) where w and V are linked via (9) or equivalently via (11).*

The drawdown equation (12) was solved previously by Cvitanić and Karatzas [9] for $w(y) = \gamma y$, $\gamma \in (0, 1)$, and recently by Elie and Touzi [13]. The use of Azéma–Yor processes simplifies considerably the proof and allows for a general w and X. We have shown that this equation has a unique strong solution and is equivalent to the Bachelier equation.

Note that we assumed X is positive. The quantity dX_t/X_{t-} has a natural interpretation as the differential of the stochastic logarithm of X. In various applications, such as financial mathematics, this logarithm process is often given directly since X is defined as a stochastic exponential in the first place.

An illustrative example. Let X be a positive max-continuous semimartingale such that $X_0 = 1$. Let U be the power utility function defined on $\mathbb{R}^+$ by $U(x) = \frac{1}{1-\gamma} x^{1-\gamma}$ with $0 < \gamma < 1$ and $u(x) = x^{-\gamma}$ its derivative. The inverse function V of U is $V(y) = ((1-\gamma)y)^{1/(1-\gamma)}$ and its derivative is $v(y) = ((1-\gamma)y)^{\gamma/(1-\gamma)}$.

Then the (power) Azéma–Yor process is

$$M_t^U(X) = Y_t = \frac{1}{1-\gamma}(\overline{X}_t)^{1-\gamma}\left(\gamma + (1-\gamma)\frac{X_t}{\overline{X}_t}\right) = \overline{Y}_t\left(\gamma + (1-\gamma)\frac{X_t}{\overline{X}_t}\right).$$

Since X is positive, $Y_t > w(\overline{Y}_t) = \gamma \overline{Y}_t$. The drawdown function w is the linear one, $w(y) = \gamma y$.

The process (Y_t) is a semimartingale (local martingale if X is a local martingale) starting from $Y_0 = 1$, and staying in the interval $[\gamma \overline{Y}_t, \overline{Y}_t]$. Since the power function U is concave, we also have an other floor process $Z_t = U(X_t)$. Both processes Z_t and $\gamma \overline{Y}_t = \gamma \overline{Z}_t$ are dominated by Y_t. They are not comparable in the sense that in general at time t either one of them can be greater. We study floor process Z in more detail in Section 5.3.

The Bachelier–drawdown equation (8)–(12) becomes here

$$
\begin{aligned}
dY_t &= \overline{X}_t^{-\gamma}\, dX_t = \big((1-\gamma)\overline{Y}_t\big)^{-\gamma/(1-\gamma)}\, dX_t \\
&= (Y_{t-} - \gamma \overline{Y}_t)\frac{dX_t}{X_{t-}}.
\end{aligned}
\tag{14}
$$

As noted above, this equation, for a class of processes X, was studied in Cvitanić and Karatzas [9]. Furthermore, in [9] authors in fact introduced processes $M^U(X)$ where U is the a power utility function, and used them to solve the portfolio optimization problem with drawdown constraint of Grossman and Zhou [17] (see also [13]). Using our methods we can simplify and generalize their results and show that the portfolio optimization problem with drawdown constraint, for a general utility function and a general drawdown function, is equivalent to an unconstrained portfolio optimization problem with a modified utility function. We develop these ideas in a separate paper.

4. Setup driven by a nonnegative local martingale converging to zero. In the previous section we investigated Azéma–Yor processes built from a nonnegative semimartingale as solutions to the drawdown equation (12). We specialize now further and study in detail Azéma–Yor processes associated with $X = N$, a nonnegative local martingale converging to zero at infinity. The maximum of N has a universal law which, together with $N_\infty = 0$, allows to write Azéma–Yor martingales explicitly from their terminal values (see Sections 4.1 and 4.2). Our study exploits tail quantiles of probability measures and is intimately linked with the average value at risk and the Hardy–Littlewood transform of a measure, as explored in Section 4.3. Finally, combining these results with Theorem 3.4, we construct Azéma–Yor martingales with prescribed terminal distributions and in particular obtain the Azéma–Yor [3] solution of the Skorokhod embedding problem.

4.1. *Universal properties of $X = N$.* A nonnegative local martingale (N_t) is a supermartingale and it is a (true) martingale if and only if $\mathbb{E}N_t = \mathbb{E}N_0$, $t \geq 0$. We also have that if N_t or N_{t-} touch zero then N_t remains in zero (see, e.g., Dellacherie and Meyer [10], Theorem VI.17).

LEMMA 4.1. *Consider a nonnegative local martingale (N_t) with $N_{0-} := N_0 > 0$. Then*

$$\tau_0(N) = \inf\{t : N_t = 0 \textit{ or } N_{t-} = 0\} = \inf\{t : N_t = 0\} \tag{15}$$

and $N_u \equiv 0$, $u \geq \tau_0(N)$.

This yields an immediate simplification of the w-DD condition. In fact in Definition 3.2, and the definition of ζ_w^Y before Theorem 3.4, it suffices to compare $w(\overline{Y}_t)$ with Y_t instead of Y_t *and* Y_{t-}.

COROLLARY 4.2. *Let w be a drawdown function of Definition 3.2 and (Y_t) a max-continuous local martingale with $Y_\zeta = w(\overline{Y}_\zeta)$ a.s. on $\{\zeta < \infty\}$, where $\zeta = \inf\{t : Y_t \le w(\overline{Y}_t)\}$. Then Y satisfies w-DD condition up to time $\zeta_w^Y = \zeta$.*

PROOF. Assume $r_w = \infty$ and let $N_t = M_t^V(Y)$ where V is given via (11). Using (9) and (10), similarly as in the proof of Theorem 3.4, and Definition 2.1, $(N_t : t \le \zeta)$ is a nonnegative max-continuous local martingale and $\zeta = \inf\{t : N_t = 0\}$. Using (15) we have $\zeta = \tau_0(N)$, and our assumptions also give $N_\zeta = 0$ on $\{\zeta < \infty\}$. It follows from Theorem 3.4 that $Y_t = M^U(X)_t$, $U = V^{-1}$ satisfies the w-DD constraint up to ζ and $\zeta = \tau_0(N) = \zeta_w^Y$. For the case $r_w < \infty$ it suffices to note that all processes are max-continuous and hence the first hitting times for $\overline{N}_t$ and $\overline{N}_{t-}$ are equal. $\square$

Throughout this and following sections, we assume that

$$(N_t : t \ge 0) \text{ is a nonnegative max-continuous local martingale,}\\ N_t \xrightarrow[t\to\infty]{} 0 \text{ a.s.} \tag{16}$$

Note that in particular $\inf\{t \ge 0 : \overline{N}_t = \overline{N}_\infty\} < \infty$ a.s.

We recall some well-known results on the distribution of the maximum of N (see Exercice III.3.12 in Revuz–Yor [30]). We assume that N_0 is a constant. If N_0 is random all results should be read conditionally on $\mathcal{F}_0$.

PROPOSITION 4.3. *Consider (N_t) satisfying (16) with $N_0 > 0$ a constant.*

(a) *The random variable $N_0/\overline{N}_\infty$ is uniformly distributed on* $[0, 1]$.

(b) *The same result holds for the conditional distribution in the following sense: let $\overline{N}_{t,\infty} = \sup_{t\le u\le\infty} N_u$ then*

$$\mathbb{P}(K > \overline{N}_{t,\infty} | \mathcal{F}_t) = (1 - N_t/K)^+;$$

that is, $\overline{N}_{t,\infty}$ has the same $\mathcal{F}_t$-conditional distribution as N_t/ξ where ξ is an independent uniform variable on $[0, 1]$.

(c) *Let $\zeta = \tau_0(N) \wedge \tau^b(N) = \inf\{t : N_t \notin (0, b)\}$, $b > N_0$. $(N_{t\wedge\zeta})$ is a bounded martingale and $N_\zeta \in \{0, b\}$. Furthermore, $\overline{N}_\zeta = \overline{N}_\infty \wedge b$ is distributed as $(N_0/\xi) \wedge b$, where ξ is uniformly distributed on* $[0, 1]$.

