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Large deviations for partition functions of directed polymers in an IID field

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Abstract. Consider the partition function of a directed polymer in $\mathbb{Z}^d$, $d \geq 1$, in an IID field. We assume that both tails of the negative and the positive part of the field are at least as light as exponential. It is well known that the free energy of the polymer is equal to a deterministic constant for almost every realization of the field and that the upper tail of the large deviations is exponential. The lower tail of the large deviations is typically lighter than exponential. In this paper we obtain sharp estimates on the lower tail of the large deviations given in terms of the distribution of the IID field. Our proofs are also applicable to the model of directed last passage percolation and (non-directed) first passage percolation.

Résumé. Considérons la fonction de partition d'un polymère dirigé dans $\mathbb{Z}^d$, $d \geq 1$, d'un champ IID. On suppose que les queues des parties positive et négative sont au moins aussi légères qu'une exponentielle. Il est bien connu que l'énergie libre du polymère est égale à une constante déterministe pour presque toute réalisation du champ et que la queue supérieure des grandes déviations est exponentielle. La queue inférieure des grandes déviations est typiquement plus légère qu'une exponentielle. Dans cet article nous obtenons des estimations précises sur la queue inférieure des grandes déviations en fonction de la distribution du champ IID. Nos preuves sont également applicables au modèle de percolation de dernier passage dirigé et à celui de percolation de premier passage (non dirigé).

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1. Introduction and statement of results

Let $V \equiv \{V(t,x)\colon (t,x) \in \mathbb{Z}_+ \times \mathbb{Z}^d\}$ denote an IID field under a probability measure Q. The corresponding expectation operator will be denoted by E^Q. We will work under the following assumption.

ASI.

(i) $V(0,0)$ *is non-degenerate.*
(ii) *There exists* $\eta_0 > 0$ *such that* $E^Q(\mathrm{e}^{\eta V(0,0)}) < \infty$ *if* $|\eta| < \eta_0$.

The random variables $V(\cdot,\cdot)$ are sometimes called weights and their are their common distribution is called the weight distribution. In some parts of our proofs it is more convenient to assume further that $E^Q(V(0,0)) = 0$, although this has no effect on the generality of our results. We write ASI$'$ for ASI with this additional condition. We endow $\mathbb{Z}^d$ with the l^1 norm, which we denote by $|\cdot|$, and is defined as the sum of the absolute values of the coordinates. Let γ

denote a nearest neighbor random walk path on $\mathbb{Z}^d$. Precisely, $\gamma:\mathbb{Z}_+\to\mathbb{Z}^d$, satisfying $|\gamma(t+1)-\gamma(t)|=1$ for all $t\in\mathbb{Z}_+$. For a path γ and $T\in\mathbb{Z}_+\backslash\{0\}$ we define $H_\gamma(T)$, the passage time or the energy of γ, by letting

$$H_\gamma(T)=\sum_{t=0}^{T-1}V\bigl(t,\gamma(t)\bigr).$$

Below, we will sometimes omit the dependence on γ and write $H(T)$ instead of $H_\gamma(T)$. For $x\in\mathbb{Z}^d$, let P_x denote the probability measure on corresponding to the symmetric nearest neighbor random walk starting from x. Let E_x denote the corresponding expectation. The partition function of a d-dimensional directed polymer, $Z(T)$, is defined by letting

$$Z(T)=E_0\bigl(\mathrm{e}^{H_\gamma(T)}\bigr).$$

We also define

$$\zeta(T)=\max_{\gamma:\gamma(0)=0}H_\gamma(T).$$

The random variable $\zeta(T)$ is known in the literature as the directed last passage percolation time. Being an expectation of an exponential function, the essential contribution to $Z(T)$ is from paths maximizing $H(T)$. Therefore $Z(T)$ can be thought of as a "penalized" analogue of $\mathrm{e}^{\zeta(T)}$.

For positive functions $q,r:\mathbb{R}_+\to\mathbb{R}_+$ or $q,r:\mathbb{Z}_+\to\mathbb{R}_+$, we say that q and r are comparable and write $q\asymp r$ if there exist constants $0<C_1\le C_2<\infty$ such that

$$C_1\le\liminf_{t\to\infty}\frac{q(t)}{r(t)}\le\limsup_{t\to\infty}\frac{q(t)}{r(t)}\le C_2.$$

We say that q and r are (asymptotically) equivalent and write $q\sim r$, if $C_1=C_2=1$.

A fundamental result is the following.

Theorem 1. *Let ASI hold.*

(i) *There exists a constant* $\lambda\in(-\infty,\infty)$ *such that*

$$\lambda=\liminf_{T\to\infty}\frac{1}{T}\ln Z(T)=\limsup_{T\to\infty}\frac{1}{T}\ln Z(T),\qquad Q\text{-almost surely.}$$

(ii) *There exists* $\varepsilon_0\in(0,\infty]$ *such that for every* $\varepsilon\in(0,\varepsilon_0)$,

$$-\ln Q\bigl(Z(T)\ge\mathrm{e}^{(\lambda+\varepsilon)T}\bigr)\asymp T.$$

Note that if ASI′ holds, then $\lambda\ge0$. The proof of the theorem is essentially due to superadditive arguments. For a proof of part (i), we refer the reader to [4], Proposition 1.5, where also (ii) was proved under additional assumptions on the weight distribution. For a proof of part (1), we refer the reader to [6], Theorem 2.11, where the analogous result for the parabolic Anderson model was established. This was done through discretization, which makes the proof essentially identical to the present model.

The analogue of Theorem 1 for ζ is the following statement. There exists a constant $\mu\in(-\infty,\infty)$ such that $\lim_{T\to\infty}\frac{1}{T}\zeta(T)=\mu$, Q-almost surely, and there exists $\varepsilon_0\in(0,\infty)$ such that for all $\varepsilon\in(0,\varepsilon_0)$, $-\ln Q(\zeta(T)\ge(\mu+\varepsilon)T)\asymp T$.

Note that Theorem 1(ii) is an upper tail large deviations result. In this paper we study the lower tail large deviations. To this end, for every $\varepsilon>0$ we define functions $R_\varepsilon, R_\varepsilon^\zeta:\mathbb{Z}_+\to[0,\infty]$ by letting

$$R_\varepsilon(T)=-\ln Q\bigl(Z(T)\le\mathrm{e}^{(\lambda-\varepsilon)T}\bigr)\quad\text{and}\quad R_\varepsilon^\zeta(T)=-\ln Q\bigl(\zeta(T)\le(\mu-\varepsilon)T\bigr).$$

As an indication of what may occur, we observe that $\zeta(T) \geq (\mu+\varepsilon)T$ if $H_\gamma(T) \geq (\mu+\varepsilon)T$ for at least one path γ, whereas $\zeta(T) \leq (\mu-\varepsilon)T$ if $H_\gamma(T) \leq (\mu-\varepsilon)T$ for all paths γ. In light of Theorem 1-(ii), this suggests that $R^\zeta_\varepsilon(T)$ (as a function of T) should be typically of smaller order than T. Due to the intuitive relation between Z and ζ mentioned above one can expect this to apply to R_ε as well. A related observation is that the superadditivity, which is tightly linked to the upper tail large deviations, is not applicable for the lower tail large deviations. This requires different techniques. It is this problem that we wish to address.

Here are some related results. In [2], Carmona and Hu showed that when $d \geq 3$ and the weight distribution is Gaussian with sufficiently small variance then, the positive part of $\lambda T - \ln Z(T)$ has tail at least as light as Gaussian. Kesten [8] studied point–point first passage percolation under the assumption that the weight distribution is bounded and later Chow and Zhang [3] studied face–face first passage percolation. Seppäläinen [9] and Deuschel and Zeitouni [7] obtained precise large deviations for the longest increasing sequences of points in the plane. The similarity between these last two models and ours is that all consider supremum (infimum in the case of first passage percolation) over some appropriate space–time paths. In a recent paper [5] Cranston, Gauthier and Mountford proved the following results.

Theorem CGM-1.1. *Suppose that for some $x_0 \in (0,\infty)$ there exists a positive, increasing function of x such that*

$$Q\bigl(V(0,0) < -x\bigr) = \mathrm{e}^{-x^{1+d} f(x)}, \quad x > x_0.$$

Then for every $\varepsilon > 0$,

$$\liminf_{T\to\infty} \frac{R^\zeta_\varepsilon(T)}{T^{1+d}} > 0$$

if and only if

$$\sum_{n=1}^{\infty} \frac{1}{f^{1/d}(2^n)} < \infty.$$

We remark that [5], Theorem 1.3, is the analogue of this theorem for the first passage percolation, which extends the result of Kesten [8], Chapter 5, to unbounded distributions. This was done through reduction to the directed model. The other case treated in that paper is the following.

Theorem CGM-1.2. *Suppose that $d = 1$ and that the weight distribution is a standard normal random variable. Then*

$$R^\zeta_\varepsilon(T) \asymp T^2/\ln T.$$

For the sake of accuracy we comment that the model considered in [5] was slightly different than ζ: The IID field was indexed by edges (bonds) rather than vertices (sites) as in our models. However, all their proofs work for ζ with some minor obvious modifications.

On one extreme, when the weight distribution is bounded from below, it is not hard to show that R^ζ_ε is comparable to T^{1+d}. Therefore, Theorem CGM-1.1 is a stability result, giving the precise conditions for this large deviations regime. On the other extreme, when the tail of the negative part of the weight distribution is sufficiently heavy, it is not hard to show that R_ε and R^ζ_ε are comparable to $-\ln Q(-V(0,0) > T)$. This forms a second large deviations regime. The models also exhibit a third, intermediate, large deviations regime, observed in Theorem CGM-1.2. In this paper we propose a new method to derive sharp estimates on R_ε and R^ζ_ε in terms of the tail of the negative part of the weight distribution. In light of the above, the major improvement here over the results of [5] is in the detailed description of intermediate large deviations regime.

The approach developed in [5] involves an elaborate construction of paths leading to estimates on the contribution of small times for a large class of weight distributions. The construction depends on the weight distribution as well as on the realization of V and also requires an a priori estimate on $R^\zeta_\varepsilon(T)$ in order to be effectively carried out. Our method is also based on a construction of paths, but our construction is a non-random choice of a subset of paths, which is a maximal tree-like subset of paths satisfying some uniformity condition. This construction is determined only by the structure of the state space of the random walk, $\mathbb{Z}^d$. This approach is also useful in obtaining the analogous results for walks on transitive graphs other than $\mathbb{Z}^d$.

Below we state and prove our results for Z. This choice is due to the fact that the proofs can be applied to ζ and the model treated in [5] with some obvious cosmetic changes and omissions, and all results below are valid when R_ε is replaced with R_ε^ζ. We will restrict our discussion to a certain family of weight distributions to be immediately described. This is because we want simple, analytically tractable estimates for R_ε otherwise sometimes harder to obtain. We recall a definition.

Definition 1. *Let $G:\mathbb{R}_+ \to \mathbb{R}_+$ and let $\alpha \in \mathbb{R}$. We say that G is regularly varying with index α if*

$$G(x) = x^\alpha L(x),$$

where $L:\mathbb{R}_+ \to \mathbb{R}_+$ is such that for all $\lambda > 0$, $\lim_{x\to\infty} L(\lambda x)/L(x) = 1$ (slowly varying). The set of regularly varying functions with index α is denoted by $RV(\alpha)$.

An additional assumption under which we will work is the following.

ASII. *There exists $x_0, \alpha \in (0,\infty)$ and a continuous function $G \in RV(\alpha)$ such that*

$$Q\bigl(-V(0,0) > x\bigr) = \mathrm{e}^{-xG(x)}, \quad x \ge x_0.$$

Let $G \in \mathrm{RV}(\alpha)$ for some $\alpha > 0$. We define the following associated functions:

$$\tilde{G}(x) = \inf_{y\ge x} G(y), \qquad \tilde{G}^{\mathrm{inv}}(y) = \inf\bigl\{x \ge 0:\ \tilde{G}(x) > y\bigr\}$$

and

$$J(z) = z^{1/d}\int_1^{\tilde{G}^{\mathrm{inv}}(z)} \tilde{G}^{-1/d}(v)\,\mathrm{d}v.$$

The properties of $\tilde{G}$, $\tilde{G}^{\mathrm{inv}}$ and J relevant to our work are listed in Lemma 1.

Our main result is the following.

Theorem 2. *Let ASI and ASII hold. Let $\varepsilon > 0$ and let $\eta:\mathbb{R}_+ \to \mathbb{R}_+$ satisfy $\limsup_{x\to\infty} J(\eta(x))/x < \infty$ and $\lim_{x\to\infty}\eta(x) = \infty$. Then $\liminf_{T\to\infty} \frac{R_\varepsilon(T)}{T\eta(T)} > 0$.*

This is complemented with a corresponding upper bound.

Theorem 3. *Let ASI and ASII hold. Let $\varepsilon > 0$ and let $\eta:\mathbb{R}_+ \to \mathbb{R}_+$ satisfy $J(\eta(x)) \asymp x$. Then there exists $\varepsilon_1 = \varepsilon_1(G,d) \in (0,\infty]$ such that for $\varepsilon \in (0,\varepsilon_1)$, $\limsup_{T\to\infty}\frac{R_\varepsilon(T)}{T\eta(T)} < \infty$. Furthermore $\varepsilon_1 = \infty$ if $\int_1^\infty G^{-1/d}(v)\,\mathrm{d}v = \infty$.*

Combining both theorems we obtain the following corollary.

