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PERCOLATION BEYOND $\mathbb{Z}^d$: THE CONTRIBUTIONS OF ODED SCHRAMM

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This paper is dedicated to the memory of Oded Schramm

Oded Schramm (1961–2008) influenced greatly the development of percolation theory beyond the usual $\mathbb{Z}^d$ setting; in particular, the case of nonamenable lattices. Here, we review some of his work in this field.

1. Introduction. Oded Schramm was born in 1961 and died in a hiking accident in 2008, in what otherwise seemed to be the middle of an extraordinary mathematical career. Although he made seminal contributions to many areas of mathematics in general and probability in particular, I will here restrict attention to his work on percolation processes taking place on graph structures more exotic than the usual $\mathbb{Z}^d$ setting. The title I have chosen alludes to the short but highly influential paper *Percolation beyond* $\mathbb{Z}^d$, *many questions and a few answers* from 1996 by Itai Benjamini and Oded Schramm [13].

I need to point out, however, that there are at least two respects in which I will fail to deliver on what my chosen title suggests. First, I will not come anywhere near an exhaustive exposition of Oded's contributions to the field. All I can offer is a personal and highly subjective selection of highlights. Second, Oded was a very collaborative mathematician, and I will make no attempt (if it even makes sense) at identifying his individual contributions as opposed to his coauthors'. Suffice it to say that everyone who worked with him knew him as a very generous person and as someone who would not put his name on a paper unless he had contributed at least his fair share. I will just quote one recollection from Oded's long-time collaborator and friend Russ Lyons:

> To me, Oded's most distinctive mathematical talent was his extraordinary clarity of thought, which led to dazzling proofs and results. Technical difficulties did not obscure his vision. Indeed, they often melted away under his gaze. At one point when the four of us [Oded, Russ, Itai Benjamini and Yuval Peres] were working on uniform spanning forests, Oded came up with a brilliant new application of the Mass-Transport Principle.

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> We were not sure it was kosher, and I still recall Yuval asking me if I believed it, saying that it seemed to be "smoke and mirrors." However, when Oded explained it again, the smoke vanished [59].

Following the spirit of the aformentioned paper [13], I will take "percolation beyond $\mathbb{Z}^d$" to mean percolation process that are not naturally thought of as embedded in d-dimensional Euclidean space. This excludes contributions by Oded not only to the theory of percolation on $\mathbb{Z}^d$ (such as [6]) but also to percolation on the triangular lattice (such as [71] and [73]) and to continuum percolation in $\mathbb{R}^d$ (such as [14]).

Other parts of Oded's work are discussed in the papers by Angel, Garban and Rohde in the present volume. For further reactions to Oded's untimely death, and memories of his life and work, see, for instance, Lyons [54], Häggström [33] and Werner [79], as well as the blog [59].

2. How it began. It will be assumed throughout that $G = (V, E)$ is an infinite but locally finite connected graph. In *i.i.d. site percolation* on G with retention parameter $p \in [0, 1]$, each vertex $v \in V$ is declared open (retained, value 1) with probability p and closed (deleted, value 0) with the remaining probability $1 - p$, and this is done independently for different vertices. Alternatively, one may consider *i.i.d. bond percolation*, which is similar except that it is the edges rather than the vertices that are declared open or closed. Write $\mathbb{P}_{p,\mathrm{site}}$ and $\mathbb{P}_{p,\mathrm{bond}}$ for the resulting probability measures on $\{0, 1\}^V$ and $\{0, 1\}^E$, respectively. The choice whether to study bond or site percolation is often (but not always) of little importance and largely a matter of taste. In either case, focus is on the connectivity structure of the resulting random subgraph of G. Of particular interest is the possible occurrence of an infinite connected component—an infinite cluster, for short. The probability under $\mathbb{P}_{p,\mathrm{site}}$ or $\mathbb{P}_{p,\mathrm{bond}}$ of having an infinite cluster is always 0 or 1, and increasing in p. This motivates defining the site percolation critical value

$$p_{c,\mathrm{site}}(G) = \inf\{p : \mathbb{P}_{p,\mathrm{site}}(\exists \text{ an infinite cluster}) > 0\}$$

and the bond percolation critical value $p_{c,\mathrm{bond}}(G)$ analogously.

By far the most studied case is where $G = (V, E)$ is the $\mathbb{Z}^d$ lattice with $d \geq 2$, meaning that $V = \mathbb{Z}^d$ and E consists of all pairs of Euclidean nearest neighbors. Some selected landmarks in the history of percolation are the 1960 result of Harris [37] that $p_{c,\mathrm{bond}}(\mathbb{Z}^2) \geq \frac{1}{2}$; the 1980 result of Kesten [41] that $p_{c,\mathrm{bond}}(\mathbb{Z}^2) = \frac{1}{2}$; the 1987 result of Aizenman, Kesten and Newman [1] establishing uniqueness of the infinite cluster for arbitrary d; and the strikingly short and beautiful alternative proof from 1989 by Burton and Keane [21] of the same result. See Grimmett [26] and Bollobás and Riordan [19] for introductions to percolation theory with emphasis on the $\mathbb{Z}^d$ case.

Benjamini and Schramm [13] were of course not the first to study percolation on more exotic graphs and lattices. The case where G is the $(d + 1)$-regular tree

$\mathbb{T}_d$ had been well understood for a long time, essentially because it can be seen as a Galton–Watson process. Lyons [50, 51] had studied percolation on general trees, and Grimmett and Newman [29] had considered percolation on the Cartesian product $\mathbb{T}_d \times \mathbb{Z}$ [the Cartesian product $G = (V, E)$ of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ has vertex set $V = V_1 \times V_2$, and an edge connecting (x_1, x_2) and (y_1, y_2) iff $x_1 = y_2$ are identical and x_2 and y_2 are neighbors or vice versa]. However, it was only with the publication of Benjamini and Schramm [13] that a systematic study of percolation beyond $\mathbb{Z}^d$ began to take off toward anything like escape velocity. Clearly, they managed to find just the right time for launching the kind of informal research programme that their paper proposes. It should be noted, however, that a large part of the meaning of "just the right time" in this context is simply "soon after Oded Schramm had been drawn into probability theory."

When, as in [13], we focus on the occurrence and properties of infinite clusters for i.i.d. site (or bond) percolation on a graph $G = (V, E)$, a first basic issue is of course whether such clusters occur at all for any nontrivial value of p, that is, whether $p_{c,\mathrm{site}}(G) < 1$ [or $p_{c,\mathrm{bond}}(G) < 1$]. Benjamini and Schramm conjecture the following, where, for $v \in V$, $B(v, n)$ denotes the set of vertices $w \in V$ such that $\mathrm{dist}_G(v, w) \le n$, and dist_G is graph-theoretic distance in G.

CONJECTURE 2.1 (Conjecture 2 in [13]). *If G is a quasi-transitive graph such that for some* (*hence any*) $v \in V$, $|B(v, n)|$ *grows faster than linearly*, *then* $p_{c,\mathrm{site}}(G) < 1$.

Here, of course, we need to define quasi-transitivity of a graph (the term used in [13] was *almost transitive*, but the mathematical community quickly decided that quasi-transitive was preferable).

DEFINITION 2.2. Let $G = (V, E)$ be an infinite locally finite connected graph. A bijective map $f : V \to V$ such that $\langle f(u), f(v)\rangle \in E$ if and only if $\langle u, v\rangle \in E$ is called a *graph automorphism* for G. The graph G is said to be *transitive* if for any $u, v \in V$ there exists a graph automorphism f such that $f(u) = v$. More generally, G is said to be *quasi-transitive* if there is a $k < \infty$ and a partitioning of V into k sets $V_1, \ldots, V_k$ such that for $i = 1, \ldots, k$ and any $u, v \in V_i$ there exists a graph automorphism f such that $f(u) = v$.

An important subclass of transitive graphs is the class of graphs arising as the Cayley graph of a finitely generated group. It may be noted that for quasi-transitive graphs (and more generally for bounded degree graphs; cf. [32]) we have $p_{c,\mathrm{bond}} < 1$ iff $p_{c,\mathrm{site}} < 1$, so Conjecture 2.1 may equivalently be phrased for bond percolation. Benjamini and Schramm found a short and elegant proof of the conjecture for the special case of so-called nonamenable graphs:

DEFINITION 2.3. The *isoperimetric constant* $h(G)$ of a graph $G = (V, E)$ is defined as

$$h(G) = \inf_S \frac{|\partial S|}{|S|},$$

where the infimum ranges over all finite nonempty subsets of V, and $\partial S = \{u \in V \setminus S : \exists v \in S \text{ such that } \langle u, v\rangle \in E\}$. The graph G is said to be *amenable* if $h(G) = 0$; otherwise, it is said to be *nonamenable*.

(Sometimes, as in [13], $h(G)$ is also called the Cheeger constant.)

THEOREM 2.4 (Theorem 2 in [13]). *Any nonamenable graph G satisfies $p_{c,\text{site}}(G) < 1$. In fact,*

$$p_{c,\text{site}}(G) \le \frac{1}{h(G)+1}. \tag{1}$$

PROOF. Fix p and a vertex $\rho \in V$, and consider the following sequential procedure for searching the open cluster containing ρ. If ρ is open, set $S_1 = \{\rho\}$, otherwise stop. At each integer time n, check the status (open or closed) of some thitherto unchecked vertex v in ∂S_{n-1}; if v is open we set $S_n = S_{n-1} \cup \{v\}$, if v is closed we set $S_n = S_{n-1}$, while if no such v can be found the procedure terminates. Define $X_0 = 0$ and $X_n = |S_n|$ for $n \ge 1$, and note that $\{X_0, X_1, \ldots\}$ is a random walk whose i.i.d. increments take value 0 with probability $1 - p$ and 1 with probability p, stopped at some random time. If G has isoperimetric constant $h(G)$, then the random walk can stop only when $\frac{n-X_n}{X_n} \ge h(G)$, that is, when

$$\frac{X_n}{n} \le \frac{1}{h(G)+1}. \tag{2}$$

But the random walk has drift p, so when $p > \frac{1}{h(G)+1}$ the Strong Law of Large Numbers implies that with positive probability $\frac{X_n}{n}$ never satisfies (2), in which case the walk never stops and ρ belongs to an infinite cluster. $\square$

It may be noted that the result is sharp in the sense that the bound (1) holds with equality for the tree $\mathbb{T}_d$, for which $p_{c,\text{site}}(\mathbb{T}_d) = \frac{1}{d}$ and $h(\mathbb{T}_d) = d - 1$.

Once $p_{c,\text{site}}(G) < 1$, we know that i.i.d. site percolation on G has two distinct phases: for $p < p_{c,\text{site}}$ there is no infinite cluster, while for $p > p_{c,\text{site}}$ there is. But what happens *at* the critical value? This has long been a central issue in percolation theory. When G is the $\mathbb{Z}^d$ lattice with $d \ge 2$, the consensus belief among percolation theorists is that there is no infinite cluster at criticality; this is known for $d = 2$ (Russo [67]) and $d \ge 19$ (Hara and Slade [36]), but the general case remains open. Benjamini and Schramm suggest that "it might be beneficial to study the problem in other settings."

