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Kolmogorov equation associated to the stochastic reflection problem on a smooth convex set of a Hilbert space II

Viorel Barbu[a,1], Giuseppe Da Prato[b,2] and Luciano Tubaro[c,2]

[a]*University Al. I. Cuza and Institute of Mathematics Octav Mayer, Blvd. Carol, 11, 700506 Iasi, Romania. E-mail:* vb41@uaic.ro
[b]*Scuola Normale Superiore, Piazza dei Cavalieri, 7, 56126 Pisa, Italy. E-mail:* daprato@sns.it
[c]*Department of Mathematics, University of Trento, Via Sommarive, 14, 38050 Povo (Trento), Italy. E-mail:* tubaro@science.unitn.it

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Abstract. This work is concerned with the existence and regularity of solutions to the Neumann problem associated with a Ornstein–Uhlenbeck operator on a bounded and smooth convex set K of a Hilbert space H. This problem is related to the reflection problem associated with a stochastic differential equation in K.

Résumé. Dans cet article nous étudions l'existence et la régularité des solutions d'un problème de Neumann associé à un opérateur de Ornstein–Uhlenbeck défini sur un domaine convexe K, borné et régulier dans un espace de Hilbert H. Le problème est lié à un problème de réflexion associé à une équation différentielle stochastique dans le domaine K.

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1. Introduction

We are given a non-degenerate Gaussian measure $\mu = N_Q$ with mean 0 and covariance operator Q in a separable Hilbert space H (with scalar product $\langle\cdot,\cdot\rangle$ and norm $|\cdot|$). We fix $\alpha \in [0,1]$ and consider the following Neumann problem on a regular convex subset K of H,

$$\begin{cases} \lambda\varphi - L_\alpha\varphi = f & \text{in } K, \\ \frac{\partial\varphi}{\partial n} = 0 & \text{on } \Sigma, \end{cases} \tag{1.1}$$

where $\lambda > 0$, Σ is the boundary of K, $f : H \to \mathbb{R}$ is a given function on H and L is the Ornstein–Uhlenbeck operator

$$L_\alpha\varphi := \frac{1}{2}\operatorname{Tr}\big[Q^{1-\alpha}D^2\varphi\big] - \frac{1}{2}\big\langle x, Q^{-\alpha}D\varphi\big\rangle. \tag{1.2}$$

We shall denote by A the self-adjoint operator $A := Q^{-1}$. Since μ is not degenerate, there exists $\delta > 0$ such that $\langle Ax, x\rangle \ge \delta|x|^2$, $\forall x \in D(A)$ for some $\delta > 0$. Of course we have also that $\operatorname{Tr} A^{-1} < \infty$.

Concerning K, we shall assume that

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Hypothesis 1.1. *There exists a convex C^∞-function $g : H \to [0, \infty)$ with $g(0) = 0$, $g'(0) = 0$ and $D^2 g$ positively defined, i.e., $\langle D^2 g(x)h, h\rangle \geq \kappa |h|^2$, $\forall h \in H, x \in H$, where $\kappa > 0$, such that*

$$K = \{x \in H : g(x) \leq 1\}, \qquad \Sigma = \{x \in H : g(x) = 1\}.$$

Moreover, we also suppose that $D^2 g$ is bounded on K and that g and all its derivatives grow at infinity at the most polynomially.

We denote by μ_Σ the surface measure induced by μ on Σ (see [5,11,12]) and by $\mathbf{n}(y)$ the inner normal to K at y, that is

$$\mathbf{n}(y) = \frac{Dg(y)}{|Dg(y)|} \quad \forall y \in \Sigma. \tag{1.3}$$

By Hypothesis 1.1 it follows that

Lemma 1.2. *K is convex, closed and bounded. Moreover there are $\gamma, \rho, \delta > 0$ such that*

$$\langle Dg(x), x\rangle \geq \gamma |x|^2 \quad \forall x \in H, \qquad |Dg(x)| \leq \delta \quad \forall x \in K, \tag{1.4}$$

$$g(x) \geq \frac{\gamma}{2}|x|^2 \quad \forall x \in H, \tag{1.5}$$

$$|Dg(x)| \geq \rho \quad \forall x \in \Sigma. \tag{1.6}$$

Proof. We have

$$Dg(x) = \int_0^1 D^2 g(tx)x \, \mathrm{d}t \quad \forall x \in H.$$

Therefore

$$\langle Dg(x), x\rangle = \int_0^1 \langle D^2 g(tx)x, x\rangle \mathrm{d}t \geq \kappa |x|^2 \quad \forall x \in H,$$

which implies the first estimate in (1.4) and also that Dg is bounded on K.

Similarly by

$$g(x) = \int_0^1 \langle Dg(tx), x\rangle \mathrm{d}t \quad \forall x \in H$$

and (1.4) it follows (1.5). This implies that K is bounded and $0 \in \mathring{K}$, where $\mathring{K}$ is the interior of K. Finally by (1.4) it follows (1.6) otherwise there is $\{x_n\} \subset \Sigma$ such that $Dg(x_n) \to 0$. Taking into account that $0 < g(x) \leq \langle Dg(x), x\rangle$ and that $\{x_n\}$ is bounded the latter implies that $1 = g(x_n) \to 0$ which is of course absurd. □

It is easy to see that μ is the unique invariant measure of the Ornstein–Uhlenbeck process in H,

$$\begin{cases} \mathrm{d}X(t) + \frac{1}{2} A^\alpha X(t)\, \mathrm{d}t = A^{(\alpha-1)/2}\, \mathrm{d}W(t), \\ X(0) = x \in H, \end{cases} \tag{1.7}$$

where W is a cylindrical Wiener process in a filtered probability space

$$(\Omega, \mathscr{F}, \mathbb{P}, \{\mathscr{F}_t\}_{t\geq 0})$$

of the form

$$\langle W(t), z\rangle = \sum_{k=1}^\infty \beta_k(t)\langle z, e_k\rangle, \quad t \geq 0\ \forall z \in H.$$

Here $\{\beta_k\}$ is a sequence of mutually independent real Brownian motions on $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\ge 0}, \mathbb{P})$ (see, e.g., [9]) and $\{e_k\}$ is an orthonormal basis in H which will be taken as a system of eigen-functions for A for simplicity, i.e.,

$$Ae_k = a_k e_k \quad \forall k \in \mathbb{N},$$

where obviously $a_k \ge \delta$.

Let us describe the results of the paper. First we consider the symmetric Dirichlet form

$$a(\varphi, \psi) = \int_K \langle A^{(\alpha-1)/2} D\varphi, A^{(\alpha-1)/2} D\psi\rangle \, \mathrm{d}\nu \quad \forall \varphi, \psi \in C^1(K), \tag{1.8}$$

where $\nu = \frac{1}{\mu(K)}\mu$ and show that a is closable (equivalently continuous) in the space $W^{1,2}_{A^{\alpha-1}}(K, \nu)$ (see Section 2). We notice that for $\alpha = 0$ this space reduces to the Malliavin space $D^{1,2}(K, \nu)$. Here we use a recent result about an integration by parts formula on K proved in [4].

Then we define a weak solution of the Neumann problem (1.1) in the usual way as a solution $\varphi \in W^{1,2}_{A^{\alpha-1}}(K, \nu)$ of the equation

$$\lambda \int_H \varphi\psi \, \mathrm{d}\mu + \frac{1}{2} a(\varphi, \psi) = \int_H f\psi \, \mathrm{d}\nu \quad \forall \psi \in W^{1,2}_{A^{\alpha-1}}(K, \nu), \tag{1.9}$$

where $f \in L^2(K, \nu)$.

If we denote by N the Kolmogorov operator corresponding to the Dirichlet form (1.8) then (1.9) can be equivalently written as $\lambda\varphi - N\varphi = f$. The second-order regularity of φ as well as the proof that it satisfies the Neumann boundary condition on Σ in the sense of trace is one of the main results of this work (Theorem 3.5). In the previous work [4] this result was proved in the case $\alpha = 1$. It should be emphasized that, though the treatment closely follows [4], there are, however, some notable differences which will be mentioned later on. The nice feature of problem (1.1) is that for all α the corresponding Ornstein–Uhlenbeck operators (1.7) have the same invariant measure $\mu = N_Q$ and this allows a unified treatment. Moreover, since the trace assumption on $A^{-\alpha}$ is weaker than that on A^{-1} we can treat into this general functional setting reflection problem not treatable for $\alpha = 1$.

We note that in specific situations A is a linear elliptic operator with suitable boundary conditions on a bounded and open subset $\mathscr{O}$ of $\mathbb{R}^d$. (See Section 5 below.)

The second part of the paper is devoted to the construction of a process $X(t, x)$ such that the semigroup P_t generated by N is expressed as $P_t\varphi(x) = \mathbb{E}[\varphi(X(t, x))]$ where X is formally the solution to the following stochastic variational inequality

$$\begin{cases} \mathrm{d}X + \frac{1}{2}A^\alpha X \, \mathrm{d}t + A^{\alpha-1} N_K(X) \, \mathrm{d}t \ni A^{(\alpha-1)/2} \, \mathrm{d}W_t, \\ X(0) = x, \end{cases} \tag{1.10}$$

where N_K is the normal cone to K, i.e.,

$$\begin{cases} N_K(x) = \varnothing & \text{if } x \in \mathring{K}, \\ N_K(x) = \{\lambda \mathbf{n}(x), \lambda \ge 0\} & \text{if } x \in \Sigma. \end{cases}$$

When $\alpha = 1$ this problem is known in literature as the stochastic reflection problem on convex set K and was studied in finite-dimensional spaces H by [2,3,6,8]. If H is infinite-dimensional, however, no results concerning existence and uniqueness of strong solutions with the notable exception of the 1992 work of Nualart and Pardoux [14] which treats this problem in $H = L^2(0, 1)$ and for $K = \{y \in L^2(0, 1)\colon\ y \ge 0 \text{ a.e. in } (0, 1)\}$.

The transition semigroup

$$(P_t\varphi)(x) = \mathbb{E}\big[\varphi\big(X(t, x)\big)\big] \quad \forall \varphi \in C_b(K), t \ge 0 \tag{1.11}$$

formally relates the Neumann problem (1.1) and Eq. (1.10) but no rigorous proof of this conjecture exists except the cases mentioned above (see also [16]). However, in [1] this is proven for $\alpha = 1$ via some sharp arguments involving theory of Langrangian flows. In particular, it is proven the existence and uniqueness of a martingale solution in sense of Stroock and Varadhan.

When $\alpha \in [0, 1)$ the operator $A^{\alpha-1} N_K$ is not monotone in H, so no existence results in the literature for Eq. (1.10) seems to be available. The second part of the paper is concerned with representation of semigroup P_t as a transition Markov semigroup in the special case where K is a ball and $\mathrm{Tr}[A^{2\delta-1}] < \infty$ for some $\delta > 0$. The proof of existence of the process is constructive and relies on some sharp BV-estimates on solutions to approximating equation associated with (1.10) and the Skorohod theorem.

2. Notations and preliminary results

Everywhere in the following $D\varphi$ is the derivative of a function $\varphi : H \to \mathbb{R}$. By $D^2\varphi : H \to L(H, H)$ we shall denote the second derivative of φ. We shall denote also by $C_b(H)$ and $C_b^k(H)$, $k \in \mathbb{N}$, the spaces of all continuous and bounded functions on H and, respectively, of k-times differentiable functions with continuous and bounded derivatives. The space $C^k(K)$, $k \in \mathbb{N}$, is defined as the space of restrictions of functions of $C_b^k(H)$ to the subset K. Also we refer to [7,9] for notations and basic results on infinite-dimensional processes.

We denote by $\{e_k\}$ the orthonormal basis in H of eigenfunctions of Q, i.e.

$$Qe_k = \lambda_k e_k \quad \forall k \in \mathbb{N}, \tag{2.1}$$

where $\lambda_k = \frac{1}{a_k}$ with $\{a_k, k \in \mathbb{N}\}$ the eigenvalues of A, by D_k the derivative in the direction e_k and set $x_k = \langle x, e_k \rangle$ for all $x \in H$, $k \in \mathbb{N}$. We denote by $\mathscr{E}(H)$ the linear span of all exponential functions $\{e^{\langle x, e_h \rangle}, h \in \mathbb{N}\}$.

Then we recall a basic integration by parts formula in H.

$$\int_H D_k\varphi \, \mathrm{d}\mu = \frac{1}{\lambda_k} \int_H x_k \varphi \, \mathrm{d}\mu \quad \forall k \in \mathbb{N}, \varphi \in C_b^1(H). \tag{2.2}$$

We denote by M_α: $C_b^1(H) \subset L^2(H, \mu) \to L^2(H, \mu; H)$

$$M_\alpha \varphi := A^{(\alpha-1)/2} D\varphi, \quad \varphi \in C_b^1(H).$$

Here M_0 is the Malliavin derivative [12]. It is well known (and easy to show thanks to (2.2)) that M_α is closable. We shall denote its closure by M_α and also by $A^{(\alpha-1)/2} D$.