Corollary 1. *Let ASI and ASII hold. Let ε_1 be as in Theorem 3 and let $\varepsilon \in (0,\varepsilon_1)$. Then:*

(i) *Suppose that $\eta:\mathbb{R}_+ \to \mathbb{R}_+$. Then $R_\varepsilon(T) \asymp T\eta(T)$ if and only if $J(\eta(x)) \asymp x$.*
(ii) *$R_\varepsilon(T) \asymp T^{1+d}$ if and only if $\int_1^\infty G^{-1/d}(v)\,\mathrm{d}v < \infty$.*
(iii) *$R_\varepsilon(T) \asymp TG(T)$ if and only if $\alpha < d$. Otherwise, $R_\varepsilon(T) = \mathrm{o}(TG(T))$, as $T \to \infty$.*

Corollary 1(ii) extends Theorem CGM-1.1 and also provides a lower bound. Corollary 1(iii) provides a necessary and sufficient condition for $R_\varepsilon(T)$ to be comparable to the exponential tail of the negative part of $V(0,0)$. Note that a necessary condition for the integral in part (ii) of the corollary to be finite is $\alpha \ge d$. Thus, a phase transition occurs at the value $\alpha = d$. Here are some concrete straightforward calculations.

Example. *Let ASI and ASII hold. Let $\varepsilon \in (0, \varepsilon_1)$, where ε_1 is the constant from Theorem 3.*

(i) *If $G(x) \sim x^{\alpha}$, then*

$$R_\varepsilon(T) \asymp \begin{cases} T^{1+\alpha}, & \alpha < d; \\ T^{1+d}/\ln^d T, & \alpha = d; \\ T^{1+d}, & \alpha > d. \end{cases}$$

(ii) *If $G(x) \sim x^d \ln^\beta x$, then*

$$R_\varepsilon(T) \asymp \begin{cases} T^{1+d}/\ln^d T, & \beta < d; \\ T^{1+d}/(\ln\ln T)^d, & \beta = d; \\ T^{1+d}, & \beta > d. \end{cases}$$

Due to standard estimates on Gaussian tails, when the weight distribution is Gaussian, the condition in part (i) of the example holds with $\alpha = 1$. This extends Theorem CGM-1.2 to all dimensions.

Note that ASII does not cover the case $G(x) \asymp 1$. For this we include the following.

Proposition 1. *Let ASI hold. Suppose that $-\ln Q(-V(0,0) > x) \asymp x$. Let $\varepsilon > 0$. Then $R_\varepsilon(T) \asymp T$.*

Finally, we mention two extensions. One is the case, where $\gamma(T)$ is pinned to some point, usually depending linearly in T. Here the argument we have used to treat the pinning at time 0 needs to be repeated at times near T, as well as considering an appropriate set of drifted paths. Another extension is to the first passage percolation model. In order to adapt Theorem 3, we essentially need to consider one component of the space as time and then only consider paths in which this component does not decrease. Further, Lemma 5 should be replaced with the corresponding result for the first passage percolation, [3], Lemma 3.1. The adaptation of Theorem 2 is straightforward.

The rest of the paper is organized as follows. In Section 2 we prove some preliminary technical results. The proof of Theorem 2 is given in Section 3. Theorem 3 is proved in Section 4. Corollary 1 and Proposition 1 are proved in Section 5. In the Appendix we prove a lemma used in Section 3, which is a an adaption of an analogous result in [5].

2. Preliminaries

Lemma 1. *Let $G \in RV(\alpha)$ for some $\alpha \in \mathbb{R}$ be continuous. Then:*

(i) *$\tilde{G} \sim G$ and $\tilde{G}$ is continuous. In particular $\tilde{G} \in RV(\alpha)$.*
(ii) *$\tilde{G}^{\mathrm{inv}} \in RV(1/\alpha)$ and is strictly increasing.*
(iii) *$\tilde{G}^{\mathrm{inv}}(\tilde{G}(x)) \sim \tilde{G}(\tilde{G}^{\mathrm{inv}}(x)) = x$.*
(iv) *If $\alpha \le d$, $\lim_{z\to\infty} \frac{\tilde{G}^{\mathrm{inv}}(z)}{J(z)} = 1 - \frac{\alpha}{d}$. If $\alpha > d$, $J(z) \asymp z^{1/d}$.*

We claim that in order to prove Theorems 2 and 3 there is no loss of generality replacing ASI with ASI$'$. To see that, suppose that ASI and ASII hold. Let $l = E^Q(V(0,0))$. Consider the zero-mean field $V' := \{V(t,x) - l\colon (t,x) \in \mathbb{Z}_+ \times \mathbb{Z}^d\}$. To distinguish the quantities related to V from those related to V', we attach an apostrophe to the latter. Clearly, $G'(x) = G(x+l)$ for all x large enough. Therefore $\tilde{G}'(x) = \tilde{G}(x+l)$ and $\tilde{G}'^{\mathrm{inv}}(x) = \tilde{G}^{\mathrm{inv}}(x) - l$. For $\delta \in (0,1)$ and x large enough

$$\tilde{G}^{-1/d}\big(x(1+\delta)\big) \le \tilde{G}'^{-1/d}(x) \le \tilde{G}^{-1/d}\big(x(1-\delta)\big)$$

and

$$(1-\delta)\tilde{G}^{\mathrm{inv}}(x) \le \tilde{G}'^{\mathrm{inv}}(x) \le (1+\delta)\tilde{G}^{\mathrm{inv}}(x).$$

Arguing as in the proof of Lemma 1(iv), these inequalities imply

$$J' \asymp J. \tag{2.1}$$

Since $Z'(T) = Z(T)\mathrm{e}^{-lT}$ and $\lambda' = \lambda - l$, we also have

$$R'_\varepsilon(T) = -\ln Q\bigl(Z'(T) \le \mathrm{e}^{(\lambda'-\varepsilon)T}\bigr) = R_\varepsilon(T). \tag{2.2}$$

The claim is then an immediate consequence of (2.1) and (2.2). Below we will always replace ASI with ASI′.

Lemma 2. *Let ASI′ and ASII hold. Then*:

(i) *There exists a constant* $\beta_0 = \beta_0(G) \in (0,\infty)$ *such that for all* $\beta \ge \beta_0$.

$$E^Q\bigl(\mathrm{e}^{-\beta V(0,0)}\bigr) \le \mathrm{e}^{2\beta \tilde{G}^{\mathrm{inv}}(2\beta)}.$$

Furthermore the function $\beta \to E^Q(\mathrm{e}^{-\beta V(0,0)})$ *is non-decreasing on* $[0,\infty)$.

(ii) *There exists a constant* $q \in [0,\infty)$ *and a non-increasing function* $\delta : \mathbb{R}_+ \to \mathbb{R}_+$ *with* $\lim_{x\to\infty} \delta(x) = 1$, *such that*

$$Q\bigl(-V(0,0) \ge x\bigr) \ge \mathrm{e}^{-q}\mathrm{e}^{-x\tilde{G}(x)\delta(x)}, \quad x \ge 0.$$

For $C > 0$ and $z \ge C^{-1}$ we define $I_C(z)$ by letting

$$I_C(z) = \sum_{t=0}^{\lfloor (Cz)^{1/d}-1\rfloor} \tilde{G}^{\mathrm{inv}}\bigl(z(1+t)^{-d}\bigr). \tag{2.3}$$

The series $I_C(z)$, with some particular choice of z, will appear in both our upper and lower bounds estimates.

Lemma 3. *Under the assumption of Lemma 1 the following hold*:

(i) *Let* $C \in (0,\infty)$. *Then* $I_C(z) \asymp J(z)$.

(ii) *Let* $\eta_1, \eta_2 : \mathbb{R}_+ \to \mathbb{R}_+$ *and suppose that* $J(\eta_1(x)) \asymp x$. *Then* $J(\eta_2(x)) \asymp x$ *if and only if* $\eta_1 \asymp \eta_2$.

Proof of Lemma 1.

(i) The first claim is [1], Theorem 1.5.3(ii), p. 23. Since $\tilde{G}$ is non-decreasing, in order to prove the second claim, it is enough to show that the range of $\tilde{G}$ is an (unbounded) interval. Let $y \ge \inf_{x\ge 0} G(x)$. Since G is continuous and $\lim_{x\to\infty} G(x) = \infty$, the set $\{x\colon G(x) = y\}$ is non-empty and compact. Let x' denote its maximal element. Note that $G(x) > y$ for all $x > x'$. Hence, $\tilde{G}(x') = G(x') = y$.

(ii), (iii) The first claim of (ii) as well as the first claim of (iii) are [1], Theorem 1.5.12, p. 28. For the remaining two claims, note that

$$\tilde{G}^{\mathrm{inv}}(y) = \inf\bigl\{x \ge 0\colon \tilde{G}(x) > y\bigr\} = \min\bigl\{x \ge 0\colon \tilde{G}(x) = y\bigr\}, \tag{2.4}$$

because the boundary of the relatively open set $\{x \ge 0\colon \tilde{G}(x) > y\} \subset [0,\infty)$ is the compact set $\{x \ge 0\colon \tilde{G}(x) = y\}$. Hence $\tilde{G}(\tilde{G}^{\mathrm{inv}}(y)) = y$. Finally, $\tilde{G}^{\mathrm{inv}}$ is non-decreasing by definition. If $y_1 < y_2$, then $\{x \ge 0\colon \tilde{G}(x) = y_1\}$ and $\{x \ge 0\colon \tilde{G}(x) = y_2\}$ are disjoint compact sets, therefore by (2.4), $\tilde{G}^{\mathrm{inv}}(y_1) < \tilde{G}^{\mathrm{inv}}(y_2)$.

(iv) Karamata's theorem [1], Theorem 1.5.11(i), p. 28, states that if $\alpha \le d$, then

$$\lim_{x\to\infty} \frac{x\tilde{G}^{-1/d}(x)}{\int_1^x \tilde{G}^{-1/d}(v)\,\mathrm{d}v} = 1 - \frac{\alpha}{d}.$$

Substituting x with $\tilde{G}^{\mathrm{inv}}(z)$ we obtain

$$1 - \frac{\alpha}{d} = \lim_{z\to\infty} \frac{\tilde{G}^{\mathrm{inv}}(z)\tilde{G}^{-1/d}(\tilde{G}^{\mathrm{inv}}(z))}{\int_1^{\tilde{G}^{\mathrm{inv}}(z)} \tilde{G}^{-1/d}(v)\,\mathrm{d}v} \underset{\text{part (iii)}}{=} \lim_{z\to\infty} \frac{\tilde{G}^{\mathrm{inv}}(z)}{J(z)},$$

and the first claim follows.

The second claim is a direct consequence of the definitions and we omit the details. □

Proof of Lemma 2. Set $Y = -V(0,0)$.

(i) For every $\beta, \rho > 0$,

$$E^Q\big(\mathrm{e}^{\beta Y}\big) = E^Q\big(\mathrm{e}^{\beta Y}(\mathbf{1}_{\{Y\le\rho\}} + \mathbf{1}_{\{Y>\rho\}})\big) \le \mathrm{e}^{\beta\rho} + 1 + \beta\int_\rho^\infty Q(Y \ge x)\mathrm{e}^{\beta x}\,\mathrm{d}x.$$

Let x_0 be as in ASII. Pick $\beta_1 > 0$ such that $\tilde G^{\mathrm{inv}}(2\beta) \ge x_0$ for all $\beta \ge \beta_1$. Let $\beta \ge \beta_1$ and let $\rho = \tilde G^{\mathrm{inv}}(2\beta) \ge x_0$. Then,

$$E^Q\big(\mathrm{e}^{\beta Y}\big) \le \mathrm{e}^{\beta\tilde G^{\mathrm{inv}}(2\beta)} + 1 + \beta\int_\rho^\infty \mathrm{e}^{-x\beta(\tilde G(x)/\beta - 1)}\,\mathrm{d}x,$$

where we have used the fact for $x \ge x_0$, $Q(Y \ge x) = \mathrm{e}^{-xG(x)} \le \mathrm{e}^{-x\tilde G(x)}$. By Lemma 1(iii), $\tilde G(\rho) = \tilde G(\tilde G^{\mathrm{inv}}(2\beta)) = 2\beta$. Therefore when $x \ge \rho$, $\frac{\tilde G(x)}{\beta} - 1 \ge 2 - 1 = 1$. This gives

$$E^Q\big(\mathrm{e}^{\beta Y}\big) \le \mathrm{e}^{\beta\tilde G^{\mathrm{inv}}(2\beta)} + 1 + \beta\int_\rho^\infty \mathrm{e}^{-x\beta}\,\mathrm{d}x = \mathrm{e}^{\beta\tilde G^{\mathrm{inv}}(2\beta)} + 1 + \mathrm{e}^{-\beta\tilde G^{\mathrm{inv}}(2\beta)}.$$

Since $\lim_{x\to\infty}\tilde G^{\mathrm{inv}}(x) = \infty$, the first claim follows by choosing $\beta_0 \ge \beta_1$ large enough.

