Brazilian Journal of Probability and Statistics
2010, Vol. 24, No. 1, 42–56
DOI: 10.1214/08-BJPS012

Estimation of the generalized lambda distribution from censored data

Joseph Mercy[a] and M. Kumaran[b]

[a]*Govt. College Kasaragod*
[b]*Nehru Arts & Science College*

Abstract. Decision making based on censored data is a problem of serious concern as such data can provide only limited information compared to the corresponding uncensored data. In this article we describe a very general method of estimating the distribution of type I and II singly censored data. We derive single unified expressions for the probability weighted moments and L-moments of all univariate continuous distributions using a four-parameter family of distributions known as the Generalised Lambda Distribution (GLD) and use them to estimate the density of complete and censored data.

1 Introduction

Many statistical techniques are based on the use of probability weighted moments (PWMs), introduced by Greenwood et al. (1979), and L-moments, introduced by Hosking (1990), which are generalizations of the usual moments of a probability distribution. The theory founded on PWMs and L-moments, which are linear functions of each other, covers the summarization and description of theoretical probability distributions and of observed data samples, nonparametric estimation of the underlying distribution of an observed sample, estimation of parameters and quantiles of probability distributions, hypothesis tests for probability distributions, etc. Both moment types offer measures of distributional location (mean), scale (variance), skewness (shape), and kurtosis. The main advantage of these moments over conventional moments is that, being linear functions of the data, they suffer less from the effects of sampling variability. These moments are more robust than conventional moments to outliers in the data, enable more secure inferences to be made from small samples about an underlying probability distribution, and frequently yield more efficient parameter estimates than the conventional moment estimates.

As the PWMs can be expressed as the linear combinations of L-moments, the procedures based on PWMs and on L-moments are equivalent. More details about PWMs can be seen in Greis and Wood (1981), Hosking, Wallis and Wood (1985), and Hosking and Wallis (1987). L-moments are more convenient as they are more

Key words and phrases. Censored data, density estimation, Generalized Lambda Distribution family, L-moments, partial probability weighted moments.
Received October 2007; accepted May 2008.

directly interpretable as measures of the scale and shape of probability distributions. The use of L-moments in various inference procedures are given in Hosking and Wallis (1997).

As PWMs and L-moments are very useful for the analysis of both complete and censored distributions and data and its expressions may not be always simple, in this article we introduce a single unified expression for the PWMs and L-moments of all univariate continuous (complete and censored) distributions and use them to estimate the density of censored data. In Section 2 we give a brief description of the GLD family. PWMs and L-moments of GLD for complete and censored distribution are given in Sections 3 and 4. Some illustrative examples are also included to establish the results.

2 The family of generalized lambda distribution (GLD)

The GLD is a family of distributions that can take on a very wide range of shapes within one distributional form. It was originally proposed by Ramberg and Schmeiser (1974) and is a four-parameter generalization of Tukey's Lambda family [Hastings et al. (1947)] that has proved useful in a number of different applications. Its main use has been in fitting distributions to the empirical data, and in the computer generation of different distributions. The various characteristics of the GLD family, its applications, and parameter estimation are explained in detail in Karian and Dudewicz (2000).

Distributions belonging to the GLD family are specified in terms of their quantile function given by

$$x(p) = \lambda_1 + \frac{p^{\lambda_3} - (1-p)^{\lambda_4}}{\lambda_2}, \tag{2.1}$$

where $0 \le p \le 1$ and $p = P(X \le x) = F(x)$. λ_1, λ_2 are, respectively, the location and scale parameters and λ_3, λ_4 are the shape parameters which jointly determine skewness and kurtosis. The probability density function is given by

$$f(x) = \frac{\lambda_2}{\lambda_3 p^{(\lambda_3-1)} + \lambda_4 (1-p)^{(\lambda_4-1)}}.$$

When $\lambda_3 = \lambda_4$, the distribution will be symmetric about λ_1. The family contains a distribution corresponding to any admissible pair of values of skewness and kurtosis. For more details refer to Karian and Dudewicz (2000). The expressions for the PWMs and L-moments of GLD are derived below.

2.1 Estimation of parameters

Since the GLD family contains distributions of a wide variety of shapes, it offers risk managers great flexibility in modeling a broad range of data arising in several fields. Due to its versatility, estimating the appropriate parameters for the GLD is a challenging problem. Several methods for estimating the parameters of GLD have been reported in the literature. The most common among them are the method of moment matching by Ramberg et al. (1979), the method of quantiles, the method of least squares by Ozturk and Dale (1985), and the starship method by King and MacGillivray (1999).

3 PWMs and L-moments for GLD

Let X be a real-valued random variable with cumulative distribution function (CDF) $F(x)$ and quantile function $x(p)$. Greenwood et al. (1979) defined PWMs to be the quantities

$$M_{k,r,s} = E\{X^k[F(x)]^r[1 - F(x)]^s\},$$

where k, r, s are real numbers. PWMs are likely to be most useful when the inverse distribution function $x(p)$ can be written in closed form.