The domain of the closure of M_α will be denoted by $W^{1,2}_{A^{\alpha-1}}(H, \mu)$. It is a Hilbert space with the inner product

$$\begin{aligned}
\langle \varphi, \psi \rangle_{W^{1,2}_{A^{\alpha-1}}(H,\mu)} &= \int_H \varphi\psi \, \mathrm{d}\mu + \int_H \big\langle A^{(\alpha-1)/2} D\varphi, A^{(\alpha-1)/2} D\varphi \big\rangle \, \mathrm{d}\mu \\
&= \int_H \varphi\psi \, \mathrm{d}\mu + \sum_{k=1}^\infty \int_H \lambda_k^{\alpha-1} D_k\varphi D_k\psi \, \mathrm{d}\mu.
\end{aligned}$$

Denote by $L^2(H, \mu)$ and $L^2(K, \nu)$ the space of μ-square integrable functions (ν-square integrable functions) on H and K, respectively.

In a similar way we define the space $W^{2,2}_{A^{\alpha-1}}(H, \nu)$. The corresponding inner product is defined by (see [4,7,10])

$$\begin{aligned}
\langle \varphi, \psi \rangle_{W^{2,2}_{A^{\alpha-1}}(H,\mu)} &= \langle \varphi, \psi \rangle_{W^{1,2}_{A^{\alpha-1}}(H,\mu)} + \int_H \mathrm{Tr}\big[A^{2(\alpha-1)} D^2\varphi D^2\psi\big] \, \mathrm{d}\mu \\
&= \langle \varphi, \psi \rangle_{W^{1,2}_{A^{\alpha-1}}(H,\mu)} + \sum_{h,k=1}^\infty \int_H \lambda_h^{1-\alpha} \lambda_k^{1-\alpha} D^2_{h,k}\varphi D^2_{h,k}\psi \, \mathrm{d}\mu.
\end{aligned}$$

2.1. *The integration by parts formula on K*

The following result is proved in [4]. For reader's convenience we recall it here, deferring to the Appendix for a proof (Theorem A.2).

Lemma 2.1. *Let $K = \{x \in H : g(x) \le 1\}$ where $g \in C^2(H)$ is convex and $|Dg(x)|^{-1} \in L^p(H,\mu)$ for all $p \ge 1$. Then*

$$\int_K D_h\varphi(x)\mu(\mathrm{d}x) = \frac{1}{\mu(K)}\int_\Sigma n_h(y)\varphi(y)\mu_\Sigma(\mathrm{d}y) + \frac{1}{\lambda_h}\int_K x_h\varphi(x)\mu(\mathrm{d}x) \quad \forall h \in H, \varphi \in C_b^1(H), \tag{2.3}$$

where $n_h(y) = \langle \mathbf{n}(y), e_h\rangle$.

With the help of this result we can define the spaces $W^{1,2}_{A^{\alpha-1}}(K,\nu)$ and $W^{2,2}_{A^{\alpha-1}}(K,\nu)$ as in [4].

Moreover, we can define the trace of a function $\varphi \in W^{1,2}_{A^{\alpha-1}}(K,\nu)$ thanks to the following result.

Proposition 2.2. *For any $\varphi \in C_b^1(H)$ we have*

$$\int_\Sigma \left|Q^{1/2}\mathbf{n}(y)\right|^2\varphi^2(y)\mu_\Sigma(\mathrm{d}y) \le C\left(\int_K \varphi^2(x)\mu(\mathrm{d}x) + \int_K \left|Q^{1/2}D\varphi(x)\right|^2\mu(\mathrm{d}x)\right). \tag{2.4}$$

Proof. Let $\varphi \in C_b^1(H)$ and $h \in \mathbb{N}$. Replacing in (2.3) φ with $\lambda_h D_h g\varphi^2$ and then $D_h\varphi$ with $2\lambda_h D_h g\varphi D_h\varphi + \lambda_h D_h^2 g\varphi^2$, yields

$$2\int_K \lambda_h D_h g\varphi D_h\varphi\,\mathrm{d}\mu + \int_K \lambda_h D_h^2 g\varphi^2\,\mathrm{d}\mu = \frac{1}{\mu(K)}\int_\Sigma \lambda_h n_h(y) D_h g\varphi^2\,\mathrm{d}\mu_\Sigma + \int_K x_h D_h g\varphi^2\,\mathrm{d}\mu.$$

Summing up on h yields

$$2\int_K \langle QD\varphi, Dg\rangle\varphi\,\mathrm{d}\mu + \int_K \mathrm{Tr}\left[QD^2g\right]\varphi^2\,\mathrm{d}\mu = \frac{1}{\mu(K)}\int_\Sigma \left\langle Q\mathbf{n}(y), Dg\right\rangle\varphi^2\,\mathrm{d}\mu_\Sigma + \int_K \langle x, Dg\rangle\varphi^2\,\mathrm{d}\mu.$$

But, taking into account (1.3), (1.6) we have

$$\left\langle Q\mathbf{n}(y), Dg(y)\right\rangle = \left|Dg(y)\right|\left\langle Q\mathbf{n}(y), \mathbf{n}(y)\right\rangle \ge \rho\left\langle Q\mathbf{n}(y), \mathbf{n}(y)\right\rangle \quad \forall y \in \Sigma.$$

Substituting in the previous identity yields

$$\frac{1}{\rho\mu(K)}\int_\Sigma \left\langle Q\mathbf{n}(y), \mathbf{n}(y)\right\rangle\varphi^2\,\mathrm{d}\mu_\Sigma + \int_K \langle x, Dg\rangle\varphi^2\,\mathrm{d}\mu \le 2\int_K \langle QD\varphi, Dg\rangle\varphi\,\mathrm{d}\mu + \int_K \mathrm{Tr}\left[QD^2g\right]\varphi^2\,\mathrm{d}\mu.$$

Taking into account that K is bounded and that Dg, D^2g are bounded on K, the conclusion follows. □

We can now define the trace of a function $\varphi \in W^{1,2}_{A^{\alpha-1}}(K,\nu)$. Let $\{\varphi_j\} \subset C^1_b(K)$ be such that

$$\begin{cases} \lim_{n\to\infty} \varphi_j = \varphi & \text{in } L^2(K,\nu), \\ \lim_{n\to\infty} A^{(\alpha-1)/2} D\varphi_j = A^{(\alpha-1)/2} D\varphi & \text{in } L^2(K,\nu). \end{cases}$$

Then by (2.4) it follows that the sequence $\{|Q^{1/2}\mathbf{n}(y)|\gamma_0(\varphi_j)\}$, where $\gamma_0(\varphi_j)$ denotes the trace of φ_j, is convergent in $L^2(\Sigma,\mu_\Sigma)$ to a function $\psi \in L^2(\Sigma,\mu_\Sigma)$. Then we define the trace $\gamma_0(\varphi)$ of φ as

$$\gamma_0(\varphi) = \frac{\psi}{|Q^{1/2}\mathbf{n}(y)|}.$$

2.2. *Trace of the normal derivative*

Proposition 2.3. *Assume that $\varphi \in W^{2,2}_{A^{\alpha-1}}(K,\nu)$. Then the following estimate holds,*

$$\int_\Sigma \left|Q^{1/2}\mathbf{n}(y)\right|^2 \left|A^{(\alpha-1)/2} D\varphi\right|^2(y)\mu_\Sigma(\mathrm{d}y) \\ \le C\left(\int_K \left|A^{(\alpha-1)/2} D\varphi(x)\right|^2 \mu(\mathrm{d}x) + \int_K \mathrm{Tr}\left[\left(A^{\alpha-1} D^2\varphi(x)\right)^2\right]\mu(\mathrm{d}x)\right). \tag{2.5}$$

Proof. Let $\varphi \in W^{2,2}_{A_{\alpha-1}}(K,\nu)$ and let $\{\varphi_j\} \subset C^2(K)$ be convergent to φ in $W^{2,2}_{A_{\alpha-1}}(K,\nu)$. For $i \in \mathbb{N}$ we apply (2.3) to $a_i^{(\alpha-1)/2} D_i\varphi_j$. We have

$$\int_\Sigma \left|Q^{1/2}\mathbf{n}(y)\right|^2 \left|a_i^{(\alpha-1)/2} D_i\varphi_j\right|^2(y)\mu_\Sigma(\mathrm{d}y) \\ \le C a_i^{(\alpha-1)/2}\left(\int_K \left|D_i\varphi_j(x)\right|^2 \mu(\mathrm{d}x) + a_i^{(\alpha-1)/2}\int_K \left|A^{(\alpha-1)/2} D D_i\varphi_j(x)\right|^2 \mu(\mathrm{d}x)\right).$$

Summing up on i yields

$$\int_\Sigma \left|Q^{1/2}\mathbf{n}(y)\right|^2 \left|A^{(\alpha-1)/2} D\varphi_j\right|^2(y)\mu_\Sigma(\mathrm{d}y) \\ \le C\left(\int_K \left|A^{(\alpha-1)/2} D_i\varphi_j(x)\right|^2 \mu(\mathrm{d}x) + \int_K \mathrm{Tr}\left[\left(A^{\alpha-1} D^2\varphi_j(x)\right)^2\right]\mu(\mathrm{d}x)\right).$$

Now the conclusion follows letting $j \to \infty$. □

3. The penalized problem

We are here concerned for any $\varepsilon > 0$ with the penalized equation

$$\begin{cases} \mathrm{d}X_\varepsilon(t) + \left[\frac{1}{2}A^\alpha X_\varepsilon(t) + A^{\alpha-1}\beta_\varepsilon\left(X_\varepsilon(t)\right)\right]\mathrm{d}t = A^{(\alpha-1)/2}\,\mathrm{d}W_t, \\ X_\varepsilon(0) = x, \end{cases} \tag{3.1}$$

where

$$\beta_\varepsilon(x) = \frac{1}{\varepsilon}\left(x - \Pi_K(x)\right) \quad \forall x \in H.$$

Since β_ε is Lipschitz continuous, it is easily seen that Eq. (3.1) which can be equivalently be written as

$$X_\varepsilon(t) = \mathrm{e}^{-tA^\alpha/2}x - \int_0^t A^{\alpha-1}\mathrm{e}^{-A^\alpha(t-s)/2}\beta_\varepsilon\big(X_\varepsilon(s)\big)\,\mathrm{d}s + \int_0^t \mathrm{e}^{-A^\alpha(t-s)/2}A^{(\alpha-1)/2}\,\mathrm{d}W_s$$

has a unique mild solution

$$X_\varepsilon(\cdot, x) \in L^2\big(\Omega, C\big([0,+\infty); H\big)\big).$$

Moreover, it is easy to see that there is a unique invariant probability measure ν_ε for X_ε given by

$$\nu_\varepsilon(\mathrm{d}x) = Z_\varepsilon^{-1}\mathrm{e}^{-d_K^2(x)/\varepsilon}, \tag{3.2}$$

where d_K is the distance to K and

$$Z_\varepsilon = \int_H \mathrm{e}^{-d_K^2(y)/\varepsilon}\mu(\mathrm{d}y). \tag{3.3}$$

The corresponding Kolmogorov operator reads as follows,

$$N_\varepsilon\varphi = L\varphi - \big\langle A^{\alpha-1}\beta_\varepsilon(x), D\varphi\big\rangle, \quad \varphi \in \mathscr{E}(H)\ \forall\varepsilon > 0, \tag{3.4}$$

where L is the Ornstein–Uhlenbeck operator

$$L\varphi = \frac{1}{2}\operatorname{Tr}\big[A^{\alpha-1}D^2\varphi\big] - \frac{1}{2}\big\langle x, A^\alpha D\varphi\big\rangle \quad \forall\varphi \in \mathscr{E}(H).$$

One can easily check that ν_ε (as defined in (3.2) and (3.3)) is an invariant measure for N_ε and that

$$\int_H N_\varepsilon\varphi\psi\,\mathrm{d}\nu_\varepsilon = -\frac{1}{2}\int_H \big\langle A^{\alpha-1}D\varphi, D\psi\big\rangle\mathrm{d}\nu_\varepsilon \quad \forall\varphi,\psi \in \mathscr{E}(H). \tag{3.5}$$

Moreover, since β_ε is Lipschitz continuous, the operator N_ε is essentially m-dissipative in $L^2(H,\nu_\varepsilon)$ (we still denote by N_ε its closure) and $\mathscr{E}(H)$ is a core for N_ε, see [7].

Section 3.1 below is devoted to several estimates for $(\lambda I - N_\varepsilon)^{-1}f$ where $f \in L^2(H,\nu_\varepsilon)$. Then these estimates are used in Section 3.2 to prove that $(\lambda I - N_\varepsilon)^{-1}f$ converges to $(\lambda I - N)^{-1}f$ as $\varepsilon \to 0$, where N is the self-adjoint operator corresponding to the Dirichlet form (1.8) (see (3.32) below), for any $f \in L^2(K,\nu)$. Moreover, we shall end up the section by proving a few sharp properties of the domain $D(N)$ of N.

3.1. *Estimates for* $(\lambda I - N_\varepsilon)^{-1}f$

Let $\lambda > 0$, $\varepsilon > 0$, $\varphi \in \mathscr{E}(H)$. We set

$$f_\varepsilon = \lambda\varphi - N_\varepsilon\varphi. \tag{3.6}$$

We are going to prove for later use a few estimates of the first and second derivatives of φ. To this purpose, since β_ε is not differentiable, we need a further approximation $\beta_{\varepsilon,\eta}$ of β_ε.

