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FUNCTIONAL QUANTIZATION RATE AND MEAN REGULARITY OF PROCESSES WITH AN APPLICATION TO LÉVY PROCESSES

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We investigate the connections between the mean pathwise regularity of stochastic processes and their $L^r(\mathbb{P})$-functional quantization rates as random variables taking values in some $L^p([0,T],dt)$-spaces ($0 < p \le r$). Our main tool is the Haar basis. We then emphasize that the derived functional quantization rate may be optimal (e.g., for Brownian motion or symmetric stable processes) so that the rate is optimal as a universal upper bound. As a first application, we establish the $O((\log N)^{-1/2})$ upper bound for general Itô processes which include multidimensional diffusions. Then, we focus on the specific family of Lévy processes for which we derive a general quantization rate based on the regular variation properties of its Lévy measure at 0. The case of compound Poisson processes, which appear as degenerate in the former approach, is studied specifically: we observe some rates which are between the finite-dimensional and infinite-dimensional "usual" rates.

1. Introduction. In this paper, we investigate the connection between the functional $L^r(\mathbb{P})$-quantization rate for a process $X = (X_t)_{t\in[0,T]}$ and the $L^r(\mathbb{P})$-mean pathwise regularity of the mapping $t \mapsto X_t$ from $[0,T] \to L^r(\mathbb{P})$ in an abstract setting by means of a constructive approach (we mean that all the rates are established using some explicit sequences of quantizers).

First, let us briefly recall what functional quantization is and how it was introduced. Let $(E, \|\cdot\|)$ denote a finite-dimensional ($E = \mathbb{R}$ or $\mathbb{R}^d$) or infinite-dimensional ($E = L^p([0,T],dt)$, $1 \le p < \infty$, $\mathcal{C}([0,T]),\dots$) separable Banach space (or complete quasi-normed space like $E = L^p([0,T],dt)$, $0 < p < 1$) and let $\alpha \subset E$ be a finite subset of size $\mathrm{card}(\alpha) \le N$, $N \ge 1$. The *Voronoi quantization* of an E-valued random vector $X : (\Omega, \mathcal{A}, \mathbb{P}) \to E$ *with respect to the codebook* α is simply the projection of X onto α following the nearest neighbor rule, that is

$$\widehat{X}^{\alpha} = \pi_\alpha(X),$$

where

$$\pi_\alpha = \sum_{a\in\alpha} a \mathbf{1}_{C_a(\alpha)},$$

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$(C_a(\alpha))_{a\in\alpha}$ being a Borel partition of E satisfying, for every $a \in \alpha$,

$$C_a(\alpha) \subset \left\{ u \in E : \|u - a\| \leq \min_{b\in\alpha\setminus\{a\}} \|u - b\| \right\}.$$

Then, the L^r-mean quantization error ($0 < r < \infty$) is defined by

$$\|X - \widehat{X}^\alpha\|_{L^r_E(\mathbb{P})} = \left(\mathbb{E} \min_{a\in\alpha} \|X - a\|^r\right)^{1/r}.$$

This quantity is finite provided $X \in L^r_E(\mathbb{P})$. The set α is called an *N-codebook* or *N-quantizer*. It can be shown that such random vectors $\widehat{X}^\alpha$ are the best approximation of X among all α-valued random vectors. The minimal Nth quantization error of X is then defined by

$$e_{N,r}(X, E) := \inf\left\{ \left(\mathbb{E} \min_{a\in\alpha} \|X - a\|^r\right)^{1/r} : \alpha \subset E, \operatorname{card}(\alpha) \leq N \right\}. \tag{1.1}$$

When $E = L^p([0, T], dt)$ (with its usual norm or quasi-norm denoted by $|\cdot|_{L^p_T}$ from now on), an E-valued random variable X is a (bimeasurable) stochastic process $X = (X_t)_{t\in[0,T]}$ defined on the probability space $(\Omega, \mathcal{A}, \mathbb{P})$ whose trajectories $(X_t(\omega))_{0\leq t\leq T}$ (almost) all belong to $L^p([0, T], dt)$. The L^r-integrability assumption then reads

$$\mathbb{E}\left(\left(\int_0^T |X_t|^p \, dt\right)^{r/p}\right) < +\infty.$$

It is still an open question whether L^r-optimal N-quantizers for Gaussian random vectors always exist in an abstract Banach space setting (see [15]) . However, in many situations of interest for processes, including all the $L^p([0, T], dt)$-spaces, $1 \leq p < +\infty$, the existence of at least one such L^r-optimal codebook has been established (provided $\mathbb{E}\|X\|^r < +\infty$). Note, however, that this is not the case for the space $\mathcal{C}([0, T])$ of continuous functions. For more details on the existence problem for optimal quantizers, we refer to [15].

On the other hand, optimal L^r-quantizers always exist when $E = \mathbb{R}^d$, $d \geq 1$. In this finite-dimensional setting, this problem is known as *optimal vector quantization* and has been extensively investigated since the early 1950s with some applications to signal processing and transmission (see [11] or [12]).

In d-dimensions, the convergence rate of $e_{N,r}$ is given by the so-called Zador theorem,

$$\lim_N N^{1/d} e_{N,r}(X, \mathbb{R}^d) = \tilde{J}_{r,d} \left(\int_{\mathbb{R}^d} g^{d/(d+r)}(\xi) \, d\xi\right)^{1/r+1/d}, \tag{1.2}$$

where g denotes the density of the absolutely continuous part of the distribution $\mathbb{P}_X$ of X and $\tilde{J}_{r,d} \in (0, \infty)$ (see [13]).

Since the early 2000's, much attention has been paid to the infinite-dimensional case. This is the so-called *functional quantization* problem for stochastic processes:

the aim is to quantize some processes viewed as random vectors taking values in their path spaces, supposed to be $L^p([0,T],dt)$ spaces, $1 \le p < +\infty$. Many results have been obtained for several families of processes with special attention having been paid to Gaussian processes and (Brownian) diffusion processes by several authors. Thus, in the purely Hilbert space setting ($r=2$, $E = L^2([0,T],dt)$), the sharp rate of quantization of the Brownian motion $(W_t)_{t\in[0,T]}$ is given (see (3.6) in [19]) by

$$e_{N,2}(W, L^2([0,T],dt)) \sim \frac{\sqrt{2}\,T}{\pi(\log N)^{1/2}}. \tag{1.3}$$

The existence of such a *sharp rate* for Brownian motion has been extended to $L^p([0,T],dt)$ spaces for $1 \le p \le \infty$ (see [8]). Similar sharp rates (with an explicit constant) hold for a wide class of Gaussian processes, including the fractional Brownian motions for which we have

$$e_{N,2}(W^H, L^2([0,T],dt)) \sim \frac{c(H,T)}{(\log N)^H},$$

where H denotes the Hurst parameter of the fractional Brownian motion W^H, the Ornstein–Uhlenbeck process, the Brownian sheet, and so on, in the purely Hilbert space setting (see [19]). The exact rate has also been established in [18] (Section 3) for a wider class of Gaussian processes. In [18, 19], these results are based on the (sharp or exact) asymptotic behavior of the eigenvalues of high order of the Karhunen–Loève expansion of the Gaussian process. As a byproduct, this approach provides very simple explicit sequences of rate-optimal asymptotic quantizers (provided that the Karhunen–Loève expansion of the process itself is accessible). Their numerical implementation has lead to some unexpectedly promising numerical applications in finance, especially for the pricing of path-dependent options like Asian options in several popular models of asset dynamics (Black–Scholes, stochastic volatility Heston and SABR models, etc.). For these aspects, we refer to [22] or [29]. We also mention applications of quantization to statistical clustering of data (see, e.g., [23]) and some more recent developments concerning functional data investigated in [27] and [28].

For Gaussian processes, an important connection with the small ball probability problem has been made (see [6, 14]). Some exact or sharp rates of convergence for different classes of Brownian diffusions have also recently been proven (see [7, 20]) with a rate driven by $(\log N)^{-1/2}$.

The common feature shared by all these results is that there is a one-to-one correspondence between the exponent a that controls the $(L^r(\mathbb{P}), L^p(dt))$-quantization rate of these processes in the $\log(N)^{-a}$ scale and their mean pathwise regularity, that is, the largest exponent a that satisfies

$$\forall s,\, t \in [0,T] \qquad \|X_t - X_s\|_{L^r(\mathbb{P})} \le C_r|t-s|^a. \tag{1.4}$$

Although such a correspondence is not really surprising given the connection between quantization rate and small ball probabilities in the Gaussian setting, this naturally leads to an attempt to derive a general abstract result that connects these two features of a process. This is the aim of Section 2 of this paper, in which we show that the mean pathwise regularity always provides a universal upper bound for the $(L^r(\mathbb{P}), L^p(dt))$-quantization rate $(0 < p \le r)$. We then retrieve the rate obtained by more specific approaches for all the processes mentioned above. We also extend to general Brownian diffusion processes and even general Itô processes the rate formerly obtained for specific classes of diffusions in [7, 20]. We also obtain some first quantization rates for some classes of Lévy processes. The main technique is to expand a process on the simplest wavelet basis—the Haar basis (known to be unconditional when $p > 1$)—and to use a nonasymptotic version of the Zador theorem (a slight improvement of the Pierce lemma; see [13]).

At this point, the next question is to ask conversely whether this always provides the true quantization rate. In this naïve form, the answer to this question is clearly "no" because equation (1.4) only takes into account the mean pathwise Hölder regularity of a process and one can trivially build (see [18]) some processes with smoother mean regularity (like processes with $\mathcal{C}^k$, $k \ge 1$, trajectories). We do not extend our approach in that direction, for the sake of simplicity, but there is no doubt that developing techniques similar to those used in Section 2, one can connect higher order mean pathwise regularity and quantization rate, as in the Hölder setting. This would require an appropriate wavelet basis. In fact, we point out in Section 4, devoted to general Lévy processes, that the answer may be negative—the quantization rate can be infinitely faster than the mean pathwise regularity—for different reasons in connection with the dimensionality of the process: a Poisson process is, in some sense, an almost finite-dimensional random vector which induces a very fast quantization rate which does not take place in the $(\log N)^{-a}$ scale, although the mean pathwise $L^r(\mathbb{P})$-regularity of a Poisson process is Hölder [and depends on r; see, e.g., (3.7) and (3.8)]. Conversely, we emphasize, via on several classes of examples, that the upper bound derived from mean regularity provides the true rate of quantization. This follows from a comparison with the lower bound that can be derived from small deviation results (see, e.g., [14] or the remark below Theorem 1 which elucidates the connection between functional quantization and small deviation theory). Thus, we prove that our approach yields the exact rate for a wide class of subordinated Lévy processes (including symmetric α-stable processes).

The main result of Section 4 is Theorem 2, which provides a functional quantization rate for a general Lévy process X having no Brownian component: this rate is controlled by the behavior of the Lévy measure ν around 0 (e.g., the index of X for a stable process). As an example for Lévy processes which have infinitely many small jumps, if the (infinite) Lévy measure ν (is locally absolutely continuous around 0 and) satisfies

$$\exists\, c > 0 \qquad \mathbf{1}_{\{0<|x|\le c\}}\nu(dx) \le \frac{C}{|x|^{\underline{\theta}+1}} \mathbf{1}_{\{0<|x|\le c\}}\, dx$$

for some $\underline{\theta} \in (0, 2]$, then, for every $p, r \in (0, \underline{\theta}]$ such that $0 < p \le r$ and $X_1 \in L^r(\mathbb{P})$,

$$e_{N,r}(X, L^p([0, T], dt)) = O((\log N)^{-1/\underline{\theta}}).$$

This makes a connection between quantization rate and the Blumenthal–Getoor index β of X when ν satisfies the above upper bound with $\underline{\theta} = \beta$. In fact, a more general result is established in Theorem 2: when the "0-tail function" $\underline{\nu} : x \mapsto \nu([-x, x]^c)$ has regular variation as x goes to 0, with index $-\underline{\theta}$, then $\underline{\theta} = \beta$ (see [5]) and we establish a close connection between the quantization rate of X and $\underline{\nu}$, $\underline{\theta}$. In many cases of interest, including α-stable processes and other classes of subordinated Lévy processes, we show that this general upper bound provides the exact rate of quantization; it matches the lower bound estimates derived from the connection between quantization rate and small deviation estimates (see, e.g., [14]). When the Lévy process does have a Brownian component, its exact quantization rate is $(\log N)^{-1/2}$, like Brownian motion [when $0 < p < r < 2$, $X_1 \in L^r(\mathbb{P})$].

When the Lévy measure is finite (then $\underline{\theta} = 0$), we also establish some quantization rates for the compound Poisson processes and show they are infinitely faster than the above ones. To this end, we design an explicit sequence of quantizers which can clearly be implemented for numerical purposes. In fact, the whole proof is constructive, provided the Lévy measure is "tractable" enough.

The paper is organized as follows. Section 2 is devoted to the abstract connection between mean regularity and quantization rate of processes. Section 3 is devoted to some initial applications to various families of processes. As far as we know, some of these rates are new. In several cases of interest, these rates are shown to be optimal. The main result is Theorem 1. Section 4 provides an upper bound for the quantization rate of general Lévy process in connection with the behavior of the Lévy measure around 0. The main results are Theorem 2 and Proposition 3. In Section 5.1, we provide the exact rate for a Lévy process having a Brownian component. Finally, in Section 5.2, we derive the exact quantization rate for subordinated Lévy processes.

NOTATION.

- $L^p_T := L^p([0, T], dt)$ and $|f|_{L^p_T} = (\int_0^T |f(t)|^p \, dt)^{1/p}$.
- Let $(a_n)_{n \ge 0}$ and $(b_n)_{n \ge 0}$ be two sequences of positive real numbers. $a_n \sim b_n$ means $a_n = b_n + o(b_n)$ and $a_n \approx b_n$ means $a_n = O(b_n)$ and $b_n = O(a_n)$.
- $[x]$ denotes the integral part of the real number x and $x_+ = \max(x, 0)$ its positive part.
- $\log_m(x)$ denotes the m-times iterated logarithm function.
- $\|Y\|_r := \|Y\|_{L^r(\mathbb{P})}$ for any random variable Y defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$.

- Throughout the paper, the letter C (possibly with subscripts) will denote a positive real constant that may vary from line to line.
- For a càdlàg continuous-time process $X=(X_t)_{t\geq 0}$, X_{t-} will denote its left limit and $\Delta X_t := X_t - X_{t-}$ its jump at time t.

2. Mean pathwise regularity and quantization error rate: an upper bound. In this section, we derive in full generality an upper bound for the $(L^r(\mathbb{P}), L^p_T)$-quantization error $e_{N,r}(X, L^p_T)$ based on the path regularity of the mapping $t\mapsto X_t$ from $[0,T]$ to $L^\rho(\mathbb{P})$. The main result of this section is Theorem 1 below. We will then illustrate via several examples that this rate may be optimal or not.

As a first step, we will reformulate the so-called Pierce lemma (see [13], page 82), which is the main step of the proof of Zador's Theorem for unbounded random variables. Note that the proof of its original formulation (see below) relies on random quantization.