Proposition 3.1. *The PWM $M_{k,r,s}$ of a GLD$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ family with quantile function $x(p)$ is given by*

$$M_{k,r,s} = \sum_{i=0}^{k} \binom{k}{i} \lambda_1^{k-i} \lambda_2^{-i} \sum_{j=0}^{i} (-1)^j \binom{i}{j} \beta[\lambda_3(i - j) + r + 1, \lambda_4 j + s + 1],$$

where $\beta(p, q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$.

Proof.

$$\begin{aligned}
M_{k,r,s} &= E\{X^k[F(x)]^r[1 - F(x)]^s\} \\
&= \int_0^1 [x(p)]^k p^r [1 - p]^s \, dp \\
&= \int_0^1 \left\{\lambda_1 + \frac{p^{\lambda_3} - (1 - p)^{\lambda_4}}{\lambda_2}\right\}^k p^r [1 - p]^s \, dp \\
&= \sum_{i=0}^{k} \binom{k}{i} {\lambda_1}^{k-i} {\lambda_2}^{-i} \int_0^1 (p^{\lambda_3} - (1 - p)^{\lambda_4})^i p^r [1 - p]^s \, dp \\
&= \sum_{i=0}^{k} \binom{k}{i} {\lambda_1}^{k-i} {\lambda_2}^{-i} \sum_{j=0}^{i} (-1)^j \binom{i}{j} \int_0^1 p^{\lambda_3(i-j)} (1 - p)^{\lambda_4 j} p^r [1 - p]^s \, dp \\
&= \sum_{i=0}^{k} \binom{k}{i} \lambda_1^{k-i} \lambda_2^{-i} \sum_{j=0}^{i} (-1)^j \binom{i}{j} \beta[\lambda_3(i - j) + r + 1, \lambda_4 j + s + 1].
\end{aligned}$$

$M_{0,r,0}$, $M_{0,0,s}$ and $M_{0,r,s}$ do not involve any parameters of the distribution and hence are of no practical use. The quantities $M_{k,0,0}$ $(k = 1, 2, \ldots)$ are the usual noncentral moments of X. For GLD it is given as

$$M_{k,0,0} = \sum_{i=0}^{k} \binom{k}{i} \lambda_1^{k-i} \lambda_2^{-i} \sum_{j=0}^{i} (-1)^j \binom{i}{j} \beta[\lambda_3(i-j)+1, \lambda_4 j+1].$$

Similarly

$$M_{k,r,0} = \sum_{i=0}^{k} \binom{k}{i} \lambda_1^{k-i} \lambda_2^{-i} \sum_{j=0}^{i} (-1)^j \binom{i}{j} \beta[\lambda_3(i-j)+r+1, \lambda_4 j+1],$$

$$M_{k,0,s} = \sum_{i=0}^{k} \binom{k}{i} \lambda_1^{k-i} \lambda_2^{-i} \sum_{j=0}^{i} (-1)^j \binom{i}{j} \beta[\lambda_3(i-j)+1, \lambda_4 j+s+1].$$

It is to be noted that

$$M_{1,0,r} = \frac{\lambda_1}{r+1} + \frac{1}{\lambda_2}\left\{\beta(\lambda_3+1, r+1) - \frac{1}{\lambda_4+r+1}\right\} \quad \text{and}$$
$$M_{1,r,0} = \frac{\lambda_1}{r+1} + \frac{1}{\lambda_2}\left\{\frac{1}{\lambda_3+r+1} - \beta(r+1, \lambda_4+1)\right\}. \tag{3.1}$$

When the distribution is symmetric about λ_1, that is, $\lambda_3 = \lambda_4$, then

$$M_{1,r,0} + M_{1,0,r} = \frac{2\lambda_1}{r+1}.$$

Greenwood et al. (1979) and many others [Greis and Wood (1981); Hosking and Wallis (1987), etc.] developed statistical inference procedure using PWMs $M_{1,0,s}$ and $M_{1,r,0}$. Here we consider $M_{1,r,0}$ only and denote it as β_r. From equation (3.1) we get the rth PWM β_r of a GLD$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ family as

$$\begin{aligned}\beta_r &= \frac{\lambda_1}{r+1} + \frac{1}{\lambda_2}\left\{\frac{1}{(\lambda_3+r+1)} - \beta(r+1, \lambda_4+1)\right\} \\ &= \frac{\lambda_1}{r+1} + \frac{1}{\lambda_2}\left\{\frac{1}{(\lambda_3+r+1)} - \frac{r!}{\prod_{j=0}^{r}(\lambda_4+j+1)}\right\}.\end{aligned} \tag{3.2}$$

Putting $r = 0, 1, 2, 3$ in the expression (3.2) we get

$$\beta_0 = \lambda_1 + \frac{1}{\lambda_2}\left\{\frac{1}{(\lambda_3+1)} - \beta(1, \lambda_4+1)\right\}, \tag{3.3}$$

$$\beta_1 = \frac{\lambda_1}{2} + \frac{1}{\lambda_2}\left\{\frac{1}{(\lambda_3+2)} - \beta(2, \lambda_4+1)\right\}, \tag{3.4}$$

$$\beta_2 = \frac{\lambda_1}{3} + \frac{1}{\lambda_2}\left\{\frac{1}{(\lambda_3+3)} - \beta(3, \lambda_4+1)\right\}, \tag{3.5}$$

$$\beta_3 = \frac{\lambda_1}{4} + \frac{1}{\lambda_2}\left\{\frac{1}{(\lambda_3+4)} - \beta(4, \lambda_4+1)\right\}. \tag{3.6}$$

