

A NEW MAXIMAL INEQUALITY AND INVARIANCE PRINCIPLE FOR STATIONARY SEQUENCES

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We derive a new maximal inequality for stationary sequences under a martingale-type condition introduced by Maxwell and Woodroffe [*Ann. Probab.* **28** (2000) 713–724]. Then, we apply it to establish the Donsker invariance principle for this class of stationary sequences. A Markov chain example is given in order to show the optimality of the conditions imposed.

1. Results. Let $(X_i)_{i \in \mathbb{Z}}$ be a stationary sequence of centered random variables with finite second moment ($E[X_1^2] < \infty$ and $E[X_1] = 0$). Denote by \mathcal{F}_k the σ -field generated by X_i with indices $i \leq k$, and define

$$S_n = \sum_{i=1}^n X_i, \quad W_n(t) = \frac{S_{[nt]}}{\sqrt{n}}, \quad 0 \leq t \leq 1,$$

where $[x]$ denotes the integer part of x . Finally, let $W = \{W(t) : 0 \leq t \leq 1\}$ be a standard Brownian motion. In the sequel \implies denotes the weak convergence and $\|X\| = \sqrt{E(X^2)}$.

THEOREM 1.1. *Assume that*

$$(1) \quad \sum_{n=1}^{\infty} \frac{\|E(S_n | \mathcal{F}_0)\|}{n^{3/2}} < \infty.$$

Then, $\{\max_{1 \leq k \leq n} S_k^2/n : n \geq 1\}$ is uniformly integrable and $W_n(t) \implies \sqrt{\eta}W(t)$, where η is a nonnegative random variable with finite mean $E[\eta] = \sigma^2$ and independent of $\{W(t); t \geq 0\}$. Moreover, condition (1) allows to identify the variable η from the existence of the following limit

$$(2) \quad \lim_{n \rightarrow \infty} \frac{E(S_n^2 | \mathcal{I})}{n} = \eta \quad \text{in } L_1,$$

where \mathcal{I} is the invariant sigma field. In particular, $\lim_{n \rightarrow \infty} E(S_n^2)/n = \sigma^2$.

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In the next theorem we show that, in its generality, condition (1) is optimal in the following sense.

THEOREM 1.2. *For any nonnegative sequence $a_n \rightarrow 0$, there exists a stationary ergodic discrete Markov chain $(Y_k)_{k \geq 0}$ and a functional g such that $X_i = g(Y_i)$; $i \geq 0$, $E[X_1] = 0$, $E[X_1^2] < \infty$ and*

$$(3) \quad \sum_{n=1}^{\infty} a_n \frac{\|E(S_n|Y_0)\|}{n^{3/2}} < \infty \quad \text{but} \quad \frac{S_n}{\sqrt{n}} \text{ is not stochastically bounded.}$$

In the ergodic case, Theorem 1.1 improves upon the corresponding results of Maxwell and Woodroffe (2000) [see also Derriennic and Lin (2003) and Wu and Woodroffe (2002)].

Our method of proof is based on the martingale approximation originated in Gordin (1969). Rather than considering and analyzing a perturbed solution of the Poisson equation, as it was suggested in Maxwell and Woodroffe (2000) [see also Liverani (1996)], we analyze small blocks and apply maximal inequalities to show that the sums of variables in these blocks can be approximated by stationary martingale differences.

In the proof of our key inequalities, we use a variety of techniques. The starting point is the dyadic induction found to be useful in the analysis of ρ -mixing sequences. This method goes back to Ibragimov (1975), and was further developed by many authors including Peligrad (1982), Shao (1989), Bradley and Utev (1994) and Peligrad and Utev (1997). The second tool is the modification of the Garsia (1965) telescoping sums approach to maximal inequalities as used by Peligrad (1999) and Dedecker and Rio (2000). Our maximal inequality, stated in Proposition 2.3, is new and has interest in itself. Finally, we use the subadditivity of the conditional sums of random variables.

In order to show the optimality of our results, we construct an example which is motivated by the well-known counterexample stating that, in the general ergodic case, unlike the i.i.d. case (the Kolmogorov strong law of the large numbers), $E|X| = \infty$ does not imply that the averages S_n/n diverge almost surely [see Halmos (1956), page 32; he has attributed this example to M. Gerstenhaber]. The discrete version of the example was probably introduced in Chung [(1960), Markov chains, page 92]. For the modern development and connection with Pomeau–Manneville type 1 intermittency model, we mention Isola (1999) whose detailed analysis was inspirational.

Theorem 1.1 is proved in Sections 2.1–2.4. Theorem 1.2 is proved in Sections 3.1–3.3.