To prove the second claim, note that since $E^Q(Y) = 0$, by Jensen's inequality we have

$$E^Q\big(\mathrm{e}^{\beta Y}\big) \ge \mathrm{e}^{\beta E^Q(Y)} = 1, \quad \beta \ge 0. \tag{2.5}$$

Take $\beta' \ge \beta > 0$. Then,

$$E^Q\big(\mathrm{e}^{\beta' Y}\big) = E^Q\big(\big(\mathrm{e}^{\beta Y}\big)^{\beta'/\beta}\big) \underset{\text{Jensen}}{\ge} \big(E^Q\big(\mathrm{e}^{\beta Y}\big)\big)^{\beta'/\beta} \underset{(2.5)}{\ge} E^Q\big(\mathrm{e}^{\beta Y}\big).$$

(ii) Since $Q(Y \ge x_0) > 0$, we can choose $q \in [0,\infty)$ such that $\mathrm{e}^{-q} = Q(Y \ge x_0)$. Then

$$Q(Y \ge x) \ge \mathrm{e}^{-(q + xG(x))}, \quad x \ge 0. \tag{2.6}$$

Let

$$\delta(x) = \sup_{y\ge x}\frac{G(y)}{\tilde G(y)}.$$

Then δ is non-increasing and by Lemma 1(i), $\lim_{x\to\infty}\delta(x) = 1$. Since $G(x) \le \tilde G(x)\delta(x)$ for all x, the claim follows from (2.6). □

Proof of Lemma 3.

(i) Let

$$J_C(z) = z^{1/d}\int_{\tilde G^{\mathrm{inv}}(C^{-1})}^{\tilde G^{\mathrm{inv}}(z)} \tilde G^{-1/d}(v)\,\mathrm{d}v.$$

Since $J_C(z) \asymp J(z)$, it is sufficient to prove the claim for J_C.

By elementary comparisons,

$$\int_1^{(Cz)^{1/d}} \tilde G^{\mathrm{inv}}\big(zu^{-d}\big)\,\mathrm{d}u \le I_C(z) \le \int_1^{(Cz)^{1/d}} \tilde G^{\mathrm{inv}}\big(zu^{-d}\big)\,\mathrm{d}u + \tilde G^{\mathrm{inv}}(z).$$

By Lemma 1(ii), $\tilde{G}^{\mathrm{inv}}(z) \in \mathrm{RV}(1/\alpha)$ and is strictly increasing. Therefore, there exists $c_1 = c_1(\alpha, d) \in (0, \infty)$ such that for sufficiently large z,

$$\tilde{G}^{\mathrm{inv}}(z) \le c_1 \tilde{G}^{\mathrm{inv}}\big(z2^{-d}\big) \le c_1 \int_1^2 \tilde{G}^{\mathrm{inv}}\big(zu^{-d}\big)\,\mathrm{d}u \le c_1 \int_1^{(Cz)^{1/d}} \tilde{G}^{\mathrm{inv}}\big(zu^{-d}\big)\,\mathrm{d}u.$$

Hence

$$\int_1^{(Cz)^{1/d}} \tilde{G}^{\mathrm{inv}}\big(zu^{-d}\big)\,\mathrm{d}u \le I_C(z) \le (1+c_1) \int_1^{(Cz)^{1/d}} \tilde{G}^{\mathrm{inv}}\big(zu^{-d}\big)\,\mathrm{d}u. \tag{2.7}$$

Since $\tilde{G}^{\mathrm{inv}}$ is strictly increasing and by Lemma 1(iii) $\tilde{G}(\tilde{G}^{\mathrm{inv}}(y)) = y$ for all $y \ge 0$, we may change variables by letting $v = \tilde{G}^{\mathrm{inv}}(zu^{-d})$. Furthermore, $\tilde{G}(v) = zu^{-d}$, hence $u = z^{1/d}\tilde{G}^{-1/d}(v)$. Integration by parts yields

$$\begin{aligned}\int_1^{(Cz)^{1/d}} \tilde{G}^{\mathrm{inv}}\big(zu^{-d}\big)\,\mathrm{d}u &= -z^{1/d} \int_{\tilde{G}^{\mathrm{inv}}(C^{-1})}^{\tilde{G}^{\mathrm{inv}}(z)} v\,\mathrm{d}\tilde{G}^{-1/d}(v) \\ &= z^{1/d}\left(-v\tilde{G}^{-1/d}(v)\Big|_{\tilde{G}^{\mathrm{inv}}(C^{-1})}^{\tilde{G}^{\mathrm{inv}}(z)} + \int_{\tilde{G}^{\mathrm{inv}}(C^{-1})}^{\tilde{G}^{\mathrm{inv}}(z)} \tilde{G}^{-1/d}(v)\,\mathrm{d}v\right).\end{aligned}$$

Rearranging terms, we obtain

$$\int_1^{(Cz)^{1/d}} \tilde{G}^{\mathrm{inv}}\big(zu^{-d}\big)\,\mathrm{d}u = z^{1/d}\tilde{G}^{\mathrm{inv}}\big(C^{-1}\big)C^{1/d} - \tilde{G}^{\mathrm{inv}}(z) + J_C(z). \tag{2.8}$$

Below we will show that there exists $c_2 = c_2(\alpha, d, C) \in (0, 1)$ such that

$$\tilde{G}^{\mathrm{inv}}(z) \le c_2 J_C(z). \tag{2.9}$$

Clearly, there exists $c_3 = c_3(d, G) \in (0, \infty)$ such that for all z sufficiently large,

$$\tilde{G}^{\mathrm{inv}}\big(C^{-1}\big)C^{1/d} \le c_3 \int_{\tilde{G}^{\mathrm{inv}}(C^{-1})}^{\tilde{G}^{\mathrm{inv}}(z)} \tilde{G}^{-1/d}(v)\,\mathrm{d}v.$$

Therefore, it follows from the above bounds and (2.8) that

$$(1-c_2)J_C(z) \le \int_1^{(Cz)^{1/d}} \tilde{G}^{\mathrm{inv}}\big(zu^{-d}\big)\,\mathrm{d}u \le (1+c_3)J_C(z).$$

Combining this with (2.7) completes the proof. It remains to prove (2.9). We continue according to the value of α. If $\alpha > d$, then $\tilde{G}^{\mathrm{inv}}(z) = \mathrm{o}(z^{1/d})$, as $z \to \infty$. However, by definition $\liminf_{z\to\infty} J_C(z)/z^{1/d} > 0$ and (2.9) holds. When $\alpha \le d$, (2.9) is a an immediate consequence of Lemma 1(iv).

(ii) We begin with the direct part. Suppose $J(\eta_2(x)) \asymp x$. We argue by contradiction, assuming that $\limsup_{x\to\infty} \eta_2(x)/\eta_1(x) = \infty$. We have

$$1 \ge \limsup_{x\to\infty} \frac{\int_1^{\tilde{G}^{\mathrm{inv}}(\eta_1(x))} \tilde{G}^{-1/d}(v)\,\mathrm{d}v}{\int_1^{\tilde{G}^{\mathrm{inv}}(\eta_2(x))} \tilde{G}^{-1/d}(v)\,\mathrm{d}v} = \limsup_{x\to\infty} \frac{J(\eta_1(x))\eta_1(x)^{-1/d}}{J(\eta_2(x))\eta_2(x)^{-1/d}} \underset{J(\eta_1(x))\asymp J(\eta_2(x))}{=} \infty.$$

This is an obvious contradiction. Hence $\limsup_{x\to\infty} \eta_2(x)/\eta_1(x) < \infty$. Similarly, $\limsup_{x\to\infty} \eta_1(x)/\eta_2(x) < \infty$. Thus, $\eta_1 \asymp \eta_2$.

To prove the converse, suppose that $\eta_1 \asymp \eta_2$. We only prove $\limsup_{x\to\infty} J(\eta_2(x))/J(\eta_1(x)) < \infty$, the proof that $\limsup_{x\to\infty} J(\eta_1(x))/J(\eta_2(x)) < \infty$ being identical. Let $c_4 = 2\limsup_{x\to\infty} \eta_2(x)/\eta_1(x)$. Then for x sufficiently large,

$$\begin{aligned}
J\big(\eta_2(x)\big) &\le \big(c_4\eta_1(x)\big)^{1/d}\int_1^{\tilde G^{\mathrm{inv}}(c_4\eta_1(x))}\tilde G^{-1/d}(v)\,\mathrm{d}v\\
&\le \big(c_4\eta_1(x)\big)^{1/d}\int_1^{2c_4^{1/\alpha}\tilde G^{\mathrm{inv}}(\eta_1(x))}\tilde G^{-1/d}(v)\,\mathrm{d}v\\
&\underset{w=v/(2c_4^{1/\alpha})}{=} 2c_4^{1/d+1/\alpha}\eta_1^{1/d}(x)\int_{(2c_4^{1/\alpha})^{-1}}^{\tilde G^{\mathrm{inv}}(\eta_1(x))}\tilde G^{-1/d}\big(w\big(2c_4^{1/\alpha}\big)\big)\,\mathrm{d}w\\
&\le 2c_4^{1/d+1/\alpha}2\big(2c_4^{1/\alpha}\big)^{-\alpha/d}\eta_1^{1/d}(x)\int_{(2c_4^{1/\alpha})^{-1}}^{\tilde G^{\mathrm{inv}}(\eta_1(x))}\tilde G^{-1/d}(w)\,\mathrm{d}w \asymp J\big(\eta_1(x)\big).
\end{aligned}$$

□

3. Proof of Theorem 2

We recall that a path is a function γ from $\mathbb{Z}_+$ to $\mathbb{Z}^d$ with the property and $|\gamma(t+1)-\gamma(t)|=1$, for all $t\in\mathbb{Z}_+$. Since we will be only interested in paths starting from the origin, in this section we will also require a path to satisfy $\gamma(0)=0$. For $T\in\mathbb{Z}_+$, and a path γ we define $\gamma|_T$, the T-truncation of γ, as the restriction of γ to $\{0,\dots,T\}$, namely: $\gamma|_T\colon \{0,\dots,T\}\to\mathbb{Z}^d$ and $\gamma|_T(s)=\gamma(s)$. If γ is a T-truncation and $t\in\mathbb{Z}_+$, then $\gamma|_t$ will denote its restriction to $\{0,\dots,t\wedge T\}$. If Γ is a set of paths or truncations, we let

$$\Gamma|_t=\{\gamma|_t\colon \gamma\in\Gamma\}.$$

Let Γ be a set of paths or truncations. For $t\in\mathbb{Z}_+$, we let

$$S_\Gamma(t)=\big\{x\in\mathbb{Z}^d\colon x=\gamma(t),\ \text{for some }\gamma\in\Gamma\big\}. \tag{3.1}$$

Thus, $S_\Gamma(t)$ is the set of all sites in $\mathbb{Z}^d$ reachable by paths in Γ at time t. For $t'\ge t$ and $x\in\mathbb{Z}^d$, we let

$$n_\Gamma\big(t',t,x\big)=\big|\big\{\gamma\in\Gamma|_{t'}\colon \gamma(t)=x\big\}\big|,$$

which is the number of t'-truncations in $\Gamma|_{t'}$ passing through x at time t. We have the following lemma.

Lemma 4. *There exists a set of paths Λ and constants $K_1=K_1(d), K_2=K_2(d)\in(0,\infty)$ such that*

(i) *For all $t\in\mathbb{Z}_+$, $S_\Lambda(t)\ge K_1(1+t)^d$.*
(ii) *For all $0\le t\le t'$ and $x\in\mathbb{Z}^d$, $n_\Lambda(t',t,x)\le K_2\frac{(1+t')^d}{(1+t)^d}$.*
(iii) *For every $x\in S_\Lambda(t)$ there exists a unique $\gamma_{t,x}\in\Lambda|_t$ with the property $\gamma(t)=x$.*

Let $W\in\mathbb{Z}_+$. We define

$$Z^W(T)=E_0\big(\mathrm{e}^{H_\gamma(T)}\mathbf{1}_{\{\max_{s\in\{0,\dots,T\}}|\gamma(s)|\le W\}}\big). \tag{3.2}$$

The second ingredient in our proof is the following result:

Lemma 5. *Let ASI′ hold. Let $\varepsilon>0$. Then there exist constants $W\in\mathbb{Z}_+\backslash\{0\}$ and $c\in(0,\infty)$ depending on ε, d and the weight distribution, such that*

$$Q\big(Z^W(T)\ge \mathrm{e}^{(\lambda-\varepsilon)T}\big)\ge 1-\mathrm{e}^{-cT},$$

for all T sufficiently large.

This lemma is an adaptation of an analogous result from [5]. A similar result for the first passage percolation is [3], Lemma 3.1. The lemma is proved in the Appendix.

Proof of Lemma 4. Let $\{e_j\colon\ j = 1, \dots, d\}$ denote the standard basis for $\mathbb{Z}^d$. Then every $x \in \mathbb{Z}^d$ is a unique linear combination of $\{e_j\colon\ j = 1, \dots, d\}$, $x = \sum_{j=1}^d x_j e_j$. We call x_j the jth coordinate of x.