Let $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{m:n} \leq \cdots \leq X_{n:n}$ be the order statistics of a sample of size n. The L-moments [Hosking (1990)] are expectations of linear functions of order statistics and are defined as

$$L_{r+1} = r^{-1} \sum_{k=0}^{r-1} (-1)^k \binom{r-1}{k} E(X_{r-k:r}), \qquad r = 1, 2, \ldots.$$

These can be easily expressed as a linear combination of PWMs and for any distribution the rth L-moment L_r is related to the rth PWM by

$$L_{r+1} = \sum_{k=0}^{r} \beta_k (-1)^{r-k} \binom{r}{k} \binom{r+k}{k}. \tag{3.7}$$

For example the first four L-moments are related to the PWMs as

$$\begin{aligned}
L_1 &= \beta_0, \\
L_2 &= 2\beta_1 - \beta_0, \\
L_3 &= 6\beta_2 - 6\beta_1 + \beta_0, \\
L_4 &= 20\beta_3 - 30\beta_2 + 12\beta_1 - \beta_0.
\end{aligned}$$

The first L-moment is the sample mean, a measure of location. The second L-moment is a scalar multiple of Gini's mean difference, a measure of the dispersion. By dividing the higher-order L-moments by the dispersion measure, we obtain L-moment ratios. Hosking (1990) defined L-moment ratios as $\tau_r = \frac{L_r}{L_2}$, $r = 3, 4, \ldots$. For example,

$$L\text{-}skew = \tau_3 = \frac{L_3}{L_2}, \qquad L\text{-}kurtosis = \tau_4 = \frac{L_4}{L_2}.$$

These are dimensionless quantities, independent of the units of measurement of the data. τ_3 is a measure of skewness and τ_4 is a measure of kurtosis. The L-moment analogue of the coefficient of variation is the L-cv which is obtained as

$$L\text{-}cv = \tau_2 = \frac{L_2}{L_1}.$$

The L-moments and the L-moment ratios are useful quantities for summarizing a distribution. Expressions for L-moments of generalized lambda distribution have been given by Karvanen, Eriksson and Koivunen (2002) and Asquith (2007).

By giving appropriate values of $\lambda_1, \lambda_2, \lambda_3$, and λ_4 corresponding to various distributions, in equation (3.2), we can approximate the values of their PWMs and hence L-moments from equation (3.7). The expressions for L_1, L_2, τ_3, and τ_4 of some distributions are given in Hosking (1990) and the numerical values of them obtained by direct calculation are compared with the values obtained from GLD and are given in Table 1. Uniform(0,1), Exponential(3), Normal(0,1), Pareto(1,5), Logistic(0,1), and Gumbel(0,1) are approximated, respectively, by GLD[0.5,2,1,1], GLD[0.02100, −0.0003603, −0.4072 ∗

Table 1 *Comparison of L-moments*

Distribution	$x(p)$	L-moments: Theoretical	Numerical value: Direct	Numerical value: Using GLD
		$\frac{1}{2}(\alpha+\beta)$	0.5	0.5
Uniform	$x=\alpha+(\beta-\alpha)p$	$\frac{1}{6}(\beta-\alpha)$	0.1667	0.1667
(α, β)		0	0	0
		0	0	0
		α	3	2.9993
Exp(α)	$x=-\alpha\log(1-p)$	$\alpha/2$	1.5	1.5013
		$1/3$	0.3333	0.3313
		$1/6$	0.1667	0.1670
		μ	0	0
Normal	$x=\mu+\sigma\phi^{-1}(p)$	$\pi^{-1/2}\sigma$	0.5642	0.5638
(μ,σ)		0	0	0
		$30\pi^{-1}\tan^{-1}\sqrt{2}-9$	0.1226	0.1245
		$\alpha/(1+k)$	0.25	0.25
Pareto	$x=\frac{\alpha[1-(1-p)^k]}{k}$	$\alpha/(1+k)(2+k)$	0.1389	0.1389
(α, k)		$(1-k)/(3+k)$	0.4286	0.4286
		$(1-k)(2-k)/(3+k)(4+k)$	0.2481	0.2481
		ξ	0	0
Logistic	$x=\xi+\alpha\log(\frac{p}{1-p})$	α	1	0.9986
(ξ, α)		0	0	0
		$1/6$	0.1667	0.1668
		$\xi+\gamma\alpha$	0.5772	0.5775
Gumbel	$x=\xi-\alpha\log(-\log p)$	$\alpha\log 2$	0.6931	0.6905
(ξ, α)		0.1699	0.1699	0.1742
		0.1504	0.1504	0.16
		(γ is Euler's constant)		

$10^{-5}, -0.001076]$, GLD$[0, 0.1975, 0.1349, 0.1349]$, GLD$[0, -1, 7.34512*10^{-12}, -0.2]$, GLD$[0, -0.0003637, -0.0003630, -0.0003637]$, and GLD$[-0.1857, 0.02107, 0.006696, 0.02326]$. In column 4 the values in the 1st, 2nd, 3rd, and 4th rows against each distribution give the numerical values of L_1, L_2, τ_3, and τ_4, respectively, of that distribution. The tabled values clearly justify the use of GLD for computing the PWMs and L-moments of unimodal continuous distributions. □

