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A probabilistic ergodic decomposition result

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Abstract. Let $(X, \mathfrak{X}, \mu)$ be a standard probability space. We say that a sub-σ-algebra $\mathfrak{B}$ of $\mathfrak{X}$ *decomposes* μ *in an ergodic way* if any regular conditional probability ${}^{\mathfrak{B}}P$ with respect to $\mathfrak{B}$ and μ satisfies, for μ-almost every $x \in X$, $\forall B \in \mathfrak{B}$, ${}^{\mathfrak{B}}P(x, B) \in \{0, 1\}$. In this case the equality $\mu(\cdot) = \int_X {}^{\mathfrak{B}}P(x, \cdot)\mu(\mathrm{d}x)$, gives us an integral decomposition in "$\mathfrak{B}$-ergodic" components.

For any sub-σ-algebra $\mathfrak{B}$ of $\mathfrak{X}$, we denote by $\overline{\mathfrak{B}}$ the smallest sub-σ-algebra of $\mathfrak{X}$ containing $\mathfrak{B}$ and the collection of all sets A in $\mathfrak{X}$ satisfying $\mu(A) = 0$. We say that $\mathfrak{B}$ is μ-complete if $\mathfrak{B} = \overline{\mathfrak{B}}$.

Let $\{\mathfrak{B}_i\colon i \in I\}$ be a non-empty family of sub-σ-algebras which decompose μ in an ergodic way. Suppose that, for any finite subset J of I, $\bigcap_{i\in J} \overline{\mathfrak{B}_i} = \overline{\bigcap_{i\in J} \mathfrak{B}_i}$; this assumption is satisfied in particular when the σ-algebras $\mathfrak{B}_i$, $i \in I$, are μ-complete. Then we prove that the sub-σ-algebra $\bigcap_{i\in I} \mathfrak{B}_i$ decomposes μ in an ergodic way.

Résumé. Soit $(X, \mathfrak{X}, \mu)$ un espace probabilisé standard. Nous disons qu'une sous-tribu $\mathfrak{B}$ de $\mathfrak{X}$ décompose ergodiquement μ si toute probabilité conditionnelle régulière ${}^{\mathfrak{B}}P$ relativement à $\mathfrak{B}$ et μ, vérifie, pour μ-presque tout $x \in X$, $\forall B \in \mathfrak{B}$, ${}^{\mathfrak{B}}P(x, B) \in \{0, 1\}$. Dans ce cas l'égalité $\mu(\cdot) = \int_X {}^{\mathfrak{B}}P(x, \cdot)\mu(\mathrm{d}x)$, nous donne une décomposition intégrale en composantes "$\mathfrak{B}$-ergodiques."

Pour toute sous-tribu $\mathfrak{B}$ de $\mathfrak{X}$, nous notons $\overline{\mathfrak{B}}$ la plus petite sous-tribu de $\mathfrak{X}$ contenant $\mathfrak{B}$ et tous les sous-ensembles mesurables de X de μ-mesure nulle. Nous disons que la tribu $\mathfrak{B}$ est μ-complète si $\mathfrak{B} = \overline{\mathfrak{B}}$.

Soit $\{\mathfrak{B}_i\colon i \in I\}$ une famille non vide de sous-tribus de $\mathfrak{X}$ décomposant ergodiquement μ. Supposons que, pour toute partie finie J de I, $\bigcap_{i\in J} \overline{\mathfrak{B}_i} = \overline{\bigcap_{i\in J} \mathfrak{B}_i}$; cette hypothèse est satisfaite si les tribus $\mathfrak{B}_i$, $i \in I$, sont μ-complètes. Alors la sous-tribu $\bigcap_{i\in I} \mathfrak{B}_i$ décompose ergodiquement μ.

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1. Introduction

There are several versions of ergodic decomposition theorems in the literature (cf. [3,4,6–8]) which give an integral decomposition of a probability measure μ, on a standard measurable space, in ergodic components. Most of these decompositions are based on abstract results like Choquet's theorem. A probabilistic approach which can prove to be more convenient due to the properties of the conditional expectation (see [2]) is the following. Let $(X, \mathfrak{X}, \mu)$ be a standard Borel probability space. Let $\mathfrak{B}$ be a sub-σ-algebra of $\mathfrak{X}$. We denote by ${}^{\mathfrak{B}}P$ a regular conditional probability of $\mathfrak{B}$ and μ. We say that $\mathfrak{B}$ decomposes μ in an ergodic way if for μ-almost every $x \in X$, $\forall B \in \mathfrak{B}$, ${}^{\mathfrak{B}}P(x, B) \in \{0, 1\}$. In this case, the equality $\mu(\mathrm{d}x) = \int_X {}^{\mathfrak{B}}P(x, \cdot)\mu(\mathrm{d}x)$ gives us an integral decomposition in $\mathfrak{B}$-*ergodic* components.

In [7,8] Shimomura proves that the intersection of a decreasing sequence of separable sub-σ-algebras of $\mathfrak{X}$ decomposes μ in an ergodic way. He also gives an example of a standard probability space and a suitable sub-σ-algebra for which the above decomposition is not ergodic.

Let $\{\mathfrak{B}_i\colon i \in I\}$ be a non-empty family of sub-σ-algebras which decompose μ in an ergodic way. Suppose that, for any finite subset J of I, $\bigcap_{i\in J} \overline{\mathfrak{B}_i} = \overline{\bigcap_{i\in J} \mathfrak{B}_i}$. The aim of this paper is to prove that the sub-σ-algebra $\bigcap_{i\in I} \mathfrak{B}_i$ decomposes μ in an ergodic way.

2. Preliminaries

It is not necessary to work with standard Borel spaces. We only need probability space for which any sub-σ-algebra has regular conditional probabilities. In this section we recall some results about this property. On a standard Borel space $(X, \mathfrak{X})$, it is well known that, for any probability measure μ and any sub-σ-algebra $\mathfrak{B}$ of $\mathfrak{X}$, there exists a regular conditional probability with respect to μ and $\mathfrak{B}$.