LEMMA 1 (Extended Pierce Lemma). *Let $r,\delta>0$. There exists a real constant $C_{r,\delta}$ such that, for every random variable $X:(\Omega,\mathcal{A})\to(\mathbb{R},\mathbb{B}(\mathbb{R}))$,*

$$\forall N\geq 1 \qquad e_{N,r}(X,\mathbb{R}) = \inf_{\operatorname{card}(\alpha)\leq N}\|X-\widehat{X}^{\alpha}\|_r \leq C_{r,\delta}\|X\|_{r+\delta}\,N^{-1}.$$

PROOF. It follows from the original Pierce lemma that there exists a universal real constant $C^0_{r,\delta}>0$ and an integer $N_{r,\delta}\geq 1$ such that, for any random variable $X:(\Omega,\mathcal{A})\to(\mathbb{R},\mathbb{B}(\mathbb{R}))$,

$$\forall N\geq N_{r,\delta} \qquad \inf_{\operatorname{card}(\alpha)\leq N}\mathbb{E}|X-\widehat{X}^{\alpha}|^r \leq C^0_{r,\delta}(1+\mathbb{E}|X|^{r+\delta})\,N^{-r}.$$

Using the scaling property of quantization, for every $\lambda>0$,

$$\|X-\widehat{X}^{\alpha}\|_r = \frac{1}{\lambda}\|(\lambda X)-\widehat{\lambda X}^{\lambda\alpha}\|_r,$$

where $\lambda\alpha=\{\lambda a,\ a\in\alpha\}$, one derives from the Pierce lemma, by considering $X/\|X\|_{r+\delta}$ and setting $\lambda := 1/\|X\|_{r+\delta}$, that

$$\forall N\geq N_{r,\delta} \qquad \inf_{\operatorname{card}(\alpha)\leq N}\|X-\widehat{X}^{\alpha}\|_r \leq (2C^0_{r,\delta})^{1/r}\|X\|_{r+\delta}\,N^{-1}.$$

Now, for every $N\in\{1,\dots,N_{r,\delta}-1\}$, setting $\alpha:=\{0\}$ yields

$$\inf_{\operatorname{card}(\alpha)\leq N}\|X-\widehat{X}^{\alpha}\|_r \leq \|X\|_r \leq N_{r,\delta}\|X\|_{r+\delta}\,N^{-1}.$$

Combining the last two inequalities and setting $C_{r,\delta}=\max((2C^0_{r,\delta})^{1/r}, N_{r,\delta})$ completes the proof. □

Let $(e_n)_{n\geq 0}$ denote the Haar basis, defined as the restrictions to $[0,T]$ of the following functions:

$$e_0 := T^{-1/2}\mathbf{1}_{[0,T]}, \qquad e_1 := T^{-1/2}\big(\mathbf{1}_{[0,T/2)}-\mathbf{1}_{[T/2,T]}\big),$$
$$e_{2^n+k} := 2^{n/2}e_1(2^n\cdot - kT), \qquad n\geq 0,\ k\in\{0,\dots,2^n-1\}.$$

With this normalization, it makes up an orthonormal basis of the Hilbert space $(L^2_T, (\cdot|\cdot))$, where $(f|g) = \int_0^T fg(t)\,dt$ and a (monotone) Schauder basis of L^p_T, $p \in [1, +\infty)$, that is, $(f|e_0)e_0 + \sum_{n\geq 0}\sum_{0\leq k\leq 2^n-1}(f|e_{2^n+k})e_{2^n+k}$, converges to f in L^p_T for every $f \in L^p_T$ (see [26]). Furthermore, it clearly satisfies, for every $f \in L^1_T$ and every $p > 0$,

$$\forall n \geq 0 \tag{2.1}$$
$$\int_0^T \left|\sum_{k=0}^{2^n-1}(f|e_{2^n+k})e_{2^n+k}(t)\right|^p dt = 2^{n(p/2-1)}T^{1-p/2}\sum_{k=0}^{2^n-1}|(f|e_{2^n+k})|^p.$$

The second key to establish a general connection between quantization rate and mean pathwise regularity is the following standard property of the Haar basis: for every $f \in L^1_T$,

$$\begin{aligned}
(f|e_{2^n+k}) &= 2^{n/2}T^{-1/2}\left(\int_{kT2^{-n}}^{(2k+1)T2^{-(n+1)}} f(u)\,du - \int_{(2k+1)T2^{-(n+1)}}^{(k+1)T2^{-n}} f(u)\,du\right) \\
&= 2^{n/2}T^{-1/2}\int_0^{T2^{-(n+1)}} \big(f(kT2^{-n}+u) \\
&\qquad - f\big((2k+1)T2^{-(n+1)}+u\big)\big)\,du.
\end{aligned} \tag{2.2}$$

Let $(X_t)_{t\in[0,T]}$ be a bimeasurable process defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with $\mathbb{P}$-almost all paths lying in L^1_T such that $X_t \in L^\rho(\mathbb{P})$ for every $t \in [0, T]$ for some positive real exponent $\rho > 0$. When $\rho \in (0, 1)$, we assume that X has càdlàg paths (right-continuous, left-limited) to ensure the measurability of the supremum in assumption (2.3) below.

We make the following φ-Lipschitz assumption on the map $t \mapsto X_t$ from $[0, T]$ into $L^\rho(\mathbb{P})$: there is a nondecreasing function $\varphi : \mathbb{R}_+ \to [0, +\infty]$, continuous at 0 with $\varphi(0) = 0$, such that

$$(L_{\varphi,\rho}) \equiv \begin{cases}
\text{(i)} & \forall s, t \in [0, T], \\
& \mathbb{E}|X_t - X_s|^\rho \leq (\varphi(|t-s|))^\rho, \qquad \text{if } \rho \geq 1, \\
\text{(ii)} & \forall t \in [0, T],\ \forall h \in (0, T], \\
& \mathbb{E}\Big(\sup_{t\leq s\leq (t+h)\wedge T}|X_s - X_t|^\rho\Big) \leq (\varphi(h))^\rho, \\
& \qquad\qquad\qquad\qquad \text{if } 0 < \rho < 1.
\end{cases} \tag{2.3}$$

[One may assume, without loss of generality, that φ is always finite, but that (i) and (ii) are only true for $|t-s|$ or h small enough, resp.] Note that this assumption implies that $\mathbb{E}(|X|^\rho_{L^\rho_T}) < +\infty$ so that, in particular, $\mathbb{P}(d\omega)$-a.s., $t \mapsto X_t(\omega)$ lies in L^ρ_T (which, in turn, implies that the paths lie in L^1_T if $\rho \geq 1$).

We make a *regularly varying assumption* on φ at 0 with index $b \geq 0$, that is, for every $t > 0$,

$$\lim_{x \to 0} \frac{\varphi(tx)}{\varphi(x)} = t^b. \tag{2.4}$$

In accordance with the literature (see [3]), this means that $x \mapsto \varphi(1/x)$ is regularly varying at infinity with index $-b$ (which is a more usual notion in that field). When $b = 0$, φ is said to be *slowly varying* at 0.

Let $r,\ p \in (0, \rho)$. Our aim is to evaluate the $L^r(\mathbb{P})$-quantization rate of the process X, viewed as an L_T^p-valued random variable induced by the "Haar product quantizations" of X defined by

$$\widehat{X} = \widehat{\xi}_0^{N_0} e_0 + \sum_{n \geq 0} \sum_{k=0}^{2^n - 1} \widehat{\xi}_{2^n+k}^{N_{2^n+k}} e_{2^n+k}, \tag{2.5}$$

where $\xi_k := (X|e_k) \in L^\rho(\mathbb{P})$, $k \geq 0$, and where $\widehat{\xi}^N$ denotes an N-quantization ($N \geq 1$) of the (real-valued) random variable ξ, that is, a quantization of ξ by a codebook α^N having N elements. A quantization taking finitely many values, we set $N_{2^n+k} = 1$ and $\widehat{\xi}_{2^n+k}^{N_{2^n+k}} = 0$ for large enough n (which may be a nonoptimal 1-quantizer for $\xi_{2^n+k}^{N_{2^n+k}}$).

We will see that this local behavior of φ at 0 induces an upper bound for the functional quantization error rate of X (regardless of the values of r and p, except for constants).

THEOREM 1. *Let $X = (X_t)_{t \in [0,T]}$ be a (bimeasurable) process defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ such that $X_t \in L^\rho(\mathbb{P})$ for an exponent $\rho > 0$. Assume that X satisfies (2.3) [the φ-Lipschitz assumption $(L_{\varphi,\rho})$] for this exponent ρ, where φ is regularly varying [in the sense of (2.4)] with index $b \geq 0$ at 0 [then $|X|_{L_T^\rho} \in L^1(\mathbb{P})$]. Then*

$$\forall\, r, p \in (0, \rho) \qquad e_{N,r}(X, L_T^p) \leq C_{r,p} \begin{cases} \varphi(1/\log N), & \text{if } b > 0, \\ \psi(1/\log N), & \text{if } b = 0, \end{cases}$$

with $\psi(x) = (\int_0^x (\varphi(\xi))^{r\wedge 1}\, d\xi/\xi)^{1/(r\wedge 1)}$, assuming, moreover, that $\int_0^1 (\varphi(\xi))^{r \wedge 1}\, d\xi/\xi < +\infty$ if $b = 0$. In particular, if $\varphi(u) = cu^b$, $b > 0$, then

$$e_{N,r}(X, L_T^p) = O((\log N)^{-b}). \tag{2.6}$$

PROOF. Using the two obvious inequalities

$$|f|_{L_T^p} \leq T^{1/p - 1/p'} |f|_{L_T^{p'}}, \qquad p \leq p',$$

for every Borel function $f : [0, T] \to \mathbb{R}$ and

$$\|Z\|_r \leq \|Z\|_{r'}, \qquad r \leq r',$$

for every random variable $Z:\Omega\to\mathbb{R}$, we may assume, without loss of generality, that either

$$1\le p=r<\rho \quad \text{or} \quad 0<p=r<\rho\le 1.$$

CASE 1 ($1\le p=r<\rho$). Let $N\ge 1$ be a fixed integer. We consider a Haar product quantization $\widehat{X}$ of X with a (product) codebook having at most N elements, that is, such that $N_0\times\prod_{n,k}N_{2^n+k}\le N$. Its characteristics will be specified below. Then, using (2.1), that is,

$$\begin{aligned}
|X-\widehat{X}|_{L^r_T} &\le T^{1/r-1/2}|\xi_0-\widehat{\xi}_0^{N_0}|+\sum_{n\ge 0}\left|\sum_{k=0}^{2^n-1}(\xi_{2^n+k}-\widehat{\xi}_{2^n+k}^{N_{2^n+k}})e_{2^n+k}\right|_{L^r_T}\\
&=T^{1/r-1/2}|\xi_0-\widehat{\xi}_0^{N_0}|\\
&\quad+T^{1/r-1/2}\sum_{n\ge 0}2^{n(1/2-1/r)}\left(\sum_{k=0}^{2^n-1}|\xi_{2^n+k}-\widehat{\xi}_{2^n+k}^{N_{2^n+k}}|^r\right)^{1/r}
\end{aligned}$$

so that, both $\|\cdot\|_r$ and $\|\cdot\|_1$ being norms,

$$\begin{aligned}
&\big\||X-\widehat{X}|_{L^r_T}\big\|_r\\
&\quad\le T^{1/r-1/2}\big\||\xi_0-\widehat{\xi}_0^{N_0}|\big\|_r\\
&\qquad+T^{1/r-1/2}\sum_{n\ge 0}2^{n(1/2-1/r)}\left\|\left(\sum_{k=0}^{2^n-1}|\xi_{2^n+k}-\widehat{\xi}_{2^n+k}^{N_{2^n+k}}|^r\right)^{1/r}\right\|_r\\
&\quad=T^{1/r-1/2}\|\xi_0-\widehat{\xi}_0^{N_0}\|_r\\
&\qquad+T^{1/r-1/2}\sum_{n\ge 0}2^{n(1/2-1/r)}\left\|\sum_{k=0}^{2^n-1}|\xi_{2^n+k}-\widehat{\xi}_{2^n+k}^{N_{2^n+k}}|^r\right\|_1^{1/r}\\
&\quad\le T^{1/r-1/2}\|\xi_0-\widehat{\xi}_0^{N_0}\|_r \qquad\qquad (2.7)\\
&\qquad+T^{1/r-1/2}\sum_{n\ge 0}2^{n(1/2-1/r)}\left(2^n\max_{0\le k\le 2^n-1}\big\||\xi_{2^n+k}-\widehat{\xi}_{2^n+k}^{N_{2^n+k}}|^r\big\|_1\right)^{1/r}\\
&\quad=T^{1/r-1/2}\|\xi_0-\widehat{\xi}_0^{N_0}\|_r\\
&\qquad+T^{1/r-1/2}\sum_{n\ge 0}2^{n/2}\max_{0\le k\le 2^n-1}\big\||\xi_{2^n+k}-\widehat{\xi}_{2^n+k}^{N_{2^n+k}}|^r\big\|_1^{1/r}\\
&\quad=T^{1/r-1/2}\|\xi_0-\widehat{\xi}_0^{N_0}\|_r\\
&\qquad+T^{1/r-1/2}\sum_{n\ge 0}2^{n/2}\max_{0\le k\le 2^n-1}\big\|\xi_{2^n+k}-\widehat{\xi}_{2^n+k}^{N_{2^n+k}}\big\|_r.
\end{aligned}$$

Let $\delta := \rho - r$. It follows from Lemma 1 (Pierce lemma) that, for every $N \geq 1$ and every r.v. $\xi \in L^r(\mathbb{P})$,

$$\inf_{\operatorname{card}(\alpha)\leq N} \|\xi - \widehat{\xi}^{\alpha}\|_r \leq C_{r,\rho}\|\xi\|_\rho N^{-1}. \tag{2.8}$$

Now, using the monotony in p of the L^p-norms with respect to the probability measure $2^{n+1}\mathbf{1}_{[0,2^{-(n+1)}T]}(t)\,dt/T$, Fubini's theorem, the $(L_{r,\varphi})$-Lipschitz continuity assumption (2.3)(i) and (2.2), we obtain

$$\begin{aligned}
&\mathbb{E}\,|\xi_{2^n+k}|^\rho \\
&\quad = \mathbb{E}\,|(X|e_{2^n+k})|^\rho \\
&\quad \leq 2^{(n/2)\rho}T^{-\rho/2} \\
&\qquad \times \mathbb{E}\biggl(\int_0^{2^{-(n+1)}T} \bigl|X_{(k/2^n)T+u} - X_{(2k+1)/(2^{n+1})T+u}\bigr|\,du\biggr)^\rho \\
&\quad \leq 2^{(n/2)\rho}2^{-(n+1)\rho}T^{\rho/2} \\
&\qquad \times \mathbb{E}\biggl(\int_0^{2^{-(n+1)}T} \bigl|X_{(k/2^n)T+u} - X_{(2k+1)/(2^{n+1})T+u}\bigr|^\rho\, 2^{n+1}\,du/T\biggr) \\
&\quad \leq 2^{-\rho}2^{-(n/2)\rho+n+1}T^{\rho/2-1} \\
&\qquad \times \int_0^{2^{-(n+1)}T} \mathbb{E}\bigl|X_{(k/2^n)T+u} - X_{(2k+1)/(2^{n+1})T+u}\bigr|^\rho\,du \\
&\quad \leq 2^{-(n/2)\rho+n+1-\rho}T^{\rho/2-1}\int_0^{2^{-(n+1)}T} (\varphi(T/2^{n+1}))^\rho\,du \\
&\quad \leq C_{X,T,r,\rho}2^{-(n/2)\rho}(\varphi(T/2^{n+1}))^\rho.
\end{aligned} \tag{2.9}$$

At this stage, we assume a priori that the size sequence $(N_{2^n+k})_{n\geq 0,\,k=0,\dots,2^n-1}$ of the marginal codebooks is nonincreasing as 2^n+k increases and satisfies

$$1 \leq \prod_{k\geq 0} N_k \leq N.$$

We assume that all the quantizations induced by these codebooks are L^r-optimal up to $n \leq m$, that is,

$$\begin{aligned}
&\|\xi_{2^n+k} - \widehat{\xi}_{2^n+k}\|_r \\
&\qquad = \inf_{\operatorname{card}(\alpha)\leq N_{2^n+k}} \|\xi_{2^n+k} - \widehat{\xi}^{\alpha}_{2^n+k}\|_r
\end{aligned}$$

and that $\widehat{\xi}_{2^n+k} = 0$ otherwise. Then, combining (2.7), (2.9) and (2.8) (Pierce Lemma) yields

$$\begin{aligned}
\| |X - \widehat{X}|_{L^r_T} \|_r &\le C_{X,T,r,\rho} \Biggl(\frac{1}{N_0} + \sum_{n\ge 0} \frac{\varphi(T2^{-(n+1)})}{N_{2^{n+1}}} \Biggr) \\
&\le C_{X,T,r,\rho} \Biggl(\frac{1}{N_0} + \frac{1}{T} \sum_{n\ge 0} \sum_{k=0}^{2^{n+1}-1} \frac{\Phi(2T/(2^{n+1}+k))}{N_{2^{n+1}+k}} \Biggr) \\
&= C_{X,T,r,\rho} \Biggl(\frac{1}{N_0} + \frac{1}{T} \sum_{k\ge 2} \frac{\Phi(2T/k)}{N_k} \Biggr),
\end{aligned}$$

where $\Phi(x) := x\varphi(x)$, $x \in (0, +\infty)$. This function Φ is regularly varying (at 0) with index $b+1$. This implies, in particular, that there is a real constant $c > 0$ such that $\Phi(T/k) \le c\Phi(1/(k+1))$ for every $k \ge 2$. Hence, inserting, for convenience, the term $\Phi(1/2)/N_1$ and modifying the real constant $C_{X,T,r,\rho}$ in an appropriate way finally yields

$$\| |X - \widehat{X}|_{L^r_T} \|_r \le C_{X,T,r,\rho} \sum_{k\ge 1} \frac{\Phi(1/k)}{N_{k-1}}.$$

Now, set, for convenience, $\nu_k = \Phi(1/k)$, $k \ge 1$. Note that in the case $b = 0$, the integrability condition $\int_0^1 \varphi(\xi)/\xi\, d\xi < +\infty$ implies $\sum_k \nu_k < +\infty$. Consequently, an upper bound for the quantization rate is given by the solution of the following optimal allocation problem:

$$\begin{aligned}
e_{N,r}(X, L^r_T) &\le C_{X,T,r,\rho} \min \Biggl\{ \sum_{k\ge 1} \frac{\nu_k}{N_{k-1}}, \\
&\qquad \prod_{k\ge 0} N_k \le N,\ N_0 \ge \cdots \ge N_k \ge \cdots \ge 1 \Biggr\} \\
&= C_{X,T,r,\rho} \min \Biggl\{ \sum_{k=1}^{m} \frac{\nu_k}{N_{k-1}} + \sum_{k\ge m+1} \nu_k,\ m \ge 1, \\
&\qquad \prod_{0\le k\le m-1} N_k \le N,\ N_0 \ge \cdots \ge N_{m-1} \ge 1 \Biggr\}.
\end{aligned} \tag{2.10}$$

The rest of the proof follows the approach developed in [18] [Section 4.1, especially Lemma 4.2, Theorem 4.6(i)–(iii) and its proof] and [19]. However, one must be be aware that we have had to modify some notation.