CONJECTURE 2.5 (Conjecture 4 in [13]). *For any quasi-transitive graph G with $p_{c,\text{site}}(G) < 1$, there is a.s. no infinite cluster at criticality.*

This remains open (as of course it must be as long as the $\mathbb{Z}^d$ case with $3 \le d \le 18$ stays unsolved), but Benjamini and Schramm were soon to be involved in remarkable progress toward proving it; see Section 4. The quasi-transitivity condition cannot be dropped, as it is easy to construct graphs with a nontrivial critical value which nevertheless percolate at criticality (a tree growing slightly faster than a binary tree will do; see, e.g., [51]).

Another natural next question, once the existence of infinite clusters at nontrivial values of p (i.e., $p_{c,\text{site}} < 1$) is established, concerns how many infinite clusters there can be. Benjamini and Schramm [13] noted that the argument of Newman and Schulman [61] for showing that the number of infinite clusters is, for fixed p, an a.s. constant which must equal 0, 1 or ∞ extends to the setting of quasi-transitive graphs. They furthermore saw that the argument of Burton and Keane [21] for ruling out infinitely many infinite clusters extends to amenable quasi-transitive graphs (a similar observation was made earlier in Theorem $1'$ of Gandolfi, Keane and Newman [25]). In other words, we get the following.

THEOREM 2.6. *For any amenable quasi-transitive graph G and any $p \in [0, 1]$, the number of infinite clusters produced by i.i.d. site or bond percolation on G with parameter p is either* 0 *or* 1 *a.s.*

In contrast, i.i.d. site percolation the regular tree $\mathbb{T}_d$ with $d \ge 2$ exhibits infinitely many infinite clusters for all $p \in (p_{c,\text{site}}, 1)$. Also, the $\mathbb{T}_d \times \mathbb{Z}$ example studied by Grimmett and Newman [29] exhibits the same phenomenon when p is above but sufficiently close to $p_{c,\text{site}}$ (Grimmett and Newman showed this for large d, and later Schonmann [69] indicated how to do it for all $d \ge 2$). Benjamini and Schramm [13] conjectured that the Burton–Keane argument is sharp in the sense that whenever G is quasi-transitive and nonamenable, uniqueness of the infinite cluster fails for p above but sufficiently close to $p_{c,\text{site}}$. In terms of the so-called uniqueness critical value

$$\begin{aligned} p_{u,\text{site}} &= p_{u,\text{site}}(G) \\ &= \inf\{p \in [0,1] : \text{i.i.d. site percolation on } G \text{ with parameter } p \\ &\qquad\qquad \text{produces a.s. a unique infinite cluster}\}, \end{aligned}$$

their conjecture reads as follows.

CONJECTURE 2.7 (Conjecture 6 in [13]). *For any nonamenable quasi-transitive graph G, we have $p_{c,\text{site}}(G) < p_{u,\text{site}}(G)$.*

In the spirit of the Grimmett–Newman result mentioned above, they proved that for any quasi-transitive graph G, the product graph $\mathbb{T}_d \times G$ satisfies $p_{c,\text{site}}(G) < p_{u,\text{site}}(G)$ provided d is large enough (this is Corollary 1 in [13]).

Conjecture 2.7 has stimulated further research as well. For instance, Pak and Smirnova-Nagnibeda [63] proved in the case of bond percolation that it holds for the Cayley graph of any nonamenable group provided an appropriate choice of generators. Lalley [43] and a later paper by Benjamini and Schramm [16] proved it for certain classes of nonamenable planar graphs; this will be discussed in more detail in Section 7.

By definition of $p_{u,\text{site}}$ and the Newman–Schulman 0–1–∞ law, we have infinitely many infinite clusters a.s. for any $p \in (p_{c,\text{site}}, p_{u,\text{site}})$. It would be nice to add that uniqueness holds for all $p \in (p_{u,\text{site}}, 1)$, but for this we need the monotonicity property that if $p_1 < p_2$ and uniqueness holds a.s. at parameter value p_1, then it holds at p_2 as well; this is part of Question 5 in [13]. The required monotonicity was proved by Häggström and Peres [35] for quasi-transitive graphs under the additional assumption of unimodularity (see Definition 3.5 below), and by Schonmann [68] without this additional assumption.

The other part of Question 5 in [13] concerns, in the case where G is quasi-transitive with $p_{c,\text{site}} < p_{u,\text{site}} < 1$, the number of infinite cluster at $p = p_{u,\text{site}}$: one or infinitely many? Somewhat surprisingly, the answer turned out to depend on the choice of G. For the Grimmett–Newman example, Schonmann [69] showed that there are infinitely many infinite clusters at the uniqueness critical point $p_{u,\text{site}}$, a result that Peres [65] extended to more general product graphs. In contrast, Benjamini and Schramm [16] showed that for planar nonamenable graphs with one end, there is a unique infinite cluster at $p_{u,\text{site}}$; see Section 7 again.

There is a good deal more to say about the *Percolation beyond* $\mathbb{Z}^d$ paper [13], but I must move on to some of Oded Schramm's later contributions. The paper's influence will be evident from the coming sections, but see also Benjamini and Schramm [15], Lyons [53] and Häggström and Jonasson [34] for partially overlapping surveys of what happened in the wake of the paper.

3. Invariant percolation and mass transport. Soon after finishing the *Percolation beyond* $\mathbb{Z}^d$ paper [13], Itai Benjamini and Oded Schramm joined forces with Russ Lyons and Yuval Peres (this quartet of authors will appear frequently in what follows, and will be abbreviated BLPS). In [9], they broadened the scope compared to [13] by considering percolation processes on quasi-transitive graphs in a more general situation than the i.i.d., namely *automorphism invariance*.

DEFINITION 3.1. Let $G = (V, E)$ be a quasi-transitive graph and let $\text{Aut}(G)$ denote the group of graph automorphisms of G. A $\{0, 1\}^V$-valued random object X is called a site percolation for G, and it is said to be *automorphism invariant* if for any n, any $v_1, \dots, v_n \in V$, any $b_1, \dots, b_n \in \{0, 1\}$ and any $\gamma \in \text{Aut}(G)$, we have

$$\mathbb{P}(X(\gamma v_1) = b_1, \dots, X(\gamma v_n) = b_n) = \mathbb{P}(X(v_1) = b_1, \dots, X(v_n) = b_n).$$

Automorphism invariance of a bond percolation for G is defined analogously. In fact, much of the work in [9] concerns an even more general setting, namely invariance under certain kinds of subgroups of $\mathrm{Aut}(G)$. Here, for simplicity, I will restrict attention to the case of invariance under the full automorphism group $\mathrm{Aut}(G)$.

There are plenty of automorphism invariant percolation processes beyond i.i.d. that arise naturally. Examples in the site precolation case include certain Gibbs distributions for spin systems such as the Ising model, and certain equilibrium measures for interacting particle systems such as the voter model. In the bond percolation case, they include the random-cluster model [27] as well as the random spanning forest models to be discussed in Section 8.

Amongst the most important contributions of BLPS [9] is the introduction of the so-called *Mass-Transport Principle* in percolation theory, and the beginning of a systematic exploitation of it for understanding the behavior of percolation processes. (This was partly inspired by an application in Häggström [31] of similar ideas in the special case where G is a regular tree.) As a kind of warm-up for readers unfamiliar with the mass-transport technique, let me suggest a very simple toy problem.

PROBLEM 3.2. Given a transitive graph $G = (V, E)$, does there exist an automorphism invariant bond percolation process which produces, with positive probability, some infinite open cluster consisting of a single self-avoiding path which is infinite in just one direction? (We call such a self-avoiding path *uni-infinite*.) In other words, this open cluster should consist of a single vertex of degree 1 in the cluster, while all the other (infinitely many) vertices of the cluster should have degree 2.

Call an infinite cluster *slim* if it is of the desired kind. (In a sense, a slim infinite cluster is the smallest infinite cluster there can be.) Also, given an automorphism invariant bond percolation process X taking values in $\{0, 1\}^E$, we define random variables $\{Y(v)\}_{v\in V}$ as follows. If v does not belong to a slim infinite cluster in v, we set $Y(v) = 0$; otherwise, we let $Y(v)$ be one plus the distance in the slim cluster from v to the one endpoint of this cluster. For $k = 0, 1, \dots,$ write $\alpha(v, k) = \mathbb{P}(Y(v) = k)$. Automorphism invariance ensures that this is independent of the choice of v, so we may write $\alpha(k)$ for $\alpha(v, k)$.

When $G = (V, E)$ is the $\mathbb{Z}^d$ lattice, we can argue as follows. Write Λ_n for the box $\{-n, -n + 1, \dots, n\}^d \subset V$. The expected number of vertices $v \in \Lambda_n$ with $Y(v) = 1$ is $(2n + 1)^d\alpha(1)$. But for any k, and any vertex v with $Y(v) = 1$, there must be a corresponding vertex u with $Y(u) = k$ within distance $k - 1$ from v. Hence, the expected number of vertices $v \in \Lambda_{n+k-1}$ with $Y(v) = k$ is at least $(2n + 1)^d\alpha(1)$, so

$$\bigl(2(n + k) - 1\bigr)^d\alpha(k) \geq (2n + 1)^d\alpha(1),$$

and sending $n \to \infty$ yields $\alpha(k) \geq \alpha(1)$. Similarly, $\alpha(1) \geq \alpha(k)$, so that in fact $\alpha(1) = \alpha(k)$, and since k was arbitrary we have $\alpha(1) = \alpha(2) = \alpha(3) = \cdots$. But $\sum_{k=1}^{\infty} \alpha(k) \leq 1$, so $\alpha(k)$ must be 0 for each k, whence slim infinite clusters do not occur in the $G = \mathbb{Z}^d$ case.

The crucial property of the $\mathbb{Z}^d$ lattice that makes the argument work is that $\frac{|\Lambda_{n+k}|}{|\Lambda_n|} \to 1$ as $n \to \infty$. Hence, the argument is easily extended to the more general case where G is transitive and amenable.

But what about the case where G is nonamenable? Now the argument does not generalize, and in fact the following example gives us problems. And *end* in a graph $G = (V, E)$ is an equivalence class of uni-infinite self-avoiding paths in X, with two paths equivalent if for all finite $W \subset V$ the paths are eventually in the same connected component of the graph obtained from G by deleting all $v \in W$.

EXAMPLE 3.3 (Trofimov's graph [77]). Consider the regular binary tree $\mathbb{T}_2$, and fix an end ξ in this tree. For each vertex v in the tree, there is a unique uni-infinite self-avoiding path from v that belongs to ξ. Call the first vertex after v on this path the ξ-*parent* of v, and call the other two neighbors of v its ξ-*children*. The ξ-*grandparent* of v is defined similarly in the obvious way. Let $G = (V, E)$ be the graph that arises by taking $\mathbb{T}_2$ and adding, for each vertex v, an extra edge connecting v to its ξ-grandparent.

Clearly, Trofimov's graph G is transitive, and it also inherits the nonamenability property of $\mathbb{T}_2$. It turns out that on G, it is possible to construct an automorphism invariant bond percolation exhibiting slim infinite clusters:

EXAMPLE 3.4. Let $G = (V, E)$ be Trofimov's graph, and consider the following automorphism invariant bond percolation on G: each $v \in V$ will have an open edge to exactly one of its ξ-children, and for each v independently, toss a fair coin to decide which ξ-child to connect to. All grandparent–grandchild edges are closed. (To see that this bond percolation is indeed automorphism invariant, it is necessary, but easy, to check that the end ξ can be identified by just looking at the graph structure of G.) This produces a percolation configuration in which a.s. each v sits in a slim infinite cluster. From v the open path extends downward (i.e., away from ξ) infinitely, and upward a geometric$(\frac{1}{2})$ number of steps.