Definition 2.1. *A σ-algebra on a set X is called separable if it is generated by a countable sub-algebra.*

Proposition 2.2. *Let $(X, \mathfrak{X})$ be a measurable space with a separable σ-algebra $\mathfrak{X}$. Then two positive σ-finite measures are equal if they coincide on a countable algebra generating $\mathfrak{X}$.*

Definition 2.3. *A class $\mathcal{C}$ of subsets of X is said to be compact if, for any sequence $(C_n)_{n\in\mathbb{N}}$ of elements of $\mathcal{C}$ with an empty intersection $\bigcap_{n\in\mathbb{N}} C_n$, there exists a natural integer p such that $\bigcap_{n=0}^{p} C_n = \varnothing$.*

Definition 2.4. *Let $(X, \mathfrak{X}, \mu)$ be a probability space. Let $\mathcal{C}$ be a compact subclass of $\mathfrak{X}$. We say that $\mathcal{C}$ is μ-approximating if*

$$\forall A \in \mathfrak{X} \quad \mu(A) = \sup\bigl\{\mu(C)\colon C \in \mathcal{C}, C \subset A\bigr\}.$$

Definition 2.5. *Let $(X, \mathfrak{X}, \mu)$ be a probability space. Let $\mathfrak{B}$ be a sub-σ-algebra of $\mathfrak{X}$. We call regular conditional probability with respect to $\mathfrak{B}$ and μ a map P from $X \times \mathfrak{X}$ to* $[0, 1]$ *such that*:

(i) *for any $x \in X$, $P(x, \cdot)$ is a probability measure on $\mathfrak{X}$.*

(ii) *for any $A \in \mathfrak{X}$, the map $x \in X \mapsto P(x, A)$ is a version of the conditional expectation* $\mathbb{E}_\mu[1_A|\mathfrak{B}]$; *that is, this map is $\mathfrak{B}$-measurable and, for any $B \in \mathfrak{B}$,*

$$\int_X 1_A(x) 1_B(x) \mu(\mathrm{d}x) = \int_X P(x, A) 1_B(x) \mu(\mathrm{d}x).$$

Then for any non-negative (*or bounded*) *$\mathfrak{X}$-measurable function f, the function Pf defined by $Pf(x) = \int_X f(y) \times P(x, \mathrm{d}y)$* (*expectation of f with respect to the probability $P(x, \cdot)$*) *is a version of the conditional expectation* $\mathbb{E}_\mu[f|\mathfrak{B}]$.

Theorem 2.6 ([5], corollaire Proposition V-4-4). *Let $(X, \mathfrak{X}, \mu)$ be a probability space with a separable σ-algebra $\mathfrak{X}$ containing a μ-approximating compact class.*

Then, for any sub-σ-algebra $\mathfrak{B}$ of $\mathfrak{X}$ there exists a regular conditional probability with respect to $\mathfrak{B}$ and μ.

Remarks. *Let $(X, \mathfrak{X}, \mu)$ be a probability space with a separable σ-algebra $\mathfrak{X}$ containing a μ-approximating compact class. Let $\mathfrak{B}$ be a sub-σ-algebra of $\mathfrak{X}$.*

1. *If P and Q are two regular conditional probabilities with respect to $\mathfrak{B}$ and μ then, for μ-almost every $x \in X$, the probability measures $P(x, \cdot)$ and $Q(x, \cdot)$ are equal.*
2. *If P is a regular conditional probability with respect to $\mathfrak{B}$ and μ, then for any $B \in \mathfrak{B}$, we have, for μ-almost every $x \in X$,*

$$P(x, B) = E_\mu[1_B|\mathfrak{B}](x) = 1_B(x) = \delta_x(B) \in \{0, 1\},$$

where δ_x is the Dirac measure at the point x.

When the σ-algebra $\mathfrak{B}$ is separable, from Proposition 2.2 *we can permute "for any $B \in \mathfrak{B}$" and "for μ-almost every $x \in X$."*

3. *Let $\overline{\mathfrak{B}}$ be the smallest sub-σ-algebra containing $\mathfrak{B}$ and the collection of all sets A in $\mathfrak{X}$ satisfying $\mu(A)=0$. One sees easily that any regular conditional probability with respect to $\mathfrak{B}$ and μ is a regular conditional probability with respect to $\overline{\mathfrak{B}}$ and μ. For two sub-σ-algebras $\mathfrak{B}_1$ and $\mathfrak{B}_2$ of $\mathfrak{X}$, the sub-σ-algebra $\overline{\mathfrak{B}_1}\cap\overline{\mathfrak{B}_2}$ is not necessarily equal to $\overline{\mathfrak{B}_1\cap\mathfrak{B}_2}$; consequently $\mathbb{L}^2(X,\overline{\mathfrak{B}_1}\cap\overline{\mathfrak{B}_2},\mu)$ is not necessarily equal to $\mathbb{L}^2(X,\overline{\mathfrak{B}_1\cap\mathfrak{B}_2},\mu)$.*

3. Main results

Throughout this section, we assume that $(X,\mathfrak{X},\mu)$ is a probability space with a separable σ-algebra $\mathfrak{X}$ containing a μ-approximating compact class. The preceding Remark 2 leads us to introduce the following definition.

Definition 3.1. *We say that a sub-σ-algebra $\mathfrak{B}$ of $\mathfrak{X}$ decomposes μ in an ergodic way if one (and thus all) regular conditional probability ${}^{\mathfrak{B}}P$ with respect to $\mathfrak{B}$ and μ satisfies, for μ-almost every $x\in X$, $\forall B\in\mathfrak{B}$, ${}^{\mathfrak{B}}P(x,B)\in\{0,1\}$.*

From the preceding Remarks 2 and 3 it follows that:

- Any separable sub-σ-algebra of $\mathfrak{X}$ decomposes μ in an ergodic way.
- If $\overline{\mathfrak{B}}$ decomposes μ in an ergodic way then so does $\mathfrak{B}$. (Any regular conditional probability ${}^{\mathfrak{B}}P$ with respect to $\mathfrak{B}$ and μ is a regular conditional probability with respect to $\overline{\mathfrak{B}}$ and μ. If $\overline{\mathfrak{B}}$ decomposes μ in an ergodic way then, for μ-almost every $x\in X$, ${}^{\mathfrak{B}}P(x,B)\in\{0,1\}$ for any $B\in\overline{\mathfrak{B}}$ and a fortiori for any $B\in\mathfrak{B}$. Which proves that $\mathfrak{B}$ decomposes μ in an ergodic way.)

Lemma 3.2. *Let $\mathfrak{B}$ be a sub-σ-algebra of $\mathfrak{X}$, and let ${}^{\mathfrak{B}}P$ be a regular conditional probability with respect to $\mathfrak{B}$ and μ.*

Then, for μ-almost every $x\in X$, we have the probability equalities

$$ {}^{\mathfrak{B}}P(y,\cdot)={}^{\mathfrak{B}}P(x,\cdot)\quad \text{for } {}^{\mathfrak{B}}P(x,\cdot)\text{-almost every } y\in X $$

and consequently for any $B\in\mathfrak{B}$, we have for μ-almost every $x\in X$,

$$ \forall A\in\mathfrak{X}\quad \int_X 1_A(y)1_B(y)\,{}^{\mathfrak{B}}P(x,\mathrm{d}y)=\int_X {}^{\mathfrak{B}}P(y,A)1_B(y)\,{}^{\mathfrak{B}}P(x,\mathrm{d}y). $$

The following proposition tells us that the sub-σ-algebra $\mathfrak{B}$ of $\mathfrak{X}$ decomposes μ in an ergodic way if and only if, in the last equalities, we can permute "for any $B\in\mathfrak{B}$" and "for μ-almost every $x\in X$."