PROPOSITION 1. *Assume* $\nu_k = \Phi(1/k)$, $k \ge 1$, *where* $\Phi(x) = x\varphi(x)$, $\varphi : (0, +\infty)$ *is a nondecreasing, regularly varying function at* 0 *with index* $b \ge 0$ *with* $\int_0^1 \varphi(\xi) \frac{d\xi}{\xi} < +\infty$ *when* $b = 0$. *Then*:

(i) $\lim_k \nu_k/\nu_{k+1} = 1$;
(ii) $(\prod_{k=1}^n \nu_k)^{1/n} \sim e^{b+1}\nu_n$;
(iii) $\sum_{k=n+1}^{\infty} \nu_k + n\nu_k \sim c\psi(1/n)$, *where* $c = 1 + 1/b$ *if* $b > 0$; $c = 1$ *if* $b = 0$;

$$\psi(x) = \varphi(x) \qquad \text{if } b > 0; \qquad \psi(x) := \int_0^x \varphi(\xi)\frac{d\xi}{\xi} \qquad \text{if } b = 0.$$

(*See* [18] *for a proof.*)

PROOF OF THEOREM 1 (Continued). Set

$$m = m^*(N) = \max\left\{m \geq 1 : N^{1/m}\nu_m\left(\prod_{j=1}^m \nu_j\right)^{-1/m} \geq 1\right\}$$

and

$$N_{k-1} = N_{k-1}(N) := \left\lceil N^{1/m}\nu_k\left(\prod_{j=1}^m \nu_j\right)^{-1/m}\right\rceil \geq 1, \qquad k = 1, \ldots, m.$$

It follows from Proposition 1(ii) that

$$m = m^*(N) \sim \frac{\log N}{b+1} \qquad \text{as } N \to \infty.$$

Then

$$\begin{aligned}
\sum_{k=1}^m \frac{\nu_k}{N_{k-1}} &\leq \max_{k\geq 1}(1 + 1/N_{k-1})mN^{-1/m}\left(\prod_{j=1}^m \nu_j\right)^{1/m} \\
&\leq 2mN^{-1/m}\left(\prod_{j=1}^m \nu_j\right)^{1/m} \\
&\leq 2m\nu_m.
\end{aligned}$$

Consequently, this time using (iii) in Proposition 1,

$$\begin{aligned}
\sum_{k=1}^m \frac{\nu_k}{N_{k-1}} + \sum_{k\geq m+1} \nu_k &\leq 2\left(m\nu_m + \sum_{k\geq m+1} \nu_k\right) \\
&= O\big(\psi(1/\log N)\big)
\end{aligned}$$

so that

$$\big\| |X - \widehat{X}|_{L_T^p} \big\|_r = O\big(\psi(1/\log N)\big).$$

CASE 2 ($\rho \le 1$). Here, we rely on the pseudo-triangular inequality

$$|f+g|^r_{L^r_T} \le |f|^r_{L^r_T} + |g|^r_{L^r_T},$$

which follows from the elementary inequality $(u+v)^r \le u^r + v^r$:

$$\begin{aligned}|X-\widehat{X}|^r_{L^r_T} &\le T^{1-r/2}|\xi_0-\widehat{\xi}_0^{N_0}|^r + \sum_{n\ge 0}\left|\sum_{k=0}^{2^n-1}(\xi_{2^n+k}-\widehat{\xi}^{N_{2^n+k}}_{2^n+k})e_{2^n+k}\right|^r_{L^r_T}\\ &= T^{1-r/2}|\xi_0-\widehat{\xi}_0^{N_0}|^r + T^{1-r/2}\sum_{n\ge 0}2^{n(r/2-1)}\sum_{k=0}^{2^n-1}|\xi_{2^n+k}-\widehat{\xi}^{N_{2^n+k}}_{2^n+k}|^r\end{aligned}$$

so that

$$\begin{aligned}\||X-\widehat{X}|_{L^r_T}\|^r_r &= \||X-\widehat{X}|^r_{L^r_T}\|_1\\ &\le T^{1-r/2}\||\xi_0-\widehat{\xi}_0^{N_0}|^r\|_1\\ &\quad + T^{1-r/2}\sum_{n\ge 0}2^{n(r/2-1)}\left\|\sum_{k=0}^{2^n-1}|\xi_{2^n+k}-\widehat{\xi}^{N_{2^n+k}}_{2^n+k}|^r\right\|_1\\ &\le T^{1-r/2}\|\xi_0-\widehat{\xi}_0^{N_0}\|^r_r\\ &\quad + T^{1-r/2}\sum_{n\ge 0}2^{n(r/2-1)}2^n\max_{0\le k\le 2^n-1}\||\xi_{2^n+k}-\widehat{\xi}^{N_{2^n+k}}_{2^n+k}|^r\|_1\\ &= T^{1-r/2}\|\xi_0-\widehat{\xi}_0^{N_0}\|^r_r\\ &\quad + T^{1-r/2}\sum_{n\ge 0}2^{nr/2}\max_{0\le k\le 2^n-1}\||\xi_{2^n+k}-\widehat{\xi}^{N_{2^n+k}}_{2^n+k}|^r\|_1\\ &= T^{1-r/2}\|\xi_0-\widehat{\xi}_0^{N_0}\|^r_r\\ &\quad + T^{1-r/2}\sum_{n\ge 0}2^{nr/2}\max_{0\le k\le 2^n-1}\|\xi_{2^n+k}-\widehat{\xi}^{N_{2^n+k}}_{2^n+k}\|^r_r.\end{aligned} \tag{2.11}$$

This inequality replaces (2.7). We then note that

$$\begin{aligned}\mathbb{E}|\xi_{2^n+k}|^\rho &\le 2^{(n/2)\rho}T^{-\rho/2}\big(2^{-(n+1)}T\varphi(T/2^{n+1})\big)^\rho\\ &= C_{X,T,r,\rho}2^{-(n/2)\rho}\big(\varphi(T/2^{n+1})\big)^\rho\end{aligned}$$

so that

$$\||X-\widehat{X}|_{L^r_T}\|^r_r \le C_{X,T,r,\rho}\left(\frac{1}{N_0^r}+\sum_{n\ge 0}\frac{\varphi(T2^{-(n+1)})^r}{N^r_{2^{n+1}}}\right).$$

We then set $\widetilde{\varphi}(u) = (\varphi(u))^r$, $\widetilde{N}_k = N_k^r$ and $\widetilde{N} := N^r$. We proceed for $\||X-\widehat{X}|_{L^r_T}\|^r_r$ with these "tilded" parameters as for $\||X-\widehat{X}|_{L^r_T}\|_r$ in the case $\rho > 1$. $\square$

REMARKS. *Concerning the case* $p > r$. When $p \geq \rho > r$, the $(L^r(\mathbb{P}), L_T^p)$-quantization problem remains consistent. However, there is a price to be paid for considering a p exponent greater than ρ. Thus, if φ in $(L_{(\rho,\varphi)})$ has regular variations with exponents $b > 0$ at 0 and if $b + \frac{1}{p} - \frac{1}{r} > 0$, then the same approach yields the rate

$$e_{N,r}(X, L_T^p) \leq C_{X,r,\delta,T,p}\varphi(1/\log N)(\log N)^{1/r-1/p}.$$

We do not know whether it is due to our approach or if it is the best possible universal rate.

Concerning lower bounds. In several situations, when the assumption $(L_{\rho,\varphi})$ is optimal in terms of mean regularity of a process, the upper bound for the functional quantization rate turns out to be the true rate. We have no general result in that direction so far since most lower bound results rely on a different approach, namely the small deviation theory. Thus, in [14], a connection is established between (functional) quantization and small deviation for Gaussian processes. In particular, this approach provides a method to derive a lower bound for the $(L^r(\mathbb{P}), L_T^p)$-quantization rate from some upper bound for the small ball problem. A careful reading of the paper (see the proof of Theorem 1.2 in [14]) shows that this small deviation lower bound holds for any *unimodal* (w.r.t. 0) nonzero process. To be precise, let $p \in (0, \infty)$ and assume that $\mathbb{P}_X$ is L_T^p-unimodal in the following sense: there exists a real $\varepsilon_0 > 0$ such that

$$\forall x \in L_T^p, \forall \varepsilon \in (0, \varepsilon_0] \qquad \mathbb{P}(|X - x|_{L_T^p} \leq \varepsilon) \leq \mathbb{P}(|X|_{L_T^p} \leq \varepsilon).$$

(For centered Gaussian processes, this follows for $p \geq 1$ from Anderson's inequality.) If

$$G\big(-\log\big(\mathbb{P}(|X|_{L_T^p} \leq \varepsilon)\big)\big) = \Omega(1/\varepsilon) \qquad \text{as } \varepsilon \to 0$$

for some increasing unbounded function $G : (0, \infty) \to (0, \infty)$, then

$$\forall r \in (0, \infty), \forall c > 1 \qquad e_{N,r}(X, L_T^p) = \Omega\left(\frac{1}{G(\log(cN))}\right). \tag{2.12}$$

3. Applications and examples. In this section, we give some examples which illustrate that the upper bound derived from the mean pathwise regularity may be optimal or not.

3.1. *Application to Itô processes and d-dimensional diffusion processes.* Let W denote an $\mathbb{R}^d$-valued standard Brownian motion defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and let $(\mathcal{F}_t^W)_{t\in[0,T]}$ denote its natural filtration (completed with all the $\mathbb{P}$-negligible sets). Let X be a 1-dimensional Itô process defined by

$$dX_t = G_t\,dt + H_t \cdot dW_t, \qquad X_0 = x_0 \in \mathbb{R},$$

where $(G_t)_{t\in[0,T]}$ is a real-valued process and $(H_t)_{t\in[0,T]}$ is an $\mathbb{R}^d$-valued process, both assumed $(\mathcal{F}_t^W)_{t\in[0,T]}$-progressively measurable. Assume that there exists a real number $\rho \geq 2$ such that

$$\sup_{t\in[0,T]} \mathbb{E}|G_t|^\rho + \sup_{t\in[0,T]} \mathbb{E}|H_t|^\rho < +\infty, \tag{3.1}$$

where $|\cdot|$ denotes any norm on $\mathbb{R}^d$. Then (see, e.g., [4]) the φ-Lipschitz assumption $(L_{\varphi,\rho})$(i) [i.e., (2.3)(i)] is satisfied with $\varphi(u) = cu^{1/2}$. It follows from Theorem 1 that

$$\forall r,\ p\in(0,\rho) \qquad e_{N,r}(X, L_T^p) = O((\log N)^{-1/2}).$$

Let $X = (X^1, \dots, X^d)$ be an $\mathbb{R}^d$-valued diffusion process defined by

$$dX_t = b(t, X_t)\,dt + \sigma(t, X_t)\,dW_t, \qquad X_0 = x_0 \in \mathbb{R}^d,$$

where $b : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : [0, T] \times \mathbb{R}^d \to \mathcal{M}(d \times q, \mathbb{R})$ are Borel functions satisfying

$$\forall t \in [0, T],\ \forall x \in \mathbb{R}^d \qquad |b(t, x)| + \|\sigma(t, x)\| \leq C(1 + |x|)$$

and W is an $\mathbb{R}^q$-valued standard Brownian motion. The above assumption does not imply that such a diffusion process X exists. (The existence holds provided b and σ are Lipschitz in x uniformly with respect to $t \in [0, T]$.) Then, every component X^i is an Itô process [with $G_t = b^i(t, X_t)$ and $H_t := \sigma^{i\cdot}(t, X_t)$] for which assumption (3.1) is satisfied for every $\rho > 0$ (see, e.g., [4]). On the other hand, if $(u^1, \dots, u^d)$ denotes the canonical basis of $\mathbb{R}^d$ and $|\cdot|$ denotes any norm on $\mathbb{R}^d$, then for every $p \geq 1$ and every $f := \sum_{1\leq i\leq d} f^i u^i : [0, T] \to \mathbb{R}^d$,

$$|f|_{L^p_{\mathbb{R}^d}([0,T],dt)} \leq \sum_{i=1}^{d} |f^i|_{L_T^p} |u^i|.$$

Now, we can quantize each Itô process $(X_t^i)_{t\in[0,T]}$, $i = 1, \dots, d$, using an (L^r, L_T^p)-optimal quantizer $\alpha^{(i)}$ of size $[\sqrt[d]{N}]$. It is clear that the resulting product quantizer $\prod_{i=1}^d \alpha^{(i)}$ of size $[\sqrt[d]{N}]^d \leq N$ induces an $(L^r, L^p_{\mathbb{R}^d}([0, T], dt))$-quantization error $O((\log N)^{-1/2})$ (see, e.g., [20]). Combining these obvious remarks finally yields

$$\forall r,\ p > 0 \qquad e_{N,r}(X, L^p_{\mathbb{R}^d}([0, T], dt)) = O((\log N)^{-1/2}).$$

In the "smooth" case $H \equiv 0$, the regularity assumption $(L_{\varphi,\rho})$ is satisfied with $\varphi(u) = cu$. We obtain the universal upper bound

$$\forall r,\ p\in(0,\rho) \qquad e_{N,r}(X, L_T^p) = O((\log N)^{-1}).$$

Both rates are optimal as universal rates for $p \geq 1$, as can be seen from $X = W$ and $X = \int_0^{\cdot} G_s\,ds$ with $G_t = \int_0^t (t-s)^{\beta-1/2}\,dW_s$ ($\beta > 0$ and $d = 1$), respectively (see [14]).

As far as quantization rates are concerned, this extends to general d-dimensional diffusions a result obtained in [20] by stochastic calculus techniques for a more restricted class of Brownian diffusions (which includes 1-dimensional ones). This also extends (the upper bound part of the) the result obtained in [7] for another class of (essentially 1-dimensional) Brownian diffusions. For the class investigated in [20], it is shown that under an ellipticity assumption on σ, this rate is optimal in the case $r, p \geq 1$. In [7], still with a (mild) ellipticity assumption, the rate is sharp for $p \geq 1$. This leads us to conjecture that this rate is optimal for not too degenerate Brownian diffusions.

3.2. *Application to fractional Brownian motion.* The fractional Brownian motion W^H with Hurst constant $H \in (0, 1]$ is a Gaussian process satisfying, for every $\rho > 0$,

$$\mathbb{E}|W_t^H - W_s^H|^\rho = C_{H,p}|t-s|^{\rho H} \quad \text{and} \quad (W_s^H)_{0 \leq s \leq t} \stackrel{\mathcal{L}}{\sim} t^H (W_{s/t}^H)_{0 \leq s \leq t}.$$

So, using Theorem 1, we obtain $e_{N,r}(W^H, L_T^p) = O((\log N)^{-H})$ as an $(L^r(\mathbb{P}), |\cdot|_{L_T^p})$-quantization rate for every $r, p > 0$. This rate is known to be optimal for $p \geq 1$. In fact, a sharp rate is established (see [19], when $p = r = 2$, or [8]) [i.e., the computation of the exact value of $\lim_N N(\log N)^H e_{N,r}(W^H, L_T^p)$].

3.3. *Stationary processes.* Let X be a centered weakly (square-integrable) stationary process. Then

$$\mathbb{E}|X_t - X_s|^2 = \mathbb{E}|X_{t-s} - X_0|^2 = 2\operatorname{Var}(X_0)\big(1 - c(|t-s|)\big),$$

where $c(t)$ denotes the correlation between X_t and X_0. Hence, if

$$c(u) = 1 - \kappa u^{2a} + o(u^{2a}) \qquad \text{as } u \to 0,$$

then the $L^r(\mathbb{P})$-rate for L_T^p-quantization $0 < p, r < 2$, will be $O((\log(N))^{-a})$. If, furthermore, X is a Gaussian process (like the Ornstein–Uhlenbeck process with $a = 1/2$), then this $O((\log N)^{-a})$ rate holds for any $r, p > 0$ since, for every $\rho \in \mathbb{N}^*$,

$$\mathbb{E}|X_t - X_s|^\rho = \mathbb{E}|X_{t-s} - X_0|^\rho = C_\rho\big(\operatorname{Var}(X_0)\big(1 - c(|t-s|)\big)\big)^{\rho/2}.$$

3.4. *Self-similar processes with stationary increments.* Let $X = (X_t)_{t \in [0,T]}$ be an H-self-similar process with stationary increments [$H \in (0, \infty)$]. Assume $X_1 \in L^\rho(\mathbb{P})$ for some $\rho \geq 1$. Then

$$\mathbb{E}|X_t - X_s|^\rho = C_\rho |t-s|^{\rho H}$$

for every $s,\ t \in [0, T]$. Since X is stochastically continuous, it has a bimeasurable modification. Theorem 1 then gives

$$\forall r,\ p \in (0, \rho) \qquad e_{N,r}(X, L^p_T) = O((\log N)^{-H}).$$

If, furthermore, X is α-stable, $\alpha \in (1, 2)$, then $X_1 \in L^\rho(\mathbb{P})$ for every $\rho \in [1, \alpha)$ so that

$$\forall r,\ p \in (0, \alpha) \qquad e_{N,r}(X, L^p_T) = O((\log N)^{-H}).$$

This class of examples comprises, for example, the linear H-fractional α-motions with $\alpha \in (1, 2)$, $H \in (0, 1)$ and the log-fractional α-stable motions with $\alpha \in (1, 2)$, where $H = 1/\alpha$ (see [10, 25]).