The construction. For every $t \in \mathbb{Z}_+$, we will construct a set of t-truncations, Λ_t, such that $\Lambda_t = \Lambda_{t+1}|_t$. Such a construction uniquely determines a set of paths Λ, satisfying $\Lambda|_t = \Lambda_t$. For convenience, we will write $S(t)$ meaning $S_{\Lambda_t}(t) = S_\Lambda(t)$. Let Λ_0 denote the set consisting of the (unique) 0-truncation. Then $S(0) = \{0\} \subset \mathbb{Z}^d$. We also let $k_0 = 1$ and $l_0 = 0$. We proceed continue the construction by induction. Besides for the requirement (iii), we will also impose the following condition:

(iv) For every $x, y \in S(t)$ and $j = 1, \dots, d$, $x_j - y_j \in 2\mathbb{Z}$.

Note that properties (iii) and (iv) trivially hold for Λ_0.

We now present the inductive step. For $t \in \mathbb{Z}_+$, we let $H_t = \{y \in \mathbb{Z}^d\colon\ y_{k_t} = l_t\}$, and let $S^=(t) = H_t \cap S(t)$ and $S^{\neq}(t) = S(t) \backslash H_t$. Informally, we obtain Λ_{t+1} from Λ_t by extending each t-truncation in Λ_t to a $t+1$-truncation whose position at time $t+1$ is one step away from H_t from its position at time t and truncation which are on H_t at time t will be split to two truncations, one at each side of H_t. We will also update the indices k_t and l_t determining H_t in a "fair" way, which will guarantee that the recently split truncations will be the last to be split again. More precisely, for $x \in S^{\neq}(t)$, we let

$$\tilde{\gamma}_x(s) = \begin{cases} \gamma_{t,x}(s), & s \le t; \\ x + \operatorname{sgn}(x_{k_t} - l_t) e_{k_t}, & s = t+1, \end{cases}$$

whereas for $x \in S^=(t)$ we define two $t+1$-truncations γ_x^{+1} and γ_x^{-1} by letting

$$\gamma_x^j(s) = \begin{cases} \gamma_{t,x}(s), & s \le t; \\ x + j e_{k_t}, & s = t+1. \end{cases}$$

We then let

$$\Lambda_{t+1} = \{\tilde{\gamma}_x\colon\ x \in S^{\neq}(t)\} \cup \{\gamma_x^{+1}, \gamma_x^{-1}\colon\ x \in S^=(t)\}.$$

It is easy to see that this construction guarantees properties (iii) and (iv) for Λ_{t+1}, if they hold for Λ_t. The latter is true due to the induction hypotheses.

To complete the induction step, we need to define k_{t+1} and l_{t+1}. One way to implement the "fair splitting" policy is by systematically scanning hyperplanes orthogonal to the axis as follows. If $(H_t + 3e_{k_t}) \cap S(t+1) \neq \varnothing$, then we let $k_{t+1} = k_t$, $l_{t+1} = l_t + 3$, hence $H_{t+1} = H_t + 3e_{k_t}$. The reason why we add 3 is because all elements in $S(t+1)$ whose k_tth coordinate is $l_t + 1$ correspond to truncations which have been recently split. Since $x_{k_t} - y_{k_t} \in 2\mathbb{Z}$ for every $x, y \in S(t+1)$, there are no elements in $S(t+1)$ whose k_tth coordinate is equal to $l_t + 2$, so the first value we need to check is $l_t + 3$. When $(H_t + 3e_{k_t}) \cap S(t+1) = \varnothing$, we move to the next dimension, by letting $k_{t+1} = k_t + 1$ if $k_t < d$ or $k_{t+1} = 1$ otherwise, and then letting $l_{t+1} = \min\{y_{k_{t+1}}\colon\ y \in S(t+1)\}$. This completes the construction.

Figure 1 below illustrates to construction in $d = 1$. Paths (truncations) are represented by the solid lines, running from $t = 0$ at $x = 0$ (bottom) to $t = 11$ (top). Clearly, $k_t = 1$ and $S^=(t) = H_t = \{l_t\}$, for all t. For each t, the site $x = l_t$ is marked by a circle, and is where a t-truncation is split into two $t+1$-truncations.

Proof of (i) *and* (ii). We will prove the claims only for $d = 2$, proof for higher dimensions being essentially identical, and the one-dimensional case being simpler and can be immediately derived from the proof below. We will adopt the following terminology. For $(t, x) \in \mathbb{Z}_+ \times \mathbb{Z}^d$, we say that (t', x') is a descendant of (t, x) if there exists $\gamma \in \Lambda_{t'}$ such that $\gamma(t') = x'$ and $\gamma(t) = x$ (obviously, $t \le t'$). When $(t+1, x')$ is a descendant of (t, x), we say that x' is a son of (t, x).

We begin by proving a claim on the construction from which the lemma will follow. Let $\sigma_0 = 0$ and define inductively $\sigma_{j+1} = \min\{t > \sigma_j\colon\ k_t = 1 \text{ and } l_t < l_{t-1}\}$. For $j \in \mathbb{Z}_+$, let $D_j = 2^j - 1$ and let $I_j = \{-D_j, D_j + 2, \dots, D_j - 2, D_j\}$. Note that $|I_j| = D_j + 1 = 2^j$. We claim:

$$S(\sigma_j) = \underbrace{I_j \times \cdots \times I_j}_{d}, \quad \sigma_j = d D_j. \tag{3.3}$$

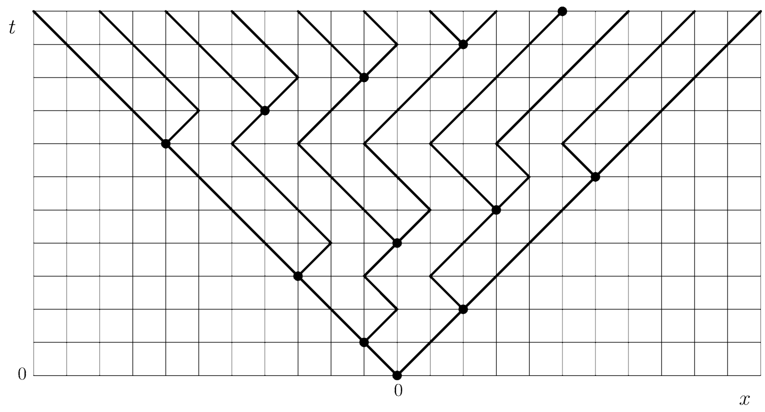

Fig. 1. $\tilde{\Gamma}$ in $d = 1$.

In particular this shows that $|S(\sigma_j)| = 2^{dj}$. We will prove the claim by induction. We begin with the base case, $j = 0$. Then $\sigma_j = 0$, $D_j = 0$ and the claim trivially holds. In order to perform the induction step, we will use nested induction. Define

$$\sigma_{j,2} = \min\{t > \sigma_j\colon\ k_t = 2\}$$

(in higher dimensions we will similarly define $\sigma_{j,k}$, $k = 2, \ldots, d$). We will prove that $\sigma_{j,2} = \sigma_j + D_j + 1$ and that

$$S(t) = \{-L_t, -L_t + 2, \ldots, L_t - 2, L_t\} \times I_j, \quad \text{where } L_t = D_j + (t - \sigma_j),\ \sigma_j \le t \le \sigma_{j,2}. \tag{3.4}$$

Starting from $t = \sigma_j$, we have $L_t = D_j$, $l_t = -D_j$, and (3.4) holds. This establishes the base step. By construction, $S(t+1)$ is obtained from $S(t)$ as follows: if $t < \sigma_{j,2}$, then all elements of $S(t)$ whose first component is $< l_t$ (resp. $> l_t$) are replaced with exactly one element in $S(t+1)$, which is one step to the left (resp. right). All elements whose first component is l_t are replaced with exactly two elements: one at one step to the left and the second at one step to the right. Thus,

$$S(t+1) = \{-L_t - 1, -L_t + 1, \ldots, L_t - 1, L_t + 1\} \times I_j = \{-L_{t+1}, -L_{t+1} + 2, \ldots, L_{t+1} - 2, L_{t+1}\} \times I_j.$$

Furthermore, note that as long as $l_t + 3 \le L_{t+1}$ the construction gives $l_{t+1} = l_t + 3$. Otherwise, $t + 1 = \sigma_{j,2}$ and then $l_{t+1} < l_t$. This completes the proof of (3.4).

Next, note that for $\sigma_j \le t < \sigma_{j,2}$, $L_{t+1} - l_{t+1} = L_t + 1 - (l_t + 3) = L_t - l_t - 2$. Therefore, at each step we decrease the difference by 2. Since by assumption $L_{\sigma_j} - l_{\sigma_j} = D_j - (-D_j) = 2D_j \in 2\mathbb{Z}_+$, the condition $l_t + 3 \le L_{t+1}$, which can be rewritten as $l_t + 2 \le L_t$, is equivalent to the condition $l_t < L_t$. Therefore the number of steps required until the latter condition fails is exactly $D_j + 1$. This proves that $\sigma_{j,2} = \sigma_j + D_j + 1$. Therefore (3.4) gives

$$S(\sigma_{j,2}) = \{-2D_j - 1, -2D_j + 1, \ldots, 2D_j - 1, 2D_j + 1\} \times I_j = I_{j+1} \times I_j.$$

Repeating the argument on the second coordinate shows that $\sigma_{j+1} = \sigma_{j,2} + D_j + 1$ and that

$$S(t) = I_{j+1} \times \{-L_t, -L_t + 2, \ldots, L_t - 2, L_t\}, \quad \text{where } L_t = D_j + (t - \sigma_{j,2}),\ \sigma_{j,2} \le t \le \sigma_{j+1}.$$

Since the proof is identical, we omit it. We only remark that in the higher dimensional analogues we do this for all d coordinates. Combining these observations, we obtain

$$\sigma_{j+1} = \sigma_j + d(D_j + 1) = dD_j + d(D_j + 1) = d(2D_j + 1) = dD_{j+1},$$

and

$$S(\sigma_{j+1}) = I_{j+1} \times I_{j+1},$$

completing the induction step in the proof of (3.3).

We will now show that

$$n_\Lambda(\sigma_{j+1}, \sigma_j, x) = 2^d \quad \text{for all } x \in S(\sigma_j). \tag{3.5}$$

This means that for every $x \in S(\sigma_j)$, (σ_j, x) has exactly 2^d descendants at time σ_{j+1}. Fix $x \in S(\sigma_j)$. By the remark below (3.3), $|S(\sigma_{j+1})| = 2^d|S(\sigma_j)|$, therefore it is enough to show that (σ_j, x), has at least two descendants at time $\sigma_{j,2}$ and that if $y \in S(\sigma_{j,2})$, then $(\sigma_{j,2}, y)$ has at least two descendants at time σ_{j+1}. Let $t_0 = (x_1 - l_{\sigma_j})/2$. It follows immediately from the construction that at time $\tilde{t} \equiv \sigma_j + t_0 < \sigma_{j,2}$, (σ_j, x) has exactly one descendant located at $(x_1 + t_0, x_2)$ and that $l_{\tilde{t}} = x_1 + t_0$. Therefore, (σ_j, x) has exactly two descendants at time $\sigma_j + t_0 + 1 \le \sigma_{j,2}$. The same argument shows that for every $y \in \sigma_{j,2}$, $(\sigma_{j,2}, y)$ will have at least two descendants at time σ_{j+1}. Therefore, (σ_j, x) has at least $4 = 2^d$ descendants at time σ_{j+1}, completing the proof of (3.5).