2. Proof of Theorem 1. Throughout the section we will use the notation

$$(4) \quad \Delta_r = \sum_{j=0}^{r-1} \left\| \frac{E(S_{2^j}|\mathcal{F}_0)}{2^{j/2}} \right\|.$$

2.1. Analysis of second-order moments of partial sums.

PROPOSITION 2.1. Let n, r be integers such that $2^{r-1} < n \leq 2^r$. Then

$$(5) \quad E(S_n^2) \leq n[\|X_1\| + \frac{1}{2}\Delta_r]^2.$$

Assume $\sum_{j=0}^\infty 2^{-j/2} \|E(S_{2^j}|\mathcal{F}_0)\| < \infty$. Then, the following limit exists in L_1 :

$$(6) \quad \eta := \lim_{n \rightarrow \infty} \frac{E(S_n^2|\mathcal{I})}{n} = E(X_1^2|\mathcal{I}) + \sum_{j=0}^\infty \frac{E[S_{2^j}(S_{2^{j+1}} - S_{2^j})|\mathcal{I}]}{2^j},$$

where \mathcal{I} is the invariant sigma field. In particular,

$$\sigma^2 := E[\eta] = E(X_1^2) + \sum_{j=0}^\infty \frac{E(S_{2^j}(S_{2^{j+1}} - S_{2^j}))}{2^j}.$$

PROOF. The last statement is an immediate consequence of (6). In order to prove (5), we shall use an induction argument. It is easy to see that (5) is true for $r = 0$ and $n = 1$. Assume (5) holds for all $n \leq 2^{r-1}$. Fix $n, 2^{r-1} < n \leq 2^r$. Starting with $S_n = S_{n-2^{r-1}} + S_n - S_{n-2^{r-1}}$ and using the Cauchy–Schwarz inequality and stationarity, we derive

$$\|S_n\|^2 \leq \|S_{n-2^{r-1}}\|^2 + \|S_{2^{r-1}}\|^2 + 2\|S_{n-2^{r-1}}\| \|E(S_{2^{r-1}}|\mathcal{F}_0)\|.$$

Now, by induction assumption, since $\|E(S_{2^{r-1}}|\mathcal{F}_0)\| = 2^{(r-1)/2}(\Delta_r - \Delta_{r-1})$, and $4(n - 2^{r-1})2^{r-1} \leq n^2$, we obtain

$$\begin{aligned} \|S_n\|^2 &\leq (n - 2^{r-1})[\|X_1\| + \frac{1}{2}\Delta_{r-1}]^2 + 2^{r-1}[\|X_1\| + \frac{1}{2}\Delta_{r-1}]^2 \\ &\quad + 2(n - 2^{r-1})^{1/2}[\|X_1\| + \frac{1}{2}\Delta_{r-1}]2^{(r-1)/2}(\Delta_r - \Delta_{r-1}) \\ &\leq n[\|X_1\| + \frac{1}{2}\Delta_{r-1} + \frac{1}{2}(\Delta_r - \Delta_{r-1})]^2 = n[\|X_1\| + \frac{1}{2}\Delta_r]^2. \end{aligned}$$

This establishes (5).

To prove (6) for the subsequence $n = 2^r$, we use the notation $E_I(Y) = E(Y|\mathcal{I})$ and $\|Y\|_I = \sqrt{E(Y^2|\mathcal{I})}$ for the corresponding norm. By recurrence, we can easily establish the representation

$$(7) \quad \begin{aligned} E_I(S_{2^r}^2) &= 2^r E_I(X_1^2) + \sum_{i=1}^r 2^i E_I[S_{2^{r-i}}(S_{2^{r-i+1}} - S_{2^{r-i}})] \\ &= 2^r \left(E_I(X_1^2) + \sum_{j=0}^{r-1} \frac{E_I(S_{2^j}(S_{2^{j+1}} - S_{2^j}))}{2^j} \right). \end{aligned}$$

We observe that

$$E[S_{2^j}(S_{2^{j+1}} - S_{2^j})|\mathcal{I}] = E\{E[S_{2^j}(S_{2^{j+1}} - S_{2^j})|\mathcal{F}_{2^j}]\|\mathcal{I}\}$$

[see, e.g., Proposition (2.2) in Bradley (2002), page 54]. Thus, by the Jensen inequality,

$$E|E[S_{2^j}(S_{2^{j+1}} - S_{2^j})|\mathcal{L}]| \leq E|E[S_{2^j}(S_{2^{j+1}} - S_{2^j})|\mathcal{F}_{2^j}]|$$

so that the Cauchy–Schwarz inequality and stationarity imply

$$E|E[S_{2^j}(S_{2^{j+1}} - S_{2^j})|\mathcal{L}]| \leq \|S_{2^j}\| \|E(S_{2^j}|\mathcal{F}_0)\|.$$

In addition, by the first part of the proposition and the summability of the series in the right-hand side of (4), we obtain

$$\sum_{j=0}^{\infty} \frac{\|S_{2^j}\| \|E(S_{2^j}|\mathcal{F}_0)\|}{2^j} \leq C \sum_{j=0}^{\infty} \left\| \frac{E(S_{2^j}|\mathcal{F}_0)}{2^{j/2}} \right\| < \infty,$$

which proves the convergence in L_1 of the series

$$E(X_1^2|\mathcal{L}) + \sum_{j=0}^{\infty} \frac{E[S_{2^j}(S_{2^{j+1}} - S_{2^j})|\mathcal{L}]}{2^j} = \eta.$$

This relation and (7) show that the convergence in (6) holds along the subsequence $n = 2^r$, that is,

$$\lim_{r \rightarrow \infty} \frac{E[S_{2^r}^2|\mathcal{L}]}{2^r} = \eta.$$

To treat the whole sequence S_n , for $1 \leq n < 2^r$, we start with the binary expansion

$$n = \sum_{k=0}^{r-1} 2^k a_k \quad \text{where } a_{r-1} = 1 \text{ and } a_k \in \{0, 1\}.$$

Then, we apply the following representation:

$$S_n = \sum_{j=0}^{r-1} T_{2^j} a_j \quad \text{where } T_{2^j} = \sum_{i=n_{j-1}+1}^{n_j} X_i, n_j = \sum_{k=0}^j 2^k a_k, n_{-1} = 0.$$

Clearly, for $a_j = 0$, $T_{2^j} = 0$. For $a_j = 1$, the conditional distribution of T_{2^j} given \mathcal{L} is equally distributed as the conditional distribution of S_{2^j} given \mathcal{L} .