3.5. *Lévy processes*: *a first approach.* A (càdlàg) Lévy process $X = (X_t)_{t \in \mathbb{R}_+}$—or Process with Stationary Independent Increments (*PSII*)—is characterized by its so-called *local characteristics* appearing in the Lévy–Khintchine formula (for an introduction to Lévy processes, we refer to [2, 16, 24]). These characteristics depend on the way the "big" jumps are truncated. We will adopt, in the following, the convention that the truncation occurs at size 1. So that, for every $t \in \mathbb{R}_+$,

$$\mathbb{E}(e^{iuX_t}) = e^{-t\psi(u)}$$

$$\text{where } \psi(u) = -iua + \tfrac{1}{2}\sigma^2 u^2 - \int_{\mathbb{R}\setminus\{0\}} (e^{iux} - 1 - iux\mathbf{1}_{\{|x|\le 1\}})\nu(dx),$$

where $a, \sigma \in \mathbb{R}$ and ν is a nonnegative measure on $\mathbb{R} \setminus \{0\}$ such that $\nu(x^2 \wedge 1) < +\infty$. The measure ν is called the *Lévy measure* of the process. It can be shown that a Lévy process is a compound Poisson process if and only if ν is a finite measure and has finite variation if and only if $\int_{\{|x|\le 1\}} |x|\nu(dx) < +\infty$. Furthermore,

$$X_t \in L^\rho(\mathbb{P}) \quad \text{if and only if} \quad \int_{\{|x|\ge 1\}} |x|^\rho \nu(dx) < +\infty.$$

We will extensively use the following Compensation Formula (see, e.g., [2] page 7):

$$\mathbb{E}\sum_{s\ge 0} F(s, X_{s-}, \Delta X_s)\mathbf{1}_{\{\Delta X_s \ne 0\}} = \mathbb{E}\int_{\mathbb{R}_+} ds \int_{\mathbb{R}\setminus\{0\}} F(s, X_{s-}, \xi)\nu(d\xi), \tag{3.2}$$

where $F : \mathbb{R}_+ \times \mathbb{R}^2 \to \overline{\mathbb{R}}_+$ is a Borel function. As concerns assumption (2.3), note that the very definition of a Lévy process implies that

$$\mathbb{E}|X_t - X_s|^\rho = \mathbb{E}|X_{t-s}|^\rho \quad \text{and} \quad \mathbb{E} \sup_{s\in[t,t+h]} |X_t - X_s|^\rho = \mathbb{E} \sup_{s\in[0,h]} |X_s|^\rho,$$

so we may focus on the distribution of X_t and $X^*_t := \sup_{s\in[0,t]} |X_s|$. Finally, note that it follows from the usual symmetry principle (see [24]) that for any Lévy process, $\mathbb{P}(X^*_t > u + v) \le \mathbb{P}(|X_t| > u)/\mathbb{P}(X^*_t \le v/2)$ so that $\mathbb{E}|X_t|^r$ and $\mathbb{E}|X^*_t|^r$ are simultaneously finite or infinite when $r > 0$.

The following result is established in [21].

LEMMA 2 (Millar's Lemma). *Assume* $\sigma = 0$. *If there exists a real number* $\rho \in (0, 2]$ *such that* $\int_{\mathbb{R}\setminus\{0\}} |x|^\rho \nu(dx) < +\infty$, *then there exist some real constants* $a_\rho \in \mathbb{R}$ *and* $C_\rho > 0$ *such that*

$$\forall t \geq 0 \qquad \mathbb{E}\Big(\sup_{s\in[0,t]} |X_s - a_\rho s|^\rho\Big) \leq C_\rho t. \tag{3.3}$$

Furthermore, *one may set* $a_\rho = 0$ *if* $\rho \geq 1$.

Hence, it follows as a consequence of Theorem 1 that

$$\forall r,\ p \in (0, \rho) \qquad e_{N,r}(X, L_T^p) = O((\log N)^{-1/\rho}). \tag{3.4}$$

This follows from the following straightforward remark: if $\beta \subset L_T^p$ is an N-quantizer and $\xi \in L_T^p$ [here $\xi(t) = a_\rho t$], then

$$\big\| |X - \widehat{X}^\beta|_{L_T^p} \big\|_r = \big\| |(X + \xi) - \widehat{(X + \xi)}^{\xi+\beta}|_{L_T^p} \big\|_r \qquad \text{with } \xi + \beta = \{\xi + f,\ f \in \beta\}.$$

However, rate (3.4) may be suboptimal, as illustrated below with α-stable processes and Poisson processes. In Section 4, we establish two improvements of this rate under some natural hypotheses (see Theorem 2 for a broad class of Lévy processes with infinite Lévy measure and Proposition 3 for compound Poisson processes).

The α-stable processes. The (strictly) α-stable processes are families of Lévy processes indexed by $\alpha \in (0, 2)$ satisfying a self-similarity property, namely

$$\forall t \in \mathbb{R}_+ \qquad X_t \stackrel{\mathcal{L}}{\sim} t^{1/\alpha} X_1 \quad \text{and} \quad \sup_{0\leq s\leq t} |X_s| \stackrel{\mathcal{L}}{\sim} t^{1/\alpha} \sup_{0\leq s\leq 1} |X_s|.$$

Furthermore,

$$\sup\Big\{r : \mathbb{E}\Big(\sup_{0\leq s\leq 1} |X_s|^r\Big) < +\infty\Big\} = \alpha \quad \text{and} \quad \mathbb{E}|X_1|^\alpha = +\infty.$$

Consequently, it follows from Theorem 1, applied with $\varphi(u) := u^{1/\alpha}$, that

$$\forall p,\ r \in (0, \alpha) \qquad e_{N,r}(X, L_T^p) = O\Big(\frac{1}{(\log N)^{1/\alpha}}\Big). \tag{3.5}$$

In the symmetric case, an α-stable process X being subordinated to a Brownian motion ($X_t = W_{A_t}$ with A a one-sided $\alpha/2$-stable process) has a unimodal distribution by the Anderson inequality (see Section 5.2 below, entirely devoted to subordinated Lévy processes). Substituting into (2.12) the small deviation estimates established in [17] shows the rate optimality of our upper bound for $e_{N,r}$ when $p \geq 1$, that is,

$$\forall r \in (0, \alpha),\ \forall p \in [1, \alpha) \qquad e_{N,r}(X, L_T^p) \approx (\log N)^{-1/\alpha}. \tag{3.6}$$

The Γ*-processes*. These are subordinators (nondecreasing Lévy processes) whose distribution $\mathbb{P}_{X_t}$ at time t is a $\gamma(\alpha, t)$-distribution,

$$\mathbb{P}_{X_t}(dx) = \frac{\alpha^t}{\Gamma(t)} \mathbf{1}_{(0,\infty)}(x) x^{t-1} e^{-\alpha x}\, dx.$$

So, easy computations show that for every $\rho > 0$,

$$\mathbb{E}|X_t|^\rho = \frac{\Gamma(t+\rho)}{\alpha^\rho \Gamma(t+1)} t \sim \frac{\Gamma(\rho)}{\alpha^\rho \Gamma(1)} t \qquad \text{as } t \to 0.$$

Consequently, it follows from Theorem 1 that

$$\forall\, p \in (0, +\infty),\ \forall\, r \in (0, p] \qquad e_{N,r}(X, L_T^p) = O\left(\frac{1}{(\log(N))^{1/p-\varepsilon}}\right) \qquad \forall\, \varepsilon > 0.$$

Compound Poisson processes from the mean regularity viewpoint. One considers a compound Poisson process

$$X_t = \sum_{k=1}^{K_t} U_k,$$

where $K = (K_t)_{t\in[0,T]}$ denotes a standard Poisson process with intensity $\lambda = 1$ defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and $(U_k)_{k\geq 1}$ an i.i.d. sequence of random variables defined on the same probability space, with $U_1 \in L^\rho(\mathbb{P})$ for some $\rho > 0$. Then, standard computations show that

$$\mathbb{E} \sup_{0\leq s\leq t} \left|\sum_{k=1}^{K_s} U_k\right|^\rho \leq \mathbb{E} \sum_{k=1}^{K_t} |U_k|^\rho = t\, \|U_1\|_\rho^\rho \qquad \text{if } 0 < \rho \leq 1, \tag{3.7}$$

$$\mathbb{E}\left|\sum_{k=1}^{K_t} U_k\right|^\rho \leq t \|U_1\|_\rho^\rho \times \left[e^{-t} \sum_{k\geq 1} \frac{t^{k-1} k^\rho}{k!}\right] \qquad \text{if } \rho > 1. \tag{3.8}$$

Consequently, assumption (2.3) is fulfilled with $\varphi(u) = cu^b$, where $b = 1/\rho$ and c is a positive real constant. Theorem 2 then yields

$$\forall\, r,\ p \in (0, \rho) \qquad e_{N,r}(X, L_T^p) = O((\log N)^{-1/\rho}).$$

Note that when $\rho \leq 2$, this is a special case of (3.3). These rates are very far from optimality, as will be seen further on (in Section 4, some faster rates are established by a completely different approach based on the almost finite-dimensional feature of the paths of such elementary jump processes). This will emphasize the fact that the mean regularity of $t \mapsto X_t$ does not always control the quantization rate.

4. A quantization rate for general Lévy processes without Brownian component. The aim of this section is to provide a general result for Lévy processes without Brownian component, with special attention being paid to compound Poisson processes which appear as a critical case of the main theorem. Before stating the main results, we need some further notation related to Lévy processes. Set

$$\underline{\theta} := \inf\left\{\theta > 0 : \int_{\{|x|\le 1\}} |x|^{\theta}\nu(dx) < +\infty\right\} \in [0, 2], \tag{4.1}$$

$$r^* := \sup\left\{r > 0 : \int_{\{|x|> 1\}} |x|^{r}\nu(dx) < +\infty\right\} \le +\infty. \tag{4.2}$$

The exponent $\underline{\theta}$ is known as the *Blumenthal–Getoor index* of X [and is often denoted $\beta(X)$ in the literature]. We define on $(0,\infty)$ the tail function of the Lévy measure $\nu : u \mapsto \underline{\nu}(u) := \nu([-u,u]^c)$. Finally, we set, for every $\underline{\theta} > 0$, $\underline{\ell}(t) := t\underline{\nu}(t^{1/\underline{\theta}})$ and, for every $\rho > 0$,

$$\Lambda_\rho(t) := (\underline{\ell}(t))^{1/2} + (\underline{\ell}(t))^{1/\rho} + (\underline{\ell}(t))^{2/\rho}\mathbf{1}_{\underline{\theta}\in(1,2]\cup IV(1)},$$

where $IV(1) = \varnothing$ if $\underline{\theta} = 1$ and $\nu(|x|) < +\infty$, and $IV(1) = \{1\}$ if $\underline{\theta} = 1$ and $\nu(|x|) = +\infty$.

THEOREM 2. *Let $X = (X_t)_{t\in[0,T]}$ be a Lévy process with Lévy measure ν and without Brownian component. Assume r^*, $\underline{\theta} > 0$.*

(a) *Assume $\underline{\theta} \in (0,2] \setminus \{1\}$. If $\int_{\{|x|\le 1\}} |x|^{\underline{\theta}}\nu(dx) < +\infty$ (i.e., $\underline{\theta}$ holds as a minimum) or if the Lévy measure satisfies*

$$\exists c \in (0,1], \exists C > 0 \qquad \mathbf{1}_{\{0<|x|\le c\}}\nu(dx) \le \frac{C}{|x|^{\underline{\theta}+1}}\mathbf{1}_{\{0<|x|\le c\}}\,dx, \tag{4.3}$$

then

$$\forall r,\ p \in (0, \underline{\theta} \wedge r^*) \qquad e_{N,r}(X, L^p_T) = O((\log N)^{-1/\underline{\theta}}). \tag{4.4}$$

(b) *Assume $\underline{\theta} \in (0,2) \setminus \{1\}$. If the tail function of the Lévy measure ν has regular variation with index $-b$ at 0, then $b = \underline{\theta}$ and the function $\underline{\ell}$ is slowly varying at 0. If, furthermore, the functions $t \mapsto t^{1/\underline{\theta}}\Lambda_\rho(t)$ are nondecreasing in a neighborhood of 0, then*

$$\begin{aligned}&\forall r,\ p \in (0, \underline{\theta} \wedge r^*) \qquad e_{N,r}(X, L^p_T) = O((\log N)^{-1/\underline{\theta}}\Lambda_\rho((\log N)^{-1}))\\ &\qquad \forall \rho \in (r \vee p, \underline{\theta}).\end{aligned} \tag{4.5}$$

(c) *Assume $\underline{\theta} < r^*$. For every $r \in [\underline{\theta}, r^*)$ and every $p \in (0, r]$,*

$$e_{N,r}(X, L^p_T) = O((\log N)^{-1/r+\eta}) \qquad \forall \eta > 0. \tag{4.6}$$

(d) *When $\underline{\theta} = 1$, if ν is symmetric or $\nu(|x|) < +\infty$, then the above rates* (4.4) *and* (4.5) *are still valid.*

REMARKS. The conclusion in (a) remains valid for any $\underline{\theta} \in (0,2]$ satisfying $\int_{\{|x|\le 1\}} |x|^{\underline{\theta}} \nu(dx) < +\infty$ or (4.3), not only for the Blumenthal–Getoor index. In particular, with $\underline{\theta}=2$ we obtain

$$\forall r,\ p \in (0, 2\wedge r^*) \qquad e_{N,r}(X, L_T^p) = O((\log N)^{-1/2}).$$

When $\underline{\theta} \in \{1,2\}$, some rates can also be derived [even when ν is not symmetric and $\nu(|x|) = +\infty$]. Thus, in item (a), if $\underline{\theta} = 1$, we can show, by adapting the proof of case $\underline{\theta} \in (1,2)$ in Proposition 2 below, that

$$e_{N,r}(X, L_T^p) = O\left(\frac{\log\log N}{\log N}\right).$$

In most natural settings, there is a dominating term in the definition of the function Λ_ρ. Thus, in (4.5), we may set

$$\Lambda_\rho(t) = \begin{cases} (\underline{\ell}(t))^{1/\rho}\mathbf{1}_{\{\underline{\theta}\in(0,1]\setminus IV(1)\}} + (\underline{\ell}(t))^{2/\rho}\mathbf{1}_{\{\underline{\theta}\in(1,2]\cup IV(1)\}}, \\ \qquad\qquad \text{when } \lim_{t\to 0} \underline{\ell}(t) = +\infty, \\ (\underline{\ell}(t))^{1/2}, \qquad \text{when } \lim_{t\to 0} \underline{\ell}(t) = 0. \end{cases}$$

Note that this theorem provides no rate when $\underline{\theta} = 0$, which is the case of an important class of Lévy processes including compound Poisson processes. In fact, for these processes, the quantization rate is not ruled by the mean regularity of their paths, as emphasized in Section 4.1.

The proof of this theorem relies on Theorem 1, that is, on the mean pathwise regularity of X, hence the critical value $\underline{\theta}$ for ρ cannot be overcome by such an approach since assumption $(L_{\varphi,\rho})$ for $\rho > \underline{\theta}$ would imply that X has a pathwise continuous modification by the Kolmogorov criterion.

EXAMPLES. Note that for *α-stable processes*, $r^* = \underline{\theta} = \alpha$, ν satisfies (4.3) and $\lim_{u\to 0} \underline{\ell}(u) \in (0,\infty)$ so that both rates obtained from (4.4) and (4.5) coincide with that obtained in Section 3.5, that is, $O((\log N)^{-1/\alpha})$. This rate is most likely optimal.