We are in a position to prove the lemma. Let

$$\begin{aligned} j(t) &= \max\{j \in \mathbb{Z}_+ \colon \sigma_j \le t\} = \max\bigl\{j \in \mathbb{Z}_+ \colon d\bigl(2^j - 1\bigr) \le t\bigr\} \\ &= \left\lfloor \log_2 \frac{t+d}{d} \right\rfloor. \end{aligned}$$

Since $t \to |S(t)|$ is increasing, it follows that

$$\bigl|S(t)\bigr| \ge \bigl|S(\sigma_{j(t)})\bigr| = 2^{dj(t)} = 2^{d\lfloor \log_2(d+t)/d\rfloor} \ge K_1(1+t)^d.$$

Next, let $t' > t$ and let $j'(t') = \min\{j \in \mathbb{Z}_+ \colon \sigma_j \ge t'\}$. Clearly, $j'(t') = \lceil \log_2(t'+d)/d\rceil$. Suppose that $x \in S(t)$. Then (t, x) is a descendant $(\sigma_{j(t)}, \tilde{x})$ for some $\tilde{x} \in S(\sigma_{j(t)})$. The number of descendants of (t, x) at time t' is less than or equal to the number of descendants of $(\sigma_{j(t)}, \tilde{x})$ at time $\sigma_{j'(t')} \ge t'$. Thanks to (3.5), the latter quantity is equal to $2^{d(j'(t') - j(t))}$. Therefore,

$$n_\Lambda\bigl(t', t, x\bigr) \le 2^{d(\lceil \log_2(t'+d)/d\rceil - \lfloor \log_2(t+d)/d\rfloor)} \le K_2\bigl(1+t'\bigr)^d(1+t)^{-d}. \qquad \square$$

Proof of Theorem 2. As explained in Section 2, there is no loss of generality proving the theorem under ASI′. Note that in particular, $\lambda \ge 0$. We first fix notation, definitions and choice of constants. For $t \le t'$ and a path γ Let

$$H_\gamma\bigl(t, t'\bigr) = \sum_{k=t}^{t'-1} V\bigl(k, \gamma(k-t)\bigr). \tag{3.6}$$

In this definition we do not require $\gamma(0) = 0$. Let β_0 be the constant appearing in Lemma 2(i). We set $C = \frac{1}{2}\min(\beta_0^{-1}, 1)$ and denote the function I_C defined in (2.3) simply by I. Let W and c be the constants appearing in Lemma 5 and let K_1 and K_2 be the constants appearing in Lemma 4. We let $\tilde{\varepsilon} = \varepsilon/3$ and let $K = \frac{\tilde{\varepsilon}}{8K_2K_1^{-1}(9W)^d}$. We let $\tilde{\eta} = \rho\eta$, where ρ is some constant in $(0, 1]$. We claim that we can choose ρ small enough, such that for all sufficiently large T

$$I\bigl(\tilde{\eta}(T)\bigr) \le KT \quad \text{and} \tag{3.7}$$

$$\tilde{\eta}^{1/d}(T)\max\bigl(1, \tilde{\varepsilon}^{-1}(\lambda - \tilde{\varepsilon} + 2\ln d)\bigr) \le T. \tag{3.8}$$

Indeed, (3.7) is an immediate consequence of Lemma 3(i). The condition (3.8) could be also guaranteed, because

$$\limsup_{T\to\infty} \frac{\eta^{1/d}(T)}{T} = \limsup_{T\to\infty} \frac{J(\eta(T))}{T\int_1^{\tilde{G}^{\mathrm{inv}}(\eta(T))} \tilde{G}^{-1/d}(v)\,\mathrm{d}v} < \infty.$$

Next let $M(T) = \lfloor (C\tilde{\eta}(T))^{1/d}\rfloor$ and let $\tilde{\Lambda} = \Lambda|_{M(T)}$, where Λ is the set of paths constructed in Lemma 4. For convenience we will denote the set $S_{\tilde{\Lambda}}(M(T))$ defined in (3.1) by $S(M(T))$. For $x \in S(M(T))$ we let γ_x denote the unique truncation in $\tilde{\Lambda}$ satisfying $\gamma_x(M(T)) = x$. Define

$$N(T) = \{x \in S(M(T)) \colon H_{\gamma_x}(M(T)) \geq -\tilde{\varepsilon} T\} \quad \text{and} \tag{3.9}$$
$$E(T) = \{|N(T)| \geq (1-(9W)^{-d})|S(M(T))|\}.$$

We will prove below that for all T sufficiently large,

$$Q(E(T)^c) \leq \mathrm{e}^{-K/2T\tilde{\eta}(T)}. \tag{3.10}$$

Let $B_W = [-2W, 2W)^d \cap \mathbb{Z}^d$. For $z \in B_W$, let $D_z = z + 4W\mathbb{Z}^d = \{z + 4Wy \colon y \in \mathbb{Z}^d\}$. Since $|B_W| = (4W)^d$, it follows that $\{D_z \colon z \in B_W\}$ is a decomposition of $\mathbb{Z}^d$ into $(4W)^d$ disjoint subsets, and each of which has the property that the distance between any two of its elements is bounded below by $4W$ (in the l^1-norm). Thus, $S(M(T)) = \bigcup_{z \in B_W} D_z \cap S(M(T))$. For some $z_0 \in B_W$, $|D_{z_0} \cap S(M(T))| \geq |S(M(T))|/(4W)^d$. Fix such z_0 and let $A(T) = D_{z_0} \cap S(M(T))$. For all T sufficiently large

$$|A(T)| \geq \frac{|S(M(T))|}{(4W)^d} \underset{\text{Lemma 4(i)}}{\geq} \frac{CK_1}{2(4W)^d}\tilde{\eta}(T). \tag{3.11}$$
$$\tilde{Z}_x(T) = E_x\left(\mathrm{e}^{H_\gamma(M(T),T)}\mathbf{1}_{\{\max_{s\in\{0,\dots,T-M(T)\}}|\gamma(s)-\gamma(0)|\leq W\}}\right).$$

Note that $\tilde{Z}_x(T)$ has the same distribution as $Z^W(T - M(T))$ defined in (3.2). Set

$$\tilde{N}(T) = \{x \in A(T) \colon \tilde{Z}_x(T) \geq \mathrm{e}^{(\lambda-\tilde{\varepsilon})(T-M(T))}\} \quad \text{and} \tag{3.12}$$
$$\tilde{E}(T) = \left\{|\tilde{N}(T)| > \frac{1}{2}|A(T)|\right\} = \left\{\sum_{x\in A(T)} \mathbf{1}_{\tilde{N}(T)}(x) > \frac{1}{2}|A(T)|\right\}.$$

By definition, for every $x \in A(T)$ the event $\{x \in \tilde{N}(T)\}$ depends only on $\{V(t,y) \colon t \geq M(T), |y-x| \leq W\}$. Hence the stationarity of V implies that the probability of the above event is independent of x. Since every two distinct points in $A(T)$ are at least $4W$ units apart, $\{\mathbf{1}_{\tilde{N}(T)}(x) \colon x \in A(T)\}$ forms an IID sequence of Bernoulli random variables. Clearly,

$$\tilde{E}(T)^c = \left\{\sum_{x\in A(T)} \mathbf{1}_{\tilde{N}(T)}(x) \leq \frac{1}{2}|A(T)|\right\}$$
$$= \bigcup_{B\subset A(T),|B|\geq 1/2|A(T)|} \{\mathbf{1}_{\tilde{N}(T)} = 0 \text{ on } B\}.$$

Let $x \in A(T)$. Then for B as above

$$Q(\mathbf{1}_{\tilde{N}(T)} = 0 \text{ on } B) = Q(\mathbf{1}_{\tilde{N}(T)}(x) = 0)^{|B|} \underset{\text{Lemma 5}}{\leq} \mathrm{e}^{-c(T-M(T))|B|} \leq \mathrm{e}^{-c(T-M(T))\lfloor 1/2|A(T)|\rfloor}.$$

By (3.8), $M(T) \leq C^{1/d}\tilde{\eta}(T)^{1/d} < C^{1/d}T$. We recall that $C = \frac{1}{2}\min(\beta_0^{-1}, 1) \leq \frac{1}{2}$. Hence $T - M(T) > (1 - C^{1/d})T > 0$. Let $c_1 = \frac{c(1-C^{1/d})CK_1}{8(4W)^d}$. Then $c_1 > 0$ and for all T sufficiently large

$$Q(\tilde{E}(T)^c) \leq \sum_{B\subset A(T),\ |B|\geq\lfloor 1/2|A(T)|\rfloor} \mathrm{e}^{-c(1-C^{-1/d})T\lfloor 1/2|A(T)|\rfloor}$$
$$\leq 2^{|A(T)|}\mathrm{e}^{-1/3c(1-C^{1/d})T|A(T)|} \leq \mathrm{e}^{-1/4c(1-C^{1/d})T|A(T)|}$$
$$\underset{(3.11)}{\leq} \mathrm{e}^{-c_1T\tilde{\eta}(T)}. \tag{3.13}$$

Consider the events $E(T)$ and $\tilde{E}(T)$ defined in (3.9) and (3.12), respectively. The former depends on $\{V(t,x)\colon t \le M(T)-1\}$ and the latter depends on $\{V(t,x)\colon t \ge M(T)\}$. Therefore they are independent. From (3.10) and (3.13) we obtain

$$\begin{aligned} Q\big(E(T)\cap \tilde{E}(T)\big) &\ge \big(1-\mathrm{e}^{-K/2T\tilde{\eta}(T)}\big)\big(1-\mathrm{e}^{-c_1T\tilde{\eta}(T)}\big) \\ &\ge 1-2\mathrm{e}^{-\min(K/2,c_1)T\tilde{\eta}(T)} \ge 1-\mathrm{e}^{-c_2T\tilde{\eta}(T)}, \end{aligned} \tag{3.14}$$

where $c_2 = 1/2\min(K/2, c_1)$.

By definition, on $E(T)\cap \tilde{E}(T)$

$$\big|N(T)\big| + \big|\tilde{N}(T)\big| \ge \big|S\big(M(T)\big)\big|\left(1-\frac{1}{(9W)^d}+\frac{1}{2(4W)^d}\right) > \big|S\big(M(T)\big)\big|.$$

Since both $N(T)$ and $\tilde{N}(T)$ are subsets of $S(M(T))$, this shows that their intersection is not empty on $E(T)\cap\tilde{E}(T)$. On this event

$$\begin{aligned} \ln Z(T) &= \ln E_0\big(\mathrm{e}^{H(T)}\big) \\ &\ge \ln\left(\sum_{x\in N(T)\cap\tilde{N}(T)} 2^{-dM(T)}\mathrm{e}^{H_{\gamma x}(M(T))}E_x\mathrm{e}^{H(M(T),T)}\right) \\ &\ge -dM(T)\ln 2 - \tilde{\varepsilon}T + (\lambda-\tilde{\varepsilon})\big(T-M(T)\big) \\ &= (\lambda-3\tilde{\varepsilon})T + \tilde{\varepsilon}T - M(T)(\lambda-\tilde{\varepsilon}+d\ln 2). \end{aligned}$$

We claim that the right-hand side is bounded below by $(\lambda-3\tilde{\varepsilon})T$. If $\lambda-\tilde{\varepsilon}+d\ln 2\le 0$, then the claim trivially holds. Otherwise, the condition (3.8) guarantees that $M(T)(\lambda-\tilde{\varepsilon}+d\ln 2) < \tilde{\varepsilon}T$ and the claim holds as well. Concluding the argument, we obtain

$$Q\big(Z(T)\ge \mathrm{e}^{(\lambda-3\tilde{\varepsilon})T}\big) \ge Q\big(E(T)\cap\tilde{E}(T)\big) \underset{(3.14)}{\ge} 1-\mathrm{e}^{-c_2T\tilde{\eta}(T)}.$$

Thus,

$$-R_{3\tilde{\varepsilon}}(T) = \ln Q\big(Z(T)\le \mathrm{e}^{(\lambda-3\tilde{\varepsilon})}\big) \le -c_2T\tilde{\eta}(T).$$

Since $\tilde{\varepsilon}=\varepsilon/3$ and $\tilde{\eta}=\rho\eta$, the theorem follows.

It remains to establish (3.10). We will obtain a lower bound on $Q(E(T))$ by rephrasing the statement on paths $E(T)$ into a statement on weighted sums IID random variables. By the definition of $E(T)$ and $N(T)$ in (3.9)

$$E(T)^c = \big\{\exists A\subset\tilde{\Lambda},\ |A| > (9W)^{-d}\big|S\big(M(T)\big)\big| \text{ and for all } \gamma\in A,\ H_\gamma\big(M(T)\big) < -\tilde{\varepsilon}T\big\}.$$

If there exists $A\subset\tilde{\Lambda}$ satisfying $|A| > (9W)^{-d}|S(M(T))|$ and $H_\gamma(M(T)) < -\tilde{\varepsilon}T$ for all $\gamma\in A$, then clearly for any $A'\subset A$ satisfying $|A'| = \lfloor (9W)^{-d}|S(M(T))|\rfloor$ we have $H_\gamma(M(T)) < -\tilde{\varepsilon}T$ for all $\gamma'\in A'$. Therefore

$$\begin{aligned} E(T)^c &\subset \big\{\exists A\subset\tilde{\Lambda},\ |A| = \big\lfloor (9W)^{-d}\big|S\big(M(T)\big)\big|\big\rfloor \text{ and for all } \gamma\in A,\ H_\gamma\big(M(T)\big) < -\tilde{\varepsilon}T\big\} \\ &= \bigcup_{A\subset\tilde{\Lambda},|A|=\lfloor (9W)^{-d}|S(M(T))|\rfloor}\ \bigcap_{\gamma\in A}\big\{-H_\gamma\big(M(T)\big) > \tilde{\varepsilon}T\big\} \\ &\subseteq \bigcup_{A\subset\tilde{\Lambda},|A|=\lfloor (9W)^{-d}|S(M(T))|\rfloor}\left\{\sum_{\gamma\in A} -H_\gamma\big(M(T)\big) > |A|\tilde{\varepsilon}T\right\}. \end{aligned}$$

Consequently,

$$Q\big(E(T)^c\big) \le \sum_{A\subset\tilde{\Lambda},|A|=\lfloor (9W)^{-d}|S(M(T))|\rfloor} Q\bigg(\sum_{\gamma\in A} -H_\gamma\big(M(T)\big) > |A|\tilde{\varepsilon}T\bigg). \tag{3.15}$$

Below we let A be a subset of $\tilde{\Lambda}$ with $|A| = \lfloor (9W)^{-d}|S(M(T))|\rfloor$. We have

$$-\sum_{\gamma\in A} H_\gamma\big(M(T)\big) = -\sum_{t=0}^{M(T)-1}\sum_x n_A\big(M(T),t,x\big)V(t,x).$$