Proposition 3.3. *Let $\mathfrak{B}$ be a sub-σ-algebra of $\mathfrak{X}$, and let ${}^{\mathfrak{B}}P$ be a regular conditional probability with respect to $\mathfrak{B}$ and μ.*

Then the two following assertions are equivalent:

(i) *$\mathfrak{B}$ decomposes μ in an ergodic way*;
(ii) *For μ-almost every $x\in X$, ${}^{\mathfrak{B}}P$ is a regular conditional probability with respect to $\mathfrak{B}$ and ${}^{\mathfrak{B}}P(x,\cdot)$.*

In this case, for any sub-σ-algebra $\mathfrak{C}$ of $\mathfrak{B}$ and any regular conditional probability ${}^{\mathfrak{C}}P$ with respect to $\mathfrak{C}$ and μ, for μ-almost every $x\in X$, ${}^{\mathfrak{B}}P$ is a regular conditional probability with respect to $\mathfrak{B}$ and ${}^{\mathfrak{C}}P(x,\cdot)$; that is, for μ-almost every $x\in X$,

$$ \forall(A,B)\in\mathfrak{X}\times\mathfrak{B}\quad \int_X {}^{\mathfrak{B}}P(y,A)1_B(y)\,{}^{\mathfrak{C}}P(x,\mathrm{d}y)=\int_X 1_A(y)1_B(y)\,{}^{\mathfrak{C}}P(x,dy). $$

Moreover, this assertion is true for any sub-σ-algebra $\mathfrak{C}$ of $\mathfrak{X}$ such that, for μ-almost every $x\in X$,

$$ {}^{\mathfrak{C}}P\,{}^{\mathfrak{B}}P(x,\cdot)\stackrel{\mathrm{def}}{=}\int_X {}^{\mathfrak{C}}P(x,\mathrm{d}y)\,{}^{\mathfrak{B}}P(y,\cdot)={}^{\mathfrak{C}}P(x,\cdot). $$

This last property is satisfied when $\mathfrak{C}$ is a sub-σ-algebra of $\mathfrak{B}$.

Theorem 3.4. *Let $(X, \mathfrak{X}, \mu)$ be a probability space with a separable σ-algebra $\mathfrak{X}$ containing a μ-approximating compact class. Let $\{\mathfrak{B}_i\colon i \in I\}$ be a non-empty family of sub-σ-algebras which decompose μ in an ergodic way. We suppose that, for any finite subset J of I, $\bigcap_{i\in J} \overline{\mathfrak{B}_i} = \overline{\bigcap_{i\in J} \mathfrak{B}_i}$.*

Then the sub-σ-algebra $\bigcap_{i\in I} \mathfrak{B}_i$ decomposes μ in an ergodic way.

4. Proof of the results

Throughout this section, we assume that $(X, \mathfrak{X}, \mu)$ is a probability space with a separable σ-algebra $\mathfrak{X}$ containing a μ-approximating compact class. For any sub-σ-algebra $\mathfrak{B}$ of $\mathfrak{X}$, we denote by ${}^{\mathfrak{B}}P$ a regular conditional probability with respect to $\mathfrak{B}$ and μ.

4.1. *Proof of Lemma* 3.2

Let $A \in \mathfrak{X}$. The functions $g(x) = {}^{\mathfrak{B}}P(x, A)$ and $(g(x))^2$ are $\mathfrak{B}$-measurable. Therefore, for μ-almost every $x \in X$,

$$ {}^{\mathfrak{B}}Pg(x) = \mathbb{E}_\mu[g|\mathfrak{B}](x) = g(x) \quad \text{and} \quad {}^{\mathfrak{B}}Pg^2(x) = g^2(x) = \left({}^{\mathfrak{B}}Pg(x)\right)^2. $$

From the Cauchy–Schwarz equality it follows that, for μ-almost every $x \in X$,

$$ {}^{\mathfrak{B}}P(x, A) = g(x) = g(y) = {}^{\mathfrak{B}}P(y, A) \quad \text{for } {}^{\mathfrak{B}}P(x, \cdot)\text{-almost every } y \in X. $$

The first assertion of the lemma is then a consequence of Proposition 2.2.

For any $B \in \mathfrak{B}$ and for μ-almost every $y \in X$,

$$ {}^{\mathfrak{B}}P(y, B) = \mathbb{E}_\mu[1_B|\mathfrak{B}](y) = 1_B(y). $$

As $\mu(\mathrm{d}y) = \int_X {}^{\mathfrak{B}}P(x, \mathrm{d}y)\mu(\mathrm{d}x)$, for μ-almost every $x \in X$,

$$ 1_B(y) = {}^{\mathfrak{B}}P(y, B) \quad \text{for } {}^{\mathfrak{B}}P(x, \cdot)\text{-almost every } y \in X. $$

Hence, for any $A \in \mathfrak{X}$ and for μ-almost every $x \in X$,

$$ \begin{aligned} \int_X 1_A(y)1_B(y)\,{}^{\mathfrak{B}}P(x, \mathrm{d}y) &= \int_X 1_A(y)\,{}^{\mathfrak{B}}P(y, B)\,{}^{\mathfrak{B}}P(x, \mathrm{d}y) \\ &= \int_X 1_A(y)\,{}^{\mathfrak{B}}P(x, B)\,{}^{\mathfrak{B}}P(x, \mathrm{d}y) \quad \text{(first assertion)} \\ &= {}^{\mathfrak{B}}P(x, A)\,{}^{\mathfrak{B}}P(x, B) \\ &= \int_X {}^{\mathfrak{B}}P(x, A)1_B(y)\,{}^{\mathfrak{B}}P(x, \mathrm{d}y) \\ &= \int_X {}^{\mathfrak{B}}P(y, A)1_B(y)\,{}^{\mathfrak{B}}P(x, \mathrm{d}y) \quad \text{(first assertion).} \end{aligned} \tag{1} $$

The Proposition 2.2 allows us to permute “for any $A \in \mathfrak{X}$” and “for μ-almost every $x \in X$.”

4.2. *Proof of Proposition* 3.3

Let X_0 be a measurable subset of X such that $\mu(X_0) = 1$ and for any $x \in X_0$,

$$ {}^{\mathfrak{B}}P(y, \cdot) = {}^{\mathfrak{B}}P(x, \cdot) \quad \text{for } {}^{\mathfrak{B}}P(x, \cdot)\text{-almost every } y \in X. $$

(i) $\Rightarrow$ (ii) If the σ-algebra $\mathfrak{B}$ decomposes μ in an ergodic way, then there exists a measurable subset X_1 of X such that $\mu(X_1)=1$ and for any $x \in X_1$,

$$\forall B \in \mathfrak{B} \quad {}^{\mathfrak{B}}P(x,B) \in \{0,1\}.$$

For $x \in X_0 \cap X_1$, we have for any $(A,B) \in \mathfrak{X} \times \mathfrak{B}$,

$$\int_X 1_A(y) 1_B(y)\, {}^{\mathfrak{B}}P(x,\mathrm{d}y) = {}^{\mathfrak{B}}P(x,A)\, {}^{\mathfrak{B}}P(x,B) = \int_X {}^{\mathfrak{B}}P(y,A) 1_B(y)\, {}^{\mathfrak{B}}P(x,\mathrm{d}y),$$

which shows that, for any $x \in X_0 \cap X_1$, ${}^{\mathfrak{B}}P$ is a regular conditional probability with respect to $\mathfrak{B}$ and ${}^{\mathfrak{B}}P(x,\cdot)$.