To prove (6), we start with the representation

$$E[S_n^2|\mathcal{L}] = \left(\sum_{i=1}^{r-1} a_i E[S_{2^i}^2|\mathcal{L}] \right) + \left(\sum_{i \neq j=1}^{r-1} a_i a_j E[T_{2^i} T_{2^j}|\mathcal{L}] \right) \equiv I_n + J_n.$$

By the above convergence, $E[S_{2^j}^2|\mathcal{L}]/2^j \rightarrow \eta$, which implies the convergence

$$\frac{I_n}{n} \rightarrow \eta \quad \text{in } L_1.$$

It remains to prove that $\frac{E|J_n|}{n} \rightarrow 0$. Let $i < j < r$. Then, as before,

$$\begin{aligned} E|E[T_{2^i} T_{2^j} | \mathcal{L}]| &\leq E|E[T_{2^i} E(T_{2^j} | \mathcal{F}_{n_i})]| \leq \|S_{2^i}\| \|E(S_{2^j} | \mathcal{F}_0)\| \\ &\leq C2^{i/2} \sqrt{n} \left\| \frac{E(S_{2^j} | \mathcal{F}_0)}{2^{j/2}} \right\| \end{aligned}$$

and, thus,

$$E|J_n| \leq 2 \sum_{1=i < j \leq r-1} E|E[T_{2^i} T_{2^j} | \mathcal{L}]| \leq 2C\sqrt{n} \sum_{i=1}^{r-2} 2^{i/2} \sum_{j=i+1}^r \left\| \frac{E(S_{2^j} | \mathcal{F}_0)}{2^{j/2}} \right\|,$$

which implies $E|J_n|/n \rightarrow 0$ because

$$\sum_{j=i}^{\infty} \left\| \frac{E(S_{2^j} | \mathcal{F}_0)}{2^{j/2}} \right\| \rightarrow 0 \quad \text{as } i \rightarrow \infty. \quad \square$$

2.2. Maximal inequalities. We start by establishing first an auxiliary lemma:

LEMMA 2.2. *Let $(Y_i)_{1 \leq i \leq n}$ be a random vector of square integrable random variables such that for each i , $1 \leq i \leq n$, Y_i is measurable with respect to $\mathcal{F}_i = \sigma(X_j, j \leq i)$, where (X_i) is a stationary sequence introduced before. Let $n \leq 2^r$. If for all $1 \leq a \leq b \leq n$, and a positive constant C ,*

$$E \left(\sum_{l=a}^b Y_l \right)^2 \leq C(b - a + 1) \quad \text{then} \quad \left| E \sum_{l=1}^{n-1} Y_l (S_n - S_l) \right| \leq \frac{1}{2} Cn \Delta_r.$$

PROOF. We shall prove this lemma by induction. It is easy to see the result of this lemma is true for $n = 2$. Assume the lemma holds for all $n \leq 2^{r-1}$. Fix now n , $2^{r-1} < n \leq 2^r$, and begin by writing

$$\begin{aligned} \sum_{l=1}^{n-1} Y_l (S_n - S_l) &= \sum_{l=1}^{n-2^{r-1}-1} Y_l (S_{n-2^{r-1}} - S_l) \\ &\quad + \sum_{l=n-2^{r-1}}^{n-1} Y_l (S_n - S_l) + \sum_{l=1}^{n-2^{r-1}-1} Y_l (S_n - S_{n-2^{r-1}}). \\ &= I_1 + I_2 + I_3. \end{aligned}$$

By using the Cauchy–Schwarz inequality, along with the conditions of this lemma and stationarity, we easily obtain

$$|EI_3| \leq C[2^{r-1}(n - 2^{r-1})]^{1/2} (\Delta_r - \Delta_{r-1}) \leq \frac{1}{2} Cn [\Delta_r - \Delta_{r-1}].$$

By the induction assumption, $|EI_1| \leq \frac{1}{2} C(n - 2^{r-1}) \Delta_{r-1}$ and $|EI_2| \leq \frac{1}{2} C2^{r-1} \times \Delta_{r-1}$, so

$$|EI_1| + |EI_2| + |EI_3| \leq \frac{1}{2} Cn \Delta_{r-1} + \frac{1}{2} Cn [\Delta_r - \Delta_{r-1}] = \frac{1}{2} Cn \Delta_r,$$

proving the lemma. \square

We are ready to state and prove our key maximal inequality.

PROPOSITION 2.3. *Let $\{X_i : i \in Z\}$ be a stationary sequence of random variables. Let n, r be integers such that $2^{r-1} < n \leq 2^r$. Then we have*

$$E \left[\max_{1 \leq i \leq n} S_i^2 \right] \leq n(2\|X_1\| + (1 + \sqrt{2})\Delta_r)^2.$$

PROOF. Denote by

$$M_n = \max_{1 \leq i \leq n} |S_i| \quad \text{and} \quad K_m = \max_{1 \leq j \leq m} \frac{1}{j} E \left[\max_{1 \leq i \leq j} S_i^2 \right].$$

We first prove that, for any positive integer n ,

$$(8) \quad E \left[\max_{1 \leq i \leq n} S_i^2 \right] \leq n[2K_n^{1/2}\Delta_r + 4[\|X_1\| + \frac{1}{2}\Delta_r]^2].$$

By the fact that K_l is nondecreasing in l , from (8), we derive

$$K_n \leq 2K_n^{1/2}\Delta_r + 4[\|X_1\| + \frac{1}{2}\Delta_r]^2,$$

which implies $K_n^{1/2} \leq 2\|X_1\| + (1 + \sqrt{2})\Delta_r$, hence, the result.