By assumption, $\lim_{T\to\infty} M(T) = \infty$. Hence, for all T sufficiently large $|A| \ge (9W)^{-d}|S(M(T))|/2$, and we obtain

$$\begin{aligned}\bigg\{\sum_{\gamma\in A} -H_\gamma\big(M(T)\big) > |A|\tilde{\varepsilon}T\bigg\} &\subseteq \bigg\{-\sum_{t=0}^{M(T)-1}\sum_x n_A\big(M(T),t,x\big)V(t,x) > \frac{(9W)^{-d}|S(M(T))|}{2}\tilde{\varepsilon}T\bigg\}\\ &= \bigg\{-\sum_{t=0}^{M(T)-1}\sum_x \frac{n_A(M(T),t,x)}{|S(M(T))|}V(t,x) > \frac{\tilde{\varepsilon}T}{2(9W)^d}\bigg\}.\end{aligned}$$

Combining this with (3.15), we have

$$Q\big(E(T)^c\big) \le \sum_{A\subset\tilde{\Lambda},|A|=\lfloor (1-r)|S(M(T))|\rfloor} Q\bigg(-\sum_{t=0}^{M(T)-1}\sum_x \frac{n_A(M(T),t,x)}{|S(M(T))|}V(t,x) > \frac{\tilde{\varepsilon}T}{2(9W)^d}\bigg). \tag{3.16}$$

It follows from Chebyshev that for every $\beta > 0$,

$$\begin{aligned}&Q\bigg(-\sum_{t=0}^{M(T)-1}\sum_x \frac{n_A(M(T),t,x)}{|S(M(T))|}V(t,x) > \frac{\tilde{\varepsilon}T}{2(9W)^d}\bigg)\\ &\quad\le \bigg(\prod_{t=0}^{M(T)-1}\prod_{\{x\in\mathbb{Z}^d:n_A(M(T),t,x)>0\}} E^Q\big(\mathrm{e}^{-\beta n_A(M(T),t,x)/|S(M(T))|V(0,0)}\big)\bigg)\mathrm{e}^{-\beta\tilde{\varepsilon}T/2(9W)^d}.\end{aligned} \tag{3.17}$$

Since $A\subset\tilde{\Lambda}$, we have $n_A(M(T),t,x) \le n_{\tilde{\Lambda}}(M(T),t,x) = n_\Lambda(M(T),t,x)$. Therefore by Lemma 4(i)–(ii)

$$\frac{n_A(M(T),t,x)}{|S(M(T))|} \le \frac{K_2(1+M(T))^d(1+t)^{-d}}{K_1(1+M(T))^d} = K_2K_1^{-1}(1+t)^{-d}.$$

By Lemma 2(i) the mapping $s \to E^Q(\mathrm{e}^{-sV(0,0)})$ is non-decreasing on $[0,\infty)$. Therefore

$$E^Q\big(\mathrm{e}^{-\beta n_A(M(T),t,x)/|S(M(T))|V(0,0)}\big) \le E^Q\big(\mathrm{e}^{-\beta K_2K_1^{-1}(1+t)^{-d}V(0,0)}\big).$$

Choose now $\beta = \frac{\tilde{\eta}(T)}{2K_2K_1^{-1}}$. Then for $t\in\{0,\dots,M(T)-1\}$,

$$\frac{\beta K_2K_1^{-1}}{(1+t)^d} = \frac{\tilde{\eta}(T)}{2(1+t)^d} \ge \frac{\tilde{\eta}(T)}{2M(T)^d} \ge \frac{1}{2C} \ge \beta_0$$

and by Lemma 2(i) we obtain

$$Q\big(\mathrm{e}^{-\beta n_A(M(T),t,x)/|S(M(T))|V(0,0)}\big) \le \exp\bigg(\frac{\tilde{\eta}(T)}{(1+t)^d}G^{\mathrm{inv}}\bigg(\frac{\tilde{\eta}(T)}{(1+t)^d}\bigg)\bigg). \tag{3.18}$$

With this choice of β, we have $\frac{\beta\tilde{\varepsilon}T}{2(9W)^d} = 2KT\tilde{\eta}(T)$. Therefore, plugging (3.18) into (3.17) gives

$$
\begin{aligned}
&Q\left(-\sum_{t=0}^{M(T)-1}\sum_{x}\frac{n_A(M(T),t,x)}{|S(M(T))|}V(t,x) > \frac{\tilde{\varepsilon}T}{2(9W)^d}\right)\\
&\quad\le \left(\prod_{t=0}^{M(T)-1}\prod_{\{x\in\mathbb{Z}^d:n_A(M(T),t,x)>0\}}\exp\left(\frac{\tilde{\eta}(T)}{(1+t)^d}G^{\mathrm{inv}}\left(\frac{\tilde{\eta}(T)}{(1+t)^d}\right)\right)\right)\mathrm{e}^{-2KT\tilde{\eta}(T)}\\
&\quad\le \left(\prod_{t=0}^{M(T)-1}\exp\left(\left|S(t)\right|\frac{\tilde{\eta}(T)}{(1+t)^d}G^{\mathrm{inv}}\left(\frac{\tilde{\eta}(T)}{(1+t)^d}\right)\right)\right)\mathrm{e}^{-2KT\tilde{\eta}(T)}\\
&\quad\le \exp\left(\sum_{t=0}^{M(T)-1}(1+t)^d\frac{\tilde{\eta}(T)}{(1+t)^d}G^{\mathrm{inv}}\left(\frac{\tilde{\eta}(T)}{(1+t)^d}\right) - 2KT\tilde{\eta}(T)\right)\\
&\quad= \exp\big(\tilde{\eta}(T)I\big(\tilde{\eta}(T)\big) - 2KT\tilde{\eta}(T)\big) \underset{(3.7)}{\le} \mathrm{e}^{-KT\tilde{\eta}(T)}.
\end{aligned}
$$

Plugging the last estimate into (3.16) gives

$$
Q\big(E(T)^c\big) \le \sum_{A\subset\tilde{\Lambda}:|A|=\lfloor(9W)^{-d}|S(M(T))|\rfloor}\mathrm{e}^{-KT\tilde{\eta}(T)} \le \sum_{A\subset\tilde{\Lambda}}\mathrm{e}^{-KT\tilde{\eta}(T)} = 2^{|\tilde{\Lambda}|}\mathrm{e}^{-KT\tilde{\eta}(T)}.
$$

Recall that $\tilde{\Lambda}$ is a subset of $M(T)$-truncations in Λ and by Lemma 4(iii), there are no more than $(1+M(T))^d \le (2M(T))^d$ $M(T)$-truncations in Λ. Therefore $|\tilde{\Lambda}| \le (2M(T))^d = \mathrm{O}(\tilde{\eta}(T)) = \mathrm{o}(T\tilde{\eta}(T))$ and (3.10) follows. □

4. Proof of Theorem 3

As explained in Section 2 there is no loss of generality proving the theorem under ASI′ instead of ASI. To simplify notation, we denote the function I_1 defined in (2.3) simply by I. By Lemma 3(i), there exist constants $0 < c_1 \le c_2 < \infty$ such that for all x sufficiently large,

$$c_1J(x) \le I(x) \le c_2J(x).$$

We let $\varepsilon_1 = \frac{\int_1^\infty \tilde{G}^{-1/d}(v)\,\mathrm{d}v}{2c_2c_1^{-1}} \in (0,\infty]$. Let $\varepsilon \in (0,\varepsilon_1)$. Let $\tilde{\eta}:\mathbb{R}_+ \to \mathbb{R}_+$ satisfy $J(\tilde{\eta}(x)) = 2c_1^{-1}\varepsilon x$. For all T sufficiently large,

$$2\varepsilon T \le I\big(\tilde{\eta}(T)\big) \le 2c_2c_1^{-1}\varepsilon T \tag{4.1}$$

and

$$\frac{\tilde{\eta}^{1/d}(T)}{T} = \frac{J(\tilde{\eta}(T))}{T\int_1^{\tilde{G}^{\mathrm{inv}}(\tilde{\eta}(T))}\tilde{G}^{-1/d}(v)\,\mathrm{d}v} \le \frac{2c_2c_1^{-1}\varepsilon}{\int_1^{\tilde{G}^{\mathrm{inv}}(\tilde{\eta}(T))}\tilde{G}^{-1/d}(v)\,\mathrm{d}v} \underset{T\to\infty}{\to} \frac{\varepsilon}{\varepsilon_1} < 1. \tag{4.2}$$

Furthermore, by Lemma 3(ii),

$$\tilde{\eta} \asymp \eta. \tag{4.3}$$

Let Γ denote the set of all nearest neighbor random walk paths on $\mathbb{Z}^d$ starting from the origin. Denote the set $S_\Gamma(t)$ defined in (3.1) by $S(t)$. Clearly, $|S(t)| = (1+t)^d$. Let $T \in \mathbb{Z}_+$ and let $M(T) = \lfloor(\tilde{\eta}(T))^{1/d}\rfloor$. We define two events:

$$A(T) = \bigcap_{t=0}^{M(T)-1}\bigcap_{x\in S(t)}\left\{-V(t,x) \ge \tilde{G}^{\mathrm{inv}}\left(\frac{\tilde{\eta}(T)}{(1+t)^d}\right)\right\}$$

and

$$B(T)=\bigcap_{x\in S(M(T))}\left\{E_x\left(\mathrm{e}^{H(M(T),T)}\right)\le \mathrm{e}^{(\lambda+\varepsilon)(T-M(T))}\right\},$$

where $H(M(T),T)$ was defined in (3.6). Note that $A(T)$ and $B(T)$ are independent, because they are determined by disjoint subsets of the field V. On the event $A(T)$,

$$H_\gamma\big(M(T)\big)\le -\sum_{t=0}^{M(T)-1}\tilde G^{\mathrm{inv}}\big(\tilde\eta(T)(1+t)^{-d}\big)=-I\big(\tilde\eta(T)\big)\underset{(4.1)}{\le}-2\varepsilon T,$$

for all paths γ. Since $(\lambda+\varepsilon)(T-M(T))\le(\lambda+\varepsilon)T$, we observe that on $B(T)$

$$E_x\left(\mathrm{e}^{H(M(T),T)}\right)\le \mathrm{e}^{(\lambda+\varepsilon)T}\quad\text{for all } x\in S\big(M(T)\big).$$

Clearly,

$$Z(T)=E_0\left(\mathrm{e}^{H(T)}\right)\le E_0\left(\mathrm{e}^{H(M(T))}\right)\max_{x\in S(M(T))}E_x\left(\mathrm{e}^{H(M(T),T)}\right).$$

Therefore on $A(T)\cap B(T)$, $Z(T)\le \mathrm{e}^{(\lambda-\varepsilon)T}$. Next,

$$\begin{aligned}-R_\varepsilon(T)&=\ln Q\big(Z(T)\le \mathrm{e}^{(\lambda-\varepsilon)T}\big)\ge \ln Q\big(A(T)\cap B(T)\big)=\ln Q\big(A(T)\big)+\ln Q\big(B(T)\big)\\&\ge \ln Q\big(A(T)\big)+\big(1+M(T)\big)^d\ln Q\big(Z\big(T-M(T)\big)\le \mathrm{e}^{(\lambda+\varepsilon)(T-M(T))}\big),\end{aligned}\tag{4.4}$$

where the second equality on the first line follows from the independence of $A(T)$ and $B(T)$ and the second line follows from the FKG inequality, applied to $B(T)$, the intersection of $(1+M(T))^d$ identically distributed decreasing events of the IID field V.