4 PWMs and L-moments for type I and II singly censored data

Observed data sets, containing values above or below the analytical threshold of measuring equipment are referred to as censored. Such data are frequently encountered in reliability theory and quality and quantity monitoring applications of water, air, and soil. Censored data are categorized as either type I censoring, where the measurement threshold is fixed and the number of censored data points varies, or type II censoring, where the number of censored data points is fixed and the implicit threshold varies. A more complicated form of type I censoring occurs when each item has its own specific censoring time. If the censoring time is the same for all items we usually refer to it as type I singly censoring. Type II censoring is also often used in life testing, where, for example, a total of n items are placed on test, but instead of continuing until all n items have failed, the test is terminated at the time of the mth item failure.

In this section, we introduce single unified expressions for the PWMs and L-moments of the type I and II singly censored (left or right) distribution and a method of fitting distributions to a censored data set using GLD.

4.1 Case I: right censoring

Let $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{m:n} \leq \cdots \leq X_{n:n}$ be the order statistics of a sample of size n. Type I right censoring occurs when m of these values are observed ($m \leq n$) and the remaining $n-m$ are censored above a known threshold T. Since the censoring threshold T is fixed in type I censoring, m is a random variable with a binomial distribution. Otherwise type II censoring results, and T becomes the random variable.

The PWMs and L-moments evaluated only over a part of the range of the random variable are often known as the partial PWMs (PPWMs) and partial L-moments. In right censoring, the censored observations are greater than the measurement threshold. Hosking (1995) introduced two different (PPWMs) for use in the analysis of right-censored observations. Hosking's "type A" PPWM, β_r^A, is equivalent to the probability weighted moment (PWM) of the uncensored observations. His type B PPWM, β_r^B, is equal to the PWM of the completed sample, where the censored observations above the censoring threshold T are set equal to the censoring threshold in type I censoring and the censored observations above the mth order statistics are set equal to $X_{m:n}$ in type II censoring. For the estimation of the parameters of a distribution usually "B" type PPWMs are preferred [Hosking (1995)] and we are considering only that type of PPWMs. The rth order type B PPWM of a right-censored distribution with CDF $F(x) = p$ and quantile function $x(p)$, with the censoring threshold $T = x(c)$ satisfying $P(X \leq T) = c$ is [Hosking (1995)]

$$\begin{aligned} \beta_r^B &= \int_{-\infty}^{T} x[F(x)]^r \, dF(x) + \frac{T[1-\{F(T)\}^{r+1}]}{r+1} \\ &= \int_0^c p^r x(p) \, dp + \frac{1-c^{r+1}}{r+1} x(c). \end{aligned} \tag{4.1}$$

Proposition 4.1. *The type B PPWMs of a GLD*$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ *family for singly right censoring are given by*

$$\beta_r^B = \frac{\lambda_1 c^{r+1}}{r+1} + \frac{c^{\lambda_3+r+1}}{\lambda_2(\lambda_3+r+1)} - \frac{1}{\lambda_2}\beta_c(r+1, \lambda_4+1) + \frac{1-c^{r+1}}{r+1}x(c), \quad (4.2)$$

where

$$\beta_c(m,n) = \int_{p\le c} p^{m-1}(1-p)^{n-1}\,dp \qquad \textit{for } 0<c<1;\ p>0$$

is the incomplete beta function.

Proof.

$$\begin{aligned}
\beta_r^B &= \int_0^c p^r x(p)\,dp + \frac{1-c^{r+1}}{r+1}x(c)\\
&= \int_0^c p^r\left(\lambda_1 + \frac{p^{\lambda_3}-(1-p)^{\lambda_4}}{\lambda_2}\right)dp + \frac{1-c^{r+1}}{r+1}x(c)\\
&\quad \text{(by doing integration by parts we get)}\\
&= \frac{\lambda_1 c^{r+1}}{r+1} + \frac{c^{\lambda_3+r+1}}{\lambda_2(\lambda_3+r+1)} - \frac{1}{\lambda_2}\beta_c(r+1,\lambda_4+1) + \frac{1-c^{r+1}}{r+1}x(c)\\
&= \frac{\lambda_1 c^{r+1}}{r+1} + \frac{c^{\lambda_3+r+1}}{\lambda_2(\lambda_3+r+1)}\\
&\quad + \frac{1}{\lambda_2}\left\{\sum_{j=0}^{r}\frac{r!}{(r-j)!}\frac{c^{(r-j)}(1-c)^{(\lambda_4+j+1)}}{\prod_{i=0}^{j}(\lambda_4+i+1)} - \frac{r!}{\prod_{j=0}^{r}(\lambda_4+j+1)}\right\}\\
&\quad + \frac{1-c^{r+1}}{r+1}x(c).
\end{aligned}$$