So perhaps slim infinite clusters can arise as soon as G is nonamenable? In fact, no. It turns out that the mass-transport method of BLPS [9] applies to rule out slim infinite clusters also in the nonamenable case, as long as the graphs satisfy the additional assumption of *unimodularity*. Unimodularity holds for all specific examples considered so far except for Trofimov's graph. It holds for Cayley graphs in general, and I daresay it tends to hold for most transitive graphs that are not constructed for the explicit purpose of being nonunimodular.

DEFINITION 3.5. Let $G = (V, E)$ be a quasi-transitive graph with automorphism group $\mathrm{Aut}(G)$. For $v \in V$, the *stabilizer* of v is defined as $\mathrm{Stab}(v) = \{\gamma \in \mathrm{Aut}(G) : \gamma v = v\}$. The graph G is said to be *unimodular* if for all $u, v \in V$ in the same orbit of $\mathrm{Aut}(G)$ we have the symmetry

$$|\mathrm{Stab}(u)v| = |\mathrm{Stab}(v)u|.$$

[Note how Trofimov's graph fails to be unimodular: for two vertices u and v such that u is the ξ-parent of v, we get $|\mathrm{Stab}(u)v| = 2$ but $|\mathrm{Stab}(v)u| = 1$. Each vertex has two children but just one parent.]

PROPOSITION 3.6. *In automorphism invariant bond percolation on a quasi-transitive unimodular graph G, there is a.s. no slim infinite cluster.*

This, we will find, is an easy consequence of the Mass-Transport Principle of BLPS [9]. For an automorphism invariant site (or bond) percolation on a quasi-transitive graph $G = (V, E)$, let μ be the corresponding probability measure on $\{0, 1\}^V$ (or on $\{0, 1\}^E$). Consider a nonnegative function $m(u, v, \omega)$ of three variables: two vertices $u, v \in V$ and the percolation configuration ω taking values in $\Omega = \{0, 1\}^V$ (or $\Omega = \{0, 1\}^E$). Intuitively, we should think of $m(u, v, \omega)$ as the mass transported from u to v given the configuration ω. We assume that $m(u, v, \omega) = 0$ unless u and v are in the same orbit of $\mathrm{Aut}(G)$, and furthermore that $m(\cdot, \cdot, \cdot)$ is invariant under the diagonal action of $\mathrm{Aut}(G)$, meaning that $m(u, v, \omega) = m(\gamma u, \gamma v, \gamma \omega)$ for all u, v, ω and $\gamma \in \mathrm{Aut}(G)$.

THEOREM 3.7 (The Mass-Transport Principle, Section 3 in [9]). *Given G, μ and $m(\cdot, \cdot, \cdot)$ as above, let*

$$M(u, v) = \int_\Omega m(u, v, \omega)\, d\mu(\omega)$$

for any $u, v \in V$. If G is unimodular, then the expected total mass transported out of any vertex v equals the expected mass transported into v, that is,

$$\sum_{u \in V} M(v, u) = \sum_{u \in V} M(u, v). \tag{3}$$

The Mass-Transport Principle as stated here fails if G is not unimodular. [To see this for Trofimov's graph, we can consider the the mass transport in which each vertex simply sends unit mass to its ξ-parent, regardless of the percolation configuration. Then each vertex sends mass 1 but receives mass 2, thus violating (3).] In fact, BLPS [9] did state a version of the Mass-Transport Principle that holds also in the nonunimodular case; this involves a reweighting of the mass sent from u to v by a factor that depends on $\frac{|\mathrm{Stab}(u)v|}{|\mathrm{Stab}(v)u|}$. But it is in the unimodular case that the Mass-Transport Principle has turned out most useful, and for simplicity we stick to this case.

The proof of the Mass-Transport Principle is particularly simple in the case where G is the Cayley graph of a finitely generated group H, so here I will settle for that case only:

PROOF OF THEOREM 3.7 IN THE CAYLEY GRAPH CASE. For $u, v \in V$, we also have that u and v are group elements of H, and that there is a unique element $h = uv^{-1} \in H$ such that $u = hv$. This gives

$$\begin{aligned}\sum_{u \in V} M(v, u) &= \sum_{h \in H} M(v, hv) = \sum_{h \in H} M(h^{-1}v, v) \\ &= \sum_{h' \in H} M(h'v, v) = \sum_{u \in V} M(u, v),\end{aligned}$$

where the second equality follows from automorphism invariance. □

PROOF OF PROPOSITION 3.6. Consider the mass transport in which each vertex v sitting in a slim infinite cluster sends unit mass to the unique endpoint of this cluster. Vertices not sitting in a slim infinite cluster send no mass at all. Then the expected mass sent from a vertex is at most 1, while if slim infinite clusters exist with positive probability then some vertices will receive infinite mass with positive probability, so that the expected mass received is infinite, contradicting (3). □

Proposition 3.6 is just an illustrative example, but BLPS [9] proved several other more interesting reuslts using the Mass-Transport Principle, which turns out to be quite a potent tool in nonamenable settings where classical density arguments and ergodic averages are not available in the same way as in the amenable case. The Mass-Transport Principle in itself is not especially deep or difficult. Rather, in the words of BLPS [10], "the creative element in applying the mass-transport method is to make a judicious choice of the transport function $m(u, v, \omega)$." We will see some examples in this section and the next. (For a remarkable recent development of the mass-transport method, see Aldous and Lyons [3] and Schramm [72].)

The following result characterizes amenability of Cayley graphs (and more generally of unimodular transitive graphs) in terms of a certain percolation threshold for invariant percolation. For a transitive graph $G = (V, E)$ and a automorphism invariant site percolation on G with distribution μ on $\{0, 1\}^V$, write $\pi(\mu)$ for the marginal probability that a given vertex is open.

THEOREM 3.8 [9]. *Let G be a unimodular transitive graph, and define $p_{c,\mathrm{inv}}(G)$ is the infimum over all $p \in [0, 1]$ such that any automorphism invariant site percolation μ on G with $\pi(\mu) = p$ is guaranteed to produce at least one infinite cluster with positive probability. Then $p_{c,\mathrm{inv}}(G) < 1$ if and only if G is nonamenable.*

Quantitative estimates for $p_{c,\mathrm{inv}}(G)$ in the nonamenable case are also provided in [9]. Theorem 3.9 below gives such a bound in the bond percolation case. For site percolation, BLPS [9] show that if $\pi(\mu) \geq \frac{d(G)}{d(G)+h(G)}$, where $d(G)$ is the degree of a vertex in G, and $h(G)$ as before is the isoperimetric constant, then there is at least one infinite cluster with positive probability. Similar bonds are given for the quasi-transitive case as well. For the case $G = \mathbb{T}_d$ the bounds go back to Häggström [31], where they were established using a precursor of the mass-transport method, and also shown to be sharp. The following bound in the bond percolation case is in terms of the *edge-isoperimetric constant* $h_E(G)$, defined by

$$h_E(G) = \inf_S \frac{|\partial_E S|}{|S|},$$

where as in Definition 2.3 the infimum ranges over all finite $S \in V$, while $\partial_E S = \{\langle u, v\rangle \in E : u \in S, v \in V \setminus S\}$. Clearly, $h(G) \leq h_E(G) \leq (d(G) - 1)h(G)$ when G is transitive, so such a G is amenable in the sense of Definition 2.3 if and only if $h_E(G) = 0$.

THEOREM 3.9 [9]. *Let $G = (V, E)$ be transitive and unimodular, and consider an automorphism invariant bond percolation on G such that for each edge $e \in E$ we have*

$$\mathbb{P}(e \text{ is open}) \geq \frac{d(G) - h_E(G)}{d(G)}. \tag{4}$$

Then the percolation produces an infinite cluster with positive probability.

The proof is worth exhibiting here, but in order to be able to follow the elegant argument from the expository follow-up paper BLPS [10], I will be content with considering the case where (4) holds with strict inequality.

PROOF OF (ALMOST) THEOREM 3.9. For a finite subgraph $G' = (V', E')$ of G, define its average internal degree

$$h_E^*(G') = \frac{2|E'|}{|V'|}$$

and set

$$h_E^*(G) = \sup_{G'} h_E^*(G'), \tag{5}$$

where the supremum is over all finite subgraphs G' of G. For any given such G', we have

$$2|E'| + |\{\langle u, v\rangle \in E : u \in V', v \in V \setminus V'\}| \leq d(G)|V'|$$

with equality if and only if all $e \in E$ with both endpoints in V' are also in E'. Hence, $h_E^*(G) + h_E(G) = d(G)$, so the right-hand side of (4) equals $\frac{h_E^*(G)}{d(G)}$. Now consider a $\{0, 1\}^E$-valued automorphism invariant bond percolation X on G such that

$$\mathbb{P}(e \text{ is open}) > \frac{h_E^*(G)}{d(G)} \tag{6}$$

for each $e \in E$, and assume for contradiction that it a.s. produces no infinite cluster. We may define a mass transport where each vertex sitting in a finite open cluster counts the number of open edges incident to it, sends out exactly this amount of mass, and distributes it equally among all the vertices sitting in its connected component in the percolation process. In other words, we take the transport function to be

$$m(u, v, \omega) = \begin{cases} \dfrac{d_\omega(u)}{|K(u)|}, & \text{if } u \text{ is in a finite component of } \omega \text{ and } v \in K(u), \\ 0, & \text{otherwise,} \end{cases}$$

where $d_\omega(u)$ is the degree of u in X, and $K(u)$ is the set of vertices having an open path in ω to u. Then (6) and the assumption that X produces no infinite clusters give

$$\begin{aligned} \mathbb{E}\Bigl[\sum_{v\in V} m(u, v, X)\Bigr] &\geq d(G) \min_{e\in E} \mathbb{P}(e \text{ is open}) \\ &> d(G)\frac{h_E^*(G)}{d(G)} = h_E^*(G), \end{aligned}$$

while the amount $\sum_{v\in V} m(v, u, X)$ received at u is the average internal degree of its connected component, which is bounded by $h_E^*(G)$. Hence,

$$\mathbb{E}\Bigl[\sum_{v\in V} m(u, v, X)\Bigr] > \mathbb{E}\Bigl[\sum_{v\in V} m(v, u, X)\Bigr]$$

contradicting the Mass-Transport Principle. $\square$

The next result from BLPS [9] concerns the expected degree of a vertex given that it belongs to an infinite cluster. Here it seems most natural to consider the bond percolation case. For G transitive and an invariant bond percolation on G with distribution μ that produces at least one infinite cluster with positive probability, define $\beta(G, \mu)$ as the expected degree of a vertex given that it belongs to an infinite cluster, and define $\beta(G) = \inf_\mu \beta(G, \mu)$ where the infimum ranges over all such automorphism invariant bond percolation processes on G.