To prove (8), we denote by $S_0 = 0$,

$$M_n^+ = \max_{1 \leq j \leq n} S_j^+ = \max(0, S_1, \dots, S_n)$$

and

$$M_n^- = \max_{1 \leq j \leq n} (-S_j^-) = \max(0, -S_1, \dots, -S_n).$$

We shall use the following simplified version of an interesting inequality in Dedecker and Rio (2000) [see (3.4) in Dedecker and Rio (2000) or (3.5) in Rio (2000)], which was obtained by using Garsia’s (1965) telescoping sum approach to the maximal inequality

$$(9) \quad (M_n^+)^2 \leq 4(S_n^+)^2 - 4 \sum_{k=1}^n M_{k-1}^+ X_k.$$

By adding to this relation the similar one for M_n^- , we obtain

$$(M_n)^2 \leq 4(S_n)^2 - 4 \sum_{k=1}^n (M_{k-1}^+ - M_{k-1}^-)(X_k).$$

We now write $X_k = (S_n - S_{k-1}) - (S_n - S_k)$ and derive

$$(10) \quad (M_n)^2 \leq 4(S_n)^2 - 4 \sum_{k=1}^{n-1} D_k (S_n - S_k),$$

where $D_k = (M_k^+ - M_{k-1}^+) - (M_k^- - M_{k-1}^-)$.

It is easy to see that

$$\left| \sum_{k=a+1}^b D_k \right| \leq \max[(M_b^+ - M_a^+), (M_b^- - M_a^-)] \leq \max_{a \leq i \leq b} |S_i - S_a|.$$

Taking the expectation, we get, by stationarity,

$$E \left(\sum_{k=a+1}^b D_k \right)^2 \leq E \left(\max_{1 \leq i \leq b-a} S_i^2 \right) = (b-a)K_{b-a} \leq (b-a)K_n.$$

Next, by Lemma 2.2 applied with $Y_k = D_k$ for $k \geq 1$, $C = K_n^{1/2}$, we obtain

$$\left| E \sum_{k=1}^{n-1} D_k (S_n - S_k) \right| \leq \frac{1}{2}n[K_n^{1/2} \Delta_r].$$

By substituting this estimate in (10) together with (5) on $E(S_n^2)$, we obtain (8) and, hence, the proposition. \square

REMARK 2.4. The inequality in Proposition 2.3 is an extension of the Doob maximal inequality for martingales, giving also an alternative proof of this famous theorem. Notice that, for the martingale case, our inequality gives the same constant as in the Doob inequality, a constant that cannot be improved. A natural question that arises is the optimality of the constant in front of Δ_r and further study is needed to determine the best constants in this inequality.

2.3. Analysis of certain series involving conditional sums.

(a) *Key result.* Let $X = (X_i)_{i \in \mathbb{Z}}$ be a stationary sequence of random variables with finite second moment. Denote by

$$S_n = \sum_{i=1}^n X_i, \quad V_n = V_n(X) = \|E(S_n | \mathcal{F}_0)\|,$$

where as before, \mathcal{F}_k is the σ -field generated by X_i with indices $i \leq k$.

The main condition (1) of Theorem 1.1 is $\sum V_n/n^{3/2} < \infty$. On the other hand, various inequalities derived in Sections 2.1 and 2.2 have used the condition $\sum V_{2^r}/2^{r/2} < \infty$. In this section we show that these conditions are equivalent and, in addition, we prove the following proposition, which is useful in establishing the martingale approximation in Theorem 1.1.

PROPOSITION 2.5. Under condition (1),

$$\frac{\|E(S_m | \mathcal{F}_0)\|}{\sqrt{m}} \rightarrow 0 \quad \text{and} \quad \frac{1}{\sqrt{m}} \sum_{j=0}^{\infty} \left\| \frac{E(S_{m2^j} | \mathcal{F}_0)}{2^{j/2}} \right\| \rightarrow 0$$

as $m \rightarrow \infty$.

PROOF. In order to prove this result, we shall analyze in Lemma 2.6 the conditional variance of sums and then, in Lemma 2.7, some related series. By Lemma 2.6, the sequence $V_m = \|E(S_m|\mathcal{F}_0)\|$ is subadditive. Then, we have only to apply Lemma 2.8 to conclude the proof of this proposition. \square

(b) *Conditional variances of sums form a subadditive sequence.* The starting point of our analysis is the following simple observation.

LEMMA 2.6. V_n is a subadditive sequence.