Putting r = 0, 1, 2, 3 in the expression (4.2) we get

$$\begin{aligned}
\beta_0^B &= \lambda_1 c + \frac{c^{\lambda_3+1}}{\lambda_2(\lambda_3+1)} - \frac{1}{\lambda_2}\beta_c(1,\lambda_4+1) + (1-c)x(c)\\
&= \lambda_1 c + \frac{c^{\lambda_3+1}}{\lambda_2(\lambda_3+1)} + \frac{(1-c)^{(\lambda_4+1)}}{\lambda_2(\lambda_4+1)} - \frac{1}{\lambda_2(\lambda_4+1)} + (1-c)x(c),
\end{aligned} \quad (4.3)$$

$$\begin{aligned}
\beta_1^B &= \frac{\lambda_1 c^2}{2} + \frac{c^{\lambda_3+2}}{\lambda_2(\lambda_3+2)} - \frac{1}{\lambda_2}\beta_c(2,\lambda_4+1) + \frac{1-c^2}{2}x(c)\\
&= \frac{\lambda_1 c^2}{2} + \frac{c^{\lambda_3+2}}{\lambda_2(\lambda_3+2)} + \frac{1}{\lambda_2}\left\{\frac{c(1-c)^{\lambda_4+1}}{\lambda_4+1} + \frac{(1-c)^{\lambda_4+2}-1}{(\lambda_4+1)(\lambda_4+2)}\right\} \quad (4.4)\\
&\quad + \frac{1-c^2}{2}x(c),
\end{aligned}$$

$$
\begin{aligned}
\beta_2^B &= \frac{\lambda_1 c^3}{3} + \frac{c^{\lambda_3+3}}{\lambda_2(\lambda_3+3)} - \frac{1}{\lambda_2}\beta_c(3, \lambda_4+1) + \frac{1-c^3}{3}x(c) \\
&= \frac{\lambda_1 c^3}{3} + \frac{c^{\lambda_3+3}}{\lambda_2(\lambda_3+3)} \\
&\quad + \frac{1}{\lambda_2}\left\{\frac{c^2(1-c)^{\lambda_4+1}}{\lambda_4+1} + \frac{2c(1-c)^{\lambda_4+2}}{(\lambda_4+1)(\lambda_4+2)} + \frac{2(1-c)^{\lambda_4+3}-2}{(\lambda_4+1)(\lambda_4+2)(\lambda_4+3)}\right\} \\
&\quad + \frac{1-c^3}{3}x(c),
\end{aligned} \tag{4.5}
$$

$$
\begin{aligned}
\beta_3^B &= \frac{\lambda_1 c^4}{4} + \frac{c^{\lambda_3+4}}{\lambda_2(\lambda_3+4)} - \frac{1}{\lambda_2}\beta_c(4, \lambda_4+1) + \frac{1-c^4}{4}x(c) \\
&= \frac{\lambda_1 c^4}{4} + \frac{c^{\lambda_3+4}}{\lambda_2(\lambda_3+4)} \\
&\quad + \frac{1}{\lambda_2}\left\{\frac{c^3(1-c)^{\lambda_4+1}}{\lambda_4+1} + \frac{3c^2(1-c)^{\lambda_4+2}}{(\lambda_4+1)(\lambda_4+2)}\right. \\
&\qquad \left. + \frac{6c(1-c)^{\lambda_4+3}}{(\lambda_4+1)(\lambda_4+2)(\lambda_4+3)} + \frac{6((1-c)^{\lambda_4+4}-1)}{(\lambda_4+1)(\lambda_4+2)(\lambda_4+3)(\lambda_4+4)}\right\} \\
&\quad + \frac{1-c^4}{4}x(c).
\end{aligned} \tag{4.6}
$$

□

4.2 Case 2: left censoring

The type I left censoring results when the observations below a fixed threshold T are censored. $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{m-1:n} \leq T$ are censored and $T \leq X_{m:n} \leq \cdots \leq X_{n-1:n} \leq X_{n:n}$ are observed where the number of the censored values ($m-1 = n-k$) is a random variable. The number of observed values is k. Similarly type II left censoring occurs when the smallest $m-1$ observations are censored.

The rth order type B PPWMs of a left-censored distribution with CDF $F(x) = p$ and quantile function $x(p)$, with the censoring threshold T satisfying $F(T) = c$, is given [Zafirakou-Kulouris et al. (1998)] by

$$
\beta_r^{B'} = \int_c^1 p^r x(p)\,dp + \frac{c^{r+1}}{r+1}x(c). \tag{4.7}
$$

Proposition 4.2. *The type B PPWMs of a GLD$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ family for left censoring are given by*

$$
\begin{aligned}
\beta_r^{B'} &= \frac{\lambda_1(1-c^{r+1})}{r+1} + \frac{(1-c^{\lambda_3+r+1})}{\lambda_2(\lambda_3+r+1)} \\
&\quad - \frac{1}{\lambda_2}\{\beta(r+1, \lambda_4+1) - \beta_c(r+1, \lambda_4+1)\} + \frac{c^{r+1}}{r+1}x(c)
\end{aligned} \tag{4.8}
$$

$$= \lambda_1 \frac{(1-c^{r+1})}{r+1} + \frac{(1-c^{\lambda_3+r+1})}{\lambda_2(\lambda_3+r+1)}$$
$$- \frac{1}{\lambda_2}\left\{\sum_{j=0}^{r} \frac{r!}{(r-j)!} \frac{c^{(r-j)}(1-c)^{(\lambda_4+j+1)})}{\prod_{i=0}^{j}(\lambda_4+i+1)}\right\} + \frac{c^{r+1}}{r+1}x(c).$$

Proof. Substituting for $x(p)$ from equation (2.1) and doing integration by parts we get the result.