(ii) $\Rightarrow$ (i) Assume there exists a measurable subset X_2 of X such that $\mu(X_2)=1$ and for any $x \in X_2$,

$$\forall A \in \mathfrak{X} \quad {}^{\mathfrak{B}}P(y,A) = \mathbb{E}_{{}^{\mathfrak{B}}P(x,\cdot)}[1_A|\mathfrak{B}](y) \quad \text{for } {}^{\mathfrak{B}}P(x,\cdot)\text{-almost every } y \in X.$$

Then for $x \in X_0 \cap X_2$, we have, for any $B \in \mathfrak{B}$,

$${}^{\mathfrak{B}}P(x,B) = {}^{\mathfrak{B}}P(y,B) = \mathbb{E}_{{}^{\mathfrak{B}}P(x,\cdot)}[1_B|\mathfrak{B}](y) = 1_B(y) \quad \text{for } {}^{\mathfrak{B}}P(x,\cdot)\text{-almost every } y \in X.$$

Hence the assertion (i).

To prove the last assertion of the proposition we need the following lemma.

Lemma 4.1. *Let $\mathfrak{B}$ and $\mathfrak{C}$ be two sub-σ-algebras of $\mathfrak{X}$ such that $\mathfrak{C} \subset \mathfrak{B}$. Then for μ-almost every $x \in X$, we have the probability equalities*

$${}^{\mathfrak{C}}P\,{}^{\mathfrak{B}}P(x,\cdot) = {}^{\mathfrak{B}}P\,{}^{\mathfrak{C}}P(x,\cdot) = {}^{\mathfrak{C}}P(x,\cdot).$$

Proof. For any $A \in \mathfrak{X}$, we have the classical μ-almost everywhere equalities

$$\mathbb{E}_\mu\big[\mathbb{E}_\mu[1_A|\mathfrak{B}]|\mathfrak{C}\big] = \mathbb{E}_\mu\big[\mathbb{E}_\mu[1_A|\mathfrak{C}]|\mathfrak{B}\big] = \mathbb{E}_\mu[1_A|\mathfrak{C}].$$

Moreover, if f and g are non-negative (or bounded) measurable functions, we know that: $f=g$ μ-a.e. $\Rightarrow \mathbb{E}_\mu[f|\mathfrak{B}] = \mathbb{E}_\mu[g|\mathfrak{B}]$ μ-a.e. It follows that, for any $A \in \mathfrak{X}$,

$${}^{\mathfrak{C}}P\,{}^{\mathfrak{B}}P(x,A) = {}^{\mathfrak{B}}P\,{}^{\mathfrak{C}}P(x,A) = {}^{\mathfrak{C}}P(x,A) \quad \text{for } \mu\text{-almost every } x \in X.$$

Then the result follows from Proposition 2.2. □

Assume (ii), for μ-almost every $z \in X$, we have: for any $(A,B) \in \mathfrak{X} \times \mathfrak{B}$,

$$\int_X 1_A(y) 1_B(y)\, {}^{\mathfrak{B}}P(z,\mathrm{d}y) = \int_X {}^{\mathfrak{B}}P(y,A) 1_B(y)\, {}^{\mathfrak{B}}P(z,\mathrm{d}y).$$

As $\mu(\mathrm{d}z) = \int_X {}^{\mathfrak{C}}P(x,\mathrm{d}z)\mu(\mathrm{d}x)$, for μ-almost every $x \in X$ and ${}^{\mathfrak{C}}P(x,\cdot)$-almost every $z \in X$, for any $(A,B) \in \mathfrak{X} \times \mathfrak{B}$,

$$\int_X 1_A(y) 1_B(y)\, {}^{\mathfrak{B}}P(z,\mathrm{d}y) = \int_X {}^{\mathfrak{B}}P(y,A) 1_B(y)\, {}^{\mathfrak{B}}P(z,\mathrm{d}y).$$

Integration by ${}^{\mathfrak{C}}P(x,\mathrm{d}z)$ gives us, for μ-almost every $x \in X$,

$$\forall (A,B) \in \mathfrak{X} \times \mathfrak{B} \quad \int_X 1_A(y) 1_B(y)\, {}^{\mathfrak{C}}P\,{}^{\mathfrak{B}}P(x,\mathrm{d}y) = \int_X {}^{\mathfrak{B}}P(y,A) 1_B(y)\, {}^{\mathfrak{C}}P\,{}^{\mathfrak{B}}P(x,\mathrm{d}y).$$

Then the result follows from Lemma 4.1.

4.3. *Proof of Theorem 3.4*

Case of two σ-algebras
We need the following result.

Theorem 4.2. *Let $(\Omega, \mathfrak{F}, \mathbb{P})$ be a probability space. Let $\mathfrak{F}_1$ [resp. $\mathfrak{F}_2$] be a sub-σ-algebra of $\mathfrak{F}$; we call P_1 [resp. P_2] the operator of conditional expectation relative to $\mathfrak{F}_1$ [resp. $\mathfrak{F}_2$] on the space $\mathbb{L}^1(\Omega, \mathfrak{F}, \mathbb{P})$.*

Then, for $f \in \mathcal{L}^1(\Omega, \mathfrak{F}, \mathbb{P})$, the sequences of functions

$$\left(\frac{1}{n}\sum_{k=0}^{n-1}(P_1P_2)^k f\right)_{n\geq 1} \quad \text{and} \quad \left(\frac{1}{n}\sum_{k=0}^{n-1}(P_2P_1)^k f\right)_{n\geq 1}$$

converge $\mathbb{P}$-almost everywhere and in norm $\mathbb{L}^1(\mathbb{P})$ towards $\mathbb{E}_{\mathbb{P}}[f|\overline{\mathfrak{F}}_1 \cap \overline{\mathfrak{F}}_2]$.