More precisely, for any $\varepsilon > 0$, $\eta > 0$ we consider the penalized equation

$$\begin{cases} \mathrm{d}X_{\varepsilon,\eta}(t) + \big(\frac{1}{2}A^\alpha X_{\varepsilon,\eta}(t) + A^{\alpha-1}\beta_{\varepsilon,\eta}\big(X_{\varepsilon,\eta}(t)\big)\big)\,\mathrm{d}t = A^{-(1-\alpha)/2}\,\mathrm{d}W_t, \\ X_{\varepsilon,\eta}(0) = x, \end{cases} \tag{3.7}$$

where $\beta_{\varepsilon,\eta}$ is the regularization of β_ε given by the infinite-dimensional mollifier

$$\beta_{\varepsilon,\eta}(x) = \mathrm{e}^{-\eta A}\int_H \beta_\varepsilon\big(\mathrm{e}^{-\eta A}x + y\big)\mu_\eta(\mathrm{d}y), \quad x \in H, \eta > 0. \tag{3.8}$$

Here μ_η is the Gaussian measure on H with mean 0 and covariance operator

$$Q_\eta := \frac{1}{2}A^{-1}\big(1-\mathrm{e}^{-2\eta A}\big).$$

Notice that $\beta_{\varepsilon,\eta}$ is of class C^∞ and its derivatives of all order are bounded. Moreover, $\beta_{\varepsilon,\eta}$ is a monotone mapping in H and

$$\lim_{\eta\to\infty}\beta_{\varepsilon,\eta}(x)=\beta_\varepsilon(x) \quad \text{in } H\ \forall\varepsilon>0, x\in H. \tag{3.9}$$

Since $\beta_{\varepsilon,\eta}$ is Lipschitz, Eq. (3.7) has a unique mild solution $X_{\varepsilon,\eta}(t,x)$. Moreover, it is easy to see that there is a unique invariant probability measure $\nu_{\varepsilon,\eta}$ for (3.7) given by

$$\nu_{\varepsilon,\eta}(\mathrm{d}x)=Z_{\varepsilon,\eta}^{-1}\mathrm{e}^{-d^2_{K,\eta}(x)/\varepsilon}, \tag{3.10}$$

where

$$Z_{\varepsilon,\eta}=\int_H \mathrm{e}^{-d^2_{K,\eta}(y)/\varepsilon}\mu(\mathrm{d}y). \tag{3.11}$$

$\frac{1}{2\varepsilon}d^2_{K,\eta}$ is the potential associated with $\beta_{\varepsilon,\eta}$, that is

$$\frac{1}{2\varepsilon}Dd^2_{K,\eta}(x)=\beta_{\varepsilon,\eta}(x) \quad \forall x\in H, \tag{3.12}$$

equivalently

$$\frac{1}{2\varepsilon}d^2_{K,\eta}(x)=\int_0^1\big\langle\beta_{\varepsilon,\eta}(tx),x\big\rangle\mathrm{d}t \quad \forall x\in H.$$

The corresponding Kolmogorov operator reads as follows,

$$N_{\varepsilon,\eta}\varphi=L\varphi-\big\langle A^{\alpha-1}\beta_{\varepsilon,\eta}(x),D\varphi\big\rangle, \quad \varphi\in\mathscr{E}(H), \varepsilon>0, \tag{3.13}$$

where L is the Ornstein–Uhlenbeck operator introduced before. Then $\nu_{\varepsilon,\eta}$ is an invariant measure for $N_{\varepsilon,\eta}$ and

$$\int_H N_{\varepsilon,\eta}\varphi\psi\,\mathrm{d}\nu_{\varepsilon,\eta}=-\frac{1}{2}\int_H\big\langle A^{\alpha-1}D\varphi,D\psi\big\rangle\mathrm{d}\nu_{\varepsilon,\eta} \quad \forall\varphi,\psi\in\mathscr{E}(H). \tag{3.14}$$

Moreover, since $\beta_{\varepsilon,\eta}$ is Lipschitz continuous, the operator $N_{\varepsilon,\eta}$ is essentially m-dissipative in $L^2(H,\nu_{\varepsilon,\eta})$ and $\mathscr{E}(H)$ is a core for $N_{\varepsilon,\eta}$ (see [10]). We shall denote again by $N_{\varepsilon,\eta}$ the closure of $N_{\varepsilon,\eta}$ in $L^2(H,\nu_{\varepsilon,\eta})$. Moreover, we have

$$\lim_{\eta\to 0}\big|X_{\varepsilon,\eta}(t,x)-X_\varepsilon(t,x)\big|=0 \quad \forall t\geq 0, x\in H, \mathbb{P}\text{-a.s.} \tag{3.15}$$

Indeed by (3.1) and (3.7) we have for all $t\geq 0$, $\varepsilon>0$, $\eta>0$,

$$\begin{aligned}
&X_{\varepsilon,\eta}(t,x)-X_\varepsilon(t,x)\\
&\quad=-\int_0^t A^{1-\alpha}\mathrm{e}^{-A^\alpha(t-s)/2}\big(\beta_{\varepsilon,\eta}\big(X_{\varepsilon,\eta}(t,x)\big)-\beta_\varepsilon\big(X_\varepsilon(t,x)\big)\big)\,\mathrm{d}s \quad \mathbb{P}\text{-a.s.}
\end{aligned}$$

and this yields

$$\begin{aligned}
\big|X_{\varepsilon,\eta}(t,x)-X_\varepsilon(t,x)\big| &\leq C\int_0^t\big|\beta_{\varepsilon,\eta}\big(X_{\varepsilon,\eta}(t,x)\big)-\beta_\varepsilon\big(X_\varepsilon(t,x)\big)\big|\,\mathrm{d}s\\
&\quad+C\int_0^t\big|X_{\varepsilon,\eta}(t,x)-X_\varepsilon(t,x)\big|\,\mathrm{d}s \quad \forall t\geq 0, \varepsilon,\eta>0\ \mathbb{P}\text{-a.s.},
\end{aligned}$$

because

$$\|\beta_{\varepsilon,\eta}\|_{\mathrm{Lip}} \le \frac{1}{\varepsilon} \quad \forall \eta > 0.$$

Since

$$\lim_{\eta\to 0} \beta_{\varepsilon,\eta}\big(X_\varepsilon(t,x)\big) = \beta_\varepsilon\big(X_\varepsilon(t,x)\big),$$

we obtain by Gronwall's lemma that (3.15) holds.

Lemma 3.1. *Let $\lambda > 0, \varepsilon > 0, \eta > 0, \varphi \in \mathscr{E}(H)$ and set*

$$f_{\varepsilon,\eta} = \lambda\varphi - N_{\varepsilon,\eta}\varphi. \tag{3.16}$$

Then the following estimates hold

$$\int_H \varphi^2 \,\mathrm{d}\nu_{\varepsilon,\eta} \le \frac{1}{\lambda^2}\int_H f_{\varepsilon,\eta}^2 \,\mathrm{d}\nu_{\varepsilon,\eta}, \tag{3.17}$$

$$\int_H \big|A^{(\alpha-1)/2} D\varphi\big|^2 \,\mathrm{d}\nu_{\varepsilon,\eta} \le \frac{2}{\lambda}\int_H f_{\varepsilon,\eta}^2 \,\mathrm{d}\nu_{\varepsilon,\eta}, \tag{3.18}$$

$$\begin{aligned}\lambda \int_H \big|A^{(\alpha-1)/2} D\varphi\big|^2 \,\mathrm{d}\nu_{\varepsilon,\eta} &+ \frac{1}{2}\int_H \mathrm{Tr}\big[\big(A^{\alpha-1} D^2\varphi\big)^2\big] \,\mathrm{d}\nu_{\varepsilon,\eta} \\ &+ \frac{1}{2}\int_H \big|A^{\alpha/2} D\varphi\big|^2 \,\mathrm{d}\nu_{\varepsilon,\eta} \le 4\int_H f_{\varepsilon,\eta}^2 \,\mathrm{d}\nu_{\varepsilon,\eta}.\end{aligned} \tag{3.19}$$

Proof. Multiplying both sides of (3.16) by φ, taking into account (3.14) and integrating in $\nu_{\varepsilon,\eta}$ over H, yields

$$\lambda \int_H \varphi^2 \,\mathrm{d}\nu_{\varepsilon,\eta} + \frac{1}{2}\int_H \big|A^{(\alpha-1)/2} D\varphi\big|^2 \,\mathrm{d}\nu_{\varepsilon,\eta} = \int_H \varphi f_{\varepsilon,\eta} \,\mathrm{d}\nu_{\varepsilon,\eta}. \tag{3.20}$$

Now (3.17) and (3.18) follow easily from the Hölder inequality. To prove (3.19) we differentiate both sides of (3.16) in the direction of e_k and obtain that

$$\lambda D_k\varphi - N_{\varepsilon,\eta} D_k\varphi + \frac{1}{2} a_k D_k\varphi + \sum_{h=1}^{\infty} \langle D_k\beta_{\varepsilon,\eta} e_h, e_k\rangle D_h\varphi = D_k f_\varepsilon.$$

Next we multiply both sides of latter equation by $a_k^{\alpha-1} D_k\varphi$. Taking into account (3.14), integrating in $\nu_{\varepsilon,\eta}$ over H and summing up over k, yields

$$\begin{aligned}\lambda \int_H \big|A^{(\alpha-1)/2} D\varphi\big|^2 \,\mathrm{d}\nu_{\varepsilon,\eta} &+ \frac{1}{2}\int_H \mathrm{Tr}\big[\big(A^{\alpha-1} D^2\varphi\big)^2\big] \,\mathrm{d}\nu_{\varepsilon,\eta} \\ &+ \frac{1}{2}\int_H \big|A^{\alpha/2} D\varphi\big|^2 \,\mathrm{d}\nu_{\varepsilon,\eta} + \int_{K^c} \big\langle D\beta_{\varepsilon,\eta} A^{(\alpha-1)/2} D\varphi, A^{(\alpha-1)/2} D\varphi\big\rangle \,\mathrm{d}\nu_{\varepsilon,\eta} \\ &= \int_H \big\langle A^{(\alpha-1)/2} D\varphi, A^{(\alpha-1)/2} Df_{\varepsilon,\eta}\big\rangle \,\mathrm{d}\nu_{\varepsilon,\eta}.\end{aligned} \tag{3.21}$$

Noting finally that, again in view of (3.14),

$$\begin{aligned}&\int_H \big\langle A^{(\alpha-1)/2} D\varphi, A^{(\alpha-1)/2} Df_{\varepsilon,\eta}\big\rangle \,\mathrm{d}\nu_{\varepsilon,\eta} \\ &\quad = 2\int_H f_\varepsilon^2 \,\mathrm{d}\nu_{\varepsilon,\eta} - 2\lambda \int_H f_{\varepsilon,\eta}\varphi \,\mathrm{d}\nu_{\varepsilon,\eta} \le 4\int_H f_{\varepsilon,\eta}^2 \,\mathrm{d}\nu_{\varepsilon,\eta},\end{aligned}$$

the conclusion follows. □

Taking into account (3.15) and that

$$\lim_{\eta\to 0} N_{\varepsilon,\eta}\varphi(x) = N_\varepsilon\varphi(x) \quad \forall\varepsilon > 0,$$

letting $\eta \to 0$ we obtain the following result.

Corollary 3.2. *Let $\lambda > 0, \varepsilon > 0, \varphi \in \mathscr{E}(H)$ and let*

$$f_\varepsilon = \lambda\varphi - N_\varepsilon\varphi. \tag{3.22}$$

Then the following estimates hold

$$\int_H \varphi^2 \, d\nu_\varepsilon \le \frac{1}{\lambda^2}\int_H f_\varepsilon^2 \, d\nu_\varepsilon, \tag{3.23}$$

$$\int_H \left|A^{(\alpha-1)/2} D\varphi\right|^2 d\nu_\varepsilon \le \frac{2}{\lambda}\int_H f_\varepsilon^2 \, d\nu_\varepsilon, \tag{3.24}$$

$$\begin{aligned}&\lambda\int_H \left|A^{(\alpha-1)/2} D\varphi\right|^2 d\nu_\varepsilon + \frac{1}{2}\int_H \mathrm{Tr}\left[\left(A^{\alpha-1}D^2\varphi\right)^2\right] d\nu_\varepsilon\\ &\quad + \frac{1}{2}\int_H \left|A^{\alpha/2} D\varphi\right|^2 d\nu_\varepsilon \le 4\int_H f_\varepsilon^2 \, d\nu_\varepsilon.\end{aligned} \tag{3.25}$$

Now we are able to prove.