Putting $r = 0, 1, 2, 3$ in the expression (4.8) we get

$$\begin{aligned}\beta_0^{B'} &= \lambda_1(1-c) + \frac{1-c^{\lambda_3+1}}{\lambda_2(\lambda_3+1)} \\ &\quad - \frac{1}{\lambda_2}\{\beta(1, \lambda_4+1) - \beta_c(1, \lambda_4+1)\} + cx(c),\end{aligned} \tag{4.9}$$

$$\begin{aligned}\beta_1^{B'} &= \frac{\lambda_1(1-c^2)}{2} + \frac{1-c^{\lambda_3+2}}{\lambda_2(\lambda_3+2)} \\ &\quad - \frac{1}{\lambda_2}\{\beta(2, \lambda_4+1) - \beta_c(2, \lambda_4+1)\} + \frac{c^2}{2}x(c),\end{aligned} \tag{4.10}$$

$$\begin{aligned}\beta_2^{B'} &= \frac{\lambda_1(1-c^3)}{3} + \frac{1-c^{\lambda_3+3}}{\lambda_2(\lambda_3+3)} \\ &\quad - \frac{1}{\lambda_2}\{\beta(3, \lambda_4+1) - \beta_c(3, \lambda_4+1)\} + \frac{c^3}{3}x(c),\end{aligned} \tag{4.11}$$

$$\begin{aligned}\beta_3^{B'} &= \frac{\lambda_1(1-c^4)}{4} + \frac{1-c^{\lambda_3+4}}{\lambda_2(\lambda_3+4)} \\ &\quad - \frac{1}{\lambda_2}\{\beta(4, \lambda_4+1) - \beta_c(4, \lambda_4+1)\} + \frac{c^4}{4}x(c).\end{aligned} \tag{4.12}$$

□

4.3 L-moments for censored distributions using GLD

The same relation (3.7) holds for PPWMs and partial L-moments [Hosking (1995)]. The population L-moments L_r^B and $L_r^{B'}$ for right and left-censored distributions are given in terms of β_r^B and $\beta_r^{B'}$ by relation (3.7).

Hosking (1995) gives the expressions of the PPWMs or partial L-moments of certain distributions. For a Pareto distribution with quantile function $x(p) = \alpha[1-(1-p)^k]/k$ the first four L-moments are given by

$$\begin{aligned}L_1^B &= \alpha m_1, \\ L_2^B &= \alpha(m_1 - m_2), \\ L_3^B &= \alpha(m_1 - 3m_2 + 2m_3),\end{aligned}$$

Table 2 *L moments of Pareto distribution for different censoring fraction c*

Distribution	c	Method	L_1^B	L_2^B	L_3^B	L_4^B
Pareto	0.99	direct	0.2437	0.1326	0.0533	0.0283
$\alpha = 1/5$,		gld	0.2437	0.1326	0.0533	0.0283
$k = -1/5$	0.9	direct	0.2104	0.1010	0.0250	0.0043
		gld	0.2104	0.1010	0.0250	0.0043
	0.8	direct	0.1810	0.0760	0.0074	−0.0050
		gld	0.1810	0.0760	0.0074	−0.0050
	0.7	direct	0.1546	0.0562	−0.0026	−0.0064
		gld	0.1546	0.0562	−0.0026	−0.0064
	0.6	direct	0.1299	0.0401	−0.0075	−0.0043
		gld	0.1299	0.0401	−0.0075	−0.0043
	0.5	direct	0.1064	0.0272	−0.0089	−0.0013
		gld	0.1064	0.0272	−0.0089	−0.0013

$$L_4^B = \alpha(m_1 - 6m_2 + 10m_3 - 5m_4),$$

where $m_r = [1 - (1 - c)^{(r+k)}]/(r + k)$.

In Table 2 the numerical values of the first four L-moments of Pareto distribution for different censoring values are compared with the values obtained by the corresponding GLD approximation.

For a gamma distribution with cumulative distribution function $F(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha} \int_0^x x^{\alpha-1} e^{-x/\beta}\, dx$, with $x \geq 0$ the first two type A PPWMs are given by

$$\beta_0^A = \alpha\beta - \frac{T^\alpha e^{-T/\beta}}{\beta^{\alpha-1} c\Gamma(\alpha)}$$

and

$$\beta_1^A = \frac{1}{2}\alpha\beta - \frac{T^\alpha e^{-T/\beta}}{\beta^{\alpha-1} c\Gamma(\alpha)} + \frac{\beta\Gamma(2\alpha, 2T/\beta)}{2^{2\alpha} c^2 [\Gamma(\alpha)]^2}.$$

From this we can obtain type B PPWMs using the relation [Hosking (1995)]

$$\beta_r^B = c^{(r+1)} \beta_r^A + \frac{1 - c^{r+1}}{r + 1} x(c).$$

In Table 3 the numerical values of the first two type B PPWMs of gamma distribution obtained by the direct method are compared with the values obtained by using the GLD. Tables 2 and 3 strongly recommend the use of GLD for modeling univariate continuous distributions using their PWMs and L-moments even for censored observations.