Proof. It's a consequence of the classical ergodic theorem of E. Hopf (see [5], Proposition V-6-3). To identify the limit we note that: P_2P_1 is the dual operator of P_1P_2 and, as the operators P_1 and P_2 are idempotent, the common limit is P_1- and P_2-invariant (see also [1]). □

Let $\mathfrak{B}_1$ and $\mathfrak{B}_2$ be two sub-σ-algebras of $\mathfrak{X}$ decomposing μ in an ergodic way which satisfy $\overline{\mathfrak{B}_1} \cap \overline{\mathfrak{B}_2} = \overline{\mathfrak{B}_1 \cap \mathfrak{B}_2}$. We set $\mathfrak{C} = \mathfrak{B}_1 \cap \mathfrak{B}_2$.

The above theorem tells us that, for any $f \in \mathcal{L}^1(X, \mathfrak{X}, \mu)$ the sequence of functions $(\frac{1}{n}\sum_{k=0}^{n-1}({}^{\mathfrak{B}_1}P\,{}^{\mathfrak{B}_2}P)^k f)_{n\geq 1}$ converges μ-a.e. and in $\mathbb{L}^1(\mu)$-norm towards ${}^{\overline{\mathfrak{C}}}Pf = {}^{\mathfrak{C}}Pf$.

As $\mu(\cdot) = \int_X {}^{\mathfrak{C}}P(x, \cdot)\mu(\mathrm{d}x)$, it follows that, for any $f \in \mathcal{L}^1(X, \mathfrak{X}, \mu)$ and for μ-almost every $x \in X$, the sequence of functions $(\frac{1}{n}\sum_{k=0}^{n-1}({}^{\mathfrak{B}_1}P\,{}^{\mathfrak{B}_2}P)^k f)_{n\geq 1}$ converges ${}^{\mathfrak{C}}P(x, \cdot)$-a.e. towards ${}^{\mathfrak{C}}Pf$.

We call X_0 a measurable subset of X, such that $\mu(X_0) = 1$ and for any $x \in X_0$, for $i \in 1, 2$, ${}^{\mathfrak{B}_i}P$ is a regular conditional probability with respect to $\mathfrak{B}_i$ and ${}^{\mathfrak{C}}P(x, \cdot)$ (Proposition 3.3).

The same theorem tells us that, for any $x \in X_0$ and for any $f \in \mathcal{L}^1(X, \mathfrak{X}, {}^{\mathfrak{C}}P(x, \cdot))$, the sequence of functions $(\frac{1}{n}\sum_{k=0}^{n-1}({}^{\mathfrak{B}_1}P\,{}^{\mathfrak{B}_2}P)^k f)_{n\geq 1}$ converge ${}^{\mathfrak{C}}P(x, \cdot)$-a.e. and in $\mathbb{L}^1({}^{\mathfrak{C}}P(x, \cdot))$-norm towards $\mathbb{E}_{{}^{\mathfrak{C}}P(x,\cdot)}[f|\widetilde{\mathfrak{B}_1} \cap \widetilde{\mathfrak{B}_2}]$ where, for $i = 1$ or 2, $\widetilde{\mathfrak{B}_i}$ is the ${}^{\mathfrak{C}}P(x, \cdot)$-completed σ-algebra of $\mathfrak{B}_i$.

Let $\mathcal{X}$ be a countable subalgebra of $\mathfrak{X}$ generating $\mathfrak{X}$. From above and Lemma 3.2, it follows that, for μ-almost any $x \in X$, for any $A \in \mathcal{X}$,

$$\text{for } {}^{\mathfrak{C}}P(x, \cdot)\text{-almost every } y \in X, \quad {}^{\mathfrak{C}}P(x, A) = {}^{\mathfrak{C}}P(y, A) = \mathbb{E}_{{}^{\mathfrak{C}}P(x,\cdot)}[1_A|\widetilde{\mathfrak{B}_1} \cap \widetilde{\mathfrak{B}_2}](y).$$

We deduce that, for μ-almost every $x \in X$,

$$\forall (A, C) \in \mathcal{X} \times (\widetilde{\mathfrak{B}_1} \cap \widetilde{\mathfrak{B}_2}) \quad \int_X 1_A(y)1_C(y)\,{}^{\mathfrak{C}}P(x, \mathrm{d}y) = \int_X {}^{\mathfrak{C}}P(y, A)1_C(y)\,{}^{\mathfrak{C}}P(x, \mathrm{d}y).$$

These equalities extend to the couples $(A, C) \in \mathfrak{X} \times (\widetilde{\mathfrak{B}_1} \cap \widetilde{\mathfrak{B}_2})$ (Proposition 2.2).

Since $\mathfrak{C} = \mathfrak{B}_1 \cap \mathfrak{B}_2 \subset \widetilde{\mathfrak{B}_1} \cap \widetilde{\mathfrak{B}_2}$, the above equalities show that, for μ-almost every $x \in X$, ${}^{\mathfrak{C}}P$ is a regular conditional probability with respect to $\mathfrak{C}$ and ${}^{\mathfrak{C}}P(x, \cdot)$. From the Proposition 3.3, the σ-algebra $\mathfrak{C}$ decomposes μ in an ergodic way.

Case of a sequence of σ-algebras
Let $(\mathfrak{B}_n)_{n\geq 1}$ be a sequence of sub-σ-algebras of $\mathfrak{X}$ which decompose μ in an ergodic way and satisfy the hypothesis of Theorem 3.4.

For any $n \geq 2$, we have

$$\overline{\bigcap_{1\leq i\leq n} \mathfrak{B}_i} \subset \overline{\bigcap_{1\leq i\leq n-1} \mathfrak{B}_i} \cap \overline{\mathfrak{B}_n} \subset \bigcap_{1\leq i\leq n} \overline{\mathfrak{B}_i}.$$

From our hypothesis, it follows that

$$\overline{\bigcap_{1\le i\le n}\mathfrak{B}_i}=\bigcap_{1\le i\le n}\overline{\mathfrak{B}_i}$$

and consequently

$$\overline{\bigcap_{1\le i\le n}\mathfrak{B}_i}=\overline{\bigcap_{1\le i\le n-1}\mathfrak{B}_i}\cap\overline{\mathfrak{B}_n}=\bigcap_{1\le i\le n}\overline{\mathfrak{B}_i}.$$

We set

$$\forall n\ge 1\quad \mathfrak{C}_n=\bigcap_{k=1}^{n}\mathfrak{B}_k\quad\text{and}\quad \mathfrak{C}=\bigcap_{k\ge 1}\mathfrak{B}_k.$$

From the case treated previously, we prove by induction that, for any $n\ge 1$, the σ-algebra $\mathfrak{C}_n$ decomposes μ in an ergodic way. From Proposition 3.3, for μ-almost every $x\in X$,

$$\forall (A,C)\in\mathfrak{X}\times\mathfrak{C}_n\quad \int_X 1_A(y)1_C(y)\,{}^{\mathfrak{C}}P(x,\mathrm{d}y)=\int_X {}^{\mathfrak{C}_n}P(y,A)1_C(y)\,{}^{\mathfrak{C}}P(x,\mathrm{d}y).$$

The decreasing martingale theorem implies that, for any $A\in\mathfrak{X}$ and for μ-almost every $x\in X$, $\mathbb{E}_\mu[1_A|\mathfrak{C}_n](x)\underset{n\to+\infty}{\longrightarrow}\mathbb{E}_\mu[1_A|\mathfrak{C}](x)$. Consequently, for any $A\in\mathfrak{X}$ and for μ-almost every $x\in X$, ${}^{\mathfrak{C}_n}P(x,A)\underset{n\to+\infty}{\longrightarrow}{}^{\mathfrak{C}}P(x,A)$.