Proposition 3.3. *Let $\lambda > 0$, $f \in L^2(H, \nu_\varepsilon)$ and let φ_ε be the solution of the equation*

$$\lambda\varphi_\varepsilon - N_\varepsilon\varphi_\varepsilon = f. \tag{3.26}$$

Then $\varphi_\varepsilon \in W^{2,2}_{A^{\alpha-1}}(H, \nu_\varepsilon)$, $A^{\alpha/2}D\varphi_\varepsilon \in L^2(H, \nu_\varepsilon)$ and the following estimates hold

$$\int_H \varphi_\varepsilon^2 \, d\nu_\varepsilon \le \frac{1}{\lambda^2}\int_H f^2 \, d\nu_\varepsilon, \tag{3.27}$$

$$\int_H \left|A^{(\alpha-1)/2} D\varphi_\varepsilon\right|^2 d\nu_\varepsilon \le \frac{2}{\lambda}\int_H f^2 \, d\nu_\varepsilon, \tag{3.28}$$

$$\begin{aligned}&\lambda\int_H \left|A^{(\alpha-1)/2} D\varphi_\varepsilon\right|^2 d\nu_\varepsilon + \frac{1}{2}\int_H \mathrm{Tr}\left[\left(A^{\alpha-1}D^2\varphi_\varepsilon\right)^2\right] d\nu_\varepsilon\\ &\quad + \frac{1}{2}\int_H \left|A^{\alpha/2} D\varphi_\varepsilon\right|^2 d\nu_\varepsilon \le 4\int_H f^2 \, d\nu_\varepsilon.\end{aligned} \tag{3.29}$$

Proof. Inequality (3.27) is obvious since by (3.5), N_ε is dissipative in $L^2(H, \nu_\varepsilon)$. Let us prove (3.28). Let $\lambda > 0$, $f \in L^2(H, \nu_\varepsilon)$ and let $\varphi_{\varepsilon,}$ be the solution to Eq. (3.26). Since $\mathscr{E}(H)$ is a core for N_ε there exists a sequence $\{\varphi_{\varepsilon,n}\}_{n\in\mathbb{N}} \subset \mathscr{E}(H)$ such that

$$\lim_{n\to\infty} \varphi_{\varepsilon,n} \to \varphi_\varepsilon, \qquad \lim_{n\to\infty} N_\varepsilon\varphi_{\varepsilon,n} \to N_\varepsilon\varphi_\varepsilon \quad \text{in } L^2(H, \nu_\varepsilon).$$

We set $f_{\varepsilon,n} = \lambda\varphi_{\varepsilon,n} - N_\varepsilon\varphi_{\varepsilon,n}$. Clearly, $f_{\varepsilon,n} \to f$ in $L^2(H, \nu_\varepsilon)$ as $n \to \infty$. We claim that $\varphi_\varepsilon \in W^{1,2}_{A^{\alpha-1}}(H, \nu_\varepsilon)$ and that

$$\lim_{n\to\infty} A^{(\alpha-1)/2} D\varphi_{\varepsilon,n} \to A^{(\alpha-1)/2} D\varphi_\varepsilon \quad \text{in } L^2(H, \nu_\varepsilon; H),$$

which will imply (3.28).

Let $m, n \in \mathbb{N}$; then by (3.24) it follows that

$$\int_H \left|A^{(\alpha-1)/2} D\varphi_{\varepsilon,n} - A^{(\alpha-1)/2} D\varphi_{\varepsilon,m}\right|^2 \mathrm{d}\nu_\varepsilon \le \frac{2}{\lambda}\int_H |f_{\varepsilon,n} - f_{\varepsilon,m}|^2 \,\mathrm{d}\nu_\varepsilon.$$

Therefore the sequence $\{\varphi_{\varepsilon,n}\}_{n\in\mathbb{N}}$ is Cauchy in $W^{1,2}_{A^{\alpha-1}}(H, \nu_\varepsilon)$ and the conclusion follows. The estimate (3.29) follows similarly by (3.25). □

We conclude this subsection with an integration by parts formula needed later. We set

$$V := \left\{\psi \in C_b^1(K)\colon |Q^{1/2}\mathbf{n}(y)|^{-1}\psi \in C_b(K)\right\}. \tag{3.30}$$

Lemma 3.4. *Let $\varphi \in D(N_\varepsilon)$ and $\psi \in V$. Then the following identity holds.*

$$\begin{aligned}\int_K N_\varepsilon\varphi\psi \,\mathrm{d}\nu &= -\frac{1}{2}\int_K \left\langle A^{(\alpha-1)/2} D\varphi, A^{(\alpha-1)/2} D\psi\right\rangle \mathrm{d}\nu \\ &\quad + \frac{1}{\mu(K)}\int_\Sigma \left\langle A^{\alpha-1}\gamma(D\varphi), \mathbf{n}(y)\right\rangle \psi \,\mathrm{d}\mu_\Sigma. \end{aligned}\tag{3.31}$$

Proof. We first notice that the last integral in (3.31) is meaningful since

$$\begin{aligned}&\left|\int_\Sigma \left\langle A^{\alpha-1}\gamma(D\varphi), \mathbf{n}(y)\right\rangle \psi \,\mathrm{d}\mu_\Sigma\right|^2 \\ &\quad \le \left\|A^{\alpha-1}\right\| \int_\Sigma \left|A^{(\alpha-1)/2}\gamma(D\varphi)\right|^2 \left|Q^{1/2}\mathbf{n}(y)\right|^2 \mathrm{d}\mu_\Sigma \int_\Sigma \psi^2 \left|Q^{1/2}\mathbf{n}(y)\right|^{-2} \mathrm{d}\mu_\Sigma < \infty\end{aligned}$$

by (2.5).

Now, taking in account that $\mathscr{E}(H)$ is a core for N_ε, it is sufficient to prove (3.31) for $\varphi \in \mathscr{E}(H)$. By the basic integration by parts formula (2.2) we deduce, for any $i \in \mathbb{N}$ and $\psi \in V$ that

$$\begin{aligned}\int_K D_i\varphi D_i\psi \,\mathrm{d}\nu &= -\int_K D_i^2\varphi\psi \,\mathrm{d}\nu + \frac{1}{\mu(K)}\int_\Sigma \gamma(D_i\varphi)\big(\mathbf{n}(y)\big)_i \psi \,\mathrm{d}\mu_\Sigma \\ &\quad + \frac{1}{\lambda_i}\int_K x_i D_i\varphi\psi \,\mathrm{d}\nu.\end{aligned}$$

It follows that

$$\begin{aligned}a_i^{\alpha-1}\int_K D_i\varphi D_i\psi \,\mathrm{d}\nu &= -a_i^{\alpha-1}\int_K D_i^2\varphi\psi \,\mathrm{d}\nu \\ &\quad + \frac{1}{\mu(K)}a_i^{\alpha-1}\int_\Sigma \gamma(D_i\varphi)\big(\mathbf{n}(y)\big)_i \psi \,\mathrm{d}\mu_\Sigma + \frac{1}{2}a_i^\alpha \int_K x_i D_i\varphi\psi \,\mathrm{d}\nu.\end{aligned}$$

Now, summing up on i yields

$$\begin{aligned}\int_K \left\langle A^{(\alpha-1)/2} D\varphi, A^{(\alpha-1)/2} D\psi\right\rangle \mathrm{d}\nu &= -\int_K \mathrm{Tr}\left[A^{\alpha-1}D^2\varphi\right]\psi \,\mathrm{d}\nu \\ &\quad + \frac{1}{\mu(K)}\int_\Sigma \left\langle A^{\alpha-1}\gamma(D\varphi), \mathbf{n}(y)\right\rangle \mathrm{d}\mu_\Sigma + 2\int_K \left\langle x, A^\alpha D\varphi\right\rangle \psi \,\mathrm{d}\nu,\end{aligned}$$

which is precisely Eq. (3.31). □

3.2. *Convergence of* $\{\varphi_\varepsilon\}$ *as* $\varepsilon \to 0$

Let $N : D(N) \subset L^2(K,\nu) \to L^2(K,\nu)$ be the operator defined by

$$\begin{cases} \langle N\varphi, \psi\rangle_{L^2(K,\nu)} = -\frac{1}{2}a(\varphi,\psi) \quad \forall \psi \in W^{1,2}_{A^{(\alpha-1)/2}}(K,\nu), \varphi \in D(N), \\ D(N) = \big\{\varphi \in W^{1,2}_{A^{(\alpha-1)/2}}(K,\nu)\colon \big|a(\varphi,\psi)\big| \le C|\varphi|_{L^2(K,\nu)}|\psi|_{L^2(K,\nu)},\ \forall \psi \in W^{1,2}_{A^{(\alpha-1)/2}}(K,\nu)\big\}. \end{cases} \tag{3.32}$$

The operator L is self-adjoint in $L^2(K,\nu)$ and the Neumann problem (1.1) (or equivalently (1.9)) reduces to

$$\lambda\varphi - N\varphi = f. \tag{3.33}$$

We are going to show that for each $f \in L^2(K,\nu)$ and $\varepsilon \to 0$, $\varphi_\varepsilon = (\lambda I - N_\varepsilon)^{-1} f$ is convergent in $L^2(K,\nu)$ to $\varphi = (\lambda I - N)^{-1} f$ and derive so, via the estimate proven in Proposition 3.3, high order regularity properties for the solution φ to (3.33).

We first note that for $f \in C_b(H)$ we have

$$\varphi_\varepsilon(x) = \mathbb{E}\int_0^\infty e^{-\lambda t} f\big(X_\varepsilon(t,x)\big)\,dt \quad \forall x \in H. \tag{3.34}$$

Now, by a standard argument it follows that from (3.34) if $f \in C_b^1(H)$ we have

$$\sup_{x\in H}\big|D\varphi_\varepsilon(x)\big| \le \frac{1}{\lambda}\|Df\|_{C_b(H)} \quad \forall \varepsilon, \lambda > 0. \tag{3.35}$$

Theorem 3.5 below is the main result of this section.

Theorem 3.5. *Let* $\lambda > 0$, $f \in L^2(K,\nu)$ *and let* φ_ε *be the solution of Eq.* (3.26). *Then* $\{\varphi_\varepsilon\}$ *is strongly convergent in* $L^2(K,\nu)$ *to* $\varphi = (\lambda I - N)^{-1} f$ *where* N *is defined by* (3.32).

Moreover, the following statements hold.

(i) $\lim_{\varepsilon\to 0} A^{(\alpha-1)/2} D\varphi_\varepsilon = A^{(\alpha-1)/2} D\varphi$ *in* $L^2(K,\nu;H)$,
(ii) $\varphi \in W^{2,2}_{A^{\alpha-1}}(K,\nu)$ *and* $|A^{\alpha/2}D\varphi| \in L^2(K,\nu)$,
(iii) φ *fulfills the Neumann condition*

$$\big\langle A^{\alpha-1}\gamma\big(D\varphi(x)\big), \mathbf{n}(x)\big\rangle = 0, \quad \mu_\Sigma \text{ a.e. on } \Sigma, \tag{3.36}$$

where $\gamma(D\varphi(x))$ *is defined by Proposition* 2.3.

In particular, since N is dissipative Theorem 3.5 amounts to say that for each $f \in L^2(K,\nu)$ the equation $\lambda\varphi - N\varphi = f$ has a unique solution φ satisfying (ii), (iii).

Proof of Theorem 3.5. Without danger of confusion we shall denote again by f the restriction $f|_K$ of f to K. In fact each $f \in L^2(K,\nu)$ can be extended by 0 outside K to a function in $L^2(H,\nu)$. By this convention, everywhere in the sequel $(\lambda I - N)^{-1} f$ for $f \in L^2(H,\nu)$ means $(\lambda I - N)^{-1} f|_K$.

Step 1. We have

$$\lim_{\varepsilon\to 0}\varphi_\varepsilon = (\lambda I - N)^{-1} f \quad \text{in } L^2(K,\nu). \tag{3.37}$$

In fact by (3.28), (3.29) it follows that there exist a sequence $\{\varepsilon_k\} \to 0$ and $\varphi \in W^{1,2}_{A^{\alpha-1}}(K,\nu)$ such that

$$\varphi_{\varepsilon_k} \to \varphi, \quad \text{weakly in } L^2(K,\nu),$$
$$A^{(\alpha-1)/2} D\varphi_{\varepsilon_k} \to A^{(\alpha-1)/2} D\varphi, \quad \text{weakly in } L^2(K,\nu;H).$$

Let $\psi \in C_b^1(H)$ and notice that by (3.5) and by (3.26) we have the identity

$$\frac{1}{2}\int_H \langle A^{(\alpha-1)/2} D\varphi_\varepsilon, A^{(\alpha-1)/2} D\psi\rangle \, \mathrm{d}\nu_\varepsilon = \int_H (f-\lambda\varphi_\varepsilon)\psi \, \mathrm{d}\nu_\varepsilon.$$

Equivalently

$$\begin{aligned}&\frac{1}{2}\int_K \langle A^{(\alpha-1)/2} D\varphi_\varepsilon, A^{(\alpha-1)/2} D\psi\rangle \, \mathrm{d}\nu + \frac{1}{2}\int_{K^c} \langle A^{(\alpha-1)/2} D\varphi_\varepsilon, A^{(\alpha-1)/2} D\psi\rangle \, \mathrm{d}\nu_\varepsilon \\ &\quad = \int_H (f-\lambda\varphi_\varepsilon)\psi \, \mathrm{d}\nu_\varepsilon. \end{aligned} \tag{3.38}$$

Since by (3.28) we have

$$\begin{aligned}\left|\int_{K^c} \langle A^{(\alpha-1)/2} D\varphi_\varepsilon, A^{(\alpha-1)/2} D\psi\rangle \, \mathrm{d}\nu_\varepsilon\right|^2 &\le \int_H \left|A^{(\alpha-1)/2} D\varphi_\varepsilon\right|^2 \mathrm{d}\nu_\varepsilon \int_{K^c} \left|A^{(\alpha-1)/2} D\psi\right|^2 \mathrm{d}\nu_\varepsilon \\ &\le \frac{2}{\lambda}\int_H f^2 \, \mathrm{d}\nu_\varepsilon \int_{K^c} \left|A^{(\alpha-1)/2} D\psi\right|^2 \mathrm{d}\nu_\varepsilon \to 0,\end{aligned}$$

as $\varepsilon \to 0$, it follows by (3.38) that

$$\frac{1}{2}\int_K \langle A^{(\alpha-1)/2} D\varphi, A^{(\alpha-1)/2} D\psi\rangle \, \mathrm{d}\nu = \int_K (f-\lambda\varphi)\psi \, \mathrm{d}\nu \quad \forall \psi \in C_b^1(K).$$

Obviously, this identity extends to all $\psi \in W^{1,2}_{A^{\alpha-1}}(K,\nu)$, which implies that $\varphi_\varepsilon \to (\lambda I - N)^{-1} f$ weakly in $L^2(K,\nu)$ as $\varepsilon \to 0$.