REMARK. (a) Given the event $\{\overline{N}_\zeta < b\} = \{N_\zeta = 0\}$, $N_0/\overline{N}_\zeta$ is uniformly distributed on $(N_0/b, 1]$. The probability of the event $\{\overline{N}_\zeta = b\}$ is N_0/b.

(b) Any nonnegative martingale N stopped at ζ, with $N_\zeta \in \{0, b\}$ a.s., may be extended into a local martingale (still denoted by N) satisfying (16), by putting $N_t := N_\zeta + \mathbf{1}_{\{\overline{N}_\zeta = b\}}(N'_t - N'_\zeta)$, $t > \zeta$, where N' is another local martingale as in (16).

PROOF OF PROPOSITION 4.3. (a) Let us consider the Azéma–Yor martingale associated with (N_t) and the function $U(x) = (K - x)^+$, where $K \geq N_0$ is a fixed real. Thanks to the positivity of (N_t), the martingale $M^U(N)$ is bounded by K,

$$0 \leq M_t^U(N) = (K - \overline{N}_t)^+ + \mathbf{1}_{\{K > \overline{N}_t\}}(\overline{N}_t - N_t) = \mathbf{1}_{\{K > \overline{N}_t\}}(K - N_t) \leq K.$$

So $M_t^U(N)$ is a uniformly integrable martingale, and $\mathbb{E}M_\infty^U(N) = M_0^U(N)$. Since $N_t \to 0$ as $t \to \infty$, $\overline{N}_\infty < \infty$ and we have $M_\infty^U(N) = K\mathbf{1}_{K > \overline{N}_\infty}$ and the previous equality can be written as $K\mathbb{P}(K > \overline{N}_\infty) = K - N_0$, or equivalently $\mathbb{P}(\frac{N_0}{\overline{N}_\infty} \leq \frac{N_0}{K}) = \frac{N_0}{K}$. That is exactly the desired result.

(b) This result is the conditional version of the previous one. The reference process is now the process $(N_{t+h} : h \geq 0)$ adapted to the filtration $\mathcal{F}_{t+h}$, local martingale for the conditional probability measure $\mathbb{P}(\cdot|\mathcal{F}_t)$.

(c) From (15) and since N is nonnegative and max-continuous it follows that $\tau_0(N) \wedge \tau^b(N) = \inf\{t : N_t \notin (0, b)\}$ and that $N_\zeta = b$, or 0. Then, we have that $\overline{N}_\zeta = \overline{N}_\infty \wedge b$ since when $N_\zeta = b$, the maximum $\overline{N}_\zeta$ is also equal to b. □

Remark about last passage times. Recently, for a continuous local martingale N, Madan, Roynette and Yor [24] have interpreted the event $\{K > \overline{N}_{T,\infty}\}$ in terms of the last passage time $g_K(N)$ over the level K, as $\{K > \overline{N}_{T,\infty}\} = \{g_K(N) < T\}$. Our last proposition yields immediately their result: the normalized put pay-off is the conditional probability of $\{g_K(N) < T\} : (1 - N_T/K)^+ = \mathbb{P}(g_K(N) < T|\mathcal{F}_T)$. In particular we obtain the whole dynamics of the put prices,

$$\mathbb{E}[(K - N_T)^+|\mathcal{F}_t] = K\mathbb{P}\big(g_K(N) < T|\mathcal{F}_t\big), \qquad t \leq T,$$

and the initial prices ($t = 0$) are deduced from the distribution of g_K. In the geometrical Brownian motion framework with $N_0 = 1$, the Black–Scholes formula just computes the distribution of $g_1(N)$ as $\mathbb{P}(g_1 < t) = \mathcal{N}(\sqrt{t}/2) - \mathcal{N}(-\sqrt{t}/2) = \mathbb{P}(4B_1^2 \leq t)$, where B_1 is a standard Gaussian random variable and $\mathcal{N}(x) = \mathbb{P}(B_1 \leq x)$ the Gaussian distribution function (see Profeta, Roynette and Yor [29] for a detailed study).

Financial framework. Assume S to be a max-continuous nonnegative submartingale whose instantaneous return by time unit is an adapted process $\lambda_t \geq 0$ defined on a filtered probability spaced $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$. For instance, S is the current price of a stock under the risk neutral probability in a financial market with short rate λ_t. Put differently, $\tilde{S}_t = \exp(-\int_0^t \lambda_s\,ds)S_t$ is an $(\mathcal{F}_t)$-martingale. We assume that $\int_0^\infty \lambda_s\,ds = \infty$ a.s. Let ζ be an additional r.v. with conditional tail function $\mathbb{P}(\zeta \geq t|\mathcal{F}_\infty) = \exp(-\int_0^t \lambda_s\,ds)$. Then $X_t = S_t\mathbf{1}_{t<\zeta}$ is a positive martingale with negative jump to zero at time ζ with respect to the enlarged filtration $\mathcal{G}_t = \sigma(\mathcal{F}_t, \zeta \wedge t)$. Since the $\mathbb{G}$-martingale X goes to zero at ∞, the random variable $\overline{X}_\zeta = \overline{S}_\zeta$ is distributed as S_0/ξ, where ξ is uniformly distributed on [0, 1]. In particular, for any bounded function h

$$\mathbb{E}[h(\overline{S}_\zeta)] = \mathbb{E}\left[\int_0^\infty e^{-\int_0^\alpha \lambda_s\,ds} h(\overline{S}_\alpha)\lambda_\alpha\,d\alpha\right] = \int_0^1 h(S_0/y)\,dy.$$

In consequence we have access to the law of the properly discounted maximum of the positive submartingale S. We stress that this is in contrast to the more usual setting when one only has access to the maximum of the discounted price process (cf. Grossman and Zhou [17]). We could also derive a conditional version of the equation above representing $U(S_t)$ as a potential of the future maximum $\overline{S}_{t,u}$. Such representation find natural applications in financial mathematics (see Bank and El Karoui [6]).

4.2. *Azéma–Yor martingales with given terminal values.* We describe now all local martingales whose terminal values are Borel functions of the maximum of some nonnegative local martingale. This will be used in subsequent sections, in particular to construct Azéma–Yor martingales with given terminal distribution and solve the Skorokhod embedding problem. We start with a simple lemma about solutions to a particular ODE.

LEMMA 4.4. *Let h be a locally bounded Borel function defined on $\mathbb{R}^+$, such that $h(x)/x^2$ is integrable away from zero. Let U be the solution of the ordinary differential equation* (*ODE*),

$$\text{(17)}\quad \forall x > 0 \qquad U(x) - xU'(x) = h(x) \qquad \text{such that } \lim_{x\to\infty} U(x)/x = 0.$$

(a) *The solution U is given by*

$$\text{(18)}\qquad U(x) = x\int_x^\infty \frac{h(s)}{s^2}\,ds = \int_0^1 h\left(\frac{x}{s}\right)ds, \qquad x > 0.$$

(b) *Let $h_b(x) := h(x \wedge b)$ be constant on (b,∞). The associated solution $U_b(x) = \int_0^1 h(\frac{x}{s} \wedge b)\,ds = U_b(x \wedge b)$ is constant on (b,∞), and $U_b(x) = h_b(x) = h(b)$.*

(c) *Let $h(m,x) = h(x \vee m)$ be constant on $(0,m)$. The associated solution $U(m,x)$ is affine $U(m,x) = U(m) - U'(m)(m-x)$ for $x \in (0,m)$.*

REMARK 4.5. We considered here U on $(0,\infty)$ but naturally if h is only defined for $x > a > 0$ then we consider U also only for $x > a > 0$. Note that to define U_b it suffices to have a locally integrable h defined on $(0,b]$. We then put $h(x) = h(b)$, $x > b$.

PROOF OF LEMMA 4.4. Formula (18) is easy to obtain using the transformation $(U(x)/x)' = -h(x)/x^2$ and the asymptotic condition in (17). Both (b) and (c) follow simply from (18). $\square$

This analytical lemma allow us to characterize Azéma–Yor martingales from their terminal values. This extends, in more detail, the ideas presented in El Karoui and Meziou [12], Proposition 5.8.