By (4.2) we have $T-M(T)\asymp T$. Then, Theorem 1(ii) shows that there exists a constant $c_0=c_0(\varepsilon,G,d)\in(0,\infty)$ such that

$$\begin{aligned}\big(1+M(T)\big)^d\ln Q\big(Z\big(T-M(T)\big)\le \mathrm{e}^{(\lambda+\varepsilon)(T-M(T))}\big)&\le (2T)^d\ln\big(1-\mathrm{e}^{-c_0(T-M(T))}\big)\\&\asymp-(2T)^d\mathrm{e}^{-c_0(T-M(T))}\underset{T\to\infty}{\to}0.\end{aligned}\tag{4.5}$$

We now estimate $\ln Q(A(T))$. By independence and stationarity of the field V,

$$\begin{aligned}Q\big(A(T)\big)&=\prod_{t=0}^{M(T)-1}\prod_{x\in S(t)}Q\left(-V(t,x)\ge \tilde G^{\mathrm{inv}}\left(\frac{\tilde\eta(T)}{(1+t)^d}\right)\right)\\&\ge\prod_{t=0}^{M(T)-1}Q\left(-V(0,0)\ge \tilde G^{\mathrm{inv}}\left(\frac{\tilde\eta(T)}{(1+t)^d}\right)\right)^{(1+t)^d}.\end{aligned}\tag{4.6}$$

By Lemma 2(ii),

$$\begin{aligned}&Q\left(-V(0,0)\ge \tilde G^{\mathrm{inv}}\left(\frac{\tilde\eta(T)}{(1+t)^d}\right)\right)\\&\quad\ge\exp\left(-q-\tilde G^{\mathrm{inv}}\left(\frac{\tilde\eta(T)}{(1+t)^d}\right)\tilde G\left(\tilde G^{\mathrm{inv}}\left(\frac{\tilde\eta(T)}{(1+t)^d}\right)\right)\delta\left(\tilde G^{\mathrm{inv}}\left(\frac{\tilde\eta(T)}{(1+t)^d}\right)\right)\right)\\&\quad\underset{\text{Lemma 1(iii)}}{=}\exp\left(-q-\tilde G^{\mathrm{inv}}\left(\frac{\tilde\eta(T)}{(1+t)^d}\right)\frac{\tilde\eta(T)}{(1+t)^d}\delta\left(\tilde G^{\mathrm{inv}}\left(\frac{\tilde\eta(T)}{(1+t)^d}\right)\right)\right)\\&\quad\underset{\delta\text{ is non-increasing}}{\ge}\exp\left(-q-\frac{\tilde\eta(T)}{(1+t)^d}\tilde G^{\mathrm{inv}}\left(\frac{\tilde\eta(T)}{(1+t)^d}\right)\delta\big(\tilde G^{\mathrm{inv}}(1)\big)\right).\end{aligned}$$

Set $\tilde{\delta} = \delta(\tilde{G}^{\mathrm{inv}}(1))$. By (4.6)

$$\ln Q\big(A(T)\big) \geq \sum_{t=0}^{M(T)-1}\left(-q(1+t)^d - \tilde{\eta}(T)\tilde{G}^{\mathrm{inv}}\left(\frac{\tilde{\eta}(T)}{(1+t)^d}\right)\tilde{\delta}\right)$$
$$\geq -q\big(M(T)\big)^{1+d} - \tilde{\eta}(T)I\big(\tilde{\eta}(T)\big)\tilde{\delta},$$

where the second line due to the fact that $\sum_{t=0}^{N-1}(1+t)^d \leq N^{1+d}$ for all $N \geq 1$. Since $M(T)^{1+d} \leq \tilde{\eta}(T)^{1/d}\tilde{\eta}(T)$, it follows from (4.2) that $M(T)^{1+d} < T\tilde{\eta}(T)$. Hence by (4.1)

$$\ln Q\big(A(T)\big) \geq -\big(q + 2\varepsilon c_2 c_1^{-1}\tilde{\delta}\big)T\tilde{\eta}(T).$$

Combining this with (4.5) and (4.4) gives

$$\limsup_{T\to\infty}\frac{R_\varepsilon(T)}{T\tilde{\eta}(T)} \leq \big(q + 2\varepsilon c_2 c_1^{-1}\tilde{\delta}\big) < \infty.$$

The claim now follows from (4.3).

5. Proof of Corollary 1 and Proposition 1

Proof of Corollary 1.

(i) This follows directly from Theorems 2 and 3.

For the proof of the next two parts, let η be as in part (i). Then, part (i) states that $R_\varepsilon(T) \asymp T\eta(T)$.

(ii) Lemma 1(i), shows that $\int_1^\infty G^{-1/d}(v)\,\mathrm{d}v < \infty$ if and only if $\int_1^\infty \tilde{G}^{-1/d}(v)\,\mathrm{d}v < \infty$. Assume that $\int_1^\infty \tilde{G}^{-1/d}(v)\,\mathrm{d}v < \infty$. Then clearly, $J(z) \asymp z^{1/d}$. It follows that $\eta^{1/d}(T) \asymp J(\eta(T)) \asymp T$. This proves that $R_\varepsilon(T) \asymp T^{1+d}$. Suppose now that $\int_1^\infty \tilde{G}^{-1/d}(v)\,\mathrm{d}v = \infty$. Then, $\eta^{1/d}(T) = \mathrm{o}(J(\eta(T))$ as $T \to \infty$. Therefore $\eta(T) = \mathrm{o}(T^d)$ and we have $R_\varepsilon(T) = \mathrm{o}(T^{1+d})$.

(iii) Assume that $\alpha < d$. Then clearly $\int_1^\infty \tilde{G}^{-1/d}(v)\,\mathrm{d}v = \infty$. Furthermore,

$$J\big(\tilde{G}(x)\big) \underset{\text{Lemma 1(iv)}}{\asymp} \tilde{G}^{\mathrm{inv}}\big(\tilde{G}(x)\big) \underset{\text{Lemma 1(iii)}}{\sim} x.$$

Therefore it follows from Theorems 2 and 3 that $R_\varepsilon(T) \asymp T\tilde{G}(T) \sim TG(T)$, where we have used Lemma 1(i) for the second equivalence.

Assume now that $\alpha \geq d$. When $\alpha = d$, then by Lemma 1(iv), $\tilde{G}^{\mathrm{inv}}(\eta(x)) = \mathrm{o}(J(\eta(x))) = \mathrm{o}(x)$, as $x \to \infty$. It follows from the definition of a regularly varying function and Lemma 1(i), that $\tilde{G}(\tilde{G}^{\mathrm{inv}}(\eta(x))) = \tilde{G}(\mathrm{o}(x)) = \mathrm{o}(\tilde{G}(x))$. However, by Lemma 1(iii), $\tilde{G}(\tilde{G}^{\mathrm{inv}}(\eta(x)) \sim \eta(x)$. Hence, $\eta(T) = \mathrm{o}(\tilde{G}(T))$, as $T \to \infty$, and we have $R_\varepsilon(T) = \mathrm{o}(T\tilde{G}(T)) = \mathrm{o}(TG(T))$. Finally, when $\alpha > d$, $\int_1^\infty \tilde{G}^{-1/d}(v)\,\mathrm{d}v < \infty$. Therefore by part (ii) $R_\varepsilon(T) \asymp T^{1+d}$. In addition, there exists $\mu \in (0, \alpha - d]$ such that $G(T) \geq T^{d+\mu}$ for all T sufficiently large. Therefore, $T^d = \mathrm{o}(G(T))$ and it follows that $R_\varepsilon(T) = \mathrm{o}(TG(T))$. □

Proof of Proposition 1. Clearly, $Z(T) \geq Z^W(T)$, where $Z^W(T)$ was defined in (3.2). Hence it follows from Lemma 5 that $\liminf_{T\to\infty} R_\varepsilon(T)/T > 0$. Let A be the event $\{V(0,0) < -2\varepsilon T\}$. By the assumption on the negative tail of $V(0,0)$ there exists a constant c, independent of T or ε such that $Q(A) \geq \mathrm{e}^{-c\varepsilon T}$ for all large T. For $x \in \mathbb{Z}^d$, let $Z_x(T) = E_x(\mathrm{e}^{H(1,T)})$. Let $B = \bigcap_{|x|=1}\{Z_x(T) \leq \mathrm{e}^{(\lambda+\varepsilon)T}\}$. By Theorem 1(i), $\lim_{T\to\infty} Q(B) = 1$. On $A \cap B$ we have $Z(T) \leq \mathrm{e}^{(\lambda-\varepsilon)T}$. Since A and B are independent, for all large T we have $R_\varepsilon(T) \leq -\ln Q(A \cap B) \leq 2c\varepsilon T$. □

Appendix

In this section we will work under ASI′. Furthermore, all proofs below are carried out in one dimension, the extension to higher dimensions being immediate.

It follows from the analyticity of the moment generating function of $-V(0,0)$ near the origin and Taylor's expansion that

$$E^Q\bigl(\mathrm{e}^{-\beta V(0,0)}\bigr)=1+\frac{\beta^2}{2}E^Q\bigl(V(0,0)^2\bigr)+\mathrm{o}\bigl(\beta^2\bigr)\quad \text{as } \beta\to 0.$$

Letting $\sigma^2=E^Q(V(0,0)^2)$, we conclude that there exists $\beta_1>0$ such that for all $|\beta|\le\beta_1$

$$E^Q\bigl(\mathrm{e}^{-\beta V(0,0)}\bigr)\le 1+\beta^2\sigma^2\le \mathrm{e}^{\beta^2\sigma^2}. \tag{A.1}$$

For nonnegative integers L and W, let

$$B_{L,W}=\Bigl\{\gamma\colon \max_{s\in\{0,\ldots,L\}}\bigl|\gamma(s)-\gamma(0)\bigr|\le W \text{ and } \gamma(L)=\gamma(0)\Bigr\}.$$

For the proof of Lemma 5 we build on the following:

Lemma 6. *Let ASI′ hold. Let $\varepsilon>0$ and let $\delta\in(0,1)$. Then there exist constants $L,W\in\mathbb{Z}_+\backslash\{0\}$ depending on ε,δ and the weight distribution such that*

$$Q\bigl(E_0\bigl(\mathrm{e}^{H(L)}\mathbf{1}_{B_{L,W}}(\gamma)\bigr)\ge \mathrm{e}^{(\lambda-\varepsilon)L}\bigr)\ge 1-\delta.$$

The ratio L/W can be chosen to be arbitrarily large.

Proof. For convenience, we let $\tilde\varepsilon=\varepsilon/4$. We also choose $N\in\mathbb{Z}_+\backslash\{0\}$ such that

$$\tilde\varepsilon N\ge(\lambda-4\tilde\varepsilon)\quad\text{and let}\quad \tilde\delta=\frac{\delta}{2N}. \tag{A.2}$$

For $t\in\mathbb{Z}_+$, let

$$U_+(t)=E_0\bigl(\mathrm{e}^{H_\gamma(t)}\mathbf{1}_{\{\gamma(t)\ge 0\}}\bigr)\quad\text{and}\quad U_-(t)=E_0\bigl(\mathrm{e}^{H_\gamma(t)}\mathbf{1}_{\{\gamma(t)\le 0\}}\bigr).$$

Then $U_+(t)$ and $U_-(t)$ are identically distributed non-decreasing functions of the IID field V. Furthermore, $Z(t)\le U_+(t)+U_-(t)$. It follows that for $a\ge 0$,

$$\begin{aligned}Q\bigl(Z(T)\ge 2a\bigr)&\le Q\bigl(U_+(t)+U_-(t)\ge 2a\bigr)\le Q\bigl(U_+(t)\ge a \text{ or } U_-(t)\ge a\bigr)\\ &\underset{\text{inclusion exclusion}}{=}2Q\bigl(U_+(t)\ge a\bigr)-Q\bigl(U_+(t)\ge a,U_-(t)\ge a\bigr)\\ &\underset{\text{FKG}}{\le}Q\bigl(U_+(t)\ge a\bigr)\bigl(2-Q\bigl(U_+(t)\ge a\bigr)\bigr).\end{aligned} \tag{A.3}$$

By Theorem 1(i), there exists $T_0\in\mathbb{Z}_+\backslash\{0\}$ such that if $T\ge T_0$ then

$$Q\bigl(Z(T)>2\mathrm{e}^{(\lambda-\tilde\varepsilon)T}\bigr)\ge 1-\tilde\delta^2.$$

Therefore letting $a=\mathrm{e}^{(\lambda-\tilde\varepsilon)T}$ and solving the quadratic equation in (A.3) we have

$$Q\bigl(U_+(T)\ge \mathrm{e}^{(\lambda-\tilde\varepsilon)T}\bigr)\ge 1-\tilde\delta\quad\text{for } T\ge T_0. \tag{A.4}$$

Below we adopt the notation introduced in (3.6). From now on, we assume $T \geq T_0$. Define the function sgn: $\mathbb{R} \to \{-1, +1\}$ by letting Let $\operatorname{sgn}(x) = 1$ if and only if $x \geq 0$. Set $x_0^* = 0$. We continue inductively. For $n \in \mathbb{Z}_+$, let

$$A_{n+1} = \{x \in \mathbb{Z}\colon \operatorname{sgn}(x_n^*)(x - x_n^*) \leq 0\} \quad \text{and} \quad v_{n+1} = \max_{x \in A_{n+1}} E_{x_n^*}\left(\mathrm{e}^{H(nT,(n+1)T)}\mathbf{1}_{\{\gamma(T)=x\}}\right).$$

With these, we define x_{n+1}^* through

$$x_{n+1}^* = \min\left\{x \in A_{n+1}\colon E_{x_n^*}\left(\mathrm{e}^{H(nT,(n+1)T)}\mathbf{1}_{\{\gamma(T)=x\}}\right) = v_{n+1}\right\}. \tag{A.5}$$

(In higher dimensions, we will choose x_{n+1}^* to be the minimal with respect to some prescribed well-ordering on $\mathbb{Z}^d$.)