Let $\nu^1_{a,\underline{\theta}}(dx) := \kappa|x|^{-\underline{\theta}-1}(-\log|x|)^{-a}\mathbf{1}_{(0,c]}(|x|)\,dx$, with $0<c<1$, $\kappa>0$, $a>0$. If $\underline{\theta} \in (0,2)$, then $\underline{\ell}(u) \sim \underline{\theta}^{a-1}(-\log u)^{-a}$ as $u \to 0$. If a Lévy process X has $\nu^1_{a,\underline{\theta}}$ as a (symmetric) Lévy measure, then $r^* = +\infty$ and

$$\forall r,\ p \in (0, \underline{\theta}) \qquad e_{N,r}(X, L_T^p) = O((\log N)^{-1/\underline{\theta}}(\log\log N)^{-a/2}).$$

Such a rate improves the one provided by (4.4)

Let $\nu^2_{a,\underline{\theta}}(dx) = \kappa|x|^{-\underline{\theta}-1}(-\log|x|)^{a}\mathbf{1}_{(0,c]}(|x|)\,dx$, $\kappa, a > 0$, $0<c<1$, $\underline{\theta} \in (0,2)$. Then $\underline{\ell}(u) \sim \underline{\theta}^{-a-1}(-\log u)^a$ as $u\to 0$. Note that $\nu^2_{a,\underline{\theta}}$ does not satisfy (4.3). If a Lévy process X has $\nu^2_{a,\underline{\theta}}$ as a (symmetric) Lévy measure, then

$r^* = +\infty$ and

$$\forall\, r, p \in (0, \underline{\theta})$$

$$e_{N,r}(X, L_T^p) = \begin{cases} O((\log N)^{-1/\underline{\theta}}(\log\log N)^{a/(\underline{\theta}-\eta)}), \\ \qquad\qquad \eta \in (0, \underline{\theta}),\ \text{if } \underline{\theta} < 1, \\ O((\log N)^{-1/\underline{\theta}}(\log\log N)^{2a/(\underline{\theta}-\eta)}), \\ \qquad\qquad \eta \in (0, \underline{\theta}),\ \text{if } \underline{\theta} \in [1, 2). \end{cases}$$

Hyperbolic Lévy motions have been applied to option pricing in finance (see [9]). These processes are Lévy processes whose distribution $\mathbb{P}_{X_1}$ at time 1 is a symmetric (centered) hyperbolic distribution

$$\mathbb{P}_{X_1} = Ce^{-\delta\sqrt{1+(x/\gamma)^2}}\,dx, \qquad \gamma, \delta > 0.$$

Hyperbolic Lévy processes are martingales with no Brownian component, satisfying $r^* = +\infty$. Their symmetric Lévy measure has a Lebesgue density that behaves like Cx^{-2} as $x \to 0$ [so that (4.3) is satisfied with $\underline{\theta} = 1$]. Hence, one obtains, for every $r,\ p \in (0, 1)$,

$$e_{N,r}(X, L_T^p) = O((\log N)^{-1})$$

and, for every $r \geq 1$ and every $p \in (0, r]$, $e_{N,r}(X, L_T^p) = O((\log N)^{-1/r+\eta})$, $\eta > 0$.

The proof of this theorem is divided into several steps and is deferred to Section 4.3. The reason is that it relies on the decomposition of X as the sum of a "bounded" jump and a "big" jump Lévy process. These are treated successively in the following two sections.

4.1. *Lévy processes with bounded jumps.* In this section, we consider a Lévy process X *without Brownian component* ($\sigma = 0$), with jumps bounded by a real constant $c > 0$. In terms of the Lévy measure ν of X, this means that

$$\nu([-c, c]^c) = 0. \tag{4.7}$$

Then, for every $\rho > 0$ and every $t \geq 0$, $X_t \in L^\rho(\mathbb{P})$, that is, $r^* = +\infty$. In Proposition 2 below, we establish Theorem 2 in that setting.

PROPOSITION 2. *Let* $(X_t)_{t\in[0,T]}$ *be a Lévy process satisfying* (4.7) *and* $\underline{\theta} > 0$. *Then claims* (a), (b), (c) *and* (d) *in Theorem* 2 *hold true with* $r^* = \infty$.

PROOF. The proof of this proposition is decomposed into several steps. We consider $\underline{\theta}$, as defined in Theorem 1. Note that, in the present setting, $\underline{\theta} = \inf\{\theta > 0 : \int |x|^\theta \nu(dx) < +\infty\}$ and that $\int |x|^\theta \nu(dx) < +\infty$ for every $\theta > \underline{\theta}$. The starting point is to separate the "small" and the "big" jumps of X in a non-homogeneous way with respect to the function $s \mapsto s^{1/\underline{\theta}}$. We will successively inspect the cases $\underline{\theta} \in (0, 1)$ (or when $\underline{\theta} = 1$ holds as a minimum) and $\underline{\theta} \in [1, 2]$.

STEP 1 (*Decomposition of X*). When $\underline{\theta} \in (0,1)$ or $\underline{\theta} = 1$ holds as a minimum, then

$$\mathbb{E}\left|\sum_{0<s\leq T} \Delta X_s\right| \leq \mathbb{E} \sum_{0<s\leq T} |\Delta X_s| = T \int |x|\nu(dx) < +\infty.$$

Consequently, X $\mathbb{P}$-a.s. has finite variation and we can decompose X as

$$X_t = \xi(t) + \sum_{0<s\leq t} \Delta X_s, \tag{4.8}$$

where $\xi(t) = at$ is a linear function.

Assume now that $\underline{\theta} \in [1,2]$. We may decompose X as follows:

$$\begin{aligned} X_t &= \xi(t) + X_t^{(\underline{\theta})} + M_t^{(\underline{\theta})} \quad \text{with} \\ \xi(t) &:= t\,\mathbb{E}(X_1), \\ X_t^{(\underline{\theta})} &:= \sum_{0<s\leq t} \Delta X_s \mathbf{1}_{\{|\Delta X_s| > s^{1/\underline{\theta}}\}} - \int_0^t ds \int_{\{s^{1/\underline{\theta}} < |x| \leq c\}} x\nu(dx). \end{aligned} \tag{4.9}$$

Note that $X^{(\underline{\theta})}$ has finite variations on $[0,T]$ since

$$\begin{aligned} &\int_0^t ds \int_{\{s^{1/\underline{\theta}} < |x| \leq c\}} |x|\nu(dx) \\ &\qquad = \int_{\{|x|\leq c\}} |x|(|x|^{\underline{\theta}} \wedge t)\nu(dx) \leq \int_{\{|x|\leq c\}} |x|^{1+\underline{\theta}}\nu(dx) < +\infty. \end{aligned}$$

Both $X^{(\underline{\theta})}$ and $M^{(\underline{\theta})}$ are martingales with (nonhomogeneous) independent increments. Their increasing predictable "bracket" processes are given by

$$\langle X^{(\underline{\theta})}\rangle_t = \int_0^t ds \int_{\{|x| > s^{1/\underline{\theta}}\}} x^2\nu(dx)$$

and

$$\langle M^{(\underline{\theta})}\rangle_t = \int_0^t ds \int_{\{|x| \leq s^{1/\underline{\theta}}\}} x^2\nu(dx).$$

From now on, we may consider the (supremum process of the) Lévy process

$$\widetilde{X}_t := X_t - \xi(t), \tag{4.10}$$

where ξ is the linear function defined by (4.8) and (4.9), respectively. Since the linear function ξ lies in L_T^p, it does not affect the quantization rate, which is invariant by translation.

STEP 2 [Increment estimates in $L^\rho(\mathbb{P})$]. In this step, we evaluate $\sup_{0\leq s\leq t} |\widetilde{X}_s|$ in $L^\rho(\mathbb{P})$, $\rho \in (0,2]$. Throughout this step, the c comes from (4.7).

LEMMA 3. (a) *Assume that* $\underline{\theta} \in (0,1)$ *or that* $\underline{\theta} = 1$ *holds as a minimum. For every* $\rho \in (0,1]$ *and* $t \in [0,T]$,

$$\mathbb{E}\Big(\sup_{0\le s\le t} |\widetilde{X}_s|^\rho\Big) \le C_\rho\bigg(\bigg(\int_0^t \int_{\{|x|\le s^{1/\underline{\theta}}\}} x^2\nu(dx)\bigg)^{\rho/2} + \int_0^t ds \int_{\{s^{1/\underline{\theta}}<|x|\le c\}} |x|^\rho \nu(dx) + \sup_{0\le s\le t}\bigg|\int_0^s du \int_{\{|x|\le u^{1/\underline{\theta}}\}} x\nu(dx)\bigg|^\rho\bigg). \tag{4.11}$$

(b) *Assume that* $\underline{\theta} \in [1,2]$. *For every* $\rho \in (0,2]$ *and every* $t \in [0,T]$,

$$\begin{aligned}\mathbb{E}\Big(\sup_{0\le s\le t} |\widetilde{X}_s|^\rho\Big) \le C_\rho\bigg(&\bigg(\int_0^t ds\int_{\{|x|\le s^{1/\underline{\theta}}\}} x^2\nu(dx)\bigg)^{\rho/2} \\ &+ \int_0^t ds \int_{\{s^{1/\underline{\theta}}<|x|\le c\}} |x|^\rho \nu(dx) \\ &+ \bigg(\int_0^t ds\int_{\{s^{1/\underline{\theta}}<|x|\le c\}} |x|^{\rho/2}\nu(dx)\bigg)^2\bigg) \\ &+ \sup_{0\le s\le t}\bigg|\int_0^s du \int_{\{u^{1/\underline{\theta}}<|x|\le c\}} x\nu(dx)\bigg|^\rho.\end{aligned} \tag{4.12}$$

PROOF. (a) $\widetilde{X}$ is a pure jump process (with finite variations). Using $\rho \in (0,1]$ and Doob's inequality, we obtain

$$\begin{aligned}&\mathbb{E}\sup_{0\le s\le t} |\widetilde{X}_s|^\rho \\ &\quad\le \mathbb{E}\sup_{0\le s\le t}\bigg|\sum_{0\le u\le s}\Delta X_u \mathbf{1}_{\{|\Delta X_u|\le u^{1/\underline{\theta}}\}}\bigg|^\rho + \mathbb{E}\sup_{0\le s\le t}\bigg|\sum_{0\le u\le s}\Delta X_u \mathbf{1}_{\{|\Delta X_u|> u^{1/\underline{\theta}}\}}\bigg|^\rho \\ &\quad\le \bigg(\mathbb{E}\sup_{0\le s\le t}\bigg(\sum_{0\le u\le s}\Delta X_u \mathbf{1}_{\{|\Delta X_u|\le u^{1/\underline{\theta}}\}}\bigg)^2\bigg)^{\rho/2} + \mathbb{E}\sum_{0<s\le t}|\Delta X_s|^\rho \mathbf{1}_{\{|\Delta X_s|> s^{1/\underline{\theta}}\}} \\ &\quad\le C_\rho\bigg(\bigg(\mathbb{E}\sup_{0\le s\le t}\bigg(\sum_{0\le u\le s}\Delta X_u \mathbf{1}_{\{|\Delta X_u|\le u^{1/\underline{\theta}}\}} - \int_0^s du\int_{\{|x|\le u^{1/\underline{\theta}}\}} x\nu(dx)\bigg)^2\bigg)^{\rho/2} \\ &\qquad\qquad + \sup_{0\le s\le t}\bigg|\int_0^s du\int_{\{|x|\le u^{1/\underline{\theta}}\}} x\nu(dx)\bigg|^\rho + \int_0^t ds\int_{\{s^{1/\underline{\theta}}<|x|\le c\}}|x|^\rho\nu(dx)\bigg) \\ &\quad\le C_\rho\bigg(\bigg(\int_0^t ds\int_{\{|x|\le s^{1/\underline{\theta}}\}} x^2\nu(dx)\bigg)^{\rho/2}\end{aligned}$$

$$+\sup_{0\le s\le t}\left|\int_0^s du\int_{\{|x|\le u^{1/\underline{\theta}}\}}x\nu(dx)\right|^\rho+\int_0^t ds\int_{\{s^{1/\underline{\theta}}<|x|\le c\}}|x|^\rho\nu(dx)\Bigg).$$

(b) It follows from Doob's inequality (and $0<\rho/2\le 1$) that

$$\mathbb{E}\Big(\sup_{0\le s\le t}|M_s^{(\underline{\theta})}|^\rho\Big)\le\Big[\mathbb{E}\sup_{0\le s\le t}\big(M_s^{(\underline{\theta})}\big)^2\Big]^{\rho/2}\le\bigg(4\int_0^t ds\int_{\{|x|\le s^{1/\underline{\theta}}\}}x^2\nu(dx)\bigg)^{\rho/2}.$$

On the other hand, since $\rho\in(0,2]$, we have

$$\begin{aligned}
&\sup_{0\le s\le t}\big|X_s^{(\underline{\theta})}\big|^\rho\\
&\quad\le C_\rho\Bigg(\bigg(\sum_{0<s\le t}|\Delta X_s|^{\rho/2}\mathbf{1}_{\{|\Delta X_s|>s^{1/\underline{\theta}}\}}\bigg)^2\\
&\qquad+\sup_{0\le s\le t}\left|\int_0^s du\int_{\{u^{1/\underline{\theta}}<|x|\le c\}}x\nu(dx)\right|^\rho\Bigg)\\
&\quad\le C_\rho\Bigg(\bigg(\sum_{0<s\le t}|\Delta X_s|^{\rho/2}\mathbf{1}_{\{|\Delta X_s|>s^{1/\underline{\theta}}\}}-\int_0^t ds\int_{\{|x|>s^{1/\underline{\theta}}\}}|x|^{\rho/2}\nu(dx)\bigg)^2\\
&\qquad+\bigg(\int_0^t ds\int_{\{|x|>s^{\frac{1}{\underline{\theta}}}\}}|x|^{\rho/2}\nu(dx)\bigg)^2\\
&\qquad\qquad+\sup_{0<s\le t}\left|\int_0^s du\int_{\{u^{1/\underline{\theta}}<|x|\le c\}}x\nu(dx)\right|^\rho\Bigg).
\end{aligned}$$

Hence, again using Doob's inequality,

$$\begin{aligned}
&\mathbb{E}\sup_{0\le s\le t}\big|X_s^{(\underline{\theta})}\big|^\rho\\
&\quad\le C_\rho\Bigg(\int_0^t ds\int_{\{s^{1/\underline{\theta}}<|x|\le c\}}|x|^\rho\nu(dx)+\bigg(\int_0^t ds\int_{\{s^{1/\underline{\theta}}<|x|\le c\}}|x|^{\rho/2}\nu(dx)\bigg)^2\\
&\qquad\qquad+\sup_{0<s\le t}\left|\int_0^s du\int_{\{u^{1/\underline{\theta}}<|x|\le c\}}x\nu(dx)\right|^\rho\Bigg).
\end{aligned}$$

□

LEMMA 4 (First extended Millar's lemma). (a) *Assume that* $\underline{\theta}\in(0,2]\setminus\{1\}$. *If the Lévy measure satisfies assumption* (4.3) *then*

$$\forall\rho\in(0,\underline{\theta}),\ \forall t\in[0,T]\qquad\mathbb{E}\sup_{0\le s\le t}|\widetilde{X}_s|^\rho\le C_\rho t^{\rho/\underline{\theta}}.\tag{4.13}$$

(b) *Assume that* $\underline{\theta} \in (0,2) \setminus \{1\}$ *and that the function* $u \mapsto \underline{\nu}(u)$ *has regular variation with index* $-b$ *at* 0. *Then* $b = \underline{\theta}$ *and, for every* $\rho \in (0, \underline{\theta})$, *there exists* $T_\rho \in (0, T]$ *such that*

$$\forall t \in [0, T_\rho] \qquad \mathbb{E} \sup_{0 \le s \le t} |\widetilde{X}_s|^\rho \le C_\rho (t^{1/\underline{\theta}} \Lambda_\rho(t))^\rho. \tag{4.14}$$

(c) *When* $\underline{\theta} = 1$, *the above upper bounds still hold, provided* ν *is symmetric or* $\nu(|x|) < +\infty$.