PROPOSITION 4.6. *Consider* (N_t) *satisfying* (16) *with* $N_0 > 0$ *a constant.*

(a) *Let* h *be a Borel function such that* $h(x)/x^2$ *is integrable away from* 0, *and* U *the solution of the ODE* (17) *given via* (18). *Then* $h(\overline{N}_\infty)$ *is an integrable random variable and the closed martingale* $\mathbb{E}(h(\overline{N}_\infty)|\mathcal{F}_t)$, $t \geq 0$, *is the Azéma–Yor martingale* $M^U(N)$.

(b) *For a function* U *with locally bounded derivative* U' *and with* $U(x)/x \to 0$ *as* $x \to \infty$, *the Azéma–Yor local martingale* $M^U(N)$ *is a uniformly integrable martingale if and only if* $h(x)/x^2$ *is integrable away from zero, where* h *is now defined via* (17).

PROOF. We start with the proof of (a). We have

$$\mathbb{E}|h(\overline{N}_\infty)| = \int_0^1 |h(N_0/s)|\, ds = N_0 \int_{N_0}^\infty |h(s)|/s^2\, ds < \infty,$$

since we assumed that $h(x)/x^2$ is integrable away from 0. To study the martingale $H_t = \mathbb{E}(h(\overline{N}_\infty)|\mathcal{F}_t)$, we use that $\overline{N}_\infty = \overline{N}_t \vee \overline{N}_{t,\infty}$. From Proposition 4.3, the running maximum $\overline{N}_{t,\infty}$, conditionally on $\mathcal{F}_t$, is distributed as N_t/ξ, for an independent r.v. ξ uniform on [0, 1]. The martingale H_t is given by the following closed formula $H_t = \mathbb{E}(h(\overline{N}_t \vee (N_t/\xi))|\mathcal{F}_t)$, that is,

$$H_t = \int_0^1 h\big(\overline{N}_t \vee (N_t/s)\big)\, ds = U(\overline{N}_t, N_t) = U(\overline{N}_t) - U'(\overline{N}_t)(\overline{N}_t - N_t),$$

where in the last equalities, we have used (c) in Lemma 4.4.

Given (a), to prove part (b) it suffices to observe that $M_t^U(N) \to h(\overline{N}_\infty)$ a.s. and hence integrability of $h(\overline{N}_\infty)$, that is, integrability of $h(x)/x^2$ away from zero, is necessary for uniform integrability of $M^U(N)$. □

REMARK 4.7. It is not necessary to assume that N_0 is a constant in Proposition 4.6. However, if N_0 is random we have to further assume that $\mathbb{E}\int_0^1 |h(N_0/s)|\, ds = \int_1^\infty \mathbb{E}|h(xN_0)| \frac{dx}{x^2} < \infty$. This holds, for example, if N_0 is integrable and $N_0 > \varepsilon > 0$ a.s. We can apply the same reasoning to the process $(N_{t+u} : u \geq 0)$ to see that if $\mathbb{E}\int_0^1 |h(N_t/s)|\, ds < \infty$ then $U(N_t) = \mathbb{E}(h(\overline{N}_{t,\infty})|\mathcal{F}_t)$.

Finally, we note that similar consideration as in (a) above were independently made in Nikeghbali and Yor [25].

We stress that the boundary condition $U(x)/x \to 0$ as $x \to \infty$ for (17) is essential in part (a). Indeed, consider $N_t = 1/Z_t$ the inverse of a three-dimensional Bessel process. Note that N_t satisfies our hypothesis, and it is well known that N_t is a strict local martingale (cf. Exercise V.2.13 in Revuz and Yor [30]). Then for

$U(x)=x$ we have $M^U_t(N)=N_t$ is also a strict local martingale, but obviously we have $U(x)-U'(x)x=0$.

Observe that $\mathbb{P}(N_\zeta\in\{0,b\})=1$ if $\zeta=\inf\{t\geq 0: N_t\notin(0,b)\}$. Then, if h is constant on $[b,\infty)$, then $h(\overline{N}_\infty)=h(\overline{N}_\zeta)$, and the closed martingale $\mathbb{E}(h(\overline{N}_\zeta)|\mathcal{F}_{t\wedge\zeta})$, $t\geq 0$, is the Azéma–Yor martingale $M^{U_b}_t(N)=M^{U_b}_{t\wedge\zeta}(N)$, where U_b the solution of the ODE (17) given in point (b) of Lemma 4.4.

As shown in Sections 2 and 3, Azéma–Yor processes $Y=M^U(N)$ generated by an increasing function U have very nice properties based on the characterization of their maximum as $\overline{Y}=U(\overline{N})$. In particular, from Theorem 3.4, the process Y satisfies a DD-constraint and can also be characterized from its terminal value. Recall Definitions 3.2, 3.3 and the stopping time ζ^Y_w defined before Theorem 3.4.

PROPOSITION 4.8. *Let h be a right-continuous nondecreasing function such that $h(x)/x^2$ is integrable away from* 0, *and put $b=\inf\{x: h(y)=h(x)\ \forall y\geq x\}$.*

(a) *The solution U of the ODE* (17) *is then a continuous strictly increasing concave function on* $(0,b)$ *and constant and equal to $h(b)$ on* (b,∞).

(b) *Let V be the inverse of U with $V(U(b))=b$. Function $w(y)=h(V(y))$ given by* (9) *for $y<U(b)$ and by $w(y)=y$ for $y\geq U(b)$, is a right-continuous drawdown function and $r_w=U(b)=h(b)$.*

(c) *Consider* (N_t) *satisfying* (16) *with $N_0>0$ a constant. The uniformly integrable martingale $Y_t=M^U_t(N)=\mathbb{E}[h(\overline{N}_\infty)|\mathcal{F}_t]$ satisfies w-DD constraint. Furthermore, $Y_t=Y_{t\wedge\zeta^Y_w}$, $Y_{\zeta^Y_w}=w(\overline{Y}_{\zeta^Y_w})$ a.s. and $\zeta^Y_w=\inf\{t: N_t\notin(0,b)\}$.*

Conversely, let w be a right-continuous drawdown function, with functions V,U,h satisfying (a) *and* (b). *Then any uniformly integrable martingale Y, satisfying the w-DD constraint and $Y_{\zeta^Y_w}=w(\overline{Y}_{\zeta^Y_w})$ a.s., is an Azéma–Yor martingale $M^U(N)$ for some (N_t) satisfying* (16) *with $N_0=V(Y_0)>0$ and such that $N_{t\wedge\zeta^Y_w}=M^V_{t\wedge\zeta^Y_w}(Y)$ and $\zeta^Y_w=\inf\{t: N_t\notin(0,V(r_w)\}$.*

REMARK 4.9. Note that h, and consequently U, need to be defined only for $x\geq N_0$. Then $V(y)$ is defined for $U(N_0)\leq y\leq U(b)$ with $V(U(b))=b\leq\infty$ and the drawdown function $w(y)$ is defined for $y\geq U(N_0)$.

A solution U of the ODE (17) is strictly increasing if and only if $U>h$; however, only increasing and concave solutions are easy to characterize.

PROOF OF PROPOSITION 4.8. (a) When h is nondecreasing, from (17) and (18) it is clear that U is strictly increasing until that h becomes constant, and constant after that. If h is differentiable, concavity of U follows since $-xU''(x)=h'(x)$. The general case follows by regularization or can be checked directly using (18) which yields to $U'(x)=\int_x^\infty(h(s)-h(x))/s^2\,ds=\int_0^\infty(h(s)-h(x))^+/s^2\,ds$.

(b) In consequence, V is increasing and convex on $[U(0), U(b))$, and hence by (9) $w(y)$ is increasing and $w(y) < y$ for $y \in (U(0), U(b))$. We thus have $r_w = U(b)$ but note that we could have $w(U(b)-)$ both less then or equal to $U(b)$. Integrability properties of w in Definition 3.3 follow since V and U are well defined and we conclude that w is a drawdown function. Right-continuity of w follows from right-continuity of h. More precisely, from (17), $u = U'$ is right continuous, and hence also $V'(y) = 1/u(V(y))$ is right-continuous and nondecreasing.