Table 3 *Probability weighted moments of gamma distribution for different censoring fractions c*

Distribution	c	Method	β_0^B	β_1^B
Gamma	0.9	direct	14.5208	8.8788
$\alpha = 5$,		gld	14.5073	8.8585
$\beta = 3$	0.8	direct	13.9664	8.3663
		gld	13.9556	8.3481
	0.7	direct	13.3497	7.8269
		gld	13.3502	7.8172
	0.6	direct	12.6663	7.2634
		gld	12.6802	7.2624
	0.5	direct	11.9041	6.6730
		gld	11.9292	6.6782

4.4 Fitting of distributions to censored data using GLD

Hosking (1995) derived sample estimators of type B PPWMs for right-censored observations. Such estimators, denoted as b_r^B, are unbiased estimators of their theoretical counterparts, β_r^B, given in (4.1). In type I censoring let the sample size be n, of which m values are observed and $n - m$ are censored above a known threshold T. Hosking (1995) showed that

$$b_r^B = n^{-1}\left\{\sum_{j=1}^{m} \frac{(j-1)(j-2)\cdots(j-r)}{(n-1)(n-2)\cdots(n-r)} X_{j:n} + \left(\sum_{j=m+1}^{n} \frac{(j-1)(j-2)\cdots(j-r)}{(n-1)(n-2)\cdots(n-r)}\right) T\right\} \tag{4.13}$$

are unbiased estimators of β_r^B for $r = 1, 2, 3, \ldots$.

To estimate the parameters of the right-censored GLD in the case of type I single censoring, we can equate the sample and population PPWMs. As for estimation usually "B" type PPWMs are preferred [Hosking (1995)] by comparing the first four theoretical and sample moments obtained from expressions (4.2) and (4.13), we can obtain the appropriate values of the parameters $\lambda_1, \lambda_2, \lambda_3$ and λ_4.

For left-censored data the corresponding unbiased sample estimators are given [Zafirakou-Kulouris et al. (1998)] by

$$b_r^{B'} = n^{-1}\left\{\sum_{j=1}^{n-k} \frac{(j-1)(j-2)\cdots(j-r)}{(n-1)(n-2)\cdots(n-r)} T + \sum_{j=n-k+1}^{n} \frac{(j-1)(j-2)\cdots(j-r)}{(n-1)(n-2)\cdots(n-r)} X_{j:n}\right\} \tag{4.14}$$

Table 4 *Lifetimes, in weeks, of* 34 *transistors in an accelerated life test, after Wilk, Gnanadesikan and Huyett* (1962). *Three of the times, denoted by asterisks, are censored at* 52 *weeks*

3, 4, 5, 6, 6, 7, 8, 8, 9, 9, 9, 10, 10, 11, 11, 11, 13, 13, 13, 13, 13, 17,17, 19, 19, 25, 29, 33, 42, 42, 52, 52*, 52*, 52*

where $k = n - m + 1$. In the case of type II censoring "T" is to be replaced by $X_{m:n}$ in the above expressions. So by comparing the first four theoretical and sample PPWMs using expressions (4.8) and (4.14) we can fit a GLD for a left-censored data.

Example: *density estimation of a type I singly censored data using GLD*. As an example, Table 4 shows a data set consisting of the lifetimes of 34 transistors in an accelerated life test. Three of the lifetimes are censored, so the censoring fraction is $3/34$ or 8.8% . The data were given by Wilk, Gnanadesikan and Huyett (1962), who stated that "there is reason, from past experience, to expect that the gamma distribution might reasonably approximate the failure time distribution." Wilk, Gnanadesikan and Huyett (1962) and also Lawless (1982) fitted a gamma distribution with parameters $\hat{k} = 1.625$ and $\hat{\alpha} = 12.361$ to the data. If we fit a GLD for this data set it seems to be more appropriate. Using the relation (4.13) we obtained the first four type B PPWMs of the sample as 18.9117, 13.4938, 10.8436, and 9.1691, respectively. Equating them with equations (4.3), (4.4), (4.5), and (4.6) with the value of $c = 0.912$ and substituting the expression (2.1) for $x(c)$ we get the values of the parameters of the corresponding GLD as $\lambda_1 = 6.2$, $\lambda_2 = -0.105$, $\lambda_3 = -0.06$, and $\lambda_4 = -0.76$.

The Kolmogorov–Smirnov (K–S) distance between the empirical distribution function and the fitted gamma distribution is 0.2144 whereas it is only 0.0951 for the GLD. The graphical comparison of the empirical distribution function and the fitted distribution function is given in Figure 1.

Indeed, the figure and the K–S distances suggest that the GLD approximation is more appropriate than the gamma distribution for this data set.