THEOREM 3.10 [9]. *For $G = (V, E)$ transitive, we have $\beta(G) = 2$ if G is unimodular, and $\beta(G) < 2$ otherwise.*

An exact expression for $\beta(G)$ in the unimodular case is also given, namely $1+\inf_{\langle u,v\rangle\in E}\frac{|\mathrm{Stab}(u)v|}{|\mathrm{Stab}(v)u|}$, where transitivity implies that the infimum is in fact a minimum. Note how Theorem 3.10 immediately implies Proposition 3.6 above: a slim infinite cluster would have vertices both of degree 1 and of degree 2, and these are the only degrees appearing, so the average degree would have had to be strictly between 1 and 2. That average degree 2 is needed is quite intuitive, but what is more surprising is that it is possible to go below 2 in the nonunimodular case. In fact, the bond percolation on Trofimov's graph in Example 3.4 has $\beta(G,\mu)=\frac{3}{2}$, and this can be pushed down to $\beta(G,\mu)=\frac{5}{4}$ (which is sharp for Trofimov's graph) by letting the slim infinite clusters live on grandparent–grandchild rather than parent–child edges.

Another striking result in BLPS [9] concerns the number of ends of infinite components: if G is quasi-transitive and unimodular and μ is an automorphism invariant site or bond percolation, then the number of ends of any infinite component must be either 1, 2 or ∞. (In the amenable case, the argument of Burton and Keane [21] excludes also the case of infinitely many ends.) Furthermore, for infinite clusters with infinitely many ends, BLPS [9] showed that that such clusters have expected degree strictly greater than 2, and that they have critical values for site or bond percolation that are strictly greater than 1.

4. No infinite cluster at criticality. I have yet to mention what is possibly the most striking result of all from BLPS [9]. Namely, this study of automorphism invariant percolation turned out to have the following implication for i.i.d. percolation, which is a remarkable step in the direction of Conjecture 2.5.

THEOREM 4.1. *Let G be a nonamenable unimodular quasi-transitive graph, and consider i.i.d. site percolation on G at the critical value $p=p_{c,\mathrm{site}}$. Then there is a.s. no infinite cluster. The analogous statement for i.i.d. bond percolation holds as well.*

Due to the focus in [9] being on the more general setting of automorphism invariant percolation, the proof given there is not the most direct possible. The authors therefore chose to publish a separate expository note, BLPS [10], with a more direct proof which is well worth recalling here. Following [10], I will restrict to the case of bond percolation on a (nonamenable, unimodular) transitive graph; the cases of site percolation and quasi-transitive graphs require only minor modification.

The reason why unimodularity is needed in the proof is that the Mass-Transport Principle is used—in fact, it is used several times (including in the proof of Theorem 3.9 which the proof of Theorem 4.1 falls back on). But we should probably expect the result to be true also in the nonunimodular case (certainly if we trust Conjecture 2.5). See Timár [76] and Peres, Pete and Scolnicov [66] for what are perhaps the best efforts to date toward a better understanding of the nonunimodular case.

PROOF OF THEOREM 4.1 FOR BOND PERCOLATION ON TRANSITIVE GRAPHS. Let $G = (V, E)$ be a nonamenable unimodular transitive graph, and consider an i.i.d. bond percolation X on G with $p = p_{c,\mathrm{bond}}(G)$. By the Newman–Schulman 0–1–∞ law, the number of infinite clusters in X is an a.s. constant N which equals either 0, 1 or ∞, and we need to rule out the possibilities $N = 1$ and $N = \infty$.

Case I. *Ruling out* $N = 1$. Assume for contradiction that the percolation configuration $X \in \{0, 1\}^E$ has a unique infinite cluster a.s. Let $\{Y(e)\}_{e\in E}$ be an i.i.d. collection of random variables, uniformly distributed on the unit interval $[0, 1]$ and independent also of X. For each $\varepsilon \in (0, 1)$ and $e \in E$, define

$$X_\varepsilon(e) = \begin{cases} 1, & \text{if } X(e) = 1 \text{ and } Y(e) > \varepsilon, \\ 0, & \text{otherwise,} \end{cases}$$

and note that

$$X_\varepsilon \in \{0, 1\}^E \text{ is an i.i.d. bond percolation on } G \text{ with parameter } (1 - \varepsilon) p_{c,\mathrm{bond}}. \tag{7}$$

For $\varepsilon \in (0, 1)$, define yet another bond percolation $Z_\varepsilon \in \{0, 1\}^E$, not i.i.d. but automorphism invariant, as follows. As before let dist_G denote graph-theoretic distance in G, and for each $v \in V$ define $U(v)$ as the set of vertices in the infinite cluster of X that minimize $\mathrm{dist}_G(u, v)$. Note that $U(v)$ is finite for all $v \in V$. For an edge $e = \langle v, w\rangle \in E$, set

$$Z_\varepsilon(e) = \begin{cases} 1, & \text{if all vertices in } U(v) \text{ and } U(w) \text{ are} \\ & \text{in the same connected component of } X_\varepsilon, \\ 0, & \text{otherwise.} \end{cases}$$

This defines the percolation $Z_\varepsilon \in \{0, 1\}^E$. For any $\langle v, w\rangle \in E$, there exists some finite collection $T(v, w) \subset E$ of open edges in X that together connect all the vertices in $U(v)$ and $U(w)$ to each other (this is where the assumption $N = 1$ is used). For definiteness, we take $T(v, w)$ to be the edge set with minimal cardinality having this property, and with minimization of $\sum_{e\in T(v,w)} Y(e)$ acting as tie-breaker. Each edge e' in this collection has $Y(e') > 0$ a.s., and so $\min_{e'\in T(v,w)} Y(e') > 0$ so that $\lim_{\varepsilon\to 0} Z_\varepsilon(e) = 1$ a.s. Hence,

$$\lim_{\varepsilon\to 0} \mathbb{P}[Z_\varepsilon(e) = 1] = 1, \tag{8}$$

and Theorem 3.9 ensures that that the percolation process Z_ε contains an infinite cluster with positive probability. But when Z_ε contains an infinite cluster, then so does X_ε. In view of (7), this contradicts the definition of $p_{c,\mathrm{bond}}$, so we are done with Case I.

Case II. *Ruling out* $N = \infty$. This time assume for contradiction that X contains infinitely many infinite clusters a.s. Here we need the concept of *encounter points* introduced by Burton and Keane [21] in their famous short proof of uniqueness of the infinite cluster on $\mathbb{Z}^d$. An encounter point in a percolation process X is a vertex $v \in V$ that has three disjoint open paths to infinity that would fall in different connected components of X if the vertex v were to be removed, but does not have four such paths. Burton and Keane showed that if $N = \infty$, then X contains encounter points a.s.; their proof was formulated for $G = \mathbb{Z}^d$, but goes through unchanged for general (quasi-)transitive graphs.

BLPS [10] begin by noting that

(9) if $v \in V$ is an encounter point, then a.s. each of the three infinite clusters $C_1(V), C_2(v), C_3(v)$ that the removal of v would produce contains further encounter points.

To see this, consider the mass transport in which each vertex u sitting in an infinite cluster with encounter points sends unit mass to the nearest encounter point with respect to dist_X, splitting it equally in case of a tie; here dist_X means graph-theoretic distance in the open subgraph of G defined by X. Failure of (9) would cause a contradiction to the Mass-Transport Principle similarly as in the proof of Proposition 3.6.

Next, we go on to define a random graph $H = (W, F)$, whose vertex set $W \subset V$ is the set of encounter points in X, and whose edge set F, which we are about to specify, will not necessarily be a subset of E (so H is not a subgraph of G). Let $\{Y(v)\}_{v \in W}$ be i.i.d., uniformly distributed on $[0, 1]$ and independent of X. Each $v \in W$ selects three other $u_1, u_2, u_3 \in V$ to form edges to, according to the following rule: one u_i should be chosen in each of the components $C_1(v)$, $C_2(v)$ and $C_3(v)$, and in each such component u_i is chosen to minimize $\mathrm{dist}_X(v, u_i)$, with minimization of $Y(u_i)$ acting as a tie-breaker. An equivalent way to formulate this is that u_i is chosen in C_i to minimize the "distance" $\mathrm{dist}_{X,Y}(v, u_i)$ defined as $\mathrm{dist}_{X,Y}(v, u_i) = \mathrm{dist}_X(v, u_i) + Y(v) + Y(u_i)$.

Each $v \in V$ thus gets H-degree at least 3, but the H-degree may exceed 3 if v is selected by some $w \in W$ which is not among v's preferred triplet. An application of the Mass-Transport Principle shows that the expected number of vertices that choose v is exactly 3, so the expected H-degree of v (conditional on being in W) is somewhere between 3 and 6, and in particular its degree is a.s. finite.

A crucial step of the argument is now to show that

(10) the graph H has no cycles.

To see this, we first note that if $v \in W$ is in such a cycle, then its two neighbors in this cycle must belong to the same $C_i(v)$, as otherwise we would get a direct contradiction to the definition of an encounter point. Using this, it is not hard to see (or consult [10] for a more detailed argument) that any cycle $v_1 \leftrightarrow v_2 \leftrightarrow \cdots \leftrightarrow v_k \leftrightarrow v_1$ would have to satisfy either

$$\mathrm{dist}_{X,Y}(v_1, v_2) < \mathrm{dist}_{X,Y}(v_2, v_3) < \cdots < \mathrm{dist}_{X,Y}(v_k, v_1) < \mathrm{dist}_{X,Y}(v_1, v_2)$$

or

$$\mathrm{dist}_{X,Y}(v_1, v_2) > \mathrm{dist}_{X,Y}(v_2, v_3) > \cdots > \mathrm{dist}_{X,Y}(v_k, v_1) > \mathrm{dist}_{X,Y}(v_1, v_2),$$

which in either case is of course a contradiction. Hence, (10).

Now take an $\varepsilon > 0$ and consider as in Case I the ε-thinned percolation process $X_\varepsilon \in \{0, 1\}^E$. Using X_ε, we define the subgraph $H_\varepsilon = (W, F_\varepsilon)$ obtained from H by deleting each $e = \langle v, w\rangle \in F$ such that v and w fail to be in the same connected component of X_ε. By the definition of $p_{c,\mathrm{bond}}$, we have that X_ε has no infinite clusters, so that H_ε has no infinite clusters either.

For $v \in W$, write $K_\varepsilon(v)$ for the set of vertices in W that belong to the same connected component of H_ε as v. Also write $\partial_{\mathrm{int}} K_\varepsilon(v)$ (int as in "internal boundary") for the set of vertices in $K_\varepsilon(v)$ that have at least one neighbor in H which is not in $K_\varepsilon(v)$.