PROOF. (a) We need only to investigate all the integrals appearing in the right-hand side of inequalities (4.11) and (4.12) in Lemma 3. Let $\rho \in (0, \underline{\theta})$ and $t \in [0, c^{\underline{\theta}} \wedge T]$. Then, if $\underline{\theta} \in (0,2)$,

$$\begin{aligned}
\int_0^t ds \int_{\{0<|x|\le s^{1/\underline{\theta}}\}} x^2 \nu(dx) &\le C \int_0^t ds \int_{\{0<|x|\le s^{1/\underline{\theta}}\}} |x|^{1-\underline{\theta}}\, dx \\
&\le C \int_0^t s^{2/\underline{\theta}-1}\, ds = C t^{2/\underline{\theta}},
\end{aligned}$$

where the real constant C comes from (4.3). If $\underline{\theta} = 2$, then

$$\int_0^t ds \int_{\{0<|x|\le s^{1/\underline{\theta}}\}} x^2 \nu(dx) \le \int_{[-c,c]} x^2 \nu(dx) t = \int_{[-c,c]} x^2 \nu(dx) t^{2/\underline{\theta}}.$$

Then, for every $t \in [0, c^{\underline{\theta}} \wedge T]$,

$$\begin{aligned}
\int_0^t ds \int_{\{s^{1/\underline{\theta}}<|x|\le c\}} |x|^\rho \nu(dx) &\le C \int_0^t ds \int_{\{s^{1/\underline{\theta}}<|x|\le c\}} |x|^{\rho-\underline{\theta}-1}\, dx \\
&\le C/\underline{\theta} - \rho \int_0^t s^{\rho/\underline{\theta}-1}\, ds = C t^{\rho/\underline{\theta}}.
\end{aligned}$$

When $\underline{\theta} \in (0,1)$, we have

$$\begin{aligned}
\sup_{0\le s\le t} \left| \int_0^s du \int_{\{|x|\le u^{1/\underline{\theta}}\}} x \nu(dx) \right| &\le \int_0^t ds \int_{\{|x|\le s^{1/\underline{\theta}}\}} |x| \nu(dx) \\
&\le C \int_0^t \frac{s^{1/\underline{\theta}-1}}{1-\underline{\theta}}\, ds = \frac{C}{1-\underline{\theta}} t^{1/\underline{\theta}}.
\end{aligned}$$

When $\underline{\theta} = 1$ and $\int |x| \nu(dx) < +\infty$, this term is trivially upper bounded by $t \int |x| \nu(dx)$. It is 0 when ν is symmetric. Similarly, when $\underline{\theta} \in (1,2]$, for every $t \in [0, c^{\underline{\theta}} \wedge T]$, we have

$$\begin{aligned}
\sup_{0\le s\le t} \left| \int_0^s du \int_{\{u^{1/\underline{\theta}}<|x|\le c\}} x \nu(dx) \right| &\le \int_0^t ds \int_{\{|x|>s^{1/\underline{\theta}}\}} |x| \nu(dx) \\
&\le C \int_0^t \frac{s^{1/\underline{\theta}-1}}{\underline{\theta}-1}\, ds = \frac{C}{\underline{\theta}-1} t^{1/\underline{\theta}}
\end{aligned}$$

and

$$\int_0^t ds \int_{\{s^{1/\underline{\theta}}<|x|\le c\}} |x|^{\rho/2}\nu(dx) \le C \int_0^t ds \int_{\{s^{1/\underline{\theta}}<|x|\le c\}} |x|^{\rho/2-\underline{\theta}-1}\,dx$$
$$\le \frac{C}{\underline{\theta}-\rho/2}\int_0^t s^{\rho/(2\underline{\theta})-1}\,ds = Ct^{\rho/(2\underline{\theta})}.$$

It can be derived from (4.11) and (4.12) that there exists a positive real constant C_ρ such that

$$\forall t \in [0, c^{\underline{\theta}} \wedge T] \qquad \mathbb{E}\sup_{0\le s\le t} |\widetilde{X}_s|^\rho \le C_\rho t^{\rho/\underline{\theta}}.$$

This inequality holds for every $t \in [0, T]$ simply by adjusting the constant C_ρ.

(b) The fact that $b = \underline{\theta}$ was first established in [5]. We provide below a short proof, leading to our main result, for the reader's convenience. It follows from Theorem 1.4.1 in [3] that $\underline{\nu}(u) = u^{-b}\ell(u)$ where ℓ is a (nonnegative) slowly varying function. Consequently, one clearly has that, for every $\rho > 0$ and every $u > 0$,

$$u^{\rho-b}\ell(u) \le \int_{\{|x|>u\}} |x|^\rho \nu(dx).$$

Now, the left-hand side of the above inequality goes to infinity as $u \to 0$ provided $\rho < b$ since ℓ has slow variations (see Proposition 1.3.6 in [3]). Consequently, $\rho \le \underline{\theta}$. Letting θ go to b implies that $b \le \underline{\theta}$.

We will make use of the following easy identity which follows from the very definition of $\underline{\nu}$: for every nonnegative Borel function $f : \mathbb{R}_+ \to \mathbb{R}$,

$$\int_{\mathbb{R}} f(|x|)\nu(dx) = -\int_{\mathbb{R}_+} f(x)\,d\underline{\nu}(x). \tag{4.15}$$

In particular, for every $x \in (0, c]$ and every $a > 0$,

$$\int_{\{|u|\ge x\}} |u|^a \nu(du) = -\int_x^c u^a\,d\underline{\nu}(u).$$

Assume that $b < \underline{\theta}$. It then follows from Theorem 1.6.4 in [3] that for every $a \in (b, \underline{\theta})$,

$$\int_x^c u^a\,d\underline{\nu}(u) \sim \frac{b}{b-a} x^a\,\underline{\nu}(x) = \frac{b}{b-a} x^{a-b}\ell(x) \to 0 \qquad \text{as } x \to 0,$$

since ℓ is slowly varying. This contradicts $\int |u|^a \nu(du) = +\infty$. Consequently, $b = \underline{\theta}$.

Now, Theorem 1.6.5 in [3] implies that for any $a > \underline{\theta}$

$$\int_{\{|u|\le x\}} |u|^a \nu(du) = -\int_{(0,x]} u^a\,d\underline{\nu}(u) \sim \frac{\underline{\theta}}{a-\underline{\theta}} x^a\,\underline{\nu}(x) \qquad \text{as } x \to 0.$$

Since $\underline{\theta} \neq 2$, this yields

$$\int_{\{|x|\le s^{1/\underline{\theta}}\}} x^2 \nu(dx) \sim \frac{\underline{\theta}}{2-\underline{\theta}} s^{2/\underline{\theta}} \underline{\nu}(s^{1/\underline{\theta}}) \qquad \text{as } s \to 0,$$

which, in turn, implies that

$$\int_0^t ds \int_{\{|x|\le s^{1/\underline{\theta}}\}} x^2 \nu(dx) \sim \frac{\underline{\theta}}{2-\underline{\theta}} \int_0^t s^{2/\underline{\theta}} \underline{\nu}(s^{1/\underline{\theta}})\, ds \qquad \text{as } t \to 0.$$

The function $s \mapsto \underline{\nu}(s^{1/\underline{\theta}})$ has regular variation (at 0) with index -1, hence Theorem 1.6.1 in [3] implies that

$$\int_0^t ds \int_{\{|x|\le s^{1/\underline{\theta}}\}} x^2 \nu(dx) \sim C_{\underline{\theta}}\, t^{2/\underline{\theta}+1} \underline{\nu}(t^{1/\underline{\theta}}) \qquad \text{as } t \to 0.$$

Finally,

$$\left(\int_0^t ds \int_{\{|x|\le s^{1/\underline{\theta}}\}} x^2 \nu(dx)\right)^{\rho/2} \sim C_{\rho,\underline{\theta}} (t^{1/\underline{\theta}} (\underline{\ell}(t))^{1/2})^{\rho} \qquad \text{as } t \to 0. \tag{4.16}$$

When $\underline{\theta} \in (0, 1)$ and $\rho \in (0, \underline{\theta})$, the same approach leads to

$$\begin{aligned} &\sup_{0<s\le t} \left| \int_0^s du \int_{\{|x|\le u^{1/\underline{\theta}}\}} x \nu(dx)\right| \\ &\qquad \le \int_0^t ds \int_{\{|x|\le s^{1/\underline{\theta}}\}} |x| \nu(dx) \sim C_{\underline{\theta}}\, t^{1/\underline{\theta}} \underline{\ell}(t) \qquad \text{as } t \to 0. \end{aligned}$$

It then follows from Theorem 1.6.4 in [3] that, for every $\rho \in (0, \underline{\theta})$,

$$\begin{aligned} \int_{\{s^{1/\underline{\theta}} \le |x| \le c\}} |x|^{\rho} \nu(dx) &= -\int_{s^{1/\underline{\theta}}}^{c} x^{\rho}\, d\underline{\nu}(x) \\ &\sim \frac{\underline{\theta}}{\underline{\theta}-\rho} s^{\rho/\underline{\theta}} \underline{\nu}(s^{1/\underline{\theta}}) \qquad \text{as } s \to 0 \end{aligned}$$

so that

$$\begin{aligned} \int_0^t \int_{\{s^{1/\underline{\theta}} \le |x| \le c\}} |x|^{\rho} \nu(dx) &\sim \underline{\theta}/\underline{\theta} - \rho \int_0^t s^{\rho/\underline{\theta}} \underline{\nu}(s^{1/\underline{\theta}})\, ds \\ &\sim C_{\rho,\underline{\theta}}\, (t^{1/\underline{\theta}} (\underline{\ell}(t))^{1/\rho})^{\rho} \qquad \text{as } t \to 0. \end{aligned}$$

Similarly (by formally setting $\rho = 1$ in the former equation) we can shown that if $\underline{\theta} \in (1, 2]$, then

$$\begin{aligned} \sup_{0<s\le t} \left| \int_0^s du \int_{\{u^{1/\underline{\theta}} \le |x| \le c\}} x \nu(dx)\right| &\le \int_0^t ds \int_{\{s^{1/\underline{\theta}} \le |x| \le c\}} |x| \nu(dx) \\ &\sim C_{\underline{\theta}}\, t^{1/\underline{\theta}} \underline{\ell}(t) \qquad \text{as } t \to 0. \end{aligned} \tag{4.17}$$

Finally, we similarly shown, for the last term in (4.12), that when $\rho \in (0, \underline{\theta})$,

$$\left(\int_0^t \int_{\{s^{1/\underline{\theta}} \le |x| \le c\}} |x|^{\rho/2} \nu(dx)\right)^2 \sim C_{\rho,\underline{\theta}} \, (t^{1/\underline{\theta}} (\underline{\ell}(t))^{2/\rho})^\rho \qquad \text{as } t \to 0.$$

Substituting these estimates into (4.11) and (4.12) and noting that, by Young's inequality,

$$\underline{\ell}(t) \le C_\rho \big((\underline{\ell}(t))^{1/2} + (\underline{\ell}(t))^{1/\rho} \mathbf{1}_{\{\rho \le 1\}} + (\underline{\ell}(t))^{2/\rho} \mathbf{1}_{\{1 < \rho \le 2\}}\big),$$

we finally obtain that $\widetilde{X}$ satisfies the assumption $(L_{\varphi,\rho})$ with the announced function φ_ρ.

(c) When ν is symmetric (and $\underline{\theta} \in (1, 2]$), for every $s \in [0, T]$,

$$\int_0^s du \int_{\{u^{1/\underline{\theta}} \le |x| \le c\}} x \nu(dx) = 0$$

so that the condition $\underline{\theta} \ne 1$ induced by (4.17) is no longer necessary. Similarly, when $\underline{\theta} \in (0, 1]$,

$$\int_0^s du \int_{\{|x| \le u^{1/\underline{\theta}}\}} x \nu(dx) = 0. \qquad \square$$

STEP 3 (Higher moments and completion of the proof). Claims (a), when $\underline{\theta}$ holds as a minimum, and (c), when $r < 2$, straightforwardly follow from Millar's inequality (3.3) by applying Theorem 1 to the function $\varphi(u) = u^{1/\underline{\theta}}$ with $\rho = \underline{\theta}$ for claim (a) and $\varphi(u) = u^{1/\rho}$ with $\rho \in (r, 2]$ for claim (c).

Claim (a), when assumption (4.3) is fulfilled, follows from Lemma 4(a) and Theorem 1 applied with the function $\varphi(u) = u^{1/\underline{\theta}}$. Finally, claim (b) follows from Lemma 4(b) and Theorem 1.

Claim (d) follows from Lemma 4(c) and Theorem 1. At this stage, it remains to prove claim (c) when $r \ge 2$. This follows (when $r > 2$) from the extension of Millar's upper bound established in the lemma below.

LEMMA 5 (Second extended Millar's lemma). *Let $(X_t)_{t \in [0,T]}$ be a Lévy process without Brownian part such that $\nu([-c, c]^c) = 0$. For every $\rho \ge 2$, there exists a real constant $C_{\rho,T} > 0$ such that*

$$\forall t \in [0, T] \qquad \mathbb{E}\Big(\sup_{0 \le s \le t} |X_s|^\rho\Big) \le C_{\rho,T}\, t.$$

PROOF. We again consider $\widetilde{X}_t = X_t - t\,\mathbb{E}X_1$, which is a martingale Lévy process. Let $k_\rho := \max\{l : 2^l < \rho\}$. For every $k = 1, \dots, k_\rho$, we define the martingales

$$N_t^{(k)} := \sum_{0 < s \le t} |\Delta X_s|^{2^k} - t \int |x|^{2^k} \nu(dx).$$

The key technique of the proof is to apply the BDG inequality in cascade. It follows from the BDG inequality that

$$\begin{aligned}\mathbb{E}\sup_{0\le s\le t}|\widetilde{X}_s|^\rho &\le C_\rho\mathbb{E}\biggl(\sum_{0<s\le t}(\Delta X_s)^2\biggr)^{\rho/2}\\ &\le C_\rho\biggl(\mathbb{E}(N_t^{(1)})^{\rho/2}+\biggl(t\int x^2\nu(dx)\biggr)^{\rho/2}\biggr).\end{aligned}$$

Now, for every $k\in\{1,\dots,k_\rho-1\}$, still using the BDG inequality yields

$$\begin{aligned}\mathbb{E}(N_t^{(k)})^{\rho/2^k} &\le C_{\rho,k}\mathbb{E}\biggl(\sum_{0<s\le t}|\Delta X_s|^{2^{k+1}}\biggr)^{\rho/2^{k+1}}\\ &\le C_{\rho,k}\biggl(\mathbb{E}(N_t^{(k+1)})^{\rho/2^{k+1}}+\biggl(t\int|x|^{2^{k+1}}\nu(dx)\biggr)^{\rho/2^{k+1}}\biggr).\end{aligned}$$

Finally, we obtain

$$\begin{aligned}\mathbb{E}\sup_{0\le s\le t}|\widetilde{X}_s|^\rho &\le C_\rho\biggl(\sum_{k=1}^{k_\rho}\biggl(t\int|x|^{2^k}\nu(dx)\biggr)^{\rho/2^k}+\mathbb{E}\biggl(\sum_{0<s\le t}|\Delta X_s|^{2^{k_\rho+1}}\biggr)^{\rho/2^{k_\rho+1}}\biggr)\\ &\le C_\rho\biggl(\sum_{k=1}^{k_\rho}\biggl(t\int|x|^{2^k}\nu(dx)\biggr)^{\rho/2^k}+\mathbb{E}\sum_{0<s\le t}|\Delta X_s|^\rho\biggr)\\ &= C_\rho\biggl(\sum_{k=1}^{k_\rho}\biggl(t\int|x|^{2^k}\nu(dx)\biggr)^{\rho/2^k}+t\int|x|^\rho\nu(dx)\biggr)\end{aligned}$$

since $\rho/2^{k_\rho+1}\le 1$. The conclusion follows from the fact that $t^{\rho/2^k}=o(t)$. □

4.2. *Compound Poisson process.* In this section, we consider a compound Poisson process $(X_t)_t$ defined by

$$X_t:=\sum_{n\ge 1}U_n\mathbf{1}_{\{S_n\le\lambda T\}},\qquad t\ge 0,$$

where $S_n=Z_1+\cdots+Z_n$, $(Z_n)_{n\ge 1}$ is an i.i.d. sequence of $\mathcal{E}xp(1)$-distributed random variables, $(U_n)_{n\ge 1}$ is an i.i.d. sequence of random variables, independent of $(Z_n)_{n\ge 1}$ with $U_1\in L^\rho$, $\rho>0$ and $\lambda>0$ is the the jump intensity. For convenience, we also introduce the underlying standard Poisson process $(K_t)_{t\ge 0}$ defined by

$$K_t:=\sum_{n\ge 1}\mathbf{1}_{\{S_n\le\lambda T\}},\qquad t\ge 0,$$

so that (with the convention that $\sum_{\varnothing} = 0$)

$$X_t = \sum_{k=1}^{K_t} U_k. \tag{4.18}$$

PROPOSITION 3. *Let X be a compound Poisson process. Then, for every $p, r \in (0, r^*)$, $p \le r$,*

$$\forall \varepsilon > 0 \quad e_{N,r}(X, L_T^p) = O\left(\exp\left(-\frac{1}{\sqrt{r(p+1+\varepsilon)}}\sqrt{\log(N)\log_2(N)}\right)\right). \tag{4.19}$$

Furthermore, when X is a standard Poisson process, we can replace $p + 1 + \varepsilon$ by $p + \varepsilon$ in (4.19).

REMARKS. Note that (4.19) implies that

$$\forall a > 0 \qquad e_{N,r}(X, L_T^p) = o((\log N)^{-a}).$$

In fact, the rate obtained in the above proposition holds provided X has the form (4.18), where (Z_n) is as above and (U_n) is $L^r(\mathbb{P})$-bounded for every $r < r^*$, independent of $(Z_n)_{n\ge1}$.

PROOF OF PROPOSITION 3. We divide the proof into two steps, one devoted to the standard Poisson process, the other to the general case. We will assume that $r^* > 1$ throughout the proof so that, as was already emphasized in the proof of Theorem 1, we may assume without loss of generality that $r,\ p \in (0, r^*) \cap [1, +\infty)$. The case $r^* \le 1$ is left to the reader, but can be treated by replacing the "triangular" Minkowski inequality by the pseudo-triangular inequalities $|f + g|_{L_T^p}^p \le |f|_{L_T^p}^p + |g|_{L_T^p}^p$ and $\|U + V\|_r^r \le \|U\|_r^r + \|V\|_r^r$.