Step 2. We have

$$\begin{cases} \lim_{\varepsilon\to 0} \varphi_\varepsilon = \varphi & \text{in } L^2(K,\nu), \\ \lim_{\varepsilon\to 0} A^{(\alpha-1)/2} D\varphi_\varepsilon = A^{(\alpha-1)/2} D\varphi & \text{in } L^2(K,\nu;K). \end{cases}$$

We first assume that $f \in C_b^1(H)$. Let us start from the identity

$$\int_H N_\varepsilon \varphi_\varepsilon \varphi_\varepsilon \, \mathrm{d}\nu_\varepsilon = -\frac{1}{2}\int_H \left|A^{(\alpha-1)/2} D\varphi_\varepsilon\right|^2 \mathrm{d}\nu_\varepsilon \quad \forall \varphi \in D(N_\varepsilon), \tag{3.39}$$

which follows from (3.5). By (3.26) and (3.39) we see that

$$\frac{1}{2}\int_H \left|A^{(\alpha-1)/2} D\varphi_\varepsilon\right|^2 \mathrm{d}\nu_\varepsilon = -\int_H (\lambda\varphi_\varepsilon - f)\varphi_\varepsilon \, \mathrm{d}\nu_\varepsilon, \tag{3.40}$$

which implies in virtue of (3.32), (3.33)

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_K \left(\frac{1}{2}\left|A^{(\alpha-1)/2} D\varphi_\varepsilon\right|^2 + \lambda\varphi_\varepsilon^2\right) \mathrm{d}\nu_\varepsilon &= \int_K f\varphi \, \mathrm{d}\nu \\ &= -\langle N\varphi, \varphi\rangle + \lambda\int_K \varphi^2 \, \mathrm{d}\nu \\ &= \int_K \left(\frac{1}{2}\left|A^{(\alpha-1)/2} D\varphi\right|^2 + \lambda\varphi^2\right) \mathrm{d}\nu. \end{aligned}\tag{3.41}$$

Here we have used the fact that

$$\lim_{\varepsilon\to 0}\int_{K^c} \left|A^{(\alpha-1)/2} D\varphi_\varepsilon\right|^2 \mathrm{d}\nu_\varepsilon = 0$$

which follows taking into account (3.35).

Therefore, there exists a sequence $\{\varepsilon_k\} \downarrow 0$ such that

$$\begin{cases} \varphi_{\varepsilon_k} \to \varphi & \text{weakly in } L^2(K,\nu), \\ A^{(\alpha-1)/2} D\varphi_{\varepsilon_k} \to A^{(\alpha-1)/2} D\varphi & \text{weakly in } L^2(K,\nu;H), \\ \lim_{k\to\infty} \int_K \big(\lambda\varphi_{\varepsilon_k}^2 + \frac{1}{2}\big|A^{(\alpha-1)/2} D\varphi_{\varepsilon_k}\big|^2\big)\,\mathrm{d}\nu = \int_K \big(\lambda\varphi^2 + \frac{1}{2}\big|A^{(\alpha-1)/2} D\varphi\big|^2\big)\,\mathrm{d}\nu. \end{cases}$$

This implies that $\varphi_{\varepsilon_k} \to \varphi$ strongly in $L^2(K,\nu)$ and $A^{(\alpha-1)/2} D\varphi_{\varepsilon_k} \to A^{(\alpha-1)/2} D\varphi$ strongly in $L^2(K,\nu;H)$.

We finally assume that $f \in L^2(H,\nu)$. Since $C_b^1(K)$ is dense in $L^2(K,\nu)$, there exists a sequence $\{f_n\} \subset C_b^1(H)$ strongly convergent in $L^2(K,\nu)$ to f. Set $\varphi_{n,\varepsilon} = (\lambda I - N_\varepsilon)^{-1} f_n$. By (3.28) we have

$$\int_H \big|A^{(\alpha-1)/2} D\varphi_\varepsilon - A^{(\alpha-1)/2} D\varphi_{n,\varepsilon}\big|^2\,\mathrm{d}\nu_\varepsilon \le \frac{2}{\lambda}\int_K |f - f_n|^2\,\mathrm{d}\nu,$$

which implies

$$\int_K \big|A^{(\alpha-1)/2} D\varphi_\varepsilon - A^{(\alpha-1)/2} D\varphi_{n,\varepsilon}\big|^2\,\mathrm{d}\nu \le \frac{2}{\lambda}\int_K |f - f_n|^2\,\mathrm{d}\nu.$$

So, again $A^{(\alpha-1)/2} D\varphi_{\varepsilon_k} \to A^{(\alpha-1)/2} D\varphi$ strongly in $L^2(K,\nu;H)$ as claimed.

Step 3. We have $\varphi \in W^{2,2}_{A^{\alpha-1}}(K,\nu)$ and $A^{\alpha/2} D\varphi \in L^2(K,\nu)$.

By estimate (3.29) we have that $\{\varphi_\varepsilon\}$ is bounded in $W^{2,2}_{A^{\alpha-1}}(K,\nu)$. Therefore there is a subsequence, still denoted $\{\varphi_\varepsilon\}$ which converges to φ in $W^{2,2}_{A^{\alpha-1}}(K,\nu)$. In the same way we show that $A^{\alpha/2} D\varphi \in L^2(K,\nu)$.

Step 4. Checking the Neumann condition for φ.

We recall that (see from (3.31))

$$\int_K N_\varepsilon\varphi_\varepsilon\psi\,\mathrm{d}\nu = -\frac{1}{2}\int_K \big\langle A^{(\alpha-1)/2} D\varphi_\varepsilon, A^{(\alpha-1)/2} D\psi\big\rangle\,\mathrm{d}\nu + \frac{1}{\mu(K)}\int_\Sigma \psi\big\langle A^{\alpha-1}\gamma(D\varphi_\varepsilon), \mathbf{n}(y)\big\rangle\,\mathrm{d}\mu_\Sigma. \tag{3.42}$$

Recalling that for $\varepsilon \to 0$, $N_\varepsilon\varphi_\varepsilon = \lambda\varphi_\varepsilon - f \to \lambda\varphi - f = N\varphi$ in $L^2(K,\nu)$ and by Proposition 2.3 we have

$$\big|Q^{1/2}\mathbf{n}(y)\big|\big\langle A^{\alpha-1}\gamma(D\varphi_\varepsilon), \mathbf{n}(y)\big\rangle \to \big|Q^{1/2}\mathbf{n}(y)\big|\big\langle A^{\alpha-1}\gamma(D\varphi), \mathbf{n}(y)\big\rangle,$$

in $L^2(\Sigma,\mu_\Sigma)$, it follows by (3.42) that

$$\int_\Sigma \big\langle A^{\alpha-1}\gamma(D\varphi), \mathbf{n}(y)\big\rangle\psi\,\mathrm{d}\mu_\Sigma = 0 \quad \forall\psi \in V,$$

where V is defied by (3.30). Since V is dense in $L^2(\Sigma,\mu_\Sigma)$ the conclusion follows.

This completes the proof of the theorem. □

4. The process associated with the reflection problem

Throughout this section the following hypothesis will be assumed.

Hypothesis 4.1.

(i) $\alpha \in [0, \frac{1}{2}]$ *and there is* $\delta \in (0,1)$ *such that* $\mathrm{Tr}[A^{2\delta-1}] < \infty$.
(ii) $K = B(0,1) = \{x \in H\colon |x| \le 1\}$.

We are going to construct a stochastic process $X = X(t,x)$ on a probability space $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{\mathbb{P}})$ associated with the semigroup P_t generated by N on $L^2(K,\nu)$, i.e.,

$$(P_t f)(x) = \tilde{\mathbb{E}}\big[f\big(X(t,x)\big)\big] \quad \forall f \in C_b(H), x \in H.$$

The main result, Theorem 4.10 below amounts to saying that there is a cadlag H-valued process X with this property.

To this aim we need first some sharp estimates on solution $X_\varepsilon(t,x)$ to approximating Eq. (3.1), that is

$$\begin{cases} \mathrm{d}X_\varepsilon + \frac{1}{2}A^\alpha X_\varepsilon\,\mathrm{d}t + A^{\alpha-1}\beta_\varepsilon(X_\varepsilon)\,\mathrm{d}t = A^{(\alpha-1)/2}\,\mathrm{d}W_t, \quad t \ge 0, \\ X_\varepsilon(0) = x. \end{cases} \tag{4.1}$$

4.1. *Estimates for X_ε*

We set

$$|x|_a = \big|A^a x\big|, \qquad \langle x, y\rangle_a = \big\langle A^a x, A^a y\big\rangle, \quad \forall x, y \in D\big(A^a\big), 0 < a < 1$$

and

$$W_A(t) = \int_0^t \mathrm{e}^{-A^\alpha(t-s)/2} A^{(\alpha-1)/2}\,\mathrm{d}W_s, \quad t \ge 0.$$

Lemma 4.2. *The following estimates hold*

$$\mathbb{E}\Big[\sup_{t\in[0,T]} \big|W_A(t)\big|_\delta^{2m}\Big] \le CT^{m+1/m+1} \quad \forall T > 0, \tag{4.2}$$

$$\mathbb{E}\Big[\sup_{t\in[T-h,T]} \big|W_A(t) - W_A(t-h)\big|^{2m}\Big] \le Ch^\rho T^{m+1/m+1} \quad \forall T > 0, \forall h > 0, \tag{4.3}$$

where $m > 1$ and $1 < \rho < m$.

Here C is a positive constant independent of ω, T and ε.

Proof. Since the proof is identical with Theorem 2.9 in [7] we shall sketch it only for convenience. We have (see [7], p. 25)

$$W_A(t) = \frac{\sin(\pi\gamma)}{\pi} \int_0^t \mathrm{e}^{-(t-s)A^\alpha/2}(t-s)^{\gamma-1} Y(s)\,\mathrm{d}s, \tag{4.4}$$

where $0 < \gamma < 1$ and

$$Y(t) = \int_0^t \mathrm{e}^{-(t-s)A^\alpha/2}(t-s)^{-\gamma} A^{(\alpha-1)/2}\,\mathrm{d}W_s.$$

In the following we shall fix $m > \frac{1}{2\gamma}$ and $0 < \gamma < \frac{1}{2}$.

We have

$$\left|\int_0^t \mathrm{e}^{-(t-s)A^\alpha/2}(t-s)^{\gamma-1} f(s)\,\mathrm{d}s\right| \le Ct^{\gamma-1/(2m)}|f|_{L^2(0,T;H)} \tag{4.5}$$

and therefore

$$\sup_{t\in[0,T]} \big|W_A(t)\big|_\delta^{2m} \le CT^{2m(\gamma-1/(2m))} \int_0^T \big|Y(s)\big|_\delta^{2m}\,\mathrm{d}s.$$

On the other hand, under Hypothesis 4.1 we have

$$\mathbb{E}\big(\big|Y(s)\big|_\delta^{2m}\big) \le Cs^m \quad \forall s > 0$$

and this implies (4.2) as claimed.

As regards (4.3), we have by (4.4) that

$$\begin{aligned}
&W_A(t) - W_A(t-h) \\
&\quad = \frac{\sin(\pi\gamma)}{\pi}\int_0^{t-h} \mathrm{e}^{-(t-h-s)A^\alpha/2}\big[(t-s)^{\gamma-1} - (t-h-s)^{\gamma-1}\mathrm{e}^{-hA^\alpha/2}\big]Y(s)\,\mathrm{d}s \\
&\qquad + \frac{\sin(\pi\gamma)}{\pi}\int_{t-h}^{t} \mathrm{e}^{-(t-s)A^\alpha/2}(t-s)^{\gamma-1}Y(s)\,\mathrm{d}s.
\end{aligned}$$

Then by (4.5) we have that

$$\begin{aligned}
&\sup_{t\in[h,T-h]}\big|W_A(t) - W_A(t-h)\big|^{2m} \\
&\quad \le C\left(h^{2m\gamma}\int_0^T \big|Y(s)\big|^{2m}\,\mathrm{d}s + \int_0^T \big|\big(I - \mathrm{e}^{-hA^\alpha/2}\big)Y(s)\big|^{2m}\,\mathrm{d}s + h^{2m-1}\int_0^T \big|Y(s)\big|^{2m}\,\mathrm{d}s\right) \\
&\quad \le C\big(h^{2m\gamma} + h^{2m-1} + h^m\big)\int_0^T \big|Y(s)\big|^{2m}\,\mathrm{d}s
\end{aligned}$$

because $|(I - \mathrm{e}^{-hA^\alpha/2})Y| \le Ch^{1/2}|Y|_{\alpha/2}$. Then we get as above that (4.3) holds. □

In the following we set $y_\varepsilon = X_\varepsilon - W_A$ and notice that y_ε is the solution to equation

$$\begin{cases} \frac{\mathrm{d}y_\varepsilon}{\mathrm{d}t}(t) + \frac{1}{2}A^\alpha y_\varepsilon(t) + A^{\alpha-1}\beta_\varepsilon\big(y_\varepsilon(t) + W_A(t)\big) = 0, \quad t \ge 0, \quad \mathbb{P}\text{-a.s.} \\ y_\varepsilon(0) = x \end{cases} \tag{4.6}$$

Equivalently

$$\begin{cases} A^{1-\alpha}\frac{\mathrm{d}y_\varepsilon}{\mathrm{d}t}(t) + \frac{1}{2}Ay_\varepsilon(t) + \beta_\varepsilon\big(y_\varepsilon(t) + W_A(t)\big) = 0, \quad t \ge 0, \quad \mathbb{P}\text{-a.s.} \\ y_\varepsilon(0) = x \end{cases} \tag{4.7}$$

Denote by $BV([0,T];H)$ the space of all H-valued functions with bounded variation on $[0,T]$ and denote by $\|y\|_{BV([0,T];H)}$ the total variation of $y \in BV([0,T];H)$. We set $\eta = \frac{1-\alpha}{2}$.