As $\mu(\cdot)=\int_X {}^{\mathfrak{C}}P(x,\cdot)\mu(\mathrm{d}x)$, it follows that: for any $A\in\mathfrak{X}$ and for μ-almost every $x\in X$,

$$\text{for } {}^{\mathfrak{C}}P(x,\cdot)\text{-almost every } y\in X,\quad {}^{\mathfrak{C}_n}P(y,A)\underset{n\to+\infty}{\longrightarrow}{}^{\mathfrak{C}}P(y,A).$$

While limiting itself to elements C of $\mathfrak{C}$, the dominated convergence theorem implies that, for any $A\in\mathfrak{X}$ and for μ-almost every $x\in X$,

$$\forall C\in\mathfrak{C}\quad \int_X 1_A(y)1_C(y)\,{}^{\mathfrak{C}}P(x,\mathrm{d}y)=\int_X {}^{\mathfrak{C}}P(y,A)1_C(y)\,{}^{\mathfrak{C}}P(x,\mathrm{d}y).$$

Now from Proposition 2.2, we can permute "for any $A\in\mathfrak{X}$" and "for μ-almost every $x\in X$," which shows that $\mathfrak{C}$ decomposes μ in an ergodic way.

The preceding proof shows the following corollary which improves Shimomura's result.

Corollary 4.3. *Let $(\mathfrak{B}_i)_{i\in\mathbb{N}}$ be a sequence of sub-σ-algebras of $\mathfrak{X}$. If for any $n\in\mathbb{N}$ the σ-algebra $\bigcap_{i=0}^n\mathfrak{B}_i$ decomposes μ in an ergodic way, then the intersection $\bigcap_{i\in\mathbb{N}}\mathfrak{B}_i$ decomposes μ in an ergodic way.*

Case of an uncountable family of σ-algebras
We need the following lemmas.

Lemma 4.4. *Let $(\mathcal{H},\langle\cdot,\cdot\rangle)$ be a separable Hilbert space. Let $\{V_i\colon i\in I\}$ a noncountable family of closed vector subspaces of $\mathcal{H}$. Then there exists a countable subset J of I such that $\bigcap_{i\in I}V_i=\bigcap_{i\in J}V_i$.*

Proof. We easily see that the orthogonal complement $V^\perp$ of $V=\bigcap_{i\in I}V_i$ in $\mathcal{H}$ is equal to $\overline{\mathrm{Vect}(\bigcup_{i\in I}V_i^\perp)}$ (the closure of the subspace generated by $\bigcup_{i\in I}V_i^\perp$). We choose a dense sequence of vectors $(u_n)_{n\ge 1}$ of $\mathrm{Vect}(\bigcup_{i\in I}V_i^\perp)$ in $V^\perp$. The Schmidt orthonormalization process allows us to extract a maximal orthonormal system; that is, an Hilbert basis $(e_n)_{n\ge 1}$ de $V^\perp$. For any $p\ge 1$, e_p is a finite linear combination of vectors from $\{u_n\colon n\ge 1\}$; each u_n is itself a finite linear combination of vectors of $\bigcup_{i\in I}V_i^\perp$. Therefore, there exists a countable subset J of I such that, $\forall p\ge 1$, $e_p\in\mathrm{Vect}(\bigcup_{i\in J}V_i^\perp)$. Hence $V^\perp=\overline{\mathrm{Vect}(\bigcup_{i\in J}V_i^\perp)}$ and $V=\bigcap_{i\in J}V_i$. □

Lemma 4.5. *Let $\mathfrak{B}$ be a sub-σ-algebra of $\mathfrak{X}$ which decomposes μ in an ergodic way. Let $\mathfrak{C}$ be a sub-σ-algebra of $\mathfrak{B}$ such that, for any bounded $\mathfrak{B}$-measurable function f, there exists a bounded $\mathfrak{C}$-measurable function g satisfying $f = g$ μ-a.e.*

Then $\mathfrak{C}$ decomposes μ in an ergodic way.

Proof. From Proposition 3.3, for μ-almost every $x \in X$, ${}^{\mathfrak{B}}P$ is a regular conditional probability with respect to $\mathfrak{B}$ and ${}^{\mathfrak{C}}P(x,\cdot)$, that is,

$$\forall (A,B) \in \mathfrak{X} \times \mathfrak{B} \quad \int_X 1_A(y) 1_B(y) {}^{\mathfrak{C}}P(x,\mathrm{d}y) = \int_X {}^{\mathfrak{B}}P(y,A) 1_B(y) {}^{\mathfrak{C}}P(x,\mathrm{d}y).$$

In these equalities, we have to replace ${}^{\mathfrak{B}}P(y,A)$ by ${}^{\mathfrak{C}}P(y,A)$. From Proposition 2.2, we can restrict our equalities to $A \in \mathcal{X}$ a separable sub-algebra of $\mathfrak{X}$ generating $\mathfrak{X}$.

Let f be a bounded $\mathfrak{B}$-measurable function. There exists a bounded $\mathfrak{C}$-measurable function g such that $f = g$ μ-a.e. Then we have, μ-a.e.,

$$\mathbb{E}_\mu[f|\mathfrak{B}] = f = g = \mathbb{E}_\mu[g|\mathfrak{C}] = \mathbb{E}_\mu[f|\mathfrak{C}].$$

It follows that, for μ-almost every $y \in X$,

$$\forall A \in \mathcal{X} \quad {}^{\mathfrak{B}}P(y,A) = {}^{\mathfrak{C}}P(y,A).$$

Now from $\mu(\mathrm{d}y) = \int_X {}^{\mathfrak{C}}P(x,\mathrm{d}y)\mu(\mathrm{d}x)$ we deduce that, for μ-almost every $x \in X$, for ${}^{\mathfrak{C}}P(x,\cdot)$-almost every $y \in X$,

$$\forall A \in \mathcal{X} \quad {}^{\mathfrak{B}}P(y,A) = {}^{\mathfrak{C}}P(y,A)$$

and consequently, for μ-almost every $x \in X$,

$$\forall (A,B) \in \mathcal{X} \times \mathfrak{B} \quad \int_X 1_A(y) 1_B(y) {}^{\mathfrak{C}}P(x,\mathrm{d}y) = \int_X {}^{\mathfrak{C}}P(y,A) 1_B(y) {}^{\mathfrak{C}}P(x,\mathrm{d}y).$$

Hence the result. □

Lemma 4.6. *Let $\{\mathfrak{B}_n \colon n \in \mathbb{N}^*\}$ be a sequence of sub-σ-algebras of $\mathfrak{X}$ satisfying, for any $n \geq 2$, $\bigcap_{1\leq k\leq n} \overline{\mathfrak{B}_k} = \overline{\bigcap_{1\leq k\leq n} \mathfrak{B}_k}$.*

Then $\bigcap_{n\geq 1} \overline{\mathfrak{B}_k} = \overline{\bigcap_{n\geq 1} \mathfrak{B}_k}$.