STEP 1 (Standard case). One quantizes the standard Poisson K in a very natural way by setting

$$\widehat{K}_t := \sum_{n\ge1} \mathbf{1}_{\{\widehat{S}_n \le \lambda t\}}, \qquad t \ge 0,$$

with

$$\widehat{S}_n := \widehat{S_n}^{\alpha_n},$$

where $\alpha_n = \alpha_n' \cup \{\lambda T\}$, α_n' is an $L^{r'}$-optimal $(N_n - 1)$-quantization of $S_n^{tr} := S_n \mathbf{1}_{\{S_n \le \lambda T\}}$ and $r' = \frac{r}{p}$. Furthermore, we assume that the sequence (N_n) is nonincreasing and satisfies $\prod_n N_n \le N$ (so that $N_n = 1$ for large enough n). Then, for every $p \ge 1$, it follows from the (extended) Minkowski inequality that

$$|K - \widehat{K}|_{L_T^p} \le \sum_{n\ge1} \left|\mathbf{1}_{\{S_n \le \lambda\cdot\}} - \mathbf{1}_{\{\widehat{S}_n \le \lambda\cdot\}}\right|_{L_T^p}.$$

Now,

$$\big|\mathbf{1}_{\{S_n\le\lambda\cdot\}}-\mathbf{1}_{\{\widehat{S}_n\le\lambda\cdot\}}\big|^p_{L^p_T}=\int_0^T\big|\mathbf{1}_{\{S_n\le\lambda t\}}-\mathbf{1}_{\{\widehat{S}_n\le\lambda t\}}\big|^p\,dt$$
$$=\frac{1}{\lambda}\,|S_n\wedge(\lambda T)-\widehat{S}_n\wedge(\lambda T)|=\frac{1}{\lambda}\,|S_n\wedge(\lambda T)-\widehat{S}_n|.$$

Now, $\{S_n>\lambda T\}\subset\{\widehat{S}_n=\lambda T\}$ since $\max\alpha_n=\lambda T$. On the other hand, $S_n=S_n^{tr}$ on $\{S_n\le\lambda T\}$ so that

$$|S_n\wedge(\lambda T)-\widehat{S}_n|=|S_n\wedge(\lambda T)-\widehat{S}_n|\mathbf{1}_{\{S_n\le\lambda T\}}=|S_n^{tr}-\widehat{S_n^{tr}}|\mathbf{1}_{\{S_n\le\lambda T\}}\le|S_n^{tr}-\widehat{S_n^{tr}}|.$$

Also, note that when $N_n=1$, $\widehat{S}_n=\lambda T$ so that $|S_n\wedge(\lambda T)-\widehat{S}_n|=(\lambda T-S_n)_+$. Consequently, for every $r\ge 1$,

$$\begin{aligned}\big\||K-\widehat{K}|_{L^p_T}\big\|_r&\le\sum_{n\ge1}\big\||\mathbf{1}_{\{S_n\le\lambda\cdot\}}-\mathbf{1}_{\{\widehat{S}_n\le\lambda\cdot\}}|_{L^p_T}\big\|_r\\&\le\frac{1}{\lambda^{1/p}}\sum_{n\ge1}\|S_n\wedge(\lambda T)-\widehat{S}_n\|_{r'}^{1/p}\\&\le\frac{1}{\lambda^{1/p}}\Bigg(\sum_{n,N_n\ge2}\|S_n^{tr}-\widehat{S_n^{tr}}^{\alpha_n}\|_{r'}^{1/p}+\sum_{n,N_n=1}\|(\lambda T-S_n)_+\|_{r'}^{1/p}\Bigg)\\&\le\frac{1}{\lambda^{1/p}}\Bigg(\sum_{n,N_n\ge2}\|S_n^{tr}-\widehat{S_n^{tr}}^{\alpha'_n}\|_{r'}^{1/p}+\sum_{n,N_n=1}\|(\lambda T-S_n)_+\|_{r'}^{1/p}\Bigg).\end{aligned}$$

The extended Pierce lemma (Lemma 1) yields that, for every $n\ge1$ such that $N_n\ge 2$ and for every $\delta>0$,

$$\begin{aligned}\|S_n^{tr}-\widehat{S_n^{tr}}^{\alpha'_n}\|_{r'}&\le\|S_n^{tr}\|_{r'+\delta/p}C_{r,p,\delta}\,|N_n-1|^{-1}\\&\le 2\big\|S_n\mathbf{1}_{\{S_n\le\lambda T\}}\big\|_{(r+\delta)/p}C_{r,p,\delta}\,N_n^{-1}.\end{aligned}$$

Set $\mu:=r'+\delta/p=\frac{r+\delta}{p}$ so that $\mu p=r+\delta$. We then have

$$\begin{aligned}\big\||K-\widehat{K}|_{L^p_T}\big\|_r&\le C_{p,r,\delta}\frac{1}{\lambda^{1/p}}\Bigg(\sum_{n,N_n\ge2}\big\|S_n\mathbf{1}_{\{S_n\le\lambda T\}}\big\|_\mu^{1/p}\frac{1}{N_n^{1/p}}\\&\qquad+\sum_{n,N_n=1}\|(\lambda T-S_n)_+\|_\mu^{1/p}\Bigg)\qquad(4.20)\\&\le C_{p,r,\delta}T^{1/p}\Bigg(\sum_{n\ge1}\big(\mathbb{P}(S_n\le\lambda T)\big)^{1/(\mu p)}\frac{1}{N_n^{1/p}}\Bigg).\end{aligned}$$

Now, standard computations show that

$$\mathbb{P}(\{S_n \le \lambda T\}) = \frac{(\lambda T)^n}{(n-1)!}\int_0^1 u^{n-1}e^{-\lambda T u}\,du \le \frac{(\lambda T)^n}{n!}.$$

Hence, setting $A = (\lambda T)^{1/(\mu p)}$ yields

$$\big(\mathbb{P}(S_n \le \lambda T)\big)^{1/(\mu p)} \le \frac{(\lambda T)^{n/(\mu p)}}{(n!)^{1/(\mu p)}} \le \frac{A^n}{(n!)^{1/(\mu p)}}.$$

For every $x \ge 0$, let $a(x) := \frac{A^x}{\Gamma(x+1)^{1/(\mu p)}}$. This function reaches a unique maximum at some $x_0 \ge 0$ and then decreases to 0 as $x \to \infty$. We modify the function a by setting $a_0(x) := a(x) \vee a(x_0)$ so that the function a_0 becomes nonincreasing and log-concave since Γ is log-convex. Now, let

$$a_n := a_0(n), \qquad n \ge 1.$$

Finally, the quantization problem (4.20) for the standard Poisson K is "upper bounded" by the following optimal integral "bit allocation" problem:

$$\min\left\{\sum_{n\ge 1}\frac{a_n}{N_n^{1/p}},\ N_n \ge 1, \prod_{n\ge 1} N_n \le N\right\}. \tag{4.21}$$

Then, let $m \ge 2x_0 + 1$ be a temporarily fixed integer. We set, for $N \ge 1$,

$$N_n = \left[\frac{a_n^p N^{1/m}}{(\prod_{1\le k\le m} a_k)^{p/m}}\right], \qquad 1 \le n \le m,\ N_n = 1,\ n \ge m+1.$$

The sequence N_n, $1 \le n \le m$, is nonincreasing. This will ensure that

$$N_n \ge 1, \qquad 1 \le n \le m.$$

We wish to choose m as a function of N so that

$$a_m N^{1/(pm)} \ge \left(\prod_{1\le k\le m} a_k\right)^{1/m}.$$

Using log-concavity, this is clearly satisfied provided that

$$a_m N^{1/(pm)} \ge a_0\big((m+1)/2\big) = a\big((m+1)/2\big) \tag{4.22}$$

[since $(m+1)/2 \ge x_0$]. Inequality (4.22) becomes, by taking logarithms,

$$\begin{aligned}&\frac{m-1}{2}\log A + \frac{1}{pm}\log N\\ &\quad \ge \frac{1}{\mu p}\big(\log\big(\Gamma(m+1)\big) - \log\big(\Gamma\big(1+(m+1)/2\big)\big)\big).\end{aligned} \tag{4.23}$$

We will make use of the following classical inequality: for every $t \ge 1/12$,

$$0 \le \log\big(\Gamma(t+1)\big) - \log\big(\sqrt{2\pi}\big) - (t+1/2)\log t + t \le 1.$$

Then, after some easy computations, one shows that inequality (4.23) is satisfied provided

$$\frac{m-1}{2}\log A+\frac{1}{pm}\log N\ge\frac{1}{\mu p}\left(\frac{m}{2}\log m-\frac{m}{8}-\frac{1}{2}\log m+\frac{5}{2}\right).$$

If one sets (this is probably optimal)

$$m=m(N):=\left\lceil 2\sqrt{\mu\frac{\log N}{\log_2 N}}\right\rceil,$$

then the above inequality is satisfied, as well as $m(N)\ge 2x_0+1$, for every large enough N, provided that we increase the value of A. With N_n and m settled as above and using the fact that $\frac{x}{[x]}\le 2$ for every $x\ge 1$, we obtain

$$\sum_{n\ge 1}\frac{a_n}{N_n^{1/p}}\le 2^{1/p}mN^{-1/(pm)}\left(\prod_{k=1}^{m}a_k\right)^{1/m}+\sum_{n\ge m+1}a_n.$$

On the one hand, $N_m\ge 1$ gives

$$N^{-1/(pm)}\left(\prod_{k=1}^{m}a_k\right)^{1/m}\le a_m.$$

On the other hand, the log-concavity and monotony of the function a over $[x_0+1,\infty)$ (and the fact that a' is nonzero) imply that

$$\sum_{n\ge m+1}a_n\le\left|\frac{a(x_0+1)}{a'(x_0+1)}\right|a_m=o(ma_m)$$

(this follows from a straightforward adaptation of the proof of Proposition 4.4 in [18], to which we refer the reader for details). So we have

$$\begin{aligned}ma_m&=m\frac{A^m}{(m!)^{1/(\mu p)}}\\&\le\exp\left(-\frac{1}{\mu p}m\log m+O(m)\right)\\&\le C\exp\left(-\frac{1}{p\mu}\sqrt{\mu\log N\log_2 N}\left(1+O\left(\frac{\log_3 N}{\log_2 N}\right)\right)\right).\end{aligned}\tag{4.24}$$

Note that $p\sqrt{\mu}=\sqrt{p\cdot p\mu}=\sqrt{(r+\delta)p}$. Finally, this yields, in particular, that for every $\varepsilon>0$,

$$\||K-\widehat{K}|_{L_T^p}\|_r=O\left(\exp\left(-\frac{1}{\sqrt{rp+\varepsilon}}\sqrt{\log N\log_2 N}\right)\right).$$

STEP 2 (Compound case). Starting from (4.18), it is natural to quantize (X_t) by setting

$$\widehat{X}_t = \sum_{k=1}^{\widehat{K}_t} \widehat{U}_k,$$

where $\widehat{K}$ is an $N^{(1)}$-quantization of the standard Poisson process K, as described in Step 1, and, for every $n \geq 1$, $\widehat{U}_n$ is an L^r-optimal $N_n^{(2)}$-quantization of U_n with $1 \leq N_1^{(2)} \times \cdots \times N_n^{(2)} \cdots \leq N^{(2)}$ and $N^{(1)} N^{(2)} \leq N$. Then, setting $\widehat{K}_t^U := \sum_{k=1}^{\widehat{K}_t} U_k$ and $K_t^{\widehat{U}} := \sum_{k=1}^{K_t} \widehat{U}_k$, we obtain

$$|K^{\widehat{U}} - \widehat{K}^{\widehat{U}}|_{L_T^p} \leq \sum_{n \geq 1} |\widehat{U}_k| \left|\mathbf{1}_{\{S_n \leq \lambda T\}} - \mathbf{1}_{\{\widehat{S}_n \leq \lambda T\}}\right|_{L_T^p}$$

so that

$$\begin{aligned}
\| |X - \widehat{K}^U|_{L_T^p} \|_r &\leq \frac{1}{\lambda^{1/p}} \sum_{n \geq 1} \|\widehat{U}_k\|_r \|S_n \wedge (\lambda T) - \widehat{S}_n \wedge (\lambda T)\|_{r/p}^{1/p} \\
&= \frac{\sup_{n \geq 1} \|\widehat{U}_n\|_r}{\lambda^{1/p}} \sum_{n \geq 1} \|S_n \wedge (\lambda T) - \widehat{S}_n \wedge (\lambda T)\|_{r/p}^{1/p},
\end{aligned}$$

where we have used the fact that the sequences (U_n) and (S_n) are independent, as are $(\widehat{U}_n)$ and (S_n). Using

$$\|\widehat{U}_n\|_r \leq \|U_n - \widehat{U}_n\|_r + \|U_1\|_r = \left\|U_1 - \widehat{U}_1^{N_n^{(2)}}\right\|_r + \|U_1\|_r$$

shows that $\sup_{n \geq 1} \|\widehat{U}_n\|_r < +\infty$. Hence, it follows from Step 1 that, for every $c < \frac{1}{\sqrt{pr}}$,

$$\| |X - \widehat{K}^U|_{L_T^p} \|_r = O\Big(\exp\Big(-c\sqrt{\log(N^{(1)}) \log_2(N^{(1)})}\Big)\Big).$$

On the other hand, with obvious notation and using the fact that $(\widehat{U}_n - U_n)$ and (S_n) are independent, we have

$$\begin{aligned}
\| |X - K^{\widehat{U}}|_{L_T^p} \|_r &= \| |K^{U - \widehat{U}}|_{L_T^p} \|_r \\
&\leq \sum_{n \geq 1} \|U_n - \widehat{U}_n\|_r \left\| \left|\mathbf{1}_{\{S_n \leq \lambda \cdot\}}\right|_{L_T^p} \right\|_r \\
&= \frac{1}{\lambda^{1/p}} \sum_{n \geq 1} \|U_n - \widehat{U}_n\|_r \|(\lambda T - S_n)_+\|_{r'}^{1/p} \\
&\leq \frac{1}{\lambda^{1/p}} \sum_{n \geq 1} \|U_n - \widehat{U}_n\|_r \frac{(\lambda T)^{1/p + n/r}}{(n!)^{1/r}}
\end{aligned}$$

$$\leq C \sum_{n\geq 1} \|U_n - \widehat{U}_n\|_r \frac{(\lambda T)^{n/r}}{(n!)^{1/r}}.$$

It now follows from the (extended) Pierce lemma that

$$\begin{aligned}\||K^U - K^{\widehat{U}}|_{L_T^p}\|_r &\leq C_{U_1,r} \sum_{n\geq 1} \frac{(\lambda T)^{n/r}}{(n!)^{1/r} N_n^{(2)}} \\ &= O\Bigl(\exp\Bigl(-\frac{1}{\sqrt{r}}\sqrt{\log(N^{(2)})\log_2(N^{(2)})}\Bigr)\Bigr).\end{aligned}$$

The rate follows from the resolution of the optimal bit allocation problem (4.21) obtained by formally setting $\mu p = r$ and $p = 1$. Then note that, on the one hand,

$$\||X - \widehat{X}|_{L_T^p}\|_r \leq \||X - K^{\widehat{U}}|_{L_T^p}\|_r + \||K^{\widehat{U}} - \widehat{K}^{\widehat{U}}|_{L_T^p}\|_r$$

and on the other hand

$$\widehat{K}_t^{\widehat{U}} = \sum_{n\geq 1} \widehat{U}_n^{N_n^{(2)}} \mathbf{1}_{\{\widehat{S}_n^{N_n^{(1)}} \leq \lambda t\}}$$

can take at most

$$\prod_{n\geq 1} N_n^{(1)} N_n^{(2)} \leq N^{(1)} \times N^{(2)} \leq N$$

values. Let $c < \frac{1}{\sqrt{pr}}$. Setting $N^{(1)} = [N^{rc^2/(1+rc^2)}]$, $N^{(2)} = [N^{1/(1+rc^2)}]$ yields a rate

$$\||X - \widehat{X}|_{L_T^p}\|_r = O\Bigl(\exp\Bigl(-\frac{1}{\sqrt{1/c^2 + r}}\sqrt{\log(N)\log_2(N)}\Bigr)\Bigr),$$

that is,

$$\forall \varepsilon > 0 \qquad \||X - \widehat{X}|_{L_T^p}\|_r = O\Bigl(\exp\Bigl(-\frac{1}{\sqrt{r(p+1+\varepsilon)}}\sqrt{\log(N)\log_2(N)}\Bigr)\Bigr).$$

□

4.3. *Proof of Theorem* 2. Any Lévy process X can be decomposed as the sum $X = X^{(1)} + X^{(2)}$ of two (independent) Lévy processes, one having bounded jumps and the other being a compound Poisson process, according to the decomposition of its Lévy measure

$$\begin{aligned}&\nu(dx) = \nu^{(1)}(dx) + \nu^{(2)}(dx) \\ &\qquad \text{with } \nu^{(1)}(dx) := \mathbf{1}_{\{|x|\leq 1\}}\nu(dx) \text{ and } \nu^{(2)}(dx) := \mathbf{1}_{\{|x|>1\}}\nu(dx).\end{aligned} \tag{4.25}$$

Assume that $r^* > 1$. It is then clear that, for every $r, p \in (0, r^*)$,

$$\begin{aligned} e_{N,r}(X, L_T^p) &\le C_{r,p,T}\, e_{[\sqrt{N}]^2,r'}(X, L_T^{r'}) \\ &\le C_{r,p,T}\big(e_{[\sqrt{N}],r'}\big(X^{(1)}, L_T^{r'}\big) + e_{[\sqrt{N}],r'}\big(X^{(2)}, L_T^{r'}\big)\big), \end{aligned} \tag{4.26}$$

where $r' = r \vee p \vee 1$. It now follows from Proposition 3 that $e_{N,r'}(X^{(2)}, L_T^{r'}) = o(e_{N,r'}(X^{(1)}, L_T^{r'}))$ so that

$$e_{N,r}(X, L_T^p) \le C'_{r,p,T}\, e_{[\sqrt{N}],r'}\big(X^{(1)}, L_T^{r'}\big).$$

Now, using the fact that $\underline{\ell}$ has slow variations at 0, we can derive that

$$e_{[\sqrt{N}],r'}\big(X^{(1)}, L_T^{r'}\big) = O\big(e_{N,r'}\big(X^{(1)}, L_T^{r'}\big)\big).$$

Proposition 2 completes the proof of Theorem 2. When $r^* \le 1$, we use

$$\begin{aligned} e_{N,r}(X, L_T^p)^r &\le C'_{r,p,T}\, e_{[\sqrt{N}]^2,r'}(X, L_T^{r'}) \\ &\le C'_{r,p,T}\big(e_{[\sqrt{N}],r'}\big(X^{(1)}, L_T^{r'}\big)^{r'} + e_{[\sqrt{N}],r'}\big(X^{(2)}, L_T^{r'}\big)^{r'}\big), \end{aligned}$$

with $r' = r \vee p < 1$ (based on the pseudo-triangular inequality satisfied by L^s-pseudo-norms when $s < 1$).