(c) Identification of Y is given in part (a) of Proposition 4.6. The rest follows from Lemma 4.1, Theorem 3.4 and the fact that $N_\zeta \in \{0, b\}$ a.s. for $\zeta = \inf\{t : N_t \notin (0, b)\}$ upon noting that $V(r_w) = b$. $\square$

4.3. *On relations between* AVaR_μ, *Hardy–Littlewood transform and tail quantiles*. In this section we present results about probability measures, their tail quantile function, the average value at risk and the Hardy–Littlewood transform. The presentation is greatly simplified using tail quantiles of measure.

The notation and quantities now introduced will be used throughout the rest of the paper. For μ a probability measure on $\mathbb{R}$ we denote l_μ, r_μ, respectively, the lower and upper bound of the support of μ. We let $\overline{\mu}(x) = \mu([x, \infty))$ and $\overline{q}_\mu : (0, 1] \to \mathbb{R} \cup \{\infty\}$ be the tail quantile function defined as the left-continuous inverse of $\overline{\mu}$, $\overline{q}_\mu(\lambda) := \inf\{x \in \mathbb{R} : \overline{\mu}(x) < \lambda\}$. When $\overline{q}_\mu(\lambda)$ is a point of continuity of $\overline{\mu}$, then $\overline{\mu}(\overline{q}_\mu(\lambda)) = \lambda$, whereas if not, $\overline{\mu}(\overline{q}_\mu(\lambda)+) < \lambda \le \overline{\mu}(\overline{q}_\mu(\lambda))$. In particular, if $\overline{\mu}(r_\mu) > 0$, $r_\mu = \overline{q}(0+)$ is a jump of $\overline{\mu}$ and $r_\mu = \overline{q}(0+) = \overline{q}(\lambda)$, if $0 < \lambda \le \overline{\mu}(r_\mu)$.

We write $X \sim \mu$ to denote that X has distribution μ and recall that $\overline{q}_\mu(\xi) \sim \mu$ for ξ uniformly distributed on $[0, 1]$.

Assume $\int_{\mathbb{R}} |s|\mu(ds) < \infty$ and let $m_\mu = \int_{\mathbb{R}} s\mu(ds)$. We define call function[3] C_μ and barycenter function ψ_μ by

$$C_\mu(K) = \int_{\mathbb{R}} (s - K)^+ \mu(ds), \qquad \psi_\mu(x) = \frac{1}{\overline{\mu}(x)} \int_{[x,\infty)} s\mu(ds), \tag{19}$$

where $K \in \mathbb{R}, x < r_\mu$. We put $\psi_\mu(x) = x$ for $x \ge r_\mu$.

Finally, we also introduce the *average value at risk* at the level $\lambda \in (0, 1]$, given by

$$\mathrm{AVaR}_\mu(\lambda) = \frac{1}{\lambda} \int_0^\lambda \overline{q}_\mu(\eta)\, d\eta. \tag{20}$$

Observe that AVaR_μ is equal to $r_\mu = \overline{q}_\mu(0+)$ on $(0, \overline{\mu}(r_\mu)]$, is strictly decreasing on $(\overline{\mu}(r_\mu), 1)$ since its derivative is then equal to $\frac{1}{\lambda^2} \int_0^\lambda (\overline{q}_\mu(\lambda) - \overline{q}_\mu(\eta))\, d\eta < 0$, and $\mathrm{AVaR}_\mu(1) = m_\mu$.

[3] This denomination is used in financial literature while the actuarial literature uses rather the notion of stop-loss function (cf. [20]).

The average value at risk AVaR_μ is thus a quantile function of some probability measure μ^{HL} with support (m_μ, r_μ), which can be defined by

$$\mu^{\mathrm{HL}} \sim \mathrm{AVaR}_\mu(\xi), \qquad \xi \text{ uniform on } [0,1]. \tag{21}$$

This distribution, called the *Hardy–Littlewood transform* of μ has been intensively studied by many authors, starting with the famous paper of Hardy and Littlewood [18]. We will describe its prominent role in the study of distributions of maxima of martingales in Section 5 below. Recently Föllmer and Schied [14] studied properties of AVaR_μ as a coherent risk measure. Finally note that here μ is the law of losses (i.e., negative of gains) and some authors refer to $\mathrm{AVaR}_\mu(\lambda)$ as $\mathrm{AVaR}_\mu(1-\lambda)$.

As in [14], pages 179–182, page 408, Lemma A.22, it is easy to characterize the Fenchel transform of the concave function $\lambda\,\mathrm{AVaR}_\mu(\lambda)$ as the call function. From this property, we infer a nonclassical representation of the tail function $\overline{\mu}^{\mathrm{HL}}(y)$ as an infimum.

PROPOSITION 4.10. *Let μ be a probability measure on $\mathbb{R}$, $\int |s|\mu(ds) < \infty$.*

(a) *The average value at risk $\mathrm{AVaR}_\mu(\lambda)$ can be described as, $\lambda \in (0,1)$,*

$$\mathrm{AVaR}_\mu(\lambda) = \frac{1}{\lambda}C_\mu(\overline{q}_\mu(\lambda)) + \overline{q}_\mu(\lambda) = \frac{1}{\lambda}\inf_{K\in\mathbb{R}}\bigl(C_\mu(K) + \lambda K\bigr). \tag{22}$$

(b) *The call function is the Fenchel transform of $\lambda\,\mathrm{AVaR}_\mu(\lambda)$, so that*

$$C_\mu(K) = \sup_{\lambda\in(0,1)}\bigl(\lambda\,\mathrm{AVaR}_\mu(\lambda) - \lambda K\bigr), \qquad K \in \mathbb{R}. \tag{23}$$

(c) *The Hardy–Littlewood tail function $\overline{\mu}^{\mathrm{HL}}$ is given for any $y \in (m_\mu, r_\mu)$ by*

$$\overline{\mu}^{\mathrm{HL}}(y) = \inf_{z>0}\frac{1}{z}C_\mu(y-z). \tag{24}$$

(d) *The barycenter function and its right-continuous inverse are related to the average value at risk and Hardy–Littlewood tail function by*

$$\begin{aligned}\psi_\mu(x) &= \mathrm{AVaR}_\mu(\overline{\mu}(x)), \qquad x \le r_\mu,\\ \psi_\mu^{-1}(y) &= \overline{q}_\mu(\overline{\mu}^{\mathrm{HL}}(y)), \qquad y \in [m_\mu, r_\mu].\end{aligned} \tag{25}$$

REMARK 4.11. From (19) we have $\psi_\mu(x) = \mathbb{E}[X|X \ge x]$, where $X \sim \mu$. Then (25) gives $\mathrm{AVaR}_\mu(\lambda) = \mathbb{E}[X|X \ge \overline{q}_\mu(\lambda)]$, $d\overline{q}_\mu(\lambda)$-a.e., which justifies names *expected shortfall*, or *conditional value at risk* used for AVaR_μ.

PROOF OF PROPOSITION 4.10. We write $\overline{q} = \overline{q}_\mu$.

(a) The proof is based on the classical property, $\overline{q}(\xi) \sim \mu$, for ξ uniformly distributed on [0, 1]. Then

$$C_\mu(\overline{q}(\lambda)) = \int_0^1 \big(\overline{q}(\eta) - \overline{q}(\lambda)\big)^+ \, d\eta = \int_0^\lambda \big(\overline{q}(\eta) - \overline{q}(\lambda)\big) \, d\eta = \lambda\big(\mathrm{AVaR}_\mu(\lambda) - \overline{q}(\lambda)\big).$$

Moreover, the convex function $G_\lambda(K) := C_\mu(K) + \lambda K$ attains its minimum in K_λ such that $\overline{\mu}(K_\lambda) = \lambda$.

When $\overline{\mu}(\overline{q}(\lambda)) = \lambda$, $\overline{q}(\lambda)$ is a minimum of the function $G_\lambda(K)$, $\lambda\, \mathrm{AVaR}_\mu(\lambda) = G_\lambda(\overline{q}(\lambda))$, and (22) holds true.

If $\overline{\mu}(\overline{q}(\lambda)) > \lambda > \overline{\mu}(\overline{q}(\lambda)+)$, then μ has an atom in $x := \overline{q}(\lambda)$. G_λ has a minimum in x and G'_λ changes sign discontinuously in x. Then we see that $G_\lambda(\overline{q}(\lambda) = G_\lambda(x)$ is linear in $\lambda \in (\overline{\mu}(x+), \overline{\mu}(x))$.