PROOF. First, since for all n , $\mathcal{F}_{-n} \subset \mathcal{F}_0$, we observe that

$$E[E(S_k|\mathcal{F}_{-n})]^2 \leq E[E(S_k|\mathcal{F}_0)]^2 = \|E(S_k|\mathcal{F}_0)\|^2 = V_k^2.$$

Hence, by stationarity,

$$\|E(S_{i+j} - S_i|\mathcal{F}_0)\| = \sqrt{E[E(S_j|\mathcal{F}_{-i})]^2} \leq V_j.$$

Thus,

$$\begin{aligned} V_{i+j} &= \|E(S_i + [S_{i+j} - S_i]|\mathcal{F}_0)\| \leq \|E(S_i|\mathcal{F}_0)\| + \|E(S_{i+j} - S_i|\mathcal{F}_0)\| \\ &\leq V_i + V_j. \end{aligned}$$

(c) *Analysis of certain series for subadditive sequences.* Let V_n be a nonnegative subadditive sequence. For a $p > 1$, define

$$I := \sum_{j=0}^{\infty} \frac{V_{2^j}}{2^{j(p-1)}}, \quad J := \sum_{n=1}^{\infty} \frac{V_n}{n^p}, \quad W := \sum_{n=1}^{\infty} n^{-p} \max_{1 \leq i \leq n} V_i. \quad \square$$

The following lemma is a crucial step in deriving the result in Proposition 2.5.

LEMMA 2.7. *There exists two positive absolute constants C_p and K_p such that*

$$C_p I \leq J \leq W \leq K_p I.$$

PROOF. We shall start with the following simple representation:

$$W = \sum_{n=1}^{\infty} n^{-p} \max_{1 \leq i \leq n} V_i = \sum_{r=0}^{\infty} \sum_{n=2^r}^{2^{r+1}-1} n^{-p} \max_{1 \leq i \leq n} V_i.$$

Then, by the subadditivity of the sequence $\{V_n; n \geq 0\}$, for $i \leq n < 2^{r+1}$,

$$V_i \leq \sum_{j=0}^r V_{2^j} \quad \text{so that} \quad \max_{1 \leq i \leq n} V_i \leq \sum_{j=0}^r V_{2^j},$$

which implies

$$\begin{aligned}
 W &\leq \sum_{r=0}^{\infty} 2^{-pr} 2^r \sum_{k=0}^r V_{2k} = \sum_{k=0}^{\infty} V_{2k} \sum_{r=k}^{\infty} 2^{-r(p-1)} = K_p \sum_{k=0}^{\infty} 2^{-k(p-1)} V_{2k} \\
 &= K_p I,
 \end{aligned}$$

where $K_p = \frac{1}{1-2^{-(p-1)}}$. The last inequality is therefore proved.

The inequality $J \leq W$ is straightforward. Now, we need the following simple combinatorial property. Define

$$A_N = \{1 \leq i \leq N : V_i \geq V_N/2\} \quad \text{and denote by } |A| \text{ the cardinal of a set } A.$$

PROPERTY. $|A_N| \geq N/2$, that is, A_N contains at least $N/2$ elements.

PROOF. To prove it, we denote by $D_N = \{1, \dots, N\}$ and fix $1 \leq i < N$. Observe that if $i \in A_N^c = D_N - A_N$, then $N - i \in A_N$ because

$$V_{N-i} \geq V_N - V_i > V_N - V_N/2 \geq V_N/2.$$

Thus, $A_N \supseteq N - A_N^c$ and so $N = |D_N| = |A_N| + |A_N^c| \leq 2|A_N|$ and the property is proved. \square

Now, in order to continue the proof of Lemma 2.7, we write

$$J = \sum_{r=0}^{\infty} \left(\sum_{n=4^r}^{4^{r+1}-1} \frac{V_n}{n^p} \right) \geq \frac{1}{4^p} \sum_{r=0}^{\infty} 4^{-rp} \left(\sum_{n=4^r}^{4^{r+1}-1} V_n \right).$$

We are going to apply the above property with $N = 4^{r+1}$. Define

$$C_r = \{n \in \{4^r, \dots, 4^{r+1} - 1\} : V_n \geq V_N/2\} = A_N \cap \{4^r, \dots, 4^{r+1} - 1\}.$$

Clearly,

$$|C_r| \geq |\{4^r, \dots, 4^{r+1} - 1\}| - |A_N^c| = 4^{r+1} - 4^r - |A_N^c|$$

and, applying the above property, we obtain

$$|C_r| \geq 4^{r+1} - 4^r - (4^{r+1} - 1)/2 \geq 4^{r+1} - 4^r - 4^{r+1}/2 = 4^r.$$

Thus,

$$J \geq \frac{1}{2} \frac{1}{4^p} \sum_{r=0}^{\infty} 4^{-rp} V_{4^{r+1}} |C_r| \geq \frac{1}{2} \frac{1}{4^p} \sum_{r=0}^{\infty} 4^{-r(p-1)} V_{4^{r+1}} = \frac{1}{8} \sum_{r=1}^{\infty} 2^{-2r(p-1)} V_{2^{2r}},$$

which implies

$$Q := \sum_{r=0}^{\infty} 2^{-2r(p-1)} V_{2^{2r}} = V_1 + \sum_{r=1}^{\infty} 2^{-2r(p-1)} V_{2^{2r}} \leq 9J.$$