Lemma 4.3 below is the main estimate.

Lemma 4.3. *Assume that $x \in D(A^\eta)$, then there exists a constant $C > 0$ independent of $\omega \in \Omega$, $T > 0$ and ε, h such that*

$$\begin{aligned}
&\int_0^T \big|y_\varepsilon(t)\big|_{1/2}^2\,\mathrm{d}t + \sup_{t\in[0,T]}\big|y_\varepsilon(t)\big|_\eta^2 + \int_0^T \big|\beta_\varepsilon\big(y_\varepsilon(t) + W_A(t)\big)\big|\,\mathrm{d}t \\
&\quad \le C\left(|x|_{\delta/2}^2 + \frac{1}{\mu}\sup_{t\in[0,T]}\big|W_A(t)\big|_\delta^2\right)\left(1 - h^p\sup_{s,t\in[0,T]}\big|W_A(t) - W_A(s)\big||t-s|^{-p}\right. \\
&\qquad \left. - \mu^\delta\sup_{s\in[0,T]}\big|W_A(s)\big|_\delta\right)^{-1},
\end{aligned} \tag{4.8}$$

$$\|y_\varepsilon\|_{BV([0,T];H)} \le C\left(|x|_\eta + \int_0^T \big|\beta_\varepsilon\big(y_\varepsilon(t) + W_A(t)\big)\big|\,\mathrm{d}t + \left(\int_0^T \big|y_\varepsilon(t)\big|_{1/2}^2\,\mathrm{d}t\right)^{1/2} T\right), \tag{4.9}$$

where $p = \frac{\rho}{2m}$.

Proof. We have

$$\begin{aligned}
&\big\langle \beta_\varepsilon(y_\varepsilon + W_A), y_\varepsilon + W_A - \theta\big\rangle \\
&\quad = \frac{1}{\varepsilon}\left(1 - \frac{1}{|y_\varepsilon + W_A|}\right)^+ \langle y_\varepsilon + W_A, y_\varepsilon + W_A - \theta\rangle \quad \forall \theta \in H.
\end{aligned}$$

This yields

$$\big\langle \beta_\varepsilon(y_\varepsilon + W_A), y_\varepsilon + W_A - \theta\big\rangle \geq 0 \quad \forall \theta \in H \text{ such that } |\theta| \leq 1.$$

In particular, the latter holds for

$$\theta = \frac{\beta_\varepsilon(y_\varepsilon + W_A)}{|\beta_\varepsilon(y_\varepsilon + W_A)|}$$

and so we get, for any $\varepsilon > 0$ and $t \in [0, T]$

$$\int_0^t \big|\beta_\varepsilon(y_\varepsilon + W_A)\big|\,\mathrm{d}s \leq \int_0^t \big\langle \beta_\varepsilon(y_\varepsilon + W_A), y_\varepsilon + W_A\big\rangle\,\mathrm{d}s. \tag{4.10}$$

On the other hand, by (4.7) we see that

$$\int_0^t \big\langle \beta_\varepsilon(y_\varepsilon + W_A), y_\varepsilon\big\rangle\,\mathrm{d}s + \frac{1}{2}\big|y_\varepsilon(t)\big|_\eta^2 + \int_0^t \big|A^{1/2} y_\varepsilon(s)\big|^2\,\mathrm{d}s = \frac{1}{2}|x|_\eta^2 \quad \forall t \geq 0$$

and so (4.10) yields

$$\begin{aligned}
&\int_0^t \big|\beta_\varepsilon(y_\varepsilon + W_A)\big|\,\mathrm{d}s + \frac{1}{2}\big|y_\varepsilon(t)\big|_\eta^2 + \int_0^t \big|A^{1/2} y_\varepsilon(s)\big|^2\,\mathrm{d}s \\
&\quad \leq \frac{1}{2}|x|_\eta^2 + \int_0^t \big\langle \beta_\varepsilon(y_\varepsilon + W_A), W_A\big\rangle\,\mathrm{d}s.
\end{aligned} \tag{4.11}$$

Now we consider $W_\mu = (1 + \mu A)^{-1} W_A$. We have

$$\begin{aligned}
&\big|W_\mu(t) - W_\mu(s)\big| \leq \big|W_A(t) - W_A(s)\big| \quad \forall t, s > 0, \\
&\big|W_\mu(t) - W_A(t)\big| \leq \mu\big|A(1 + \mu A)^{-1} W_A\big| \leq \mu^\delta |W_A|_\delta, \\
&\big|A W_\mu(t)\big| \leq \left(1 + \frac{1}{\mu}\right)\big|W_A(t)\big| \quad \forall t \geq 0, \mu > 0.
\end{aligned} \tag{4.12}$$

Then we have

$$\begin{aligned}
&\int_0^t \big\langle \beta_\varepsilon(y_\varepsilon + W_A), W_A\big\rangle\,\mathrm{d}s \\
&\quad \leq \int_0^t \big\langle \beta_\varepsilon(y_\varepsilon + W_A), W_A - W_\mu\big\rangle\,\mathrm{d}s + \int_0^t \big\langle \beta_\varepsilon(y_\varepsilon + W_A), W_\mu\big\rangle\,\mathrm{d}s \\
&\quad \leq \sup_{s \in (0,t)} \big|W_A(s) - W_\mu(s)\big| \int_0^t \big|\beta_\varepsilon(y_\varepsilon + W_A)\big|\,\mathrm{d}s + \int_0^t \big\langle \beta_\varepsilon(y_\varepsilon + W_A), W_\mu\big\rangle\,\mathrm{d}s \\
&\quad \leq \mu^\delta \sup_{s \in (0,t)} \big|W_A(s)\big|_\delta \int_0^t \big|\beta_\varepsilon(y_\varepsilon + W_A)\big|\,\mathrm{d}s + \int_0^t \big\langle \beta_\varepsilon(y_\varepsilon + W_A), W_\mu\big\rangle\,\mathrm{d}s.
\end{aligned}$$

On the other hand, we have

$$\int_0^t \langle \beta_\varepsilon(y_\varepsilon + W_A), W_\mu \rangle \, ds = \sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} \langle \beta_\varepsilon(y_\varepsilon + W_A), W_\mu(s) - W_\mu(t_i) \rangle \, ds + \sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} \langle \beta_\varepsilon(y_\varepsilon + W_A), W_\mu(t_i) \rangle \, ds, \tag{4.13}$$

where $0 = t_0 \leq t_1 \leq \cdots \leq t_N = t$ are chosen in such a way that $\max(t_{i+1} - t_i) \leq h$. We have therefore by (4.12) that

$$\left| \sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} \langle \beta_\varepsilon(y_\varepsilon + W_A), W_\mu(s) - W_\mu(t_i) \rangle \, ds \right| \leq h^p \sup_{s, \tilde{s} \in [0,t]} \left[\left| W_A(s) - W_A(\tilde{s}) \right| |s - \tilde{s}|^{-p} \right] \int_0^t \left| \beta_\varepsilon(y_\varepsilon + W_A) \right| ds \tag{4.14}$$

and by (4.6) it follows that

$$\begin{aligned}
&\left| \sum_{i=0}^{N-1} \left\langle W_\mu(t_i), \int_{t_i}^{t_{i+1}} \beta_\varepsilon(y_\varepsilon + W_A) \, ds \right\rangle \right| \\
&\quad \leq \sum_{i=0}^{N-1} \left| \left\langle W_\mu(t_i), A^{2\eta} y_\varepsilon(t_{i+1}) - A^{2\eta} y_\varepsilon(t_i) - \frac{1}{2} \int_{t_i}^{t_{i+1}} A y_\varepsilon(s) \, ds \right\rangle \right| \\
&\quad \leq \sum_{i=0}^{N-1} \left| W_\mu(t_i) \right|_\eta \left(\left| y_\varepsilon(t_{i+1}) \right|_\eta + \left| y_\varepsilon(t_i) \right|_\eta \right) \\
&\qquad + \sum_{i=0}^{N-1} \left| W_\mu(t_i) \right|_{1/2} \int_{t_i}^{t_{i+1}} \left| y_\varepsilon(s) \right|_{1/2} ds \\
&\quad \leq 2N \left(1 + \frac{1}{\mu} \right) \sup_{s \in [0,t]} \left| W_A(s) \right| \sup_{s \in [0,t]} \left| y_\varepsilon(s) \right|_\eta \\
&\qquad + \left(1 + \frac{1}{\mu} \right) \sup_{s \in [0,t]} \left| W_A(s) \right| \int_0^t \left| y_\varepsilon(s) \right|_{1/2} ds,
\end{aligned} \tag{4.15}$$

because $|W_\mu|_\eta \leq |AW_A| \leq (1 + \frac{1}{\mu})|W_A|$.

Then substituting into (4.13) yields

$$\int_0^t \langle \beta_\varepsilon(y_\varepsilon + W_A), W_\mu \rangle \, ds \leq \frac{1}{4} \left(\sup_{s \in (0,t)} \left| y_\varepsilon(s) \right|_\eta^2 + \int_0^t \left| y_\varepsilon(s) \right|_{1/2}^2 ds \right) + C \left(1 + \frac{T}{\mu^2} \right) \sup_{s \in (0,t)} \left| W_A(s) \right|^2$$

and substituting into (4.11) we get by (4.13) that

$$\begin{aligned}
&\int_0^t \left| \beta_\varepsilon(y_\varepsilon + W_A) \right| ds + \frac{1}{4} \left(\sup_{s \in (0,t)} \left| y_\varepsilon(s) \right|_\eta^2 + \int_0^t \left| y_\varepsilon(s) \right|_{1/2}^2 ds \right) \\
&\quad \leq C \left(|x|_\eta^2 + \left(1 + \frac{T}{\mu^2} \right) \sup_{s \in (0,t)} \left| W_A(s) \right|^2 \right. \\
&\qquad \left. + \left(h^p \sup_{s, \bar{s} \in (0,t)} \left| W_A(s) - W_A(\bar{s}) \right| |s - \bar{s}|^{-p} + \mu^\delta \sup_{s \in (0,t)} \left| W_A(s) \right|_\delta \right) \int_0^t \left| \beta_\varepsilon(y_\varepsilon + W_A) \right| ds \right)
\end{aligned}$$

which implies (4.8) as claimed. By (4.6) we see that (recall that $0 \le \alpha \le \frac{1}{2}$),

$$\int_0^T \left|\frac{\mathrm{d}y_\varepsilon}{\mathrm{d}t}\right| \mathrm{d}t \le C \int_0^T \left(\left|A^\alpha y_\varepsilon\right| + \left|\beta_\varepsilon(y_\varepsilon + W_A)\right|\right) \mathrm{d}t \le C\left(\left(\int_0^T \left|y_\varepsilon(t)\right|_{1/2}^2 \mathrm{d}t\right)^{1/2} T + \int_0^T \left|\beta_\varepsilon(y_\varepsilon + W_A)\right| \mathrm{d}t\right)$$

which clearly implies (4.9).

Now combining (4.8) and (4.9) yields

$$\begin{aligned} &\sup_{t\in[0,T]} \left|y_\varepsilon(t)\right|_\eta + \|y_\varepsilon\|_{BV([0,T];H)} \\ &\quad \le C\left(|x|_\eta^2 + \frac{T^2}{\mu} \sup_{t\in[0,T]} \left|W_A(t)\right|_\delta^2\right)\left(1 - h^p H(T) - \mu^\delta H_1(T)\right), \end{aligned} \tag{4.16}$$

where

$$\begin{aligned} H(T) &= \sup_{s,t\in[0,T]} \left[\left|W_A(t) - W_A(s)\right| |t-s|^{-p}\right], \\ H_1(T) &= \sup_{t\in[0,T]} \left|W_A(t)\right|_\delta. \end{aligned} \tag{4.17}$$

□

An immediate corollary is Lemma 4.4 below.

Lemma 4.4. *For each $N > 0$ and $T > 0$ there is $\Omega_{T,N} \subset \Omega$ such that*

$$\mathbb{P}(\Omega_{T,N}) \ge 1 - \frac{C_*^1}{N} \tag{4.18}$$

and

$$\|y_\varepsilon\|_{BV([0,T];H)} + \sup_{t\in[0,T]} \left|y_\varepsilon(t)\right|_\eta^2 \le C_*^2\left(|x|_\eta^2 + N^{1/2}T^6\right) \quad \forall \omega \in \Omega_{T,N}, \tag{4.19}$$

where C_*^i, $i = 1, 2$, are independent of ε, T, N and ω.