(b) Convex duality for Fenchel transforms yields (23) from (22).

(c) Using (22) we have, for any $y > m_\mu$ and $\lambda \in (0, 1)$

$$\begin{aligned} \mathrm{AVaR}(\lambda) < y \quad &\Leftrightarrow \quad \exists K \text{ such that } y > \frac{C_\mu(K)}{\lambda} + K \\ &\Leftrightarrow \quad \exists K < y \text{ such that } \lambda > \frac{C_\mu(K)}{y - K} \\ &\Leftrightarrow \quad \lambda > \inf_{K<y} \frac{C_\mu(K)}{y - K}. \end{aligned} \tag{26}$$

The function $\inf_{K<y} \frac{C_\mu(K)}{y-K} = \inf_{z>0} \frac{1}{z} C_\mu(y - z)$ is decreasing and left continuous. We conclude that it is the left-continuous inverse function of $\mathrm{AVaR}_\mu(\lambda)$ which is $\overline{\mu}^{\mathrm{HL}}$.

(d) By definition, $\overline{\mu}(x)\, \mathrm{AVaR}_\mu(\overline{\mu}(x)) = \int_0^{\overline{\mu}(x)} \overline{q}_\mu(\eta)\, d\eta = \int_{[x,\infty)} s\mu(ds)$.

The right-continuous inverse $\psi_\mu^{-1}(y)$ of the nondecreasing left-continuous function ψ_μ is defined by $\psi_\mu^{-1}(y) = \sup\{x : \psi_\mu(x) \le y\} = \sup\{x : \mathrm{AVaR}_\mu(\overline{\mu}(x)) \le y\}$. Since, $\overline{\mu}^{\mathrm{HL}}$ is the left-continuous inverse of AVaR_μ, the following inequalities hold true for $y \in [m_\mu, r_\mu] : \psi_\mu^{-1}(y) = \sup\{x : \overline{\mu}(x) \ge \overline{\mu}^{\mathrm{HL}}(y)\} = \sup\{x : x \le \overline{q}(\overline{\mu}^{\mathrm{HL}}(y))\} = \overline{q}(\overline{\mu}^{\mathrm{HL}}(y))$. □

We now describe the relationship between μ, AVaR_μ, ψ_μ and μ^{HL} on one hand, and w_μ, solutions U_μ of (17) when $h(x) = \overline{q}_\mu(1/x)$ and the associated Azéma–Yor martingales $M^{U_\mu}(N)$ on the other hand. It turns out all these objects are intimately linked together in a rather elegant manner. Some of our descriptions below, in particular characterization of AVaR in (a), appear to be different from classical forms in the literature. We note that we start with μ and define h but equivalently we could start with a nondecresing right-continuous h and use $h(x) = \overline{q}_\mu(1/x)$ to define μ.

Recall Definitions 3.2, 3.3 and the stopping time ζ_w^Y defined before Theorem 3.4.

PROPOSITION 4.12. *Let μ be a probability measure on $\mathbb{R}$, $\int |s|\mu(ds) < \infty$.*

(a) *$U_\mu(x) := \mathrm{AVaR}_\mu(1/x)$ solves (17) with $h_\mu(x) = \overline{q}_\mu(1/x)$, $x \geq 1$, and $h_\mu(x)/x^2$ is integrable away from zero. In particular U_μ is given by (18) and $U_\mu(x) = U_\mu(x \wedge b_\mu)$ with $b_\mu = 1/\overline{\mu}(r_\mu)$. U_μ is concave and $V_\mu(y) = 1/\overline{\mu}^{\mathrm{HL}}(y)$ is the inverse function of U_μ.*

(b) *Let $w_\mu(y) = h_\mu(V_\mu(y)) = q_\mu(\overline{\mu}^{\mathrm{HL}}(y))$ be the function associated with μ by (9) for $y \in (m_\mu, r_\mu)$, and extended via $w_\mu(y) = y$ for $y \geq r_\mu$. Then w_μ is a drawdown function, $r_w = r_\mu$ and w_μ is the right-continuous inverse of the barycenter function ψ_μ. Furthermore, w_μ is the hyperbolic derivative of V_μ as defined by Kertz and Rösler [23].*

(c) *Let N satisfy (16) with $N_0 = 1$ and $Y_t = M_t^{U_\mu}(N)$. Then $Y_t \geq U_\mu(N_t)$, $Y_\infty = Y_{\zeta^Y_{w_\mu}} = \overline{q}_\mu(1/\overline{N}_\zeta)$ is distributed according to μ and $\overline{Y}_\infty = U_\mu(\overline{N}_\zeta) = \mathrm{AVaR}_\mu(1/\overline{N}_\zeta)$ is distributed according to μ^{HL}. Furthermore, the process (Y_t) is a uniformly integrable martingale which satisfies w_μ-DD constraint and $\zeta^Y_{w_\mu} = \inf\{t : N_t \notin (0, b_\mu)\} = \inf\{t : \psi_\mu(Y_t) \leq \overline{Y}_t\}$.*

The same properties hold true for any max-continuous uniformly integrable martingale Y, $Y_0 = U(1)$, satisfying the w_μ-DD constraint up to $\zeta = \zeta^Y_{w_\mu}$ and $Y_\zeta = w_\mu(\overline{Y}_\zeta)$ a.s.

PROOF. We write $\overline{q} = \overline{q}_\mu$, $r = r_\mu$, $b = b_\mu = 1/\overline{\mu}(r_\mu) = V_\mu(r_\mu-)$.

(a) From definition (20) we have $\mathrm{AVaR}_\mu(\lambda) = \int_0^1 \overline{q}(\lambda s)\, ds$ which is exactly formula (18). Note that in the case $\overline{\mu}(r) > 0$ we have $h(x) = h(b)$, $x \geq b$ with $b = 1/\overline{\mu}(r)$. U_μ is concave by Proposition 4.8. The rest follows since AVaR_μ is the tail quantile of μ^{HL} [see (21)].

(b) This follows by part (d) in Proposition 4.10 and the last statement follows from (11) and Theorem 4.3 in [23].

(c) We have $Y_t \geq U(N_t)$ from concavity of U_μ. The rest follows easily from points (a) and (b) above together with Proposition 4.8, properties of $\overline{q}$, universal law of $\overline{N}_\zeta$ given in point (c) of Proposition 4.3 and the definition of μ^{HL} in (21). $\square$

An illustrative example (*continued from Section* 3.2). We come back to the example with linear DD-constraint $w(y) = \gamma y$, $0 < \gamma < 1$, resulting from function $U(x) = \frac{1}{1-\gamma} x^{1-\gamma}$, $x \geq 1$. Using Proposition 4.12 we have $Y_\infty \sim \mu$ and $\overline{Y}_\infty \sim \mu^{\mathrm{HL}}$ which we can now easily describe. We have $\overline{\mu}^{\mathrm{HL}}(y) = 1/V(y) = ((1-\gamma)y)^{1/(\gamma-1)}$ for $y \geq m_\mu = \mathrm{AVaR}_\mu(1) = U(1) = \frac{1}{1-\gamma}$. In consequence, the random variable $\overline{Y}_\infty$ is distributed according to a *Pareto distribution*, with shape parameter $a = m_\mu = \frac{1}{1-\gamma}$ and location parameter $m = m_\mu$. The mean of $\overline{Y}_\infty$ is $am/(a-1) = (\gamma(1-\gamma))^{-1}$. Since $Y_\infty = w(\overline{Y}_\infty) = \gamma \overline{Y}_\infty$ we see that Y_∞ is still distributed according to a Pareto distribution, with the same shape parameter, and location parameter $m_1 = m\gamma = \frac{\gamma}{1-\gamma}$. Naturally, we could also describe μ using $\overline{q}_\mu(\lambda) = h(1/\lambda) = \frac{\gamma}{1-\gamma}\lambda^{\gamma-1}$ which, taking inverses, gives $\overline{\mu}(x) = (\frac{\gamma}{1-\gamma}\frac{1}{x})^{1/(1-\gamma)}$ as required.