Then, by subadditivity, $V_{2^{2r+1}} \leq 2V_{2^{2r}}$, so that

$$P := \sum_{r=0}^{\infty} 2^{-(2r+1)(p-1)} V_{2^{2r+1}} \leq \frac{2}{2^{(p-1)}} \sum_{r=0}^{\infty} 2^{-(2r)(p-1)} V_{2^{2r}} = \frac{2}{2^{(p-1)}} Q$$

and, as a consequence,

$$I = \sum_{r=0}^{\infty} \frac{V_{2^{2r}}}{2^{2r(p-1)}} + \sum_{r=0}^{\infty} \frac{V_{2^{2r+1}}}{2^{(2r+1)(p-1)}} = P + Q \leq 9 \left(\frac{2}{2^{(p-1)}} + 1 \right) J,$$

and the proof of Lemma 2.7 is complete. \square

LEMMA 2.8. *Assume that $\sum_{n=1}^{\infty} V_n n^{-3/2} < \infty$. Then,*

$$G_m = \frac{1}{\sqrt{m}} \sum_{k=0}^{\infty} \frac{V_{m2^k}}{2^{k/2}} \rightarrow 0 \quad \text{as } m \rightarrow \infty.$$

In particular, $V_m/\sqrt{m} \rightarrow 0$ as $m \rightarrow \infty$.

PROOF. By rewriting G_m , we obtain

$$\begin{aligned} G_m &= \sum_{k=0}^{\infty} \sum_{n=m2^k}^{m2^{k+1}-1} (m2^k)^{-3/2} V_{m2^k} \leq 2^{3/2} \sum_{k=0}^{\infty} \sum_{n=m2^k}^{m2^{k+1}-1} n^{-3/2} \max_{1 \leq i \leq n} V_i \\ &= 2^{3/2} \sum_{n=m}^{\infty} n^{-3/2} \max_{1 \leq i \leq n} V_i, \end{aligned}$$

which proves that $G_m \rightarrow 0$ as $m \rightarrow \infty$ by Lemma 2.7. \square

2.4. *Martingale approximation and the proof of Theorem 1.* Let m be a fixed integer and $k = [n/m]$, where, as before, $[x]$ denotes the integer part of x . We start the proof by dividing the variables in blocks of size m and making the sums in each block

$$X_i^{(m)} = m^{-1/2} \sum_{j=(i-1)m+1}^{im} X_j, \quad i \geq 1.$$

Then we construct the martingale

$$M_k^{(m)} = \sum_{i=1}^k (X_i^{(m)} - E(X_i^{(m)} | \mathcal{F}_{i-1}^{(m)})), \quad i \in \mathbb{Z},$$

where $\mathcal{F}_k^{(m)}$ denotes the σ -field generated by $X_i^{(m)}$ with indices $i \leq k$.

Notice that $M_k^{(m)}$ is a stationary martingale and, therefore, by the classical invariance principle for martingales, we derive

$$\frac{1}{\sqrt{k}}M_{[kt]}^{(m)} \implies \sqrt{\eta^{(m)}}W,$$

where $\eta^{(m)}$ is the following limit (both in L_1 and almost surely):

$$\eta^{(m)} = \lim_{k \rightarrow \infty} \frac{1}{k} \sum_{i=1}^k (X_i^{(m)} - E(X_i^{(m)} | \mathcal{F}_{i-1}^{(m)}))^2.$$

In order to prove the invariance principle for $\frac{1}{\sqrt{n}}S_{[nt]}$, together with the uniform integrability of the sequence $\max_{1 \leq k \leq n} S_k^2/n$, by the Doob maximal inequality and Theorem 4.2 in Billingsley (1968), we have only to establish that

$$(11) \quad \|\sqrt{\eta^{(m)}} - \sqrt{\eta}\| \rightarrow 0 \quad \text{as } m \rightarrow \infty$$

and

$$(12) \quad \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \left\| \sup_{0 \leq t \leq 1} \left| \frac{1}{\sqrt{n}}S_{[nt]} - \frac{1}{\sqrt{k}}M_{[kt]}^{(m)} \right| \right\| = 0.$$

Notice first that by the convergence in Proposition 2.5,

$$\lim_{m \rightarrow \infty} \frac{1}{m} E[E(S_m | \mathcal{F}_0)]^2 = 0.$$

On the other hand, by the ergodic theorem (both almost surely and in L_1),

$$\lim_{k \rightarrow \infty} \frac{1}{k} \sum_{i=1}^k (X_i^{(m)})^2 = \frac{1}{m} \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n (S_{i+m} - S_i)^2 = \frac{E[S_m^2 | \mathcal{I}] }{m},$$

where \mathcal{I} is the σ -field of invariant sets.

Therefore, by Proposition 2.1, we obtain the following convergence in L_1 :

$$\lim_{m \rightarrow \infty} \eta^{(m)} = \lim_{m \rightarrow \infty} \frac{E(S_m^2 | \mathcal{I})}{m} = \eta,$$

which implies (11).

To prove (12), we first notice that

$$\left\| \sup_{0 \leq t \leq 1} \left| \frac{1}{\sqrt{n}}S_{[nt]} - \frac{1}{\sqrt{km}}S_{[nt]} \right| \right\| \leq \left(1 - \frac{\sqrt{n}}{\sqrt{km}} \right) \left\| \frac{1}{\sqrt{n}} \max_{1 \leq j \leq n} |S_j| \right\|.$$