Now define the following mass transport on $G = (V, E)$. No vertex $v \in V$ sends any mass unless it is an encounter point, that is, unless $v \in W$. Each encounter point v sends unit mass and divides it equally amongst the vertices in $K_\varepsilon(v)$. This defines the mass transport, and the mass received at v becomes

$$\begin{cases} \dfrac{|K_\varepsilon(v)|}{|\partial_{\mathrm{int}} K_\varepsilon(v)|}, & \text{if } v \text{ is an encounter point and belongs to } \partial_{\mathrm{int}} K_\varepsilon(v), \\ 0, & \text{otherwise.} \end{cases}$$

Since F is a forest in which each vertex has degree is at least 3, its isoperimetric constant is easily seen to be at least 1 (which holds with equality on the binary tree $\mathbb{T}_2$), and similarly $|K_\varepsilon(v)|/|\partial_{\mathrm{int}} K_\varepsilon(v)| \le 2$. So the expected mass received at v is bounded by $2\mathbb{P}(v \in W, v \in \partial_{\mathrm{int}} K_\varepsilon(v))$, while the expected mass sent from $v \in V$ equals $\mathbb{P}(v \in W)$. Hence, the Mass-Transport Principle gives

$$\mathbb{P}(v \in W) \le 2\mathbb{P}\big(v \in W, v \in \partial_{\mathrm{int}} K_\varepsilon(v)\big). \tag{11}$$

But similarly as in the argument for (8), we get for any $v \in W$ that $v \notin \partial_{\mathrm{int}} K_\varepsilon(v)$ for all sufficiently small ε, so that

$$\lim_{\varepsilon \to 0} 2\mathbb{P}\big(v \in W, v \in \partial_{\mathrm{int}} K_\varepsilon(v)\big) = 0.$$

Since (11) was shown to hold for any $\varepsilon > 0$, we get $\mathbb{P}(v \in W) = 0$, which contradicts $N = \infty$, so Case II is finished and the proof is complete. $\square$

5. Random walks on percolation clusters. One of the most natural probabilistic objects, besides i.i.d. percolation, to define on a graph $G = (V, E)$, is simple random walk (SRW), which is a V-valued random process $\{Z_0, Z_1, \dots\}$ where one (typically) takes $Z_0 = \rho$ for some prespecified choice of $\rho \in V$, and then iterates the following: given $Z_0, Z_1, \dots, Z_{n-1}$, the value of Z_n is chosen uniformly among the neighbors in G of Z_{n-1}.

In contrast to the study of percolation on nonamenable Cayley graphs and related classes of graphs which began to take off only in the 1990s, the literature on SRWs on such graphs goes back much further. An early seminal contribution is the work of Kesten [39, 40] from the late 1950s showing that if G is a Cayley graph for a finitely generated group, then the return probability $\mathbb{P}[Z_n = \rho]$ decays exponentially if and only if G is nonamenable. See, for example, Woess [81] for an introduction to this field.

Another topic of considerable interest is the study of SRW on percolation clusters. For the $\mathbb{Z}^d$ case, see, for instance, papers like [23] and [17] on central limit theorems, and [28] which extends Pólya's classical $d = 2$ versus $d \geq 3$ recurrence-transience dichotomy for random walk on $\mathbb{Z}^d$ to the case of supercritical percolation on $\mathbb{Z}^d$.

Given these traditions, it was a very natural step for Schramm and his collaborators to go on to consider random walks on percolation clusters on nonamenable graphs. Their work is of two kinds: on one hand, the analysis of SRW on a percolation cluster as a worthwhile object of study in its own right, and, on the other hand, the exploitation of random walk on a percolation cluster as a means toward understanding properties of percolation clusters that do not primarily have anything to do with random walk. A remarkable application of the second kind will be described in Section 6 on so-called cluster indistinguishability, while in the present section I will recall a result of the first kind (Theorem 5.1 below) from a rich paper by Benjamini, Lyons and Schramm, henceforth BLS [12].

A natural question to ask for SRW on an infinite graph G is how fast it escapes from the starting point ρ, that is, how fast does $\mathrm{dist}_G(\rho, Z_n)$ grow? Define the *speed*

$$S = \lim_{n\to\infty} \frac{\mathrm{dist}_G(\rho, Z_n)}{n}$$

provided the limit exists. When G is the $\mathbb{Z}^d$ lattice, $\mathrm{dist}_G(\rho, Z_n)$ scales like $\sqrt{n}$, and not surprisingly $S = 0$ a.s. More generally, when G is any transitive graph, the Subadditive Ergodic Theorem immediately implies that the limit S exists and is an a.s. constant.

If we go on on to consider the speed of SRW on an infinite cluster of, say, i.i.d. bond percolation on G (still with the speed defined with respect to dist_G), then the existence of the speed S is less obvious. However, BLS [12] showed, when G is unimodular, and the percolation process is automorphism invariant, that the speed does exist a.s. and does not depend on the random walk, but only on the percolation configuration. Having come that far, it is easy to see that the speed cannot depend on where in an infinite cluster the random walk starts, so each infinite cluster has a well-defined characteristic SRW speed. For the i.i.d. bond percolation case, we can then invoke the cluster indistinguishability result of Lyons and Schramm [56] (Theorem 6.1 below) to deduce that all infinite clusters have the same SRW speed.

Of particular interest is to determine whether the speed is zero or positive. For the nonamenable unimodular case, BLS [12] found a general answer.

THEOREM 5.1. *The speed S of SRW on an infinite cluster of an i.i.d. bond percolation X on a unimodular nonamenable transitive graph G satisfies $S > 0$ a.s.*

An important step in the proof of this is the following result (also of independent interest) from [12] on the geometry of infinite clusters. That the nonamenability of G should be inherited by the infinite clusters of X is too much to hope for: a sufficient condition for a percolation cluster in X to be amenable is that it contains arbitrarily long "naked" paths, that is, paths of vertices with X-degree 2, and it can be shown that a.s. all infinite clusters arising from i.i.d. percolation on a transitive graph contain such paths. But the infinite clusters of X do contain nonamenable subgraphs.

THEOREM 5.2. *Any infinite cluster in i.i.d. bond percolation X on a unimodular nonamenable transitive graph G contains a nonamenable subgraph a.s.*

The proof in [12] of this result involves yet another application of mass transport. Both Theorems 5.2 and 5.1 are in fact proved more generally than just for i.i.d. percolation. Automorphism invariance alone does not suffice (each of Examples 3.1, 3.2 and 3.3 of Häggström [31] shows this, and moreover that the $>$ in condition (d) below cannot be replaced by a $\geq$), but if we add any of the conditions:

(a) X is i.i.d.,
(b) X has a unique infinite cluster a.s.,
(c) the infinite clusters of X have at least 3 (hence infinitely many) ends a.s., or
(d) X is ergodic with sufficiently large values of $\mathbb{P}(e \text{ is open})$, more precisely the expected degree of a vertex should strictly exceed the quantity h_E^* defined in (5),

then the conclusions of Theorems 5.2 and 5.1 hold; cf. Theorems 3.9 and 4.4 in BLS [12].

6. Cluster indistinguishability. Consider an i.i.d. bond percolation X with parameter $p > p_{c,\mathrm{bond}}(C)$ on a graph G, so that X produces one or more infinite clusters. It is then natural to ask questions about properties of these infinite clusters.

Properties that we have already discussed in earlier sections include the number of ends of an infinite cluster, whether it contains encounter points, and the speed of SRW on the cluster.

Yet another natural such property of an infinite cluster C is the value of $p_{c,\mathrm{bond}}(C)$, that is, how much further i.i.d. edge-thinning can the infinite cluster C take before it breaks apart into finite components only? For G transitive, it is known (see Häggström and Peres [35] for the unimodular case, and Schonmann [68] for the full result) that a.s. $p_{c,\mathrm{bond}}(C) = p_{c,\mathrm{bond}}(G)/p$ for all infinite clusters C of X.

All these properties are examples of *invariant properties*. For $G = (V, E)$ transitive with automorphism group $\mathrm{Aut}(G)$, a property (which may or may not hold for clusters of bond percolation on G) can be identified with a Borel measurable subset of $\{0, 1\}^E$, and a property $\mathcal{A} \subset \{0, 1\}^E$ is said to be invariant if for all $\omega \in \mathcal{A}$ and all $\gamma \in \mathrm{Aut}(G)$ we have $\gamma\omega \in \mathcal{A}$. Lyons and Schramm [56] proved the following Theorem 6.1, known as *cluster indistinguishability*. Shortly before [56], a weaker result for the case of so-called increasing invariant properties was established in [35].

THEOREM 6.1. *Let $G = (V, E)$ be a nonamenable unimodular transitive graph, and consider i.i.d. bond percolation X on G with p in the parameter regime where X produces infinitely many infinite clusters a.s. Then, for any invariant component property $\mathcal{A}$, we have a.s. that either all infinite components of X satisfy $\mathcal{A}$ or all infinite components of X satisfy $\neg\mathcal{A}$.*

Space does not permit me to give the full proof of this beautiful result, but I can explain what the main steps are.

SKETCH PROOF OF THEOREM 6.1. Let $G = (V, E)$ and the percolation $X \in \{0, 1\}^E$ be as in the theorem, and assume for contradiction that $\mathcal{A}$ is an invariant property such that with positive probability, X contains both infinite clusters with property $\mathcal{A}$ and infinite clusters with property $\neg\mathcal{A}$.

Step I. *Existence of pivotal edges.* For an infinite cluster C of X and an edge $e \in E$ with $X(e) = 0$ that has an endpoint in C, call *e pivotal* for C if either $C \in \mathcal{A}$ and switching on the edge e would create an infinite cluster (containing C) with property $\neg\mathcal{A}$, or vice versa. If there exists an $e \in E$ with $X(e) = 0$ that has one endpoint in an infinite cluster with property $\mathcal{A}$ and the other in an infinite cluster with property $\neg\mathcal{A}$, then clearly e is pivotal for one of the clusters. Otherwise, there exists such a pair of clusters within finite distance from each other, and by sequentially switching on one edge after another on a finite path between them we see that somewhere along the way one infinite cluster of type $\mathcal{A}$ must turn into $\neg\mathcal{A}$ or vice versa. (This is an example of a well-known technique in percolation theory known as *local modification*, pioneered by Newman and Schulman [61] and

Burton and Keane [21]; cf. also Coupling 2.5 in [34] for a careful explanation.) Hence, pivotal edges exist with positive probability.

Step II. *Stationarity of random walk.* Consider a SRW $\{Z_0, Z_1, \ldots\}$ on a percolation cluster of X, defined as in Section 5. It would be nice to think that the percolation configuration "as seen from the point of view of the walker," would be stationary. To make this precise, fix for each $v \in V$ a $\gamma_{v,\rho} \in \mathrm{Aut}(G)$ that maps v on ρ, where, as in Section 5, ρ is the starting point of the random walk. The idea of stationarity of the percolation configuration as seen from the random walker is that for any Borel measurable Stab(ρ)-invariant $\mathcal{B} \in \{0,1\}^E$ and any n we should have

$$\mathbb{P}(\gamma_{Z_n,\rho} X \in \mathcal{B}) = \mathbb{P}(X \in \mathcal{B}).$$

This is not true, however, and to see this, think, for example, about what happens when the starting point ρ happens to be in a finite open cluster C which is simply a path of length 2; in other words, C has three vertices, one of which has degree 2 and two of which have degree 1. Conditioned on ρ being in such a cluster, it has probability $\frac{1}{3}$ of being in the vertex of degree 2 (this follows from a straightforward mass-transport argument). But SRW on such an open cluster will spend half the time (either all even or all odd times) on that vertex, so it cannot possibly be stationary in the desired sense.

All is not lost, however. Stationarity can be recovered by a minor modification of SRW, namely the *delayed simple random walk* (*DSRW*), denoted $\{\tilde{Z}_0, \tilde{Z}_1, \ldots\}$. This is again a V-valued random process, and as with SRW we take $\tilde{Z}_0 = \rho$, but the transition mechanism is slightly different: given $\tilde{Z}_0, \ldots, \tilde{Z}_{n-1}$, a vertex w is chosen according to uniform distribution on the set of $\tilde{Z}_{n-1}$'s G-neighbors, and we set

$$\tilde{Z}_n = \begin{cases} w, & \text{if } X(\langle \tilde{Z}_{n-1}, w\rangle) = 1, \\ \tilde{Z}_{n-1}, & \text{otherwise.} \end{cases}$$

Such a DSRW has the desired stationarity property, that is,

$$\mathbb{P}(\gamma_{\tilde{Z}_n,\rho} X \in \mathcal{B}) = \mathbb{P}(X \in \mathcal{B})$$

for any $\mathcal{B}$ and any n, and this can be shown via yet another mass-transport argument.