Proof. By (4.2) and respectively (4.3) we have for all $M > 0$ and $m = 2$

$$\mathbb{P}\left(\sup_{t\in[0,T]} \left|W_A(t)\right|_\delta \le M\right) \ge 1 - \frac{C}{M^4} T^3 \tag{4.20}$$

and

$$\begin{aligned} &\mathbb{P}\left(h^p H(T) \le \frac{1}{4}\right) \ge 1 - CT^3 h^{2p} \quad \forall h > 0, \\ &\mathbb{P}\left(\mu^\delta H_1(T) \le \frac{1}{4}\right) \ge 1 - CT^3 \mu^{4\delta}. \end{aligned} \tag{4.21}$$

On the other hand, by (4.8), (4.9) and (4.16) we have

$$\begin{aligned} &\sup_{t\in[0,T]} |y_\varepsilon|_\eta^2 + \|y_\varepsilon\|_{BV([0,T];H)} \le 2C\left(|x|_\eta^2 + M^2\right) \\ &\quad \text{in } \left\{\omega\colon h^p H(T) \le \frac{1}{4}\right\} \cap \left\{\omega\colon \sup_{t\in[0,T]} \left|W_A(t)\right|_\eta \le M\right\} \cap \left\{\omega\colon \mu^\delta H_1(T) \le \frac{1}{4}\right\}. \end{aligned} \tag{4.22}$$

If we choose $M = N^{1/4}T^3$, $h = (NT^3)^{-2/p}$ and

$$\Omega_{T,N} = \Big\{\omega\colon \sup_{t\in[0,T]} \big|W_A(t)\big|_\eta \le M\Big\} \cap \Big\{\omega\colon h^p H(T) \le \frac{1}{4}\Big\} \cap \Big\{\omega\colon \mu^\delta H_1(T) \le \frac{1}{4}\Big\}$$

we obtain (4.18) and (4.19) as desired. □

The convergence in law

We denote by $BV(0,\infty;H)$ the space of H-valued functions $u:[0,\infty)\to H$ which have bounded variation on each interval $[0,T]$. This is a locally convex space with the family of seminorms

$$|u|_T = \|u\|_{BV([0,T];H)} \quad \forall T>0.$$

We shall construct below a space of cadlag trajectories which is a Polish space in an appropriate topology. To this end we consider the family of spaces $\{\mathscr{X}_N\}_{N=1}^\infty$ defined by

$$\mathscr{X}_N = \big\{u\in BV(0,\infty;H)\cap L^\infty_{\mathrm{loc}}\big(0,\infty;D\big(A^\eta\big)\big)\colon |u|_T + |u|^2_{L^\infty(0,T;D(A^\eta))} \le 2C_*^2\big(|x|_\eta^2 + N^{1/2}T^6\big)\ \forall T>0\big\}. \tag{4.23}$$

(Here C_*^2 is the constant arising in (4.19).)

Each $\mathscr{X}_N$ is a closed and bounded subset of $BV([0,T];H)$. We shall introduce on $\mathscr{X}_N$ the topology (infact a pseudo-topology) defined by the convergence in measure, i.e., we say that $u_n \Longrightarrow u$ in $\mathscr{X}_N$ if for each $T>0$

$$\lim_{n\to\infty}\int_0^T f\big(t,u_n(t)\big)\,\mathrm{d}t = \int_0^T f\big(t,u(t)\big)\,\mathrm{d}t \tag{4.24}$$

for all bounded and continuous functions $f\in C_b([0,\infty)\times H)$.

It turns out that this topology is just given by the metric

$$d(u,v) = \sum_{j=1}^\infty \frac{1}{2^j}\frac{\mathrm{d}_{T_j}(u,v)}{1+d_{T_j}(u,v)}, \tag{4.25}$$

where $\{T_j\}$ is an increasing sequence of times that goes to infinity and

$$d_{T_j}(u,v) = \sum_{k=1}^\infty \frac{1}{2^k}\frac{|\int_0^{T_j}(f_k^j(t,u(t)) - f_k^j(t,v(t)))\,\mathrm{d}t|}{1+|\int_0^{T_j}(f_k^j(t,u(t)) - f_k^j(t,v(t)))\,\mathrm{d}t|},$$

where, for each j, $\{f_k^j\}_{k=1}^\infty$ is a dense subset of $C([0,T_j]\times H)$.

Lemma 4.5. *The space $\mathscr{X}_N$ endowed with the metric d is a compact complete metric space and the convergence induced by this topology coincides with that induced by convergence in measure* (4.24).

Proof. It is immediate that d is a metric on $\mathscr{X}_N$ and that $u_n \Longrightarrow u$ if and only if $\lim_{n\to\infty} d(u_n,u) = 0$. Moreover, by the infinite-dimensional Helly theorem the set $\mathscr{X}_N$ is compact in topology $\Longrightarrow$ (or equivalently that induced by the distance d). This implies that the metric d is complete and the space $\mathscr{X}_N$ is compact and so also separable. □

Now we shall define the space $\mathring{\mathscr{X}} \subset BV(0,\infty;H)\cap L^\infty_{\mathrm{loc}}(0,\infty;D(A^\eta))$ by

$$\mathring{\mathscr{X}} = \bigcup_{N=1}^\infty \mathscr{X}_N. \tag{4.26}$$

In other words, $u \in \mathring{\mathscr{X}}$ if and only if $u \in \mathscr{X}_N$ for some $N \in \mathbb{N}$. (Recall that $\eta = \frac{1-\alpha}{2}$.)

We shall denote by $\mathscr{X}$ the completion of $\mathring{\mathscr{X}}$ in the metric (topology) d. Clearly $\mathscr{X}$ is a separable complete metric space.

For each $u \in \mathscr{X}$ we can associate its pseudo-path which is a probability law μ_u on $[0,\infty) \times H$. Then for each $f \in C_b([0,\infty) \times H)$ we have

$$\int f\big(t, u(t)\big)\,\mathrm{d}t = \int f\,\mathrm{d}\mu_u \quad \forall f \in C_b\big([0,\infty) \times H\big)$$

and so the convergence (4.24) (respectively the topology induced by it) reduces to the convergence in measure or to the so-called pseudo-path topology (see [13]). Since the space $\mathbb{D}$ of cadlag H-valued functions is closed in this topology and

$$\mathring{\mathscr{X}} \subset BV(0,\infty; H) \cap L^\infty_{\mathrm{loc}}\big(0,\infty; D\big(A^\eta\big)\big) \subset \mathbb{D}$$

we conclude that

Lemma 4.6. *Any $u \in \mathscr{X}$ is a cadlag H-valued function, i.e., u is right continuous with left limit.*

Remark 4.7. *Of course the previous analysis of cadlag function spaces refer to real valued functions but it extends mutatis mutandis to H-valued functions considering first weakly cadlag functions $u:[0,\infty) \to H$, i.e., $t \to \langle u(t), x\rangle$ is cadlag for each $x \in H$ and after to strong cadlag functions via compacity $D(A^\eta) \subset H$.*

Now we consider the family of probability measures $\{\mathfrak{P}_\varepsilon\} \subset \mathscr{P}(\mathscr{X})$ defined by

$$\mathfrak{P}_\varepsilon(\Gamma) = \mathbb{P}(X_\varepsilon \in \Gamma), \quad \Gamma \subset \mathscr{X} \text{ Borelian.} \tag{4.27}$$

Lemma 4.8. *The family $\{\mathfrak{P}_\varepsilon\}_{\varepsilon>0}$ is tight.*

Proof. Taking into account that $X_\varepsilon = y_\varepsilon + W_A$ it suffices to prove that the family $\{\tilde{\mathfrak{P}}_\varepsilon\}$, where $\tilde{\mathfrak{P}}_\varepsilon(\Gamma) = \mathbb{P}(y_\varepsilon \in \Gamma)$, is tight. By the Prohorov theorem it suffices to show that for each $\xi > 0$ there is a compact subset $K_\xi \subset \mathscr{X}$ such that

$$\mathbb{P}(y_\varepsilon \in K_\xi) \ge 1 - \xi. \tag{4.28}$$

We take

$$\begin{aligned} K_\xi = \big\{ & u \in BV(0,\infty; H) \cap L^\infty_{\mathrm{loc}}\big(0,\infty; D\big(A^\eta\big)\big): \\ & |u|_T + |u|^2_{L^\infty(0,T;D(A^\eta))} \le 2C_*^2\big(|x|^2_\eta + \big(C_*^1\xi^{-1}\big)^{1/2}T^6\big)\ \forall T > 0\big\}. \end{aligned}$$

By Lemma 3.4 we see that (4.28) holds. On the other hand, since $K_\eta \subset \mathscr{X}_N$ for $N = C_*^1\xi^{-1}$ it follows that K_η is compact in $\mathring{\mathscr{X}}$ and therefore in $\mathscr{X}$ as well. This completes the proof of Lemma 4.8. □

Then there is $\mathfrak{P} \in \mathscr{P}(\mathscr{X})$ such that on a subsequence $\varepsilon \to 0$

$$\mathfrak{P}_\varepsilon \to \mathfrak{P} \quad \text{weakly in } \mathscr{P}(\mathscr{X}).$$

Moreover, by the Skorohod theorem (see, e.g., [15]), we have

Proposition 4.9. *There is a probability space $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{\mathbb{P}})$ and a sequence $\{\tilde{X}_\varepsilon\}$ of $\mathscr{X}$-valued processes on $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{\mathbb{P}})$ and $\mathscr{X}$-valued stochastic process X such that*

$$\mathfrak{P}_\varepsilon(\Gamma) = \mathbb{P}(\tilde{X}_\varepsilon \in \Gamma), \tag{4.29}$$

$$\tilde{X}_\varepsilon \to X \quad \tilde{\mathbb{P}}\text{-a.s. in } \mathscr{X}, \tag{4.30}$$

$$\mathfrak{P}(\Gamma) = \mathbb{P}(X \in \Gamma) \tag{4.31}$$

for all Borelian set $\Gamma \subset \mathscr{X}$.

By Lemma 4.6, $X = X(t,x)$ is a cadlag H-valued process.

Let N be the Kolmogorov operator associated with the Neumann problem and let P_t the semigroup generated by N. We have

Theorem 4.10. *Let Hypothesis 4.1 holds. Let $X = X(t) : [0,\infty) \to H$ be the process defined by Proposition 4.9. Then*

$$(P_t\varphi)(x) = \int_{\tilde{\Omega}} \varphi\big(X(t,x)\big)\,\mathrm{d}\tilde{\mathbb{P}}(\omega) \quad \forall t \ge 0, x \in D\big(A^\delta\big), \varphi \in C_b(H). \tag{4.32}$$

Proof. We have by Proposition 4.9

$$\begin{aligned}
&\big(P_\varepsilon(t)\varphi\big)(x) = \tilde{\mathbb{E}}\big(\varphi\big(\tilde{X}_\varepsilon(t,x)\big)\big) = \int_{\tilde{\Omega}} \varphi\big(\tilde{X}_\varepsilon(t,x)\big)\,\mathrm{d}\tilde{\mathbb{P}}(\omega) \quad \forall t \ge 0, x \in D\big(A^\delta\big), \varphi \in C_b(H),\\
&\lim_{\varepsilon\to 0}\big(P_\varepsilon(t)\varphi\big)(x) = \int_{\tilde{\Omega}} \varphi\big(X(t,x)\big)\,\mathrm{d}\tilde{\mathbb{P}}(\omega).
\end{aligned} \tag{4.33}$$

On the other hand, we know by Theorem 3.5 that

$$(\lambda I - N)^{-1}\varphi = \lim_{\varepsilon\to 0}(\lambda I - N_\varepsilon)^{-1}\varphi = \lim_{\varepsilon\to 0}\int_0^\infty \mathrm{e}^{-\lambda t} P_\varepsilon(t)\varphi\,\mathrm{d}t \quad \forall \lambda > 0. \tag{4.34}$$

By (4.33), (4.34) we see that

$$\int_0^\infty \mathrm{e}^{-\lambda t}(P_t\varphi)(x)\,\mathrm{d}t = \int_0^\infty \mathrm{e}^{-\lambda t}\,\mathrm{d}t \int_{\tilde{\Omega}} \varphi\big(X(t,x)\big)\,\mathrm{d}\tilde{\mathbb{P}}(\omega) \quad \forall \lambda > 0$$

which clearly implies (4.32) as claimed. □

Proposition 4.11. *We have*

$$X(t,x) \in K \quad \tilde{\mathbb{P}}\text{-}a.s.\ \forall t > 0. \tag{4.35}$$

Proof. By Lemma 4.4 we have that for each N,

$$\int_0^T \big|\beta_\varepsilon\big(X_\varepsilon(t)\big)\big|\,\mathrm{d}t \le C\big(1 + N^{1/2}T^6\big) \quad \forall \omega \in \Omega_{T,N},$$

where $\mathbb{P}(\Omega_{T,N}) \ge 1 - \frac{C_*^1}{N}$.

This yields

$$\int_0^T \big|X_\varepsilon(t) - \Pi_K X_\varepsilon(t)\big|\,\mathrm{d}t \le C\varepsilon\big(1 + N^{1/2}T^6\big) \quad \forall \varepsilon > 0, \omega \in \Omega_{T,N}$$

and therefore

$$\int_0^T \big|\tilde{X}_\varepsilon(t) - \Pi_K \tilde{X}_\varepsilon(t)\big|\,\mathrm{d}t \le C\varepsilon\big(1 + N^{1/2}T^6\big) \quad \forall \varepsilon > 0, \omega \in \tilde{\Omega}_{T,N},$$

where $\tilde{\Omega}_{T,N} \subset \tilde{\Omega}$, and $\tilde{\mathbb{P}}(\tilde{\Omega}_{T,N}) \ge 1 - \frac{C_*^1}{N}$.