By taking into account Proposition 2.3 and the fact that $\lim_{n \rightarrow \infty} (1 - \frac{\sqrt{n}}{\sqrt{km}}) = 0$, the right-hand side of the above inequality tends to 0. Therefore, we have only

to estimate

$$\begin{aligned} & \left\| \sup_{0 \leq t \leq 1} \left| \frac{1}{\sqrt{km}} S_{[nt]} - \frac{1}{\sqrt{k}} M_{[kt]}^{(m)} \right| \right\| \\ & \leq \frac{1}{\sqrt{km}} \left\| \sup_{0 \leq t \leq 1} \sum_{i=[kt]_{m+1}}^{[nt]} X_i \right\| + \frac{1}{\sqrt{k}} \left\| \sup_{0 \leq t \leq 1} \left| \sum_{i=1}^{[kt]} E(X_i^{(m)} | \mathcal{F}_{i-1}^{(m)}) \right| \right\|, \end{aligned}$$

which leads to the estimate

$$\begin{aligned} & \left\| \sup_{0 \leq t \leq 1} \left| \frac{1}{\sqrt{km}} S_{[nt]} - \frac{1}{\sqrt{k}} M_{[kt]}^{(m)} \right| \right\| \\ & \leq \frac{3m}{\sqrt{km}} \left\| \max_{1 \leq i \leq n} X_i \right\| + \frac{1}{\sqrt{k}} \left\| \max_{1 \leq j \leq k} \left| \sum_{i=1}^j E(X_i^{(m)} | \mathcal{F}_{i-1}^{(m)}) \right| \right\|. \end{aligned}$$

Since for every $\epsilon > 0$,

$$E \max_{1 \leq i \leq n} X_i^2 \leq \epsilon + \sum_{i=1}^n X_i^2 I(|X_i| > \epsilon)$$

by stationarity, for any fix m , $\lim_{n \rightarrow \infty} 3m \|\max_{1 \leq i \leq n} X_i\| / \sqrt{km} = 0$.

On the other hand, by Propositions 2.3 and 2.5, we derive

$$\begin{aligned} & \frac{1}{\sqrt{k}} \left\| \max_{1 \leq j \leq k} \left| \sum_{i=1}^j E(X_i^{(m)} | \mathcal{F}_{i-1}^{(m)}) \right| \right\| \\ & \leq 2 \frac{\|E(S_m | \mathcal{F}_0)\|}{\sqrt{m}} + (1 + \sqrt{2}) \frac{1}{\sqrt{m}} \sum_{j=0}^{\infty} \left\| \frac{E(S_{m2^j} | \mathcal{F}_0)}{2^{j/2}} \right\| \rightarrow 0 \end{aligned}$$

as $m \rightarrow \infty$, uniformly in n , which completes the proof of Theorem 1.1.

3. Proof of Theorem 2.

3.1. *The countable Markov chain and its preliminary analysis.* Let $\{Y_k; k \geq 0\}$ be a discrete Markov chain with the state space Z^+ and transition matrix $P = (p_{ij})$ given by $p_{k(k-1)} = 1$ for $k \geq 1$ and $p_j = p_{0(j-1)} = P(\tau = j)$, $j = 1, 2, \dots$ (i.e., whenever the chain hits 0, $Y_t = 0$, it then regenerates with the probability p_j). When $p_1, p_2 > 0$, and, in addition, $p_{n_j} > 0$ along $n_j \rightarrow \infty$, the chain is irreducible and aperiodic. The stationary distribution exists if and only if $E[\tau] < \infty$ and it is given by

$$\pi_j = \pi_0 \sum_{i=j+1}^{\infty} p_i, \quad j = 1, 2, \dots,$$

where $\pi_0 = 1/E[\tau]$.

Let us consider now an arbitrary nonnegative sequence $a_n \rightarrow 0$ as in our Theorem 1.2. Notice that, without loss of generality, it is enough to assume that a_n is a strictly decreasing sequence of real positive numbers.

The choice of p_j further depends on this arbitrary nonnegative sequence a_n . First, we define a sequence $\{u_k; k = 1, 2, \dots\}$ of positive integers such that

$$(13) \quad \begin{aligned} u_1 = 1, \quad u_2 = 2, \quad u_k^4 + 1 < u_{k+1} \quad \text{for } k \geq 3 \quad \text{and} \\ a_t \leq k^{-2} \quad \text{for } t \geq u_k. \end{aligned}$$

Then, for $i \geq 1$, we take

$$p_i = \begin{cases} c/u_j^2, & \text{if } i = u_j \text{ for some } j \geq 1, \\ 0, & \text{if } i \neq u_j \text{ for all } j \geq 1, \end{cases}$$

that is, for each positive integer $j \geq 1$, $p_{u_j} = c/u_j^2$ and $p_i = 0$ for $u_j < i < u_{j+1}$.

Clearly,

$$(14) \quad E[\tau] < \infty \quad \text{but } E[\tau^2] = \infty.$$

As a functional g , we take $I_{(x=0)} - \pi_0$, where $\pi_0 = P_\pi(Y_0 = 0)$ under the stationary distribution denoted by P_π (E_π denotes the expectations for the process started with the stationary distribution). The stationary sequence is defined by

$$X_j = I_{(Y_j=0)} - \pi_0 \quad \text{so that } S_n = \sum_{j=1}^n X_j = \sum_{j=1}^n I_{(Y_j=0)} - n\pi_0.$$

By P_k and E_k , we denote the probability and the expectation operator when the Markov chain is started at k , that is, $P(Y_0 = k) = 1$. Let

$$v = \min\{m \geq 1 : Y_m = 0\}, \quad A_n = E_0[S_n], \quad x \wedge y = \min(x, y).$$

PROPOSITION 3.1.

$$\begin{aligned} V_n = \|E(S_n|Y_0)\| &\leq \|v \wedge n\| + \max_{1 \leq i \leq n} |A_i| \\ &\equiv I_n + J_n, \end{aligned}$$

where $\|x\|^2 = \sum_{k=0}^\infty x_k^2 \pi_k$.