Proof. Let $f \in \mathcal{L}^1(X,\mathfrak{X},\mu)$. From the decreasing martingale theorem, we have, μ-almost everywhere,

$$\begin{aligned}\mathbb{E}_\mu\left[f\middle|\bigcap_{k\geq 1}\overline{\mathfrak{B}_k}\right] &= \lim_{n\to+\infty}\mathbb{E}_\mu\left[f\middle|\bigcap_{1\leq k\leq n}\overline{\mathfrak{B}_k}\right] = \lim_{n\to+\infty}\mathbb{E}_\mu\left[f\middle|\overline{\bigcap_{1\leq k\leq n}\mathfrak{B}_k}\right] \\ &= \lim_{n\to+\infty}\mathbb{E}_\mu\left[f\middle|\bigcap_{1\leq k\leq n}\mathfrak{B}_k\right] = \mathbb{E}_\mu\left[f\middle|\bigcap_{k\geq 1}\mathfrak{B}_k\right].\end{aligned} \tag{2}$$

Hence the result. □

Consider the separable Hilbert space $\mathbb{L}^2(X,\mathfrak{X},\mu)$. It is well known that, for each sub-σ-algebra $\mathfrak{B}$ of $\mathfrak{X}$, the space $\mathbb{L}^2(X,\mathfrak{B},\mu)$ is identified to a closed subspace of $\mathbb{L}^2(X,\mathfrak{X},\mu)$ and the conditional expectation relative to $\mathfrak{B}$ is identified to the orthogonal projection onto this closed subspace.

Let $\{\mathfrak{B}_i \colon i \in I\}$ be an uncountable family of μ-complete sub-σ-algebras which decompose μ in an ergodic way and satisfy the hypothesis of Theorem 3.4. From the Lemmas 4.4 and 4.6, there exists a countable subset J of I such that

$$\mathbb{L}^2\left(X,\bigcap_{i\in I}\mathfrak{B}_i,\mu\right) \subset \bigcap_{i\in I}\mathbb{L}^2(X,\mathfrak{B}_i,\mu) = \bigcap_{i\in J}\mathbb{L}^2(X,\mathfrak{B}_i,\mu) = \mathbb{L}^2\left(X,\bigcap_{i\in J}\mathfrak{B}_i,\mu\right).$$

It follows that for any $f \in \mathcal{L}^2(X, \bigcap_{i\in I} \mathfrak{B}_i, \mu)$ there exists a function $g \in \mathcal{L}^2(X, \bigcap_{i\in J} \mathfrak{B}_i, \mu)$ such that $f = g$, μ-a.e. According to the case treated previously, we know that the sub-σ-algebra $\bigcap_{i\in J} \mathfrak{B}_i$ decomposes μ in an ergodic way. Then the result follows from Lemma 4.5.

5. Examples and applications

1. Let $(X, \mathfrak{X}, \mu)$ be a probability space with a separable σ-algebra $\mathfrak{X}$ containing a μ-approximating compact class. Let τ be an invertible bi-measurable transformation of $(X, \mathfrak{X})$ such that μ is quasi-invariant for the action of τ. We consider the sub-σ-algebra $\mathfrak{J} = \mathfrak{J}_\tau$ of $\mathfrak{X}$ defined by $\mathfrak{J} = \{B \in \mathfrak{X}\colon\ \tau^{-1}(B) = B\}$. Then the following result is well known.

Proposition 5.1. *The σ-algebra $\mathfrak{J}$ decomposes μ in an ergodic way.*

Proof. An idea of the proof is the following. We consider the contraction T of $\mathbb{L}^1(X, \mathfrak{X}, \mu)$ defined by

$$Tf(x) = f \circ \tau^{-1}(x) \frac{\mathrm{d}(\tau(\mu))}{\mathrm{d}(\mu)}(x).$$

Replacing τ by τ^{-1} we obtain the inverse operator T^{-1}.

From the Chacon–Ornstein ergodic theorem, one proves [2] that, with obvious notations: for any $f \in \mathcal{L}^1(X, \mathfrak{X}, \mu)$ and for μ-almost every $x \in X$,

$$\sum_{k=-n}^{n} T^k f(x) \Big/ \sum_{k=-n}^{n} T^k 1(x) \underset{n\to+\infty}{\longrightarrow} {}^{\mathfrak{J}}Pf(x).$$

One sees easily that there exists a measurable subset X_0 of X such that $\mu(X_0) = 1$ and for any $x \in X_0$ the probability ${}^{\mathfrak{J}}P(x, \cdot)$ is τ-quasi-invariant with $\frac{\mathrm{d}(\tau\, {}^{\mathfrak{J}}P(x,\cdot))}{\mathrm{d}\, {}^{\mathfrak{J}}P(x,\cdot)} = \frac{\mathrm{d}(\tau\mu)}{\mathrm{d}\mu}$.

Then the same ergodic theorem tells us that, for any $x \in X_0$, for any $f \in \mathcal{L}^1(X, \mathfrak{X}, {}^{\mathfrak{J}}P(x, \cdot))$ and for ${}^{\mathfrak{J}}P(x, \cdot)$-almost every $y \in X$,

$$\sum_{k=-n}^{n} T^k f(y) \Big/ \sum_{k=-n}^{n} T^k 1(y) \underset{n\to+\infty}{\longrightarrow} \mathbb{E}_{{}^{\mathfrak{J}}P(x,\cdot)}[f|\mathfrak{J}](x).$$

As in the first case of Theorem 3.4, we prove that for μ-almost every $x \in X$, ${}^{\mathfrak{J}}P$ is a regular conditional probability with respect to $\mathfrak{J}$ and ${}^{\mathfrak{J}}P(x, \cdot)$. The result follows from Proposition 3.3. □

In [3], Greshchonig and Schmidt consider the case of a Borel action of a locally compact second countable group G on a standard probability space $(X, \mathfrak{X}, \mu)$; that is, a group homomorphism $g \mapsto \tau_g$ from G into the group $\mathrm{Aut}(X)$ of Borel automorphisms of X such that the map $(g, x) \mapsto \tau_g x$ from $G \times X$ to X is Borel and μ is quasi-invariant under each τ_g, $g \in G$. They prove that the σ-algebra $\bigcap_{g\in G} \mathfrak{J}_{\tau_g}$ decomposes μ in an ergodic way.