The definition of A_{n+1} guarantees that if $x_n^* \geq 0$, then $x_{n+1}^* \leq x_n^*$, and if $x_n^* < 0$, then $x_{n+1}^* \geq x_n^*$. In addition, since $|x_{n+1}^* - x_n^*| \leq T$, it follows that for all n, $|x_n^*| \leq T$. As an immediate consequence, we note that all paths considered in the expectation on the righthand side of (A.5) satisfy $|\gamma(s)| \leq T + \frac{1}{2}T \leq 2T$ for all $s \leq T$. We also observe that $E_0(\mathrm{e}^{H(T)}\mathbf{1}_{\{\gamma(T)=x_1^*\}})$ stochastically dominates $\frac{1}{1+T}U_+(T) \geq \mathrm{e}^{-\tilde{\varepsilon}T}U_+(T)$. Therefore

$$Q\left(E_0\left(\mathrm{e}^{H(T)}\mathbf{1}_{\{\gamma(T)=x_1^*\}}\right) > \mathrm{e}^{(\lambda-2\tilde{\varepsilon})T}\right) \geq Q\left(\mathrm{e}^{-\tilde{\varepsilon}T}U_+(T) \geq \mathrm{e}^{(\lambda-2\tilde{\varepsilon})T}\right) \underset{(\text{A.4})}{\geq} 1 - \tilde{\delta}. \tag{A.6}$$

Next, we note that for every $l \in \mathbb{Z}$, if $Q(\{x_n^* = l\}) > 0$, then the random variable $E_{x_n^*}(\mathrm{e}^{H(nT,(n+1)T)}\mathbf{1}_{\{\gamma(T)=x_{n+1}^*\}})$ under $Q(\cdot|\{x_n^* = l\})$ has the same distribution as $E_0(\mathrm{e}^{H(T)}\mathbf{1}_{\{\gamma(T)=x_1^*\}})$ under Q. It easily follows that the random variables $\{E_{x_n^*}(\mathrm{e}^{H(nT,(n+1)T)}\mathbf{1}_{\{\gamma(T)=x_{n+1}^*\}})\colon n \in \mathbb{Z}_+\}$ are identically distributed. Thus,

$$\begin{aligned} Z_N &:= E_0\left(\mathrm{e}^{H(NT)}\prod_{n=1}^{N}\mathbf{1}_{\{\gamma(nT)=x_n^*\}}\right) \\ &\underset{\text{Markov property}}{=} \left(\prod_{n=0}^{N-1} E_{x_n^*}\left(\mathrm{e}^{H(nT,(n+1)T)}\mathbf{1}_{\{\gamma(T)=x_{n+1}^*\}}\right)\right). \end{aligned}$$

Observing that

$$\begin{aligned} Q\left(\bigcup_{n=0}^{N-1}\left\{E_{x_n^*}\left(\mathrm{e}^{H(nT,(n+1)T)}\mathbf{1}_{\{\gamma(T)=x_{n+1}^*\}}\right) < \mathrm{e}^{(\lambda-2\tilde{\varepsilon})T}\right\}\right) &\leq NQ\left(E_0\left(\mathrm{e}^{H(T)}\mathbf{1}_{\{\gamma(T)=x_1^*\}}\right) \leq \mathrm{e}^{(\lambda-2\tilde{\varepsilon})T}\right) \\ &\underset{(\text{A.6})}{\leq} N\tilde{\delta}, \end{aligned}$$

we obtain

$$Q\left(Z_N > \mathrm{e}^{(\lambda-2\tilde{\varepsilon})NT}\right) \geq 1 - N\tilde{\delta}. \tag{A.7}$$

Let $C_T = \{-T, -T+2, \ldots, T-2, T\}$. Since T is even and $|x_N^*| \leq T$, $x_N^* \in C_T$. For every $z \in C_T$ let γ_z denote some path such that $\gamma_z(0) = z$ and $\gamma_z(T) = 0$ and let $W_z = 2^{-T}\mathrm{e}^{H_{\gamma_z}(NT,(N+1)T)}$. Then

$$E_0\left(\mathrm{e}^{H((N+1)T)}\mathbf{1}_{B_{(N+1)T,2T}}(\gamma)\right) \geq Z_N \min_{z \in C_T} W_z. \tag{A.8}$$

The distribution of W_z is the same as the distribution of $2^{-T}\mathrm{e}^{\sum_{s=0}^{T-1}V(s,0)}$. Therefore letting $c_0 = \tilde{\varepsilon}N - \ln 2$, we get

$$Q\left(\min_{z \in C_T} W_z \leq \mathrm{e}^{-\tilde{\varepsilon}NT}\right) \leq (1+T)Q\left(\sum_{s=0}^{T-1}V(s,0) \leq -c_0 T\right). \tag{A.9}$$

Choose $\beta \in (0, \beta_1)$, where $\beta_1 > 0$ is the constant from (A.1). We will further assume that $\sigma^2\beta < c_0$. We let $c_2 = \beta(c_0 - \sigma^2\beta) > 0$. Then by Chebyshev and (A.1)

$$Q\left(-\sum_{s=0}^{T-1} V(s,0) \geq c_0 T\right) \leq \mathrm{e}^{\beta^2\sigma^2 T}\mathrm{e}^{-\beta c_0 T} = \mathrm{e}^{-c_2 T}.$$

By (A.9),

$$Q\Bigl(\min_{z\in C_T} W_z \leq \mathrm{e}^{-\tilde{\varepsilon}NT}\Bigr) \leq \mathrm{e}^{-(c_2/2)T},$$

for sufficiently large T. We are free to choose $T \in 2\mathbb{Z}_+ \cap [T_0, \infty)$, and we choose T satisfying $\mathrm{e}^{-(c_2/2)T} \leq N\tilde{\delta}$. From (A.8) and (A.7) we get

$$Q\bigl(E_0\bigl(\mathrm{e}^{H((N+1)T)}\mathbf{1}_{B_{(N+1)T,2T}}(\gamma)\bigr) \geq \mathrm{e}^{(\lambda-3\tilde{\varepsilon})NT}\bigr) \geq 1 - N\tilde{\delta} - \mathrm{e}^{-(c_2/2)T} \geq 1 - 2N\tilde{\delta} \geq 1 - \delta.$$

Furthermore,

$$(\lambda - 3\tilde{\varepsilon})NT = (\lambda - 4\tilde{\varepsilon})(N+1)T - (\lambda - 4\tilde{\varepsilon})T + \tilde{\varepsilon}NT \underset{(A.2)}{\geq} (\lambda - \varepsilon)(N+1)T.$$

Setting $L = (N+1)T$ and $W = 2T$ we have

$$Q\bigl(E_0\bigl(\mathrm{e}^{H(L)}\mathbf{1}_{B_{L,W}}(\gamma)\bigr) \geq \mathrm{e}^{(\lambda-\varepsilon)L}\bigr) \geq 1 - \delta.$$

This completes the proof, because N can be arbitrarily large. □

Proof of Lemma 5. Let $\tilde{\varepsilon} = \varepsilon/3$. For every $\delta \in \mathbb{R}$

$$(\lambda - \tilde{\varepsilon})(1 - 2\delta) = (\lambda - 2\tilde{\varepsilon}) - (\lambda - 2\tilde{\varepsilon})2\delta + \tilde{\varepsilon}(1 - 2\delta),$$

so we may choose $\delta \in (0, 1)$ such that

$$\tilde{\varepsilon} > \delta \ln 4, \quad (\lambda - \tilde{\varepsilon})(1 - 2\delta) \geq (\lambda - 2\tilde{\varepsilon}). \tag{A.10}$$

We let

$$\beta = \frac{1}{2}\min\left(\beta_1, \frac{\tilde{\varepsilon} - \delta\ln 4}{2\delta\sigma^2}\right) > 0 \quad \text{and} \quad c_1 = \beta\bigl(\tilde{\varepsilon} - \delta\ln 4 - 2\beta\delta\sigma^2\bigr), \tag{A.11}$$

where β_1 and σ^2 are the constants appearing on (A.1). Let W and L the constants from Lemma 6 corresponding to $\tilde{\varepsilon}$ and δ. The lemma states that may choose L to satisfy

$$L \geq \frac{\ln 4}{c_1}. \tag{A.12}$$

For $k \in \mathbb{Z}_+$, let

$$X_k = E_0\bigl(\mathrm{e}^{(H(kL,(k+1)L))}\mathbf{1}_{B_{L,W}}(\gamma)\bigr),$$

where $H(kL, (k+1)L)$ is defined as in (3.6). Then, by Lemma 6,

$$Q\bigl(X_k \geq \mathrm{e}^{(\lambda-\tilde{\varepsilon})L}\bigr) \geq 1 - \delta. \tag{A.13}$$

For $n \in \mathbb{Z}_+ \backslash \{0\}$ let

$$A_n = \bigcup_{B \subset \{0,\dots,n-1\}, |B| \le 2\delta n} \left\{ \prod_{k \in B} X_k \le \mathrm{e}^{-\tilde{\varepsilon} n L} \right\} \quad \text{and}$$

$$C_n = \bigcup_{G \subset \{0,\dots,n-1\}, |G| \ge (1-2\delta) n} \bigcap_{k \in G} \{ X_k \ge \mathrm{e}^{(\lambda - \tilde{\varepsilon}) L} \}. \tag{A.14}$$

On $A_n^c \cap C_n$

$$\prod_{k=0}^{n-1} X_k \ge \mathrm{e}^{(\lambda - \tilde{\varepsilon})(1-2\delta) n L} \mathrm{e}^{-\tilde{\varepsilon} n L} \underset{(A.10)}{\ge} \mathrm{e}^{(\lambda - 3\tilde{\varepsilon}) n L} = \mathrm{e}^{(\lambda - \varepsilon) n L}.$$

Since

$$E_0\big(\mathrm{e}^{H(nL)} \mathbf{1}_{B(nL,W)}(\gamma)\big) \ge \prod_{k=0}^{n-1} X_k,$$

it follows that

$$Q\big(E_0\big(\mathrm{e}^{H(nL)} \mathbf{1}_{B(nL,W)}(\gamma)\big) \ge \mathrm{e}^{(\lambda - 3\tilde{\varepsilon}) n L}\big) \ge Q\big(A_n^c \cap C_n\big) \ge Q\big(A_n^c\big) Q(C_n), \tag{A.15}$$

where the last inequality is due to the FKG inequality applied to the events A_n^c and C_n which are non-decreasing with respect to the IID field V.

Observe that C_n is the event that the number of successes in n IID Bernoulli trials is at least $(1 - 2\delta)n$, where a success in the kth trial is the event $\{X_k \ge \mathrm{e}^{(\lambda - \tilde{\varepsilon})L}\}$. By (A.13), the probability of the latter event is bounded below by $1 - \delta$. Therefore, standard large deviations estimates for IID Bernoulli sequences guarantee the existence of a constant $c_2 = c_2(\delta) > 0$, such that

$$Q(C_n) \ge 1 - \mathrm{e}^{-c_2 n}. \tag{A.16}$$

Next we estimate $Q(A_n)$. Choose any path γ satisfying $\gamma(kL) = 0$ for all $k \in \mathbb{Z}_+$. Then clearly,

$$X_k \ge 2^{-L} \mathrm{e}^{\sum_{s=kL}^{(k+1)L-1} V(s,\gamma(s))}.$$

For every $B \subset \{0, \dots, n-1\}$ with $|B| \le 2\delta n$

$$\begin{aligned}
\left\{ \prod_{k \in B} X_k < \mathrm{e}^{-\tilde{\varepsilon} n L} \right\} &\subset \left\{ \sum_{k \in B} \left(-L \ln 2 + \sum_{s=kL}^{(k+1)L-1} V\big(s, \gamma(s)\big) \right) \le -\tilde{\varepsilon} n L \right\} \\
&= \left\{ -\sum_{k \in B} \sum_{s=kL}^{(k+1)L-1} V\big(s, \gamma(s)\big) \ge \tilde{\varepsilon} n L - |B| L \ln 2 \right\} \\
&\subset \left\{ -\sum_{k \in B} \sum_{s=kL}^{(k+1)L-1} V\big(s, \gamma(s)\big) \ge (\tilde{\varepsilon} - \delta \ln 4) n L \right\}.
\end{aligned}$$

Therefore,

$$Q\left(\left\{ \prod_{k \in B} X_k < \mathrm{e}^{-\tilde{\varepsilon} n L} \right\} \right) \le Q(E_{|B|}), \quad \text{where } E_{|B|} = \left\{ -\sum_{s=0}^{L|B|-1} V(s, 0) \ge (\tilde{\varepsilon} - \delta \ln 4) n L \right\} \tag{A.17}$$

and then

$$
\begin{aligned}
Q(E_{|B|}) &\underset{\text{Chebyshev}}{\leq} E^Q\left(e^{-\beta V(0,0)}\right)^{L|B|} e^{-\beta(\tilde{\varepsilon}-\delta\ln 4)nL} \\
&\underset{E^Q(e^{-\beta V(0,0)})\geq 1}{\leq} E^Q\left(e^{-\beta V(0,0)}\right)^{2\delta nL} e^{-\beta(\tilde{\varepsilon}-\delta\ln 4)nL} \\
&\underset{(A.11),(A.1)}{\leq} e^{\beta^2\sigma^2 2\delta nL-\beta(\tilde{\varepsilon}-\delta\ln 4)} \underset{(A.11)}{=} e^{-c_1 nL}.
\end{aligned}
$$

Thus, by (A.14) and (A.17)

$$
Q(A_n) \leq \sum_{B\subset\{0,\ldots,n-1\},|B|\leq 2\delta n} Q(E_{|B|}) \leq 2^n e^{-c_1 nL} \underset{(A.12)}{\leq} e^{-n\ln 2}.
$$

With (A.16) we obtain

$$
Q\left(A_n^c\right)Q(C_n) \geq \left(1-e^{-n\ln 2}\right)\left(1-e^{-c_2 n}\right) \geq 1-2e^{-\min(\ln 2,c_2)n},
$$

for all n sufficiently large. This estimate and (A.15) prove the lemma for all T of the form nL for large n. The extension to all large T is simple and we omit the details. $\square$

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