5 Conclusion

Since GLD is a four-parameter family of distributions, consisting of a wide variety of curve shapes, the expressions for the PWMs and L-moments of it help us to find out the same for any univariate continuous (both complete and censored) distribution. Another advantage of using of GLD is that the expressions for the PWMs and L-moments, both for complete and censored data, do not change with respect to changes in the form of the distribution except for the values of the parameters. This makes both analysis and decision making much simpler. Again, the

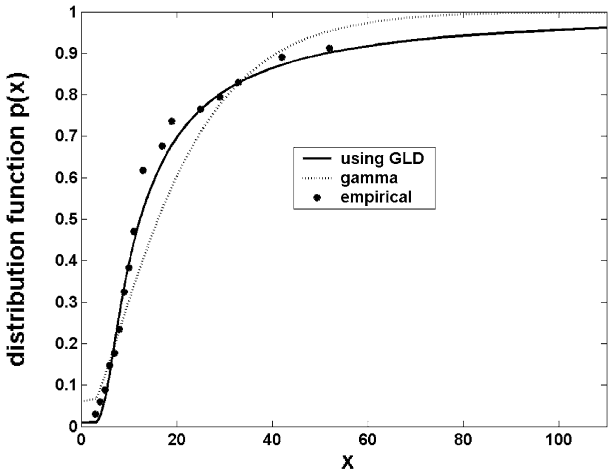

Figure 1 *Comparison of empirical distribution functions of type I censored data.*

PWMs and L-moments of all the univariate continuous distributions which are not representable in the inverse form can be approximated using GLD. We can easily fit a distribution to both censored and complete data using the method explained in Section 4.4. As GLD covers a wide class of distributions of a variety of shapes, it also provides a sound basis for most realistic modeling. As the method employed is independent of any specific distributional assumptions, it can be used for all univariate continuous distributions.

Acknowledgments

The authors gratefully appreciate the dedicated review and valuable comments of the esteemed referee, which enhanced the merit of the article substantially. We would like to place on record our sincere gratitude to the editor in chief.

References

Asquith, W. H. (2007). L-moments and TL-moments of the generalized lambda distribution. *Computational Statistics and Data Analysis* **51** 4484–4496. MR2364460

Greenwood, J. A., Landwehr, J. M., Matalas, N. C. and Wallis, J. R. (1979). Probability weighted moments: Definition and relation to parameters of several distributions expressible in inverse form. *Water Resources Research* **15** 1049–1054.

Greis, N. P. and Wood, E. F. (1981). Regional flood frequency estimation and network design. *Water Resources Research* **17** 1167–1177; corrections **19** (1983) 589–590.

Hosking, J. R. M. and Wallis, J. R. (1997). *Regional Frequency Analysis: An Approach Based on L-Moments*. Cambridge Univ. Press, New York.

Hosking, J. R. M. and Wallis J. R. (1987). Parameter and quantile estimation for the generalised Pareto distributon. *Technometrics* **29** 339–349. MR0906643

Hosking, J. R. M. (1990). L-moments: Analysis and estimation of distributions using linear combinations of order statistics. *Journal of the Royal Statistical Society, Series B* **52** 105–124. MR1049304

Hosking, J. R. M., Wallis, J. R. and Wood, E. F. (1985). Estimation of the generalized extreme-value distribution by the method of probability weighted moments. *Technometrics* **27** 251–261. MR0797563

Hosking, J. R. M. (1995). The use of L-moments in the analysis of censored data. In *Recent Advances in Life-Testing and Reliability* (N. Balakrishnan, ed.) Chap. 29, 546–564. CRC Press, Boca Raton.

Hastings, C. F. Mosteller, F., Turkey, J. W. and Winsor, C. P. (1947). Low moments for small samples: A comparitive study of statistics. *Annals of Mathematics* **18** 413–426. MR0022335

Karian, Z. A. and Dudewicz, E. J. (2000). *Fitting Statistical Distributions—The Generalised Lambda Distribution and Generalised Bootstrap Methods*. CRC press, New York. MR1850548

Karvanen, J., Eriksson, J. and Koivunen, V. (2002). Adaptive score functions for maximum likelihood ICA. *Journal of VLSI Signal Processing* **32** 83–92.

King, R. and MacGillivray, H. L. (1999). A starship estimation method for the generalized lambda distributions. *Australian and New Zealand Journal of Statistics* **41** 353–374. MR1718029

Lawless, J. F. (1982). *Statistical Models and Methods for Lifetime Data*. Wiley, New York. MR0640866

Ozturk, A. and Dale, R. (1985). Least squares estimation of the parameters of the generalized lambda distribution. *Technometrics* **27** 81–84.

Ramberg, J. S. and Schmeiser, B. W. (1974). An approximate method for generating asymmetric random variable. *Communications of the ACM* **17** 78–82. MR0331711

Ramberg, J. S., Tadikamalla, P. R., Dudewicz, E. J. and Mykytka, E. F. (1979). A probability distribution and its uses in fitting data. *Technometrics* **21** 201–214.

Wilk, M. B., Gnanadesikan, R. and Huyett, M. J. (1962). Estimation of parameters of the gamma distribution using order statistics. *Biometrika* **49** 525–545. MR0157441

Zafirakou-Kulouris, A., Richard, M. V., Sott, M. V. and Joerg, H. (1998). L-moment diagrams for censored observations. *Water Resources Research* **34** 1241–1249.

Govt. College
Kasaragod 671123
Kerala
India
E-mail: mersu@rediffmail.com

Nehru Arts & Science College
Kanhangad 671328
Kerala
India
E-mail: mknas@rediffmail.com