(Stationarity of DSRW is clearly an interesting result in its own right, and the idea was first exploited in [31]. Lyons and Schramm decided to put their most general version of the stationarity result not in [56] but in a separate paper [57]. Without unimodularity, however, stationarity fails, as is easily seen by considering DSRW in Example 3.4: here, $\tilde{Z}_0$ has probability $\frac{1}{2}$ of sitting in the "topmost" (closest to ξ) component of its open cluster, while the probability that $\tilde{Z}_n$ does so tends to 0 as $n \to \infty$.)

Step III. *Lots of encounter points.* Since X has infinitely many infinite clusters, we can modify X by changing finitely many edges so as to connect three of them to form an encounter point. Hence, encounter points exist with positive probability; this is just the local modification argument of Burton and Keane [21]. By (9), we have that every infinite cluster with encounter points must in fact have infinitely many. From this, it follows that *every* infinite cluster has infinitely many encounter points, because otherwise we could use the local modification technique to connect an infinite cluster with encounter points to one without and obtain a contradiction to (9).

Step IV. *Random walk is transient.* Given the prevalence of encounter points, infinite clusters contain, loosely speaking, a large-scale structure similar to the binary tree $\mathbb{T}_2$. This strongly suggests that DSRW on such an infinite cluster should be transient. Lyons and Schramm [56] converts this intuition into a proof by combining the result mentioned in the final paragraph of Section 3 about such infinite clusters C having $p_{c,\mathrm{bond}}(C) < 1$, with a basic comparison of random walk and percolation on trees due to Lyons [50]. (Alternatively, we could quote Theorem 5.1 here, but that would be a detour.)

Step V. *Local modification applied to a pivotal edge.* Let $\mathcal{B}$ be the event that the starting point ρ of the DSRW is in an infinite cluster with property $\mathcal{A}$. Given some small $\varepsilon > 0$, we can find a $k < \infty$ and a Stab(ρ)-invariant event $\mathcal{B}^*$ that depends only on edges within G-distance k from ρ, approximating $\mathcal{B}$ in the sense that

$$\mathbb{P}(\mathcal{B}\triangle\mathcal{B}^*) < \varepsilon. \tag{12}$$

Define, for each n,

$$Y_n = \begin{cases} 1, & \text{if } \gamma_{\tilde{Z}_n,\rho} X \in \mathcal{B}^*, \\ 0, & \text{otherwise.} \end{cases} \tag{13}$$

Then $\{Y_0, Y_1, \ldots\}$ is a stationary process, so the limit $\bar{Y} = \lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} Y_n$ exists a.s. Furthermore,

(14) the limit $\bar{Y}$ depends only on the percolation configuration and not on the DSRW.

To see (14), define a second DSRW $\{\tilde{Z}'_0, \tilde{Z}'_1, \ldots\}$ from ρ which conditionally on X is independent of $\{\tilde{Z}_0, \tilde{Z}_1, \ldots\}$, define $\{Y'_0, Y'_1, \ldots\}$ analogously as in (13), and set $\bar{Y}' = \lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} Y'_n$. Then we can define $Y_{-i} = Y'_i$ for each $i \geq 1$, and it turns out (see [56], Lemma 3.13) that the two-sided sequence $\{\ldots, Y_{-1}, Y_0, Y_1, \ldots\}$ becomes stationary. Now, if (14) failed we would with positive probability have $\bar{Y} \neq \bar{Y}'$, which however would contradict stationarity of the two-sided sequence $\{\ldots, Y_{-1}, Y_0, Y_1, \ldots\}$ in view of the ergodic theorem. Hence, (14).

Thus, we can assign each infinite cluster C a value $\alpha(C)$ as the a.s. value of $\bar{Y}$ that DSRW on that cluster would produce. By (12), we have that infinite clusters with different values of $\alpha(C)$ coexist with positive probability, provided we chose ε small enough to begin with. Call an edge $e \in E$ *α-pivotal* if it is closed $[X(e) = 0]$ with endpoints in two different infinite clusters C and C' with

$$\alpha(C) \neq \alpha(C'). \tag{15}$$

By the reasoning in Step 1, there exist α-pivotal edges in X with positive probability. In particular, there exists an $e \in E$ which with positive probability is α-pivotal for the infinite cluster containing ρ. Suppose this happens, let C be the infinite cluster containing ρ, and let C' be the other infinite cluster meeting e. Then $\bar{Y} = \alpha(C)$ a.s.

But look what happens if we apply local modification to the edge e. If we turn e on $[X(e) = 1]$ and leave the status of all other edges intact, we get a new infinite cluster C'' uniting C and C'. What will then happen with $\bar{Y}$? By transience of DSRW, we get with positive probability that the DSRW escapes to infinity without ever noticing the edge e, and we get a.s. on this event that $\bar{Y} = \alpha(C)$; this uses the fact that Y_n is a function of edges within bounded distance k from $\tilde{Z}_n$ only. On the other hand, we get with positive probability that the DSRW reaches e, crosses it, and then escapes to infinity without ever crossing it back; on this event we get a.s. $\bar{Y} = \alpha(C')$. In view of (15), this contradicts (14) and proves the theorem. $\square$

An inspection of the proof to determine which properties of i.i.d. percolation are actually used reveals that it is enough to assume automorphism invariance plus so-called *insertion tolerance*, which is the term Lyons and Schramm [56] used for a refinement of the finite energy property considered by Newman and Schulman [61], Burton and Keane [21], and others.

DEFINITION 6.2. A bond percolation X on a graph $G = (V, E)$ is called *insertion tolerant* if it admits conditional probabilities such that for every $e \in E$ and every $\xi \in \{0, 1\}^{E \setminus \{e\}}$ we have $\mathbb{P}(X(e) = 1 | X(E \setminus \{e\} = \xi) > 0$. If instead $\mathbb{P}(X(e) = 0 | X(E \setminus \{e\} = \xi) > 0$ for all such e and ξ, then X is called *deletion tolerant*.

(The conjunction of insertion tolerance and deletion tolerance is precisely the finite energy property.)

THEOREM 6.3. *Let $G = (V, E)$ be a nonamenable unimodular transitive graph, and consider an insertion tolerant automorphism invariant bond percolation X on G. Then, for any invariant component property $\mathcal{A}$, we have a.s. that either all infinite components of X are in $\mathcal{A}$ or no infinite components of X are in $\mathcal{A}$.*

In fact, the formulation in [56] is even more general than this: here, as in much of the work discussed in Section 3, invariance under the full automorphism group can be weakened to invariance under certain subgroups. Also, the result holds with site percolation in place of bond percolation.

It is worth mentioning some limitations to the scope of cluster indistinguishability. For instance, insertion tolerance cannot be replaced by deletion tolerance in Theorem 6.3, as the following example from [56] shows.

EXAMPLE 6.4. Let $G = (V, E)$ be the binary tree $\mathbb{T}_2$, recall that $p_{c,\text{bond}}(T_2) = \frac{1}{2}$, and fix p_1 and p_2 such that $\frac{1}{2} < p_1 < p_2 < 1$. Let $X \in \{0, 1\}^E$ be i.i.d. bond percolation on G with parameter p_2. Then obtain another bond percolation process $X' \in \{0, 1\}^E$ from X as follows. For each infinite cluster C of X independently, toss a fair coin. If the coin comes up heads, delete each edge of C independently with probability $1 - \frac{p_1}{p_2}$; otherwise let all of C's edges be intact. While X' is automorphism invariant and deletion tolerant, some of its infinite clusters C' will have all the characteristics of those produced by i.i.d. (p_2) percolation and thus have $p_{c,\text{bond}}(C') = \frac{1}{2p_2}$, while others will look like i.i.d. (p_1) percolation clusters and thus have $p_{c,\text{bond}}(C') = \frac{1}{2p_1}$, so cluster indistinguishability fails for X'.

How about the unimodularity assumption in Theorems 6.1 and 6.3? This cannot be dropped, not even from Theorem 6.1. To see this, let $G = (V, E)$ be Trofimov's graph (Example 3.3), and consider i.i.d. bond percolation with parameter $p \in (p_{c,\text{bond}}(G), 1)$. Each infinite cluster C then has a "topmost" vertex $w(C)$ (in the direction of the designated end ξ), and for $i = 0, 1, 2$ we may define the property $\mathcal{A}_i$ by stipulating that $C \in \mathcal{A}_i$ if $w(C)$ is directly linked to exactly i of its ξ-children via open edges. Then $\mathcal{A}_0$, $\mathcal{A}_1$ and $\mathcal{A}_2$ are $\text{Aut}(G)$-invariant properties, and the percolation will a.s. produce infinite clusters of all three kinds, so cluster indistinguishability fails.

7. In the hyperbolic plane. In their *Percolation beyond* $\mathbb{Z}^d$ paper [13], Benjamini and Schramm emphasized three properties of graphs that could be expected to be of particular relevance to the behavior of percolation processes: quasi-transitivity, (non-)amenability, and planarity. The first two I have discussed at some length, while the third was only mentioned in passing in Section 2. In this section, I will briefly make up for this.

Planarity plays a crucial role in the classical study of percolation on the $\mathbb{Z}^2$ lattice, such as in the seminal contributions by Harris [37] and Kesten [41]. A key device is the notion of planar duality: for a graph $G = (V, E)$ with a planar embedding in $\mathbb{R}^2$ (or in some other two-dimensional manifold), it is often useful to define its dual graph $G^\dagger = (V^\dagger, E^\dagger)$ by identifying $V^\dagger$ with the faces of the planar embedding of G, and including an edge $e^\dagger \in E^\dagger$ crossing each $e \in E$. If $X \in \{0, 1\}^E$ a bond percolation on G, then we can define a bond percolation $X^\dagger \in \{0, 1\}^E$ on $G^\dagger$ by declaring, for each $e \in E$, $X^\dagger(e^\dagger) = 1 - X(e)$. If X is i.i.d. (p), then $X^\dagger$ becomes i.i.d. $(1 - p)$. Furthermore, if G is the $\mathbb{Z}^2$ lattice, then $G^\dagger$ is isomorphic to G, and for $p = 1 - p = \frac{1}{2}$ the distributions of X and $X^\dagger$ will be the same; this observation is basic to proving the Harris–Kesten theorem that $p_{c,\text{bond}}(\mathbb{Z}^2) = \frac{1}{2}$.