5. Further results for Lévy processes.

5.1. *An exact rate for Lévy processes with a Brownian component.* In that case, the quantization rate of the Brownian motion controls the global rate of convergence.

PROPOSITION 4. *Let X be a Lévy process with a nonvanishing Brownian component. Let $r^* = r^*(X)$, defined by* (4.2). *Then*

$$\forall r,\ p \in (0, r^* \wedge 2) \qquad e_{N,r}(X, L_T^p) = O((\log N)^{-1/2})$$

and

$$\forall r,\ p \in (0, +\infty) \qquad e_{N,r}(X, L_T^p) = \Omega(e_{N,r}(W, L_T^p)).$$

In particular, $\forall r \in (0, +\infty)$, $\forall p \in [1, +\infty)$, $e_{N,r}(X, L_T^p) = \Omega((\log N)^{-1/2})$.

PROOF. We can decompose X as

$$X = cW + X^{(1)} + X^{(2)},$$

where $X^{(i)}$, $i = 1, 2$, have $\nu^{(i)}$ as Lévy measure, as defined in (4.25) in the above proof of Theorem 2. Then, if $r^* > 1$ and $r, p \in (0, r^* \wedge 2)$, we can easily check that, for every $N \ge 1$,

$$\begin{aligned} e_{N,r'}(X, L_T^p) &\le e_{[\sqrt[3]{N}]^3,r'}(X, L_T^{r'}) \\ &\le e_{[\sqrt[3]{N}],r'}(c\,W, L_T^{r'}) + e_{[\sqrt[3]{N}],r'}\big(X^{(1)}, L_T^{r'}\big) + e_{[\sqrt[3]{N}],r'}\big(X^{(2)}, L_T^{r'}\big), \end{aligned}$$

where $r' = r \vee p \vee 1$. It follows from Proposition 3 (see the remark immediately below) that $e_{N,r'}(X^{(2)}, L_T^{r'}) = o(e_{N,r'}(W, L_T^{r'}))$. Now, $\int_{\mathbb{R}\setminus\{0\}} x^2 \nu^{(1)}(dx) < +\infty$, hence, by Millar's lemma,

$$\mathbb{E} \sup_{s\in[0,t]} \left|X_s^{(1)}\right|^2 \le Ct.$$

We can then easily derive from (3.4) (or directly from Theorem 1) that $e_{N,r'}(X^{(1)}, L_T^{r'}) = O((\log N)^{-1/2})$. This yields the announced upper bound since $e_{N,r'}(W, L_T^{r'}) = O((\log N)^{-1/2})$. If $r^* \le 1$, we proceed as above, using the pseudo-triangular inequality for L^s-pseudo-norms (with $r' = r \vee p < 1$).

As concerns the lower bound, note that if Y and Z are L_T^r-valued independent random vectors, then for every $r,\ p > 0$,

$$\begin{aligned}
\left(e_{N,r}(Y+Z, L_T^p)\right)^r &= \inf_{\alpha\subset L_T^p,\, \mathrm{card}(\alpha)\le N} \int \mathbb{E} \min_{a\in\alpha} |Y - z - a|_{L_T^p}^r \mathbb{P}_Z(dz) \\
&\ge \int_{L_T^p} \inf_{\alpha\subset L_T^p,\, \mathrm{card}(\alpha)\le N} \mathbb{E} \min_{a\in\alpha} |Y - z - a|_{L_T^p}^r \mathbb{P}_Z(dz) \\
&= (e_{N,r}(Y, L_T^p))^r
\end{aligned}$$

so that

$$e_{N,r}(Y+Z, L_T^p) \ge \max(e_{N,r}(Y, L_T^p), e_{N,r}(Z, L_T^p)).$$

This holds true, by induction, for any finite sum of independent random variables. In particular,

$$e_{N,r}(X, L_T^p) \ge e_{N,r}(cW, L_T^p) = c e_{N,r}(W, L_T^p).$$

This completes the proof. $\square$

5.2. *Subordinated Lévy processes.* We now consider subordination of the Brownian motion, that is, Lévy processes of the form

$$X_t = W_{A_t}, \qquad t \ge 0,$$

where W denotes a standard Brownian motion and A a subordinator independent of W. A subordinator is a nondecreasing (hence nonnegative) Lévy process. What follows is borrowed from [1]. Its Lévy–Khintchine characteristics (a, σ^2, ν_A) satisfy $\sigma^2 = 0$, $\nu_A((-\infty, 0)) = 0$, $\int_0^1 x\nu_A(dx) < +\infty$ and $\gamma := a - \int_0^1 x\nu_A(dx) \ge 0$ [so that $\underline{\theta}(A) \le 1$]. Consequently, a subordinator is of the form

$$A_t = \gamma t + \sum_{s\le t} \Delta A_s, \qquad t \ge 0.$$

Its Laplace transform is given by $\mathbb{E}\, e^{-uA_t} = e^{-t\Phi(u)}$ with, for every $u \geq 0$,

$$\begin{aligned} \Phi(u) &= \gamma u + \int_0^{+\infty} (1 - e^{-ux}) \nu_A(dx) \\ &= \gamma u + u \int_0^{+\infty} e^{-ux} \underline{\nu}_A(x)\, dx, \end{aligned} \tag{5.1}$$

where $\underline{\nu}_A(x) = \nu_A((x, +\infty))$ denotes the tail of the Lévy measure ν_A and $\lim_{u\to+\infty} \frac{\Phi(u)}{u} = \gamma$. Furthermore, for every $t \geq 0$ and $u \in \mathbb{R}$,

$$\mathbb{E}(e^{iuX_t}) = \mathbb{E}\big(e^{(-u^2/2)A_t}\big) = \exp\left(-\frac{u^2}{2}\gamma - \int_0^{+\infty} (1 - e^{-(u^2/2)x}) \nu_A(dx)\right)$$

so that we can easily derive that

$$\nu_X(f) = \int_{(0,\infty)} \mathbb{E}\big(f(\sqrt{x}\, Z)\big) \nu_A(dx)$$

[with $Z \sim \mathcal{N}(0; 1)$] and that X has a Brownian component if and only if $\gamma > 0$ (see also [24], page 198).

The small deviation of subordinator has been extensively investigated in [17]. It is there established that if $\liminf_{u\to+\infty} \frac{\Phi(u)}{\log u} > 0$, then

$$\forall\, p \in [1, +\infty) \qquad -\log\big(\mathbb{P}(|X|_{L_T^p} \leq \varepsilon)\big) \approx \Phi(\varepsilon^{-2}) \qquad \text{as } \varepsilon \to 0. \tag{5.2}$$

These processes preserve a Gaussian feature which will be the key to estimate their quantization rate: they satisfy the Anderson inequality, as briefly recalled in the lemma below.

LEMMA 6. *A subordinated Lévy process is unimodal for every L_T^p-norm, for every $p \in [1, +\infty)$. The result still holds if one replaces W by, for example, any pathwise continuous centered Gaussian process (e.g., fractional Brownian motion, etc.).*

PROOF. Using the fact that A and W are independent, it suffices to show that for every nondecreasing function $a : [0, T] \to [0, \alpha(T)]$, $a(0) = 0$, and every $x \in L_T^p$,

$$\mathbb{P}\left(\int_0^T |W_{a(s)} - x(s)|^p\, ds \leq \varepsilon^p\right) \leq \mathbb{P}\left(\int_0^T |W_{a(s)}|^p\, ds \leq \varepsilon^p\right), \qquad \varepsilon > 0.$$

It is clear that $(W_{a(t)})_{t\in[0,T]}$ is a centered (bimeasurable) Gaussian process and has sample paths in L_T^p a.s. Hence, $(W_{a(t)})_{t\in[0,T]}$ can be seen as an L_T^p-valued centered Gaussian random vector and the assertion follows from the Anderson inequality. $\square$

We now make the connection between Blumenthal–Getoor indices of X and A (and between the finiteness of moments).

LEMMA 7. *$\underline{\theta}(X) = 2\underline{\theta}(A)$ and $r^*(X) = 2r^*(A)$.*

PROOF. As a consequence of the expression for ν_X, we check that for every $\theta \in (0, 2]$,

$$\int_{\{|x|\le 1\}} |x|^\theta \nu_X(dx) = \int u^{\theta/2} \int_{\{|y|\le 1/\sqrt{u}\}} |y|^\theta e^{-y^2/2} \frac{dy}{\sqrt{2\pi}}$$
$$= \int_{\{u>0\}} u^{\theta/2} \nu_A \ell_\theta(u)(du),$$

where $\ell_\theta(u) > 0$ when $u > 0$ and $\lim_{u\to 0} \ell_\theta(u) = C_\theta \in (0, +\infty)$. Hence, the first equality follows. As concerns the second equality, $r^*(X)$ coincides with the (absolute) moments of X, so it is obvious that

$$\mathbb{E}(|X_t|^r) = \mathbb{E}(|W_{A_t}|^r) = \mathbb{E}(A_t^{2r}).$$

Consequently, $\mathbb{E}(|X_t|^r) < +\infty$ iff $\mathbb{E}(A_t^{2r}) < +\infty$ so that $r^*(X) = 2r^*(A)$. □

As concerns upper bounds, we cannot apply Theorem 2 since a subordinated Lévy process may have a Brownian component. Therefore, we must return Theorem 1.

PROPOSITION 5. (a) *If $\gamma > 0$, then*

$$\forall r,\ p \in \bigl(0, r^*(X) \wedge 2)\bigr) \qquad e_{N,r}(X, L_T^p) = O((\log N)^{-1/2}).$$

(b) *If $\underline{\theta}(A) \in (0, 1)$, $\gamma = 0$ and $\nu_A(dx)\mathbf{1}_{\{0<x\le\eta\}} \le c\mathbf{1}_{\{0<x\le\eta\}} \frac{dx}{x^{1+\underline{\theta}(A)}}$ for some real constants $c, \eta > 0$, then*

$$\forall r,\ p \in (0, \underline{\theta}(X) \wedge r^*(X)) \qquad e_{N,r}(X, L_T^p) = O\bigl((\log N)^{-1/(\underline{\theta}(X))}\bigr).$$

PROOF. (a) follows from Proposition 4 since X has a Brownian component.

(b) Let $\rho < 2(\underline{\theta}(A) \wedge r^*(A))$. First, note that $\mathbb{E}(|X_t|^\rho) = \mathbb{E}A_t^{\rho/2} \le Ct^{\rho/2\underline{\theta}(A)}$, $\rho \le 2(\underline{\theta}(A) \wedge r^*(A)) = \underline{\theta}(X) \wedge r^*(X)$ (by Lemma 4 applied to A). The result then follows from Theorem 1. □

The following lower bounds follow from Lemma 6 and inequality (2.12) (see the remark immediately after Theorem 1). The main point to be noted is that *the upper and lower bounds obtained match, providing an exact quantization rate for subordinated Lévy processes.*

PROPOSITION 6. (a) *If $\gamma > 0$, then*

$$\forall r \in (0, +\infty),\ \forall p \in [1, +\infty) \qquad e_{N,r}(X, L_T^p) = \Omega((\log N)^{-1/2}).$$

(b) *If $\gamma = 0$, $\underline{\theta}(A) > 0$ and $\mathbf{1}_{\{0<x\le\eta\}}\nu_A(dx) \ge c\mathbf{1}_{\{0<x\le\eta\}} \frac{dx}{x^{1+\underline{\theta}(A)}}$ for some real constants $c, \eta > 0$, then*

$$\forall r \in (0, +\infty),\ \forall p \in [1, +\infty) \qquad e_{N,r}(X, L_T^p) = \Omega\bigl((\log N)^{1/\underline{\theta}(X)}\bigr).$$

PROOF. (a) follows from Proposition 4 since X has a Brownian component.

(b) It follows from the assumption made on ν_A that $\underline{\nu}_A(x) \geq c \int_x^\eta \xi^{-\underline{\theta}-1}\, d\xi \geq \kappa x^{-\underline{\theta}}$ for $x \in (0, \eta/2]$. Hence, it follows from (5.1) that

$$\Phi(u) \geq cu \int_0^{\eta/2} e^{-ux} \underline{\nu}_A(x)\, dx = cu^{\underline{\theta}} \int_0^{u\eta/2} e^{-y} y^{-\underline{\theta}}\, dy \geq c' u^{\underline{\theta}}$$

for large enough u (with an appropriate real constant $c' > 0$). We conclude by combining (2.12) and (5.2) since X is strongly unimodal. $\square$

EXAMPLES. If A is a tempered α-stable process with Lévy measure, then

$$\nu_A(dx) = \frac{2^\alpha \alpha}{\Gamma(1-\alpha)} x^{-(\alpha+1)} \exp\left(-\frac{1}{2}\delta^{1/\alpha}\right) \mathbf{1}_{(0,\infty)}(x)\, dx,$$

with $\alpha \in (0,1)$, $\delta > 0$, $\gamma = 0$, so that $\underline{\theta}(A) = \alpha$ and $r^*(A) + \infty$. We the obtain

$$\forall\, r \in (0, 2\alpha),\ \forall\, p \in [1, 2\alpha) \qquad e_{N,r}(X, L_T^p) \approx (\log N)^{-1/(2\alpha)}.$$

Assume that $\underline{\theta}(A) \in (0,1)$ and that the function Φ is regularly varying at ∞ with index $\alpha \in (0,1)$ such that

$$\Phi(x) \sim cx^\alpha (\log(x))^c \qquad \text{as } x \to \infty,$$

for some real constant $c > 0$. Since $\alpha < 1$, we have $\gamma = 0$. Then

$$\Gamma(1-\alpha)\underline{\nu}(x) \sim \Phi(1/x) \qquad \text{as } x \to 0$$

(see [1]) so that $\underline{\nu}$ is regularly varying at zero with index $-\alpha$. By Theorem 2, $\underline{\theta}(A) = \alpha$. Set

$$\Psi(x) = x^{1/(2\underline{\theta}(A))} (\log x)^{-c/(2\underline{\theta}(A))}$$

for large enough $x > 0$. Then $\Psi \circ \Phi(x) \sim c\sqrt{x}$ as $x \to \infty$ so that $\Psi \circ \Phi(1/\varepsilon^2) \sim c\varepsilon^{-1}$ as $\varepsilon \to 0$. Thus,

$$\forall\, r > 0, \forall\, p \in [1, +\infty)$$
$$e_{N,r}(X, L_T^p) = \Omega\big((\log N)^{-1/(2\underline{\theta}(A))} (\log\log N)^{-c/(2\underline{\theta}(A))}\big).$$

On the other hand, by Lemma 4 and remark below Theorem 1, in the case $c > 0$,

$$\mathbb{E}\, A_t^{\rho/2} \leq C t^{\rho/(2\underline{\theta}(A))} (-\log t)^c, \qquad \rho/2 < \underline{\theta}(A) \wedge r^*(A)$$

so that

$$\forall\, r,\ p \in \big(0, \underline{\theta}(X) \wedge r^*(X)\big),$$
$$e_{N,r}(X, L_T^p) = O\big((\log N)^{-1/\underline{\theta}(X)} (\log\log N)^{c/\rho}\big), \qquad \rho < \underline{\theta}(X) \wedge r^*(X).$$

In the case $r^*(X) \geq \underline{\theta}(X)$, this matches the lower bound up to a $O(\log\log N)^\varepsilon$ term, $\varepsilon > 0$.

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