As a consequence we see that if (Y_t) is a max-continuous martingale which satisfies a linear constraint $Y_t > \gamma\overline{Y}_t$ until $\zeta = \zeta_w^Y < \infty$ a.s., then, necessarily, $Y_\zeta = \gamma\overline{Y}_\zeta$ has a Pareto distribution.

4.4. *The Skorokhod embedding problem revisited.* The Skorokhod embedding problem can be phrased as follows: given a probability measure μ on $\mathbb{R}$ find a stopping time T such that X_T has the law μ, $X_T \sim \mu$. One further requires T to be *small* in some sense, typically saying that T is minimal. We refer the reader to Obłój [26] for further details and the history of the problem.

In [3] Azéma and Yor introduced the family of martingales described in Definition 2.1 and used them to give an elegant solution to the Skorokhod embedding problem for X a continuous local martingale (and μ centered). Namely, they proved that

$$T_\psi(X) = \inf\{t \geq 0 : \psi_\mu(X_t) \leq \overline{X}_t\}, \tag{27}$$

where ψ_μ is the barycenter function (19), solves the embedding problem.

We propose to rediscover their solution in a natural way using our methods, based on the observation that the process X satisfies the w_μ-DD constraint up to $T_\psi(X)$. If we show the equality $X_\zeta = w_\mu(\overline{X}_\zeta)$ at time $\zeta = T_\psi(X)$, Proposition 4.12 gives us the result.

THEOREM 4.13 (Azéma and Yor [3]). *Let (X_t) be a continuous local martingale, $X_0 \in \mathbb{R}$ a constant, $\langle X\rangle_\infty = \infty$ a.s. and μ a probability measure on $\mathbb{R} : \int |x|\mu(dx) < \infty$, $\int x\mu(dx) = X_0$. Then $T_\psi < \infty$ a.s., $(X_{t\wedge T_\psi})$ is a uniformly integrable martingale and $X_{T_\psi} \sim \mu$, $\overline{X}_{T_\psi} \sim \mu^{\mathrm{HL}}$, where T_ψ is defined via (27).*

With notation of Proposition 4.12, define $N_t = M^{V_\mu}_{t\wedge\tau^{r_\mu}(X)}(X)$. Then

$$T_\psi = \inf\{t \geq 0 : X_t \leq w_\mu(\overline{X}_t)\} = \inf\{t \geq 0 : N_t \leq 0\} \wedge \tau^{b_\mu}(N) \tag{28}$$

and $X_{t\wedge T_\psi} = M^{U_\mu}_{t\wedge T_\psi}(N)$.

PROOF. Let $\tau = \tau^{r_\mu}(X)$. $(N_t : t < \tau)$ is a continuous local martingale with $N_0 = 1$ since $U_\mu(1) = \mathrm{AVaR}_\mu(1) = X_0$. If $b_\mu < \infty$, then $r_\mu < \infty$ and $(N_t : t \leq \tau)$ is a local martingale stopped at $\inf\{t : N_t = b_\mu\} = \tau < \infty$ a.s. Suppose $b_\mu = \infty$. Then $\overline{N}_{\tau-} = \lim_{x\to r_\mu} V(x) = \infty$. This readily implies that $\langle N\rangle_{\tau-} = \infty$ a.s. and in particular $\tau_0(N) < \tau$ a.s. (cf. Proposition V.1.8 in Revuz and Yor [30]). Note that this applies both for the case r_μ finite and infinite. We conclude that $N_{t\wedge\tau_0(N)}$ is a continuous local martingale satisfying (16) stopped at $\tau_0(N) \wedge \tau^{b_\mu}(N) < \infty$ a.s. The theorem now follows from part (c) in Proposition 4.12. $\square$

REMARK 4.14. Note that in general only max-continuity of (X_t) would not be enough. More precisely we need to have $X_{T_\psi} = w_\mu(\overline{X}_{T_\psi})$ a.s. or equivalently that the process N_t crosses zero continuously. Note also that we do *not* necessarily have that $\psi_\mu(X_{T_\psi}) = \overline{X}_{T_\psi}$. Finally, we point out that the value of $X_0 = \int x\mu(dx)$ plays no special role and we do not need to assume that $X_0 = 0$.

5. On optimal properties of AY martingales related to HL transform and its inverse. In this final section we investigate the optimal properties of Azéma–Yor processes and of the Hardy–Littlewood transform $\mu \to \mu^{\mathrm{HL}}$ and its inverse operator Δ. We use two orderings of probability measures. We say that μ dominates ν in the *stochastic order* (or stochastically) if $\overline{\mu}(y) \ge \overline{\nu}(y)$, $y \in \mathbb{R}$. We say that μ dominates ν in the *increasing convex order* if $\int g(y)\mu(dy) \ge \int g(y)\nu(dy)$ for any increasing convex function g whenever the integrals are defined. Observe that the latter order is equivalent to $C_\mu(K) \ge C_\nu(K)$, $K \in \mathbb{R}$ (cf. Shaked and Shanthikumar [32], Theorem 3.A.1).

From (22) and (23) we deduce instantly that if μ, ρ are probability measures on $\mathbb{R}$ which admit first moments, then

$$\begin{aligned} &C_\mu(K) \le C_\rho(K), \qquad K \in \mathbb{R} \\ &\quad \Leftrightarrow \quad \mathrm{AVaR}_\mu(\lambda) \le \mathrm{AVaR}_\rho(\lambda), \qquad \lambda \in (0,1). \end{aligned} \tag{29}$$

By definition of μ^{HL}, $\mathrm{AVaR}_\mu(\lambda) = \overline{q}_{\mu^{\mathrm{HL}}}(\lambda)$, and hence we obtain

$$\begin{aligned} C_\mu(K) \le C_\rho(K), \qquad K \in \mathbb{R} \quad &\Leftrightarrow \quad \overline{q}_{\mu^{\mathrm{HL}}}(\lambda) \le \overline{q}_{\rho^{\mathrm{HL}}}(\lambda), \qquad \lambda \in [0,1] \\ &\Leftrightarrow \quad \overline{\mu}^{\mathrm{HL}}(y) \le \overline{\rho}^{\mathrm{HL}}(y), \qquad y \in \mathbb{R}, \end{aligned} \tag{30}$$

so that ρ^{HL} dominates μ^{HL} stochastically if and only if ρ dominates μ in the convex order.

5.1. *Optimality of Azéma–Yor stopping time and Hardy–Littlewood transformation*. The Azéma–Yor stopping time has a remarkable property that the maximum of a martingale stopped at this time is distributed according to μ^{HL} [see Theorem 4.13, also part (c) in Proposition 4.12]. The importance of this result comes from the result of Blackwell and Dubins [7] (see also the concise version of Gilat and Meljison [16]) showing:

THEOREM 5.1 (Blackwell and Dubins [7]). *Let (P_t) be a uniformly integrable martingale and μ the distribution of P_∞. Then*

$$\mathbb{P}(\overline{P}_\infty \ge y) \le \mu^{\mathrm{HL}}([y,\infty)), \qquad y \in \mathbb{R}. \tag{31}$$

In other words, any Hardy–Littlewood maximal r.v. associated with P_∞ dominates stochastically $\overline{P}_\infty$.

In fact μ^{HL} is sometimes defined as the smallest measure which satisfies (31). One then proves the representation (21).

PROOF OF THEOREM 5.1. We present a short proof of the theorem based on arguments in Brown, Hobson and Rogers [8]. Let (P_t) be a uniformly integrable martingale with terminal distribution μ. Choose $y \in (0, r_\mu)$ and observe that for any $K < y$ the following inequality holds a.s.:

$$\mathbf{1}_{\overline{P}_\infty \geq y} \leq \frac{(P_\infty - K)^+}{y - K} + \frac{y - P_\infty}{y - K}\mathbf{1}_{\overline{P}_\infty \geq y}. \tag{32}$$

If P is max-continuous then the last term on the RHS is simply $-M_\infty^F(P)$ for $F(z) = \frac{(z-y)^+}{y-K}$ and has zero expectation. In general, we can substitute the last term with a greater term $\frac{P_{\tau^y(P)} - P_\infty}{y-K}\mathbf{1}_{\overline{P}_\infty \geq y}$ which has zero expectation. Hence, taking expectations in (32) we find

$$\mathbb{P}(\overline{P}_\infty \geq y) \leq \frac{1}{y - K}\int_K^\infty (x - K)\mu(dx).$$

Taking infimum in $K < y$ and using (24) we conclude that $\mathbb{P}(\overline{P}_\infty \geq y) \leq \overline{\mu}^{\mathrm{HL}}(y)$. □

Azéma–Yor martingales, stopped appropriately, are examples of martingales which achieve equality in (31). We can reformulate the previous result in terms of optimality of the Azéma–Yor stopping time, which has been studied by several authors (Azéma and Yor [4], Gilat and Meljison [16], Kertz and Rösler [21] and Hobson [19]).