Suppose that G is infinite, planar and transitive. It is known (see Babai [5]) that such a graph, equipped with the usual distance dist_G, is quasi-isometric to exactly one of the four spaces $\mathbb{R}$, $\mathbb{R}^2$, $\mathbb{T}_2$ and the hyperbolic plane $\mathbb{H}^2$, the last of which may be defined as the open unit disk $\{z \in \mathbb{C} : |z| < 1\}$ equipped with the metric s given by

$$ds^2 = \frac{dx^2 + dy^2}{(1 - x^2 - y^2)^2}.$$

In the paper *Percolation in the hyperbolic plane* [16], Benjamini and Schramm consider the situation where G has one end, in which case it must be quasi-isometric to either $\mathbb{R}^2$ or $\mathbb{H}^2$. Which of these is determined by amenability: if G is amenable, then $\mathbb{R}^2$, while if it is nonamenable, then $\mathbb{H}^2$. The focus of [16] is on the nonamenable case; hence, the title of the paper. A happy circumstance here is that a planar nonamenable transitive graph with one end is also unimodular (Proposition 2.1 in [16]), which allows the machinery developed in BLPS [9] to come into play.

Recall the Newman–Schulman 0–1–∞ law about the number of infinite clusters on transitive graphs. This result needs the percolation process to be automorphism invariant and insertion tolerant (cf. Definition 6.2). In general, insertion tolerance cannot be dropped (not even on the $\mathbb{Z}^2$ lattice; see Burton and Keane [22]), although in the present setting, remarkably enough, it can:

THEOREM 7.1 (Theorem 8.1 in [9]). *If $G = (V, E)$ is a planar nonamenable quasi-transitive graph with one end, then a.s. any automorphism invariant bond percolation $X \in \{0, 1\}^E$ has* 0, 1, *or infinitely many infinite clusters.*

PROOF. This proof from [16] differs somewhat from the original one in [9]. There is no loss of generality in assuming that the number of infinite clusters is an a.s. constant k. Assume for contradiction that $k \in \{2, 3, \dots\}$. By randomly deleting all edges of all infinite clusters but two, chosen uniformly at random, we still preserve automorphism invariance; thus we may assume $k = 2$. Call one of them C_1 and the other C_2, using a fair coin toss to decide which is which. Then turn on each edge e which does not meet C_2 and from which there is a path in G to C_1 that does not meet C_2. This preserves automorphism invariance, and expands C_1 to a larger infinite cluster $\hat{C}_1$ that, loosely speaking, sits as close to C_2 as is possible without touching it. Now define a bond percolation $\hat{X}^\dagger$ on the dual graph $G^\dagger$ (which is also planar, nonamenable, quasi-transitive and one-ended) by turning on exactly those edges $e^\dagger \in E^\dagger$ whose corresponding $e \in E$ are pivotal for connecting $\hat{C}_1$ to C_2. Then $\hat{X}^\dagger$ is $\mathrm{Aut}(G^\dagger)$-invariant, and consists (due to one-endedness of G) of a single bi-infinite open path. Hence, $\hat{X}^\dagger$ has a unique infinite cluster $C^\dagger$ with $p_{c,\mathrm{bond}}(C^\dagger) = 1$. On the other hand, the reasoning in the proof of Theorem 4.1, Case I, shows that any unique infinite cluster arising from automorphism invariant percolation in such a graph has $p_{c,\mathrm{bond}} < 1$, and this is the desired contradiction. $\square$

From here, Benjamini and Schramm [16] go on to show a number of interesting results for percolation in the hyperbolic plane. For starters, let G be as in

Theorem 7.1, let $G^\dagger$ be its planar dual, let X be an invariant bond percolation process on G, let $X^\dagger$ be its dual, and let k and $k^\dagger$ be their (possibly random) number of infinite clusters. Then Theorem 7.1 leaves nine possibilities for the value of $(k,k^\dagger)$: $(0,0)$, $(0,1)$, $(0,\infty)$, $(1,0)$, $(1,1)$, $(1,\infty)$, $(\infty,0)$, $(\infty,1)$ and (∞,∞), but the following result rules out four of them.

THEOREM 7.2 (Theorem 3.1 in [16]). *With G, $G^\dagger$, X and $X^\dagger$ as above, we have a.s. that*

$$(k,k^\dagger) \in \{(0,1),(1,0),(1,\infty),(\infty,1),(\infty,\infty)\}. \tag{16}$$

All five outcomes in (16) can actually happen. The cases $(1,\infty)$ and $(\infty,1)$ arise in the so-called uniform spanning forest model (see Theorem 8.2 in the next section), but cannot arise in i.i.d. percolation because of a local modification argument (Theorem 3.7 in [16]) that can turn $(k,k^\dagger)=(1,\infty)$ into $(2,\infty)$, contradicting Theorem 7.1. Benjamini and Schramm show that, in fact,

(17) outcomes $(0,1)$, $(1,0)$ and (∞,∞) are exactly those that happen for i.i.d. percolation,

and they prove the following remarkable result, establishing Conjecture 2.7 for the case of planar hyperbolic graphs:

THEOREM 7.3 (Theorem 1.1 in [16]). *Let $G=(V,E)$ be a planar nonamenable transitive graph with one end. Then $0<p_{c,\text{bond}}(G)<p_{u,\text{bond}}(G)<1$. The same is true for site percolation.*

This generalizes a result of Lalley [43] who showed

$$0<p_{c,\text{site}}(G)<p_{u,\text{site}}(G)<1$$

for a more restrictive class of graphs. Together with (17), this yields

$$(k,k^\dagger)=\begin{cases}(0,1), & \text{for } p\in(0,p_{c,\text{bond}}(G)),\\ (\infty,\infty), & \text{for } p\in(p_{c,\text{bond}}(G),p_{u,\text{bond}}(G)),\\ (1,0), & \text{for } p\in(p_{u,\text{bond}}(G),1).\end{cases}$$

Concerning the behavior at the critical values, Theorem 4.1 yields $(k,k^\dagger)=(0,1)$ for $p=p_{c,\text{bond}}(G)$, and by exchanging the roles of G and $G^\dagger$ we see that $(k,k^\dagger)=(1,0)$ at $p=p_{u,\text{bond}}(G)$. As mentioned in Section 2, this uniqueness of the infinite cluster already *at* the uniqueness critical value contrasts with the behavior obtained for certain other nonamenable transitive graphs by Schonmann [69] and Peres [65].

In fact, similar considerations for the hypothetical scenario that Theorem 7.3 fails show how smoothly Theorem 7.3 follows from (17). We would then have $(k,k^\dagger)=(0,1)$ for $p<p_{c,\text{bond}}(G)$ and $(k,k^\dagger)=(1,0)$ for $p>p_{c,\text{bond}}(G)$. Hence $p_{c,\text{bond}}(G^\dagger)=1-p_{c,\text{bond}}(G)$, and Theorem 4.1 applied to both G and $G^\dagger$ yields $(k,k^\dagger)=(0,0)$ at $p=p_{c,\text{bond}}(G)$. This, however, contradicts (17).

Benjamini and Schramm [16] go on to study properties of the infinite clusters and their limit points on the boundary of the hyperbolic disk (see also Lalley [43, 44] for further results in this direction). The final part of their paper [16] concerns i.i.d. site percolation on a certain random lattice in $\mathbb{H}^2$, namely the Delaunay triangulation of a Voronoi tesselation for a Poisson process with intensity $\lambda > 0$ in $\mathbb{H}^2$. The counterpart in $\mathbb{R}^d$ of such a model has also been studied; see Bollobás and Riordan [18, 19] for a recent breakthrough in $\mathbb{R}^2$. However, the phase diagram becomes richer and more interesting in the $\mathbb{H}^2$ case, not just because of the nonuniqueness phase between $p_{c,\text{site}}$ and $p_{u,\text{site}}$, but also because it becomes a true two-parameter family of models: in $\mathbb{R}^d$, changing λ is just a trivial rescaling of the model, while in $\mathbb{H}^2$ there is no such scale invariance.

More recently, Tykesson [78] considered a different way of doing percolation in $\mathbb{H}^2$ based on a Poisson process, namely to place a ball of a fixed hyperbolic radius R around each point, and consider percolation properties of the region covered by the union of the balls. (This is the so-called Boolean model of continuum percolation, which has received a fair amount of attention in $\mathbb{R}^d$; see, e.g., Meester and Roy [58].) Equally natural is to consider percolation properties of the vacant region (i.e., the complement of the covered region). To consider percolation properties of both regions simultaneously is analogous to working with the pair $(X, X^\dagger)$ in the discrete lattice setting. The results in [78] turn out mostly analogous to those of Benjamini and Schramm [16] discussed above for the discrete lattice setting. However, in an even more recent paper by Benjamini, Jonasson, Schramm and Tykesson [7], a phenomenon which is particular to the continuum setting is revealed. Namely, are there hyperbolic lines entirely contained in the vacant region, so that someone living in $\mathbb{H}^2$ can actually "see infinity" in some (necessarily random) directions? In $\mathbb{R}^d$ the answer to the analogous question is no (this is related to Olbers' paradox in astronomy; see, e.g., Harrison [38]) while in $\mathbb{H}^2$ the answer turns out to be yes in certain parts of the parameter space. The paper [7]—which, sadly, became one of the last by Oded Schramm—also contains results on various refinements and variants of this question.

8. Random spanning forests. So far, a lot has been said about percolation on nonamenable transitive graphs under the general assumption of automorphism invariance, but hardly anything about particular examples beyond the i.i.d. cases (other than a few examples specifically designed to be counterexamples). But as mentioned in Section 3, there *are* plenty of important examples, and time has come to discuss one of them: the uniform spanning forest. Later in this section, I will go on to discuss its cousin, the *minimal* spanning forest.

A spanning tree for a finite connected graph $G = (V, E)$ is a connected subgraph containing all vertices but no cycles. A *uniform spanning tree* for G is one chosen at random according to uniform distribution on the set of possible spanning trees. Replacing G by, say, the $\mathbb{Z}^d$ lattice, the number of possible spanning trees skyrockets to ∞, and it is no longer obvious how to make sense of the uniform spanning tree. Pemantle [64] managed to make sense of it: for any infinite locally finite connected graph $G = (V, E)$, let $\{G_1 = (V_1, E_1), G_2 = (V_2, E_2), \ldots\}$ be an increasing sequence of finite connected subgraphs of G, which exhausts G in the sense that each $e \in E$ and each $v \in V$ is in all but at most finitely many of the G_i's. Then, it turns out, the uniform spanning tree measures for $G_1, G_2, \ldots$ converge weakly to a probability measure μ_G on $\{0, 1\}^E$ which is independent of the exhaustion $\{G_1, G_2, \ldots\}$. In particular, μ_G is Aut(G)-invariant. Furthermore, it is concentrated on the event that there are no open cycles and no finite open clusters, so that in other words what we get is μ_G-a.s. a forest, all of whose trees are infinite. Naively, one might expect to get a single tree, but this is not always the case. Pemantle showed for the $\mathbb{Z}^d$ case that the number of trees is an a.s. constant N satisfying

$$N = \begin{cases} 1, & \text{if } d \leq 4, \\ \infty, & \text{otherwise.} \end{cases} \tag{18}$$

This $d \leq 4$ vs. $d > 4$ dichotomy is related to the fact that two independent SRW trajectories on the $\mathbb{Z}^d$ lattice intersect a.s. if and only if $d \leq 4$ (though this innocent-looking statement hides the fact that Pemantle [64] had to use deep results by Lawler [45] on so-called loop-erased random walk; the proof was simplified in the paper [11] to be discussed below). The behavior for $d > 4$ suggests *uniform spanning forest* as a better term than uniform spanning tree when G is infinite. The analysis of the uniform spanning forest μ_G builds to a large extent on the beautiful collection of identities between uniform spanning trees, electrical networks and random walks which has a long and disperse history beginning with the 1847 paper by Kirchhoff [42]. For instance, Rayleigh's Monotonicity Principle for effective resistances in electrical networks (see, e.g., Doyle and Snell [24]) underlies the stochastic monotonicity properties of the sequence $\mu_{G_1}, \mu_{G_2}, \ldots$ that allows us to deduce the existence of the limiting measure μ_G.