The Theorem 3.4 makes it possible to find and improve this result.

Corollary 5.2. *Let $\{\tau_i\colon\ i \in I\}$ be a non-empty family of Borel automorphisms of X. Then the sub-σ-algebra $\bigcap_{i\in I} \mathfrak{J}_{\tau_i}$ of $\mathfrak{X}$ decomposes μ in an ergodic way.*

Proof. Taking into account the Proposition 5.1, it is enough to show that, for any finite subset J of I, $\bigcap_{i\in J} \overline{\mathfrak{J}_{\tau_i}} = \overline{\bigcap_{i\in J} \mathfrak{J}_{\tau_i}}$.

Let f be a $\bigcap_{i\in J} \overline{\mathfrak{J}_{\tau_i}}$-measurable function. We set $X_0 = \bigcap_{i\in J}\{f \circ \tau_i = f\}$; we have $\mu(X_0) = 1$.

We call G the algebraic subgroup of $\mathrm{Aut}(X)$ generated by the Borel automorphisms $\{\tau_i\colon\ i \in J\}$; G is a countable subset of $\mathrm{Aut}(X)$. The subset $X_1 = \bigcap_{s\in G} sX_0$ of X_0 belongs to $\bigcap_{i\in J} \mathfrak{J}_{\tau_i}$ and $\mu(X_1) = 1$. Then the function $g = f 1_{X_1}$ is $\bigcap_{i\in J} \mathfrak{J}_{\tau_i}$-measurable and $f = g\mu$-a.e. Which shows that f is $\overline{\bigcap_{i\in J} \mathfrak{J}_{\tau_i}}$-measurable. □

2. Let $(X, \mathfrak{X}, \mu, \tau)$ be a dynamical system with a polish space and a not necessarily invertible transformation. We denote by $(Y, \mathfrak{F}, \lambda, \eta)$ the natural extension of our dynamical and by π the natural projection of Y onto X. With obvious notations, one sees easily that f is $\mathfrak{J}_\eta \cap \pi^{-1}(\mathfrak{X})$-measurable (resp. $\overline{\mathfrak{J}_\eta} \cap \overline{\pi^{-1}(\mathfrak{X})}$-measurable) if and only if there exists $g \in \mathfrak{J}_\tau$ such that $f = g \circ \pi$ (resp. $f = g \circ \pi$ λ-a.e.). It follows that

$$\overline{\mathfrak{J}_\eta} \cap \overline{\pi^{-1}(\mathfrak{X})} = \overline{\mathfrak{J}_\eta \cap \pi^{-1}(\mathfrak{X})}.$$

We know that the σ-algebra $\mathfrak{J}_\eta$ decomposes λ in an ergodic way. The σ-algebra $\pi^{-1}(\mathfrak{X})$ is separable. Therefore the σ-algebra $\mathfrak{C} = \mathfrak{J}_\eta \cap \pi^{-1}(\mathfrak{X})$ decomposes λ in an ergodic way.

Let P be a regular conditional probability with respect to $\mathfrak{J}_\tau$ and μ. Let Q be a regular conditional probability with respect to $\mathfrak{C}$ and λ. For any $A \in \mathfrak{X}$ and $C \in \overline{\mathfrak{J}_\tau}$ we have:

$$\int_X P(x, A) 1_C(x) \mu(\mathrm{d}x) = \int_X 1_A(x) 1_C(x) \mu(\mathrm{d}x)$$

and therefore

$$\begin{aligned} \int_Y P\big(\pi(y), A\big) 1_C\big(\pi(y)\big) \lambda(\mathrm{d}y) &= \int_Y 1_A\big(\pi(y)\big) 1_C\big(\pi(y)\big) \lambda(\mathrm{d}y) \\ &= \int_Y \mathbb{E}_\lambda[1_A \circ \pi | \mathfrak{C}](y) 1_C\big(\pi(y)\big) \lambda(\mathrm{d}y) \\ &= \int_Y Q\big(y, \pi^{-1}(A)\big) 1_C\big(\pi(y)\big) \lambda(\mathrm{d}y). \end{aligned} \tag{3}$$

Which proves, via the Proposition 2.2, that

$$\text{for } \lambda\text{-almost every } y \in Y, \quad P\big(\pi(y), \cdot\big) = Q\big(y, \pi^{-1}(\cdot)\big)$$

and the σ-algebra $\mathfrak{J}_\tau$ decomposes μ in an ergodic way.

3. Let P be a transition probability on a measurable space $(X, \mathfrak{X})$ with a separable σ-algebra $\mathfrak{X}$ containing a μ-approximating compact class.

We denote by Π the set of P-invariant probability measures on $(X, \mathfrak{X})$:

$$\pi \in \Pi \quad \Leftrightarrow \quad \int_X f(x) \pi P(\mathrm{d}x) = \int_X P f(x) \pi(\mathrm{d}x) = \int_X f(x) \pi(\mathrm{d}x)$$

for any non-negative or bounded measurable function f on X. We assume that $\Pi \neq \varnothing$.

For any $\pi \in \Pi$, we denote by $\mathfrak{B}_\pi$ the sub-σ-algebra of $\mathfrak{X}$ defined by:

$$\mathfrak{B}_\pi = \{A \in \mathfrak{X}\colon\ P 1_A = 1_A \ \pi\text{-a.e.}\}$$

and we set $\mathfrak{B} = \bigcap_{\pi \in \Pi} \mathfrak{B}_\pi$.

Let $\pi \in \Pi$. Let f be a bounded $\mathfrak{B}_\pi$-measurable function on X. The function g, defined by

$$g(x) = \liminf \frac{1}{n} \sum_{k=0}^{n-1} P^k f(x),$$

satisfies $g = f$ π-a.e. and $Pg \le g$. From the latter inequality, it follows that, for any $\sigma \in \Pi$, $Pg = g$ σ-a.e. and g is $\mathfrak{B}$-measurable. We deduce that $\mathfrak{B}_\pi = \mathfrak{B}$ π-a.e.

From the Hopf theorem ([3], Proposition V-6-3), for any $f \in \mathcal{L}^1(X, \mathfrak{X}, \pi)$, the sequences of functions

$$\left(\frac{1}{n} \sum_{k=0}^{n-1} \frac{\mathrm{d}((f\pi)P^k)}{\mathrm{d}\pi}\right)_{n \ge 1} \quad \text{and} \quad \left(\frac{1}{n} \sum_{k=0}^{n-1} P^k f\right)_{n \ge 1}$$

converge π-almost everywhere and in norm $\mathbb{L}^1(\pi)$ towards ${}^{\mathfrak{B}_\pi}Pf = {}^{\mathfrak{B}}Pf$. As in Example 1, one deduces that $\mathfrak{B}$ decomposes π in an ergodic way.

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