COROLLARY 5.2 (Azéma and Yor [4]). *In the setup and notation of Theorem 4.13, the distribution of $\overline{X}_{T_\psi}$ is μ^{HL}. In consequence, $\overline{X}_{T_\psi}$ dominates stochastically the maximum of any other uniformly integrable martingale with terminal distribution μ.*

The result is a corollary of Theorem 5.1 and the fact that the maximum $\overline{X}_{T_\psi}$ is a Hardy–Littlewood maximal r.v. associated with μ, which follows from Proposition 4.12. Alternatively, it follows from our proof of Theorem 5.1 upon observing, from the definition of T_ψ, that $X_{T_\psi} = w_\mu(\overline{X}_{T_\psi})$ and hence, with $P_t = X_{t \wedge T_\psi}$, we have a.s. equality in (32) for $K = w_\mu(y)$ and in consequence $\mathbb{P}(\overline{X}_{T_\psi} \geq y) = \overline{\mu}^{\mathrm{HL}}(y)$.

5.2. *Optimality*: *a dual point of view.* We identified above μ^{HL} as the maximal, relative to stochastic order, possible distribution of maximum of a uniformly integrable martingale with a fixed terminal law μ. We consider now the dual problem: we look for a maximal terminal distribution of a uniformly integrable martingale with a fixed law of its maximum. We saw in (30) that stochastic order of HL transforms translates into increasing convex ordering of the underlying distributions, and we expect the solution to the dual problem to be optimal relative to increasing convex order.

We offer two viewpoints on this dual problem. First we will see it as a problem of finding the inverse operator to the Hardy–Littlewood transform. Then we will rephrase the result in martingale terms.

Let us fix a distribution ν, which the reader may think of as the distribution of the one-sided maximum of some uniformly integrable martingale. We look at measures ρ, $\int |x|\rho(dx) < \infty$, such that ρ^{HL} stochastically dominates $\nu : \overline{\nu}(x) \leq \overline{\rho}^{\mathrm{HL}}(x)$, $x \in \mathbb{R}$. We note $\mathcal{S}_\nu$ the set of such measures. Passing to the inverses, we can express the condition on $\rho \in \mathcal{S}_\nu$ in terms of tail quantiles,

$$\rho \in \mathcal{S}_\nu \quad \Leftrightarrow \quad \overline{q}_\nu(\lambda) \leq \overline{q}_{\rho^{\mathrm{HL}}}(\lambda) = \mathrm{AVaR}_\rho(\lambda) = \frac{1}{\lambda}\int_0^\lambda \overline{q}_\rho(\eta)\,d\eta, \qquad \lambda \in [0,1], \tag{33}$$

using definitions in (20) and (21). Note that for existence of $\rho \in \mathcal{S}_\nu$ it is necessary that

$$\lambda \overline{q}_\nu(\lambda) \underset{\lambda \to 0}{\longrightarrow} 0 \text{ which is equivalent to } x\overline{\nu}(x) \underset{x \to \infty}{\longrightarrow} 0, \tag{34}$$

where the equivalence follows from the change of variables $x = \overline{q}_\nu(\lambda_x)$ and $\lambda_x \overline{q}_\nu(\lambda_x) = x\overline{\nu}(x)\,dx$-a.e.

If $\nu = \mu^{\mathrm{HL}}$, then by (30) we see instantly that μ is the minimal element of $\mathcal{S}_\nu$ relative to the increasing convex order. We extend this now to general measures ν.

THEOREM 5.3. *Let ν be a probability measure on $\mathbb{R}$. The set $\mathcal{S}_\nu$ is nonempty if and only if ν satisfies* (34). *Under* (34), *$\mathcal{S}_\nu$ admits a minimal element ν_Δ relative to the increasing convex order, which is characterized by*

$$\lambda \overline{q}_{\nu_\Delta^{\mathrm{HL}}}(\lambda) = \int_0^\lambda \overline{q}_{\nu_\Delta}(\eta)\,d\eta \qquad \text{is the concave envelope of } \lambda \overline{q}_\nu(\lambda).$$

If $\nu = \mu^{\mathrm{HL}}$ for an integrable probability measure μ, then $\nu_\Delta = \mu$.

PROOF. As observed above, the case $\nu = \mu^{\mathrm{HL}}$ follows instantly from (30), which in turn used the fact that $\lambda \overline{q}_\nu(\lambda)$ is concave and equal to $\int_0^\lambda \overline{q}_\mu(\eta)\,d\eta$.

It is natural to extend the above ideas to a general case. Assume ν satisfies (34), and let $G(\lambda)$ be the concave envelope (i.e., the smallest concave majorant) of $\lambda \overline{q}_\nu(\lambda)$. If there exists a measure ν_Δ such that $G(\lambda) = \int_0^\lambda \overline{q}_{\nu_\Delta}(\eta)\,d\eta$, then clearly $\nu_\Delta \in \mathcal{S}_\nu$ by definition in (33). Furthermore, since $\int_0^\lambda \overline{q}_\rho(\eta)\,d\eta$ is a concave function, we have that

$$\int_0^\lambda \overline{q}_{\nu_\Delta}(\eta)\,d\eta \leq \int_0^\lambda \overline{q}_\rho(\eta)\,d\eta, \qquad \lambda \in [0,1],\ \forall \rho \in \mathcal{S}_\nu. \tag{35}$$

This in turn, using (29), is equivalent to ν_Δ being the infimum of $\rho \in \mathcal{S}_\nu$ relative to increasing convex ordering of measures and thus being a solution to our dual problem.

It remains to argue that ν_Δ exists for a general ν. Recall that $-\infty \le l_\nu < r_\nu \le \infty$ are, respectively, the lower and the upper bounds of the support of ν. Let $\tilde{G}(x)$ be the (formal) Fenchel transform of $\lambda\overline{q}_\nu(\lambda)$,

$$\tilde{G}(x) = \sup_{\lambda\in(0,1)} \big(\lambda\overline{q}_\nu(\lambda) - \lambda x\big), \qquad x \in [l_\nu, r_\nu]. \tag{36}$$

Observe that $\tilde{G}(x) \ge 0$ thanks to assumption (34) and by definition $\tilde{G}(x)$ is convex, decreasing and $\tilde{G}'(x) \in [-1, 0]$. This implies that there exists a probability measure ν_Δ such that $\tilde{G}(x) = \int (y-x)^+ \nu_\Delta(dy) = C_{\nu_\Delta}(y)$. In fact we simply have $\overline{\nu}_\Delta(x) := -\tilde{G}'(x-)$. Since G was the concave envelope of $\lambda\overline{q}_\nu(\lambda)$ we can recover it as the dual Fenchel transform of $\tilde{G}$ and, using (22), we have

$$G(\lambda) = \inf_{x\in[l_\nu,r_\nu]} \big(\tilde{G}(x) + x\lambda\big) = \int_0^\lambda \overline{q}_{\nu_\Delta}(\eta)\, d\eta, \qquad \lambda \in [0, 1], \tag{37}$$

as required. Note that we could also take $x \in \mathbb{R}$ above since the infimum is always attained for $x \in [l_\nu, r_\nu]$. $\square$

REMARK 5.4. Theorem 5.3 synthesizes several results from Kertz and Rösler [22, 23] as well as adds a new interpretation of Δ operator as the inverse of $\mu \to \mu^{\mathrm{HL}}$. Furthermore, we stress that in the proof we obtained, in fact, a rather explicit representation which can be used to construct ν_Δ. Namely we have $\tilde{G}(x) = C_{\nu_\Delta}(y)$ with $\tilde{G}$ defined in (36), in particular $\overline{\nu}_\Delta(x) = -\tilde{G}'(x)$. Equivalently we have $\nu_\Delta \sim \frac{1}{\xi}G(\xi)$, for a uniform r.v. ξ.