Some years after Pemantle's [64] pioneering 1991 paper, and at about the same time that BLPS [9] and BLPS [10] were written, the BLPS quartet started getting seriously interested in uniform spanning forests; see Lyons [54] for a more exact statement about the timing and relation betweens the various BLPS projects. This resulted in the magnificent paper *Uniform spanning forests* by Benjamini, Lyons, Peres and Schramm [11], which provides a unified treatment and a number of simplications of what was known on uniform spanning trees and forests, together with a host of new and important results. The methods involve, in addition to the aforementioned connections to random walks and electrical networks, also mass-transport ideas, Hilbert space projections, and Wilson's [80] substantial improvement on the Aldous–Broder algorithm [2, 20] for generating uniform spanning trees. Here, I will mention only a couple of the results from BLPS [11], but see Lyons [52] for a gentle introduction to the same topic (note that despite the inverted publication dates, the 1998 paper [52] surveys much of the original work in the 2001 paper [11]).

It turns out that there is another, equally natural, way to obtain a uniform spanning forest for an infinite graph $G = (V, E)$ via the exhaustion $\{G_1, G_2, \dots\}$ considered above, namely if for each i we consider the uniform spanning tree not on G_i but on the modified graph where an extra vertex w is introduced, together with edges $\langle v, w\rangle$ for all $v \in V_i \setminus V_{i-1}$. We think of this as a "wired" version of G_i, and the limiting measure on $\{0, 1\}^E$, which is denoted WSF_G and which exists for similar reasons as for μ_G, should be thought of as the *wired uniform spanning forest* for G. For consistency of terminology, we call μ_G the *free uniform spanning forest*, and switch to denoting it FSF_G. (The free and wired uniform spanning forests are analogous to the free and wired limiting measures for the random-cluster model; cf. [27].) The wired measure $WSF_{\mathbb{Z}^d}$ appeared implicitly in Pemantle [64] together with the result that $WSF_{\mathbb{Z}^d} = FSF_{\mathbb{Z}^d}$; this was made explicit in Häggström [30], and BLPS [11] noted that this extends to $WSF_G = FSF_G$ whenever G is transitive and amenable. On the other hand, $WSF_{\mathbb{T}_d} \neq FSF_{\mathbb{T}_d}$ for the $(d + 1)$-regular tree $\mathbb{T}_d$ when $d \geq 2$, so a first guess might be that for transitive graphs, $WSF_G = FSF_G$ is equivalent to amenability. This turns out not to be true, however, and BLPS [11] offer instead the following characterization (which does not require G to be transitive or even quasi-transitive). Recall that for a graph $G = (V, E)$ a function $f : V \to \mathbb{R}$ is called *harmonic* if for any $v \in V$ we have that $f(v)$ equals the average of $f(w)$ among all its neighbors w, and that f is called *Dirichlet* if $\sum_{\langle u,v\rangle\in E}(f(u) - f(v))^2 < \infty$.

THEOREM 8.1 (Theorem 7.3 in [11]). *For any graph G, we have $WSF_G = FSF_G$ if and only if G admits no nonconstant harmonic Dirichlet functions.*

As an example of a nonamenable transitive graph for which $WSF_G = FSF_G$ holds, we may take the Grimmett–Newman example discussed in Section 2; this follows from Theorem 8.1 in combination with the result of Thomassen [74, 75] that the Cartesian product of any two infinite graphs has no nonconstant harmonic Dirichlet functions.

A nice class of graphs where $WSF_G \neq FSF_G$ are the planar hyperbolic lattices considered in the previous section:

THEOREM 8.2 (Theorems 12.2 and 12.7 in BLPS [11]). *For any planar nonamenable transitive graph $G = (V, E)$ with one end, we have that $WSF_G \neq FSF_G$. In particular, they differ in the number of infinite clusters: FSF_G produces a unique infinite cluster a.s., while WSF_G produces infinitely many a.s. If $G^\dagger = (V^\dagger, E^\dagger)$ is the planar dual of G, and $X \in \{0, 1\}^E$ is given by WSF_G, then the dual percolation $X^\dagger \in \{0, 1\}^{E^\dagger}$ defined as in Section 7, has distribution $FSF_{G^\dagger}$.*

There is a lot more to quote from BLPS [11] concerning uniform spanning forests, but I will instead cite a result from a later paper by Benjamini, Kesten, Peres and Schramm [8]. Strictly speaking the result falls outside the scope of the present survey as I defined it in Section 1, because it concerns $\mathbb{Z}^d$, but it is so supremely beautiful that I find this inconsistency of mine to be motivated. For a bond percolation process $X \in \{0,1\}^E$ on a graph $G = (V, E)$, and two vertices $u, v \in V$, define the random variable $D_{\mathrm{closed}}(u, v)$ as the minimal number of closed edges that a path in G from u to v needs to traverse, and $D^{\max}_{\mathrm{closed}}$ as the supremum of $D_{\mathrm{closed}}(u, v)$ over all choices of $u, v \in V$. For percolation processes such as WSF_G and FSF_G where a.s. all vertices belong to infinite clusters, $D^{\max}_{\mathrm{closed}} = 0$ means precisely uniqueness of the infinite cluster, while $D^{\max}_{\mathrm{closed}} = 1$ means that uniqueness fails but that any pair of infinite clusters come within unit distance from each other somewhere in G. The dichotomy by Pemantle quoted in (18) says that for uniform spanning forests on $\mathbb{Z}^d$, we have $D^{\max}_{\mathrm{closed}} = 0$ a.s. for $d \le 4$ but $D^{\max}_{\mathrm{closed}} \ge 1$ a.s. when $d > 4$. Benjamini, Kesten, Peres and Schramm [8] proved the following refinement.

THEOREM 8.3. *For the uniform spanning forest measure $WSF_{\mathbb{Z}^d}$ (or equivalently $FSF_{\mathbb{Z}^d}$), we have, a.s.,*

$$D^{\max}_{\mathrm{closed}} = \begin{cases} 0, & \text{if } d \in \{1,2,3,4\}, \\ 1, & \text{if } d \in \{5,6,7,8\}, \\ 2, & \text{if } d \in \{9,10,11,12\}, \\ 3, & \text{if } d \in \{13,14,15,16\}, \\ & \vdots \end{cases}$$

Next, let us return briefly to the case where $G = (V, E)$ is a finite graph. Besides the uniform spanning tree, there is another, much-studied and equally natural, way of picking a random spanning tree for G: attach i.i.d. weights $\{U(e)\}_{e \in E}$ to the edges of G, and pick the spanning tree for G that minimizes the sum of the edge weights; this is the so-called *minimal spanning tree* for G. The marginal distribution of the $U(e)$'s has no effect on the distribution of the tree as long as it is free from atoms, but it turns out to facilitate the analysis to take it to be uniform on $[0, 1]$. It is easy to see that an edge $e \in E$ is included of the minimal spanning tree if and only if $U(e) < \max_{e \in C} U(e')$ for all cycles C containing e.

If instead G is infinite, this characterization can be taken as the definition, and the resulting random subgraph of G is called the *free minimal spanning forest*, the corresponding probability measure on $\{0,1\}^E$ being denoted $FMSF_G$. An alternative extension to infinite G is to include $e \in E$ if and only if $U(e) < \max_{e \in C} U(e')$ for all *generalized* cycles C containing e, where by a generalized cycle we mean a cycle or a bi-infinite self-avoiding path. This gives rise to the so-called *wired minimal spanning forest*, and a corresponding probability measure $WMSF_G$ on $\{0,1\}^E$. The study of $FMSF_G$ and $WMSF_G$ parallels the study of FSF_G and WSF_G in many ways, and I wish to draw the reader's attention to the paper *Minimal spanning forests* by Lyons, Peres and Schramm [55], where, in the words of the authors, "a key theme [is] to describe striking analogies, and important differences, between uniform and minimal spanning forests."

As for uniform spanning forests, one of the central issues is to determine when $FMSF_G = WMSF_G$. And as for uniform spanning forests, equality turns out to hold on $\mathbb{Z}^d$ and more generally on amenable transitive graphs. But moving on to the nonamenable case, the statements $FMSF_G = WMSF_G$ and $FSF_G = WSF_G$ are no longer equivalent.

THEOREM 8.4 (Proposition 3.6 in [55]). *On any G, we have $FMSF_G = WMSF_G$ if and only if for Lebesgue-a.e. $p \in [0, 1]$ it is the case that i.i.d. bond percolation on G with parameter p produces a.s. at most one infinite cluster.*

Hence we should expect to have $FMSF_G \neq WMSF_G$ for nonamenable quasi-transitive G; this is equivalent to Conjecture 2.7. One example where we do know that $FMSF_G \neq WMSF_G$ while $FSF_G = WSF_G$ is when G is the Grimmett–Newman graph $\mathbb{T}_d \times \mathbb{Z}$; cf. Theorem 8.1 and the comment following it.

Another consequence of having a nonuniqueness phase for i.i.d. bond percolation on G is that uniqueness of the infinite cluster fails for $WMSF_G$ (this is Corollary 3.7 in [12]). Determining the number of infinite clusters in $WMSF_G$ (or in $FMSF_G$) more generally is of course a central issue. On $\mathbb{Z}^d$ for $d \geq 2$, the only case that has been settled is $d = 2$, where Alexander [4] showed that the number of infinite clusters is 1. For higher dimensions, a dichotomy like Pemantle's (18) can be expected; Newman and Stein [62] conjectured that the switch from uniqueness to infinitely many infinite clusters should happen at $d = 8$ or 9.

A general difference—but at the same time a parallel—between the uniform spanning forest and the minimal spanning forest is that while the former has intimate connections to SRW on G, the latter seems equally intimately connected to i.i.d. bond percolation on G, such as in Theorem 8.4. [The key device for exposing the connections between the minimal spanning forest and i.i.d. percolation is the coupling in which we use the $U(e)$'s underlying $FMSF_G$ and $WMSF_G$ also for generating i.i.d. bond percolation for any p: an edge e is declared open on level p iff $U(e) < p$.] One striking instance of this parallel is the following. Morris [60] showed for any G that a.s. any component C of the wired uniform spanning forest has the property that SRW on it is recurrent, while Lyons, Peres and Schramm [55] showed for any G that a.s. any component C of the wired minimal spanning forest has $p_{c,\mathrm{bond}}(C) = 1$. Both results are sharp in the sense that in neither of them can the set of possible C's be narrowed further without making the statement false.

9. Postscript. The reader may have noticed that most of the work surveyed here is from the mid-to-late 1990s, and ask why this is. Is it because Oded got bored with percolation beyond $\mathbb{Z}^d$? No, it is not, and no, he did not. The main reason is Oded's discovery in [70] of SLE (Schramm–Loewner evolution, or stochastic Loewner evolution as Oded himself preferred to call it). This led him to the far-reaching and more urgent project, beginning with a now-famous series of papers with Greg Lawler and Wendelin Werner [46–49], of understanding SLE and how it arises as a scaling limit of various critical models in two dimensions. But that is a different story.

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