Letting ε tend to zero we obtain that $|X(t) - \Pi_K X(t)| = 0$, $\forall t \ge 0$, $\tilde{\mathbb{P}}$-a.s. as claimed. □

Remark 4.12. *We recall that X is a martingale solution to* (1.10), *if*

$$\tilde{\mathbb{P}}\big(X(t) \in K, \forall t \geq 0\big) = 1, \quad \tilde{\mathbb{P}}\big(X(0,x) = x\big) = 1 \tag{4.36}$$

and for any smooth function φ in a core $D(N_0)$ of N,

$$\varphi\big(X(t)\big) - \int_0^t N\varphi\big(X(s)\big)\,\mathrm{d}s - \varphi(x) =: \tilde{M}(t) \tag{4.37}$$

is a martingale with respect to natural filtration $\tilde{\mathscr{F}}_t = \sigma(X(s), s \leq t)$, $t \geq 0$.

It is easily seen by Theorem 4.10 and (3.5) that if N has a core $D(N_0)$ then the process X constructed above is the unique martingale solution to (1.1). However the existence of a core for N is still open.

5. An example

Consider the stochastic variational inequality (see (1.10))

$$\begin{aligned}
&\mathrm{d}X(t) - \Delta X(t)\,\mathrm{d}t - \Delta N_K\big(X(t)\big)\,\mathrm{d}t \ni A_0^{-1}\,\mathrm{d}W_t \quad \text{in } (0,\infty) \times \mathscr{O},\\
&X(t) = 0 \quad \text{on } (0,\infty) \times \partial\mathscr{O},\\
&X(0) = x \quad \text{in } \mathscr{O},
\end{aligned} \tag{5.1}$$

where $\mathscr{O}$ is a bounded open subset of $\mathbb{R}^d$ with smooth boundary $\partial\mathscr{O}$ and

$$K = \left\{x \in L^2(\mathscr{O}): \int_{\mathscr{O}} j\big(x(\xi)\big)\,\mathrm{d}\xi \leq 1\right\}, \tag{5.2}$$

where $j:\mathbb{R} \to \mathbb{R}$ is a C^∞-convex function such that $0 < c \leq j''(r) \leq c_1$, $\forall r \in \mathbb{R}$, $j(0) = j'(0) = 0$ and $A_0 = -\Delta$, $D(A_0) = H_0^1(\mathscr{O}) \cap H^2(\mathscr{O})$.

Formally, (5.1) reduces to the stochastic reflection problem

$$\begin{aligned}
&\mathrm{d}X(t) - \Delta X(t)\,\mathrm{d}t = A_0^{-1}\,\mathrm{d}W_t \quad \text{in } \left\{x \in L^2(\mathscr{O}): \int_{\mathscr{O}} j\big(x(\xi)\big)\,\mathrm{d}\xi < 1\right\},\\
&\mathrm{d}X(t) - \Delta X(t)\,\mathrm{d}t \in \big\{\lambda\Delta j'\big(X(t)\big)\big\}_{\lambda>0}\,\mathrm{d}t + A_0^{-1}\,\mathrm{d}W_t \quad \text{in } \left\{x \in L^2(\mathscr{O}): \int_{\mathscr{O}} j\big(x(\xi)\big)\,\mathrm{d}\xi = 1\right\},\\
&X(t) = 0 \quad \text{on } (0,\infty) \times \partial\mathscr{O},\\
&X(0) = x \quad \text{in } \mathscr{O}.
\end{aligned} \tag{5.3}$$

The results of Sections 1–3 and in particular, Theorem 3.5 apply with $\alpha = \frac{1}{2}$, $H = L^2(\mathscr{O})$, $A = \Delta^2$, $D(A) = \{u \in H^2(\mathscr{O}) \cap H_0^1(\mathscr{O}), \Delta u \in H_0^1(\mathscr{O}), \Delta^2 u \in L^2(\mathscr{O})\}$ on K defined by (5.2). Then $A^{1/2} = A_0$ and $\operatorname{Tr} A^{-1+2\delta} < \infty$ if $1 \leq d \leq 3$ and δ is small.

Then the corresponding Kolmogorov operator N defined by (3.32) satisfies the regularity properties in Theorem 3.5 and the Markov semigroup P_t generated by N is given by

$$(P_t\varphi_0)(x) = \varphi(t,x) \quad \forall t \geq 0, x \in L^2(\mathscr{O}),$$

where φ is the solution to infinite-dimensional parabolic problem

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_K \varphi(t,x)\psi(x)\nu(\mathrm{d}x) - \frac{1}{2}\int_K\left(\int_{\mathscr{O}} \Delta\varphi\big(t, X(\xi)\big)\psi\big(X(\xi)\big)\,\mathrm{d}\xi\right)\nu(\mathrm{d}x) \quad \forall t \geq 0, \forall\psi \in C^1(K),\\
&\varphi(0,x) = \varphi_0(x).
\end{aligned} \tag{5.4}$$

Moreover, if $d = 1$ and $j(r) = r^2$ then Hypothesis 4.1 holds and so by Theorem 4.10 there is a cadlag process $X(t) : [0, \infty) \to L^2(\mathscr{O})$ in a probability space $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{\mathbb{P}})$ such that

$$(P_t\varphi)(x) = \int_{\tilde{\Omega}} \varphi\big(X(t,x)\big)\, d\tilde{\mathbb{P}}(\omega) \quad \forall x \in D\big(A^\delta\big)$$

for $\delta > 0$.

As mentioned earlier we may view X as a martingale solution to problem (5.1).

Remark 5.1. *This example illustrates the fact that considering the class of problems* (1.7) *with* $\alpha \in [0, 1]$ *one might study reflection problems of the form* (5.1) *which otherwise are untractable in more dimensions.*

Appendix

We recall again the following well-known integration by parts formula for the measure μ (see, e.g., [10]). For any $\varphi, \psi \in W^{1,2}(H, \mu)$ and $z \in H$,

$$\int_H \big\langle D\varphi, Q^{1/2}z\big\rangle \psi \, d\mu = -\int_H \big\langle D\psi, Q^{1/2}z\big\rangle \varphi \, d\mu + \int_H W_z\varphi\psi \, d\mu, \tag{A.1}$$

where W_z represents the *white noise* function,

$$W_z(x) = \sum_{k=1}^{\infty} \frac{1}{\sqrt{\lambda_k}} \langle x, e_k\rangle\langle z, e_k\rangle \quad \forall z \text{ and } \mu\text{-a.e. } x \in H.$$

We recall that W_z is a Gaussian random variable in $L^2(H, \mu)$ with mean 0 and covariance $|z|^2$. We notice that, thanks to Hypothesis 1.1(ii) the surface measure μ_Σ is well defined (see [12]).

We want now to prove an integration by parts formula in a subdomain K of H which generalizes (A.1). K is defined by a function g as stated in the Introduction. It is convenient to introduce a sequence of suitable measures $\{\mu_\varepsilon\}_{\varepsilon>0}$ defined by

$$\mu_\varepsilon(dx) = \rho_\varepsilon(x)\mu(dx), \quad x \in H,$$

where

$$\rho_\varepsilon(x) = e^{-(g(x)-1)^2/\varepsilon \mathbb{1}_{g(x)\geq 1}}.$$

Notice that,

$$\lim_{\varepsilon\to 0} \rho_\varepsilon(x) = \begin{cases} 1 & \text{if } x \in K, \\ 0 & \text{if } x \notin K. \end{cases}$$

So, we have

$$\lim_{\varepsilon\to 0} \mu_\varepsilon = \mu(K)\nu \quad \text{weakly in } \mathscr{P}(H),$$

where ν is the measure introduced previously. Moreover,

$$D\rho_\varepsilon(x) = -\frac{2}{\varepsilon}\rho_\varepsilon(x)\mathbb{1}_{g(x)\geq 1} Dg(x)\big(g(x) - 1\big),$$

so that $\rho_\varepsilon \in W^{1,2}(H, \mu)$.

The integration by parts formula

Here we are going to derive from (A.1), an integration by parts formula for the measure μ_ε. Let $\varphi \in C_b^1(H)$, $z \in H$, then, since $\rho_\varepsilon \in W^{1,2}(H,\mu)$, we find from (A.1) that

$$\begin{aligned}\int_H \langle D\varphi, Q^{1/2}z\rangle \,\mathrm{d}\mu_\varepsilon &= \int_H \langle D\varphi, Q^{1/2}z\rangle \rho_\varepsilon \,\mathrm{d}\mu \\ &= -\int_H \varphi \langle D\log\rho_\varepsilon, Q^{1/2}z\rangle \,\mathrm{d}\mu_\varepsilon + \int_H W_z\varphi \,\mathrm{d}\mu_\varepsilon.\end{aligned}$$

Since,

$$D\log\rho_\varepsilon(x) = -\frac{2}{\varepsilon}\mathbb{1}_{g(x)\geq 1} Dg(x)\big(g(x)-1\big),$$

we find the formula,

$$\begin{aligned}\int_H \langle D\varphi, Q^{1/2}z\rangle \mu_\varepsilon(\mathrm{d}x) &= \frac{2}{\varepsilon}\int_H \varphi(x)\mathbb{1}_{g(x)\geq 1}\big(g(x)-1\big)\langle Dg(x), Q^{1/2}z\rangle \mu_\varepsilon(\mathrm{d}x) \\ &\quad + \int_H W_z(x)\varphi(x)\mu_\varepsilon(\mathrm{d}x).\end{aligned} \tag{A.2}$$

Lemma A.1. *Let $\varphi \in C_b^1(H)$, $z \in H$. Then there exists the limit,*

$$\begin{aligned}\lim_{\varepsilon\to 0} J_\varepsilon^z(\varphi) &:= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_H \varphi(x)\mathbb{1}_{g(x)\geq 1}\big(g(x)-1\big)\langle Dg(x), Q^{1/2}z\rangle \mu_\varepsilon(\mathrm{d}x) \\ &= \frac{1}{2}\int_\Sigma \varphi(y)\langle \mathbf{n}(y), Q^{1/2}z\rangle \mu_\Sigma(\mathrm{d}y),\end{aligned} \tag{A.3}$$

where $\mathbf{n}(y) = \frac{Dg(y)}{|Dg(y)|}$ is the exterior normal to Σ at y and μ_Σ is the surface measure on Σ induced by μ (see [12]).

Proof. First we notice that

$$J_\varepsilon^z(\varphi) = \frac{1}{\varepsilon}\int_{\{g(x)>1\}} \varphi(x)\big(g(x)-1\big)\langle Dg(x), Q^{1/2}z\rangle \mathrm{e}^{-(g(x)-1)^2/\varepsilon}\mu(\mathrm{d}x).$$

By the co-area formula (see [12], p. 140)[1] we have

$$\int_H f\mu(\mathrm{d}x) = \int_0^\infty \left[\int_{g=r} f(y)\frac{1}{|Dg(y)|}\mu_{\Sigma_r}(\mathrm{d}y)\right]\mathrm{d}r. \tag{A.4}$$

(By (1.4) we know that $|Dg(x)| \geq \gamma|x|$ and so $|Dg(x)|^{-1} \in L^p(H,\mu)$ for all $p \geq 1$.) Notice that the surface measure is defined for all $r \geq 0$ taking into account [12], Theorem 6.2, Chapter V, moreover, [12], Theorem 1.1, Corollary 6.3.2, Chapter V, give the continuity property in Theorem 6.3.1 of Chapter V of [12]. Setting in (A.4)

$$f = \mathbb{1}_{g\geq 1}\varphi(x)\big(g(x)-1\big)\langle Dg(x), Q^{1/2}z\rangle \mathrm{e}^{-(g(x)-1)^2/\varepsilon}$$

we get

$$\begin{aligned}&\int_{g\geq 1}\varphi(x)\big(g(x)-1\big)\langle Dg(x), Q^{1/2}z\rangle \mathrm{e}^{-(g(x)-1)^2/\varepsilon}\mu(\mathrm{d}x) \\ &\quad = \int_1^\infty (r-1)\mathrm{e}^{-(r-1)^2/\varepsilon}\left[\int_{g=r}\varphi(y)\langle Dg(y), Q^{1/2}z\rangle \frac{1}{|Dg(y)|}\mu_{\Sigma_r}(\mathrm{d}y)\right]\mathrm{d}r.\end{aligned}$$

[1]Here, we have extended the validity of (A.4) to functions f, continuous and in $L^p(H,\mu)$ for any $p \geq 1$, by a density argument.

Hence, setting $r = 1 + \sqrt{\varepsilon} s$, yields

$$J_\varepsilon^z(\varphi) = \int_0^\infty s e^{-s^2} ds \int_{g=1+\sqrt{\varepsilon}s} \varphi(y) \left\langle \frac{Dg(y)}{|Dg(y)|}, Q^{1/2} z \right\rangle \mu_{\Sigma_{g=1+\sqrt{\varepsilon}s}}(dy).$$

So (A.3) follows. □

We are now in position to prove the announced integration by parts formula.

Theorem A.2. *Let $\varphi \in C_b^1(H)$, $z \in H$. Then for any $z \in H$ we have*

$$\int_K \langle D\varphi(x), Q^{1/2} z \rangle \mu(dx) = \int_\Sigma \varphi(y) \langle \mathbf{n}(y), Q^{1/2} z \rangle \mu_\Sigma(dy) \tag{A.5}$$

$$+ \int_K W_z(x) \varphi(x) \mu(dx). \tag{A.6}$$

Proof. The conclusion of the theorem follows letting $\varepsilon \to 0$ in (A.2) and taking into account Lemma A.1. □

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