We offer now the second viewpoint on the problem and rephrase the results above in martingale terms.

THEOREM 5.5. *Let ν be a distribution satisfying* (34), *and $U(x)$ be the increasing concave envelope of $\overline{q}_\nu(1/x)$, $x \ge 1$. Let $Y_t = M_t^U(N)$ for some (N_t) satisfying* (16), $N_0 = 1$.

Then $\overline{Y}_\infty$ dominates ν for the stochastic order and if (P_t) is any uniformly integrable martingale such that $\overline{P}_\infty$ dominates ν for the stochastic order, then P_∞ dominates Y_∞ for the increasing convex order.

Furthermore, if $\nu = \mu^{\mathrm{HL}}$ and (P_t) as above is max-continuous with $P_\infty \sim \mu$ then P is the Azéma–Yor martingale $M^{U_\mu}(N)$ for some (N_t) satisfying (16).

PROOF. From (34) U is well defined and observe that $U(x) = xG(1/x)$, $x \ge 1$, where $G(\lambda)$ is the concave envelope of $\lambda\overline{q}_\nu(\lambda)$. From the proof of Theorem 5.3 we see that $\frac{1}{\lambda}G(\lambda) = \mathrm{AVaR}_{\nu_\Delta}(\lambda)$ and hence $Y = M_t^{U_{\nu_\Delta}}(N)$ is the Azéma–Yor martingale associated with ν_Δ by Proposition 4.12. Let $\mu \sim P_\infty$. By Corollary 5.2 the distribution of $\overline{P}_\infty$ is dominated stochastically by μ^{HL}. Hence μ^{HL}

dominates stochastically ν and $\mu \in \mathcal{S}_\nu$. The first part of theorem is then a corollary of Theorem 5.3.

It remains to argue the last statement of the theorem. Since $P_\infty \sim \mu$ and the distribution of $\overline{P}_\infty$ dominates stochastically μ^{HL} it follows from Theorem 5.1 that $\overline{P}_\infty \sim \mu^{\mathrm{HL}}$. We deduce from the proof of Corollary 5.2 that we have an a.s. equality in (32) for any $y > 0$ and $K = w_\mu(y)$ and hence

$$\{P_\infty \geq w_\mu(y)\} \supseteq \{\overline{P}_\infty > y\} \supseteq \{P_\infty > w_\mu(y)\}.$$

It follows that $P_\infty = w_\mu(\overline{P}_\infty)$. Further, from uniform integrability of (P_t),

$$\mathbb{E}P_\infty = \mathbb{E}P_{\zeta^P_{w_\mu}} \leq \mathbb{E}w_\mu(\overline{P}_{\zeta^P_{w_\mu}}) \leq \mathbb{E}w_\mu(\overline{P}_\infty).$$

In consequence $P_t = P_{t\wedge\zeta^P_{w_\mu}}$, and the statement follows with $N_t = M^{V_\mu}_{t\wedge\tau^{r_\mu}(P)}(P)$ (see Theorem 4.13 and Remark 4.14). $\square$

REMARK 5.6. Distribution ν_Δ can also be easily recovered from U since, using Proposition 4.12, $h_{\nu_\Delta}(x) = \overline{q}_{\nu_\Delta}(1/x)$ is obtained from (17) as $h_{\nu_\Delta}(x) = U(x) - xU'(x)$.

5.3. *Floor constraint and concave order.* In this final section we study how Theorem 5.5 can be used to solve different optimization problems motivated by portfolio insurance. Our insight comes in particular from constrained portfolio optimization problems discussed by El Karoui and Meziou [11].

Consider g an increasing function on $\mathbb{R}_+$ such that $\lim_{x\to\infty} g(x)/x = 0$, and let U be its increasing concave envelope. Let N_t satisfy (16) with $N_0 = 1$. In the financial context, the underlying floor is modeled by $F_t = g(N_t)$. Financial positions can be modeled with uniformly integrable martingales, and we are interested in choosing the optimal one, among all which dominate F_t for all $t \geq 0$. We note that it is quite remarkable that this pathwise domination requirement turns out to be equivalent to potentially weaker conditions of ordering of distributions.

Finally we remark that in financial context we often use the increasing concave order between two variables (rather then convex). This is simply a consequence of the fact that utility functions are typically concave.

PROPOSITION 5.7. *Let $F_t = g(N_t)$ be the floor process, and $\mathcal{M}^s_F$ denote the set of uniformly integrable martingales (P_t), with $P_0 = U(N_0)$ and $P_t \geq F_t, t \geq 0$. Then the Azéma–Yor martingale $M^U_t(N)$ belongs to $\mathcal{M}^s_F$ and is optimal for the concave order of the terminal values, that is, for any increasing concave function G and $P \in \mathcal{M}^s_F$, $\mathbb{E}G(M^U_\infty(N)) \geq \mathbb{E}G(P_\infty)$.*

In fact the same result holds in the larger set $\mathcal{M}^w_F$ of uniformly integrable martingales (P_t) with $P_0 = U(N_0)$ and $\mathbb{P}(\overline{P}_\infty \geq x) \geq \mathbb{P}(\overline{F}_\infty \geq x)$, for all $x \in \mathbb{R}$.

PROOF. Let $\nu \sim \overline{F}_\infty = g(\overline{N}_\infty)$, which can also be written as $\overline{q}_\nu(\lambda) = g(\frac{1}{\lambda})$. Note that our assumption $g(x)/x \to 0$ as $x \to \infty$ is equivalent to (34). Recall that $\lambda U(\frac{1}{\lambda})$ is the concave envelope of $\lambda g(\frac{1}{\lambda})$, $\lambda \in (0,1)$. The result now follows from Theorem 5.5. It suffices to note that, since $\mathbb{E}P_\infty = F_0 = \mathbb{E}M^U_\infty(N)$, increasing convex order, increasing concave order and convex order on P_∞ and $M^U_\infty(N)$ are all equivalent (cf. Shaked and Shanthikumar [32], Theorems 3.A.15 and 3.A.16). $\square$

If we want show the above statement only for the smaller set $\mathcal{M}^s_F$, then we can give a direct proof as in [12]. Any martingale P which dominates F_t dominates also the smallest supermartingale Z_t which dominates F_t, and it is easy to see that $Z_t = U(N_t)$. The process (Z_t) is the Snell envelope of $(g(N_t))$, as shown in Galtchouk and Mirochnitchenko [15] using that U is an affine function on $\{x : U(x) > g(x)\}$.

From Proposition 4.6 we know that $M_t = M^U_t(N) = \mathbb{E}[h(\overline{N}_\infty)|\mathcal{F}_t]$, where $h(x) = U(x) - xU'(x)$, is a uniformly integrable martingale, and we also have $\overline{M}_t = U(\overline{N}_t) = \overline{Z}_t$ (cf. Proposition 2.2). We assume G is twice continuously differentiable, the general case following via a limiting argument. Since G is concave, $G(y) - G(x) \le G'(x)(y-x)$ for all $x, y \ge 0$. In consequence

$$
\begin{aligned}
&\mathbb{E}[G(P_\infty) - G(M_\infty)]\\
&\qquad \le \mathbb{E}[G'(M_\infty)(P_\infty - M_\infty)] = \mathbb{E}[G'(h(\overline{N}_\infty))(P_\infty - M_\infty)]\\
&\qquad \le \mathbb{E}\int_0^\infty G'(h(\overline{N}_t))\,d(P_t - M_t) + \mathbb{E}\int_0^\infty (P_t - M_t)G''(h(\overline{N}_t))\,d(h(\overline{N}_t)).
\end{aligned}
$$

The first integral is a difference of two uniformly integrable martingales (note that $\overline{N}_0 > 0$) and its expectation is zero. For the second integral, recall that h is increasing and the support of $d(h(\overline{N}_t))$ is contained in the support of $d\overline{N}_t$ on which $M_t = \overline{M}_t = \overline{Z}_t = Z_t \le P_t$. As G is concave we see that the integral is a.s. negative which yields the desired inequality.

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