PROOF. We first notice that $|S_n| \leq n$ and $P_k(v = k) = 1$, so that, conditionally on $Y_0 = k$ (with $0 < k \leq n$),

$$E_k(S_n) = E_k(S_k) + E_k(S_n - S_k).$$

The first term is bounded by k and the second term is equal to $E_0(S_{n-k+1})$ since $Y_k = 0$. Thus,

$$|E_k(S_n)| \leq k \wedge n + |A_{n-k+1}|. \quad \square$$

3.2. *Proving that* $\sum a_n \|E(S_n|Y_0)\| n^{-3/2} < \infty$. By Proposition 3.1, it is enough to prove that

$$(15) \quad \sum_{n=1}^{\infty} a_n I_n/n^{3/2} + \sum_{n=1}^{\infty} a_n J_n/n^{3/2} < \infty.$$

The first sum is easily treated by a straightforward analysis. Indeed, to analyze $I = \sum a_n I_n/n^{3/2}$, we first notice that, for $u_{t-1} \leq j$,

$$\pi_j = \pi_0 \sum_{i=j+1}^{\infty} p_i \leq \pi_0 c_1/u_t^2.$$

Therefore, we write, for $u_k < n \leq u_{k+1}$,

$$\begin{aligned} I_n^2 &= E_{\pi}(v \wedge n)^2 = \sum_{j=1}^n j^2 \pi_j + n^2 \sum_{j=n+1}^{\infty} \pi_j \\ &\leq \left[\sum_{t=1}^k \left(\sum_{j=u_{t-1}+1}^{u_t} j^2 \pi_j \right) \right] + \left(\sum_{j=u_k+1}^n j^2 \pi_j \right) + n^2 \sum_{t=k+1}^{\infty} \left(\sum_{j=u_{t-1}+1}^{u_t} \pi_j \right) \\ &\leq c_2 \left[\sum_{t=1}^k u_t^{-2} \left(\sum_{j=1}^{u_t} j^2 \right) \right] + \frac{c_3}{u_{k+1}} \left(\sum_{j=u_k+1}^n j^2 \right) + c_3 n^2 \sum_{t=k+1}^{\infty} \frac{1}{u_t} \\ &\leq c_4 (u_k + n^3/u_{k+1}^2 + n^2/u_{k+1}). \end{aligned}$$

Next, write

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{a_n I_n}{n^{3/2}} &= \sum_{k=1}^{\infty} \sum_{n=u_k+1}^{u_{k+1}} \frac{a_n I_n}{n^{3/2}} \leq \sum_{k=1}^{\infty} \frac{1}{k^2} \sum_{n=u_k+1}^{u_{k+1}} I_n n^{-3/2} \\ &\leq \sqrt{c_4} \sum_{k=1}^{\infty} \frac{\sqrt{u_k}}{k^2} \sum_{n=u_k+1}^{u_{k+1}} n^{-3/2} + \sqrt{c_4} \sum_{k=1}^{\infty} \frac{1}{u_{k+1}} \frac{1}{k^2} \sum_{n=u_k+1}^{u_{k+1}} 1 \\ &\quad + \sqrt{c_4} \sum_{k=1}^{\infty} \frac{1}{\sqrt{u_{k+1}}} \frac{1}{k^2} \sum_{n=u_k+1}^{u_{k+1}} n^{-1/2} < \infty. \end{aligned}$$

To prove that the second sum is finite, we need to analyze A_n , which satisfies the renewal equation

$$A_n = E_0[S_{n \wedge v}] + \sum_{j=1}^{n-1} A_{n-j} p_j.$$

Unlike Isola (1999), we use probabilistic arguments to analyze this renewal equation.

We define

$$T_0 = 0, \quad T_k = \min\{t > T_{k-1} : Y_t = 0\}, \quad \tau_k = T_k - T_{k-1}, \quad k = 1, 2, \dots$$

Then, $\{\tau_j\}$ are independent variables equally distributed as τ . [See, e.g., Breiman (1968), page 146.] Let $\xi_j = 1 - \pi_0 \tau_j$ and introduce the stopping time

$$\nu_n = \min\{j \geq 1 : T_j \geq n\}.$$

Clearly, $S_{T_k} = \sum_{j=1}^k \xi_j$, $E_0[\xi_1] = 0$, $\nu_n \leq n$ and, thus, by the Wald identity,

$$E_0[S_{T_{\nu_n}}] = E\left[\sum_{j=1}^{\nu_n} \xi_j\right] = 0.$$

Hence, since $|S_a - S_b| \leq |a - b|$, by the definition of A_n , we obtain

$$|A_n| = |E_0[S_{T_{\nu_n}} - S_n]| \leq E_0[\tau_{\nu_n}] \leq E_0\left[\max_{1 \leq i \leq n} \tau_i\right].$$

Let us denote by

$$M_n = \max_{1 \leq i \leq n} \tau_i.$$

Then,

$$J_n = \max_{1 \leq i \leq n} |A_i| \leq E[M_n].$$

To analyze $E[M_n]$, we notice that

$$E[M_n] = \sum_{t=1}^{\infty} u_t P(M_n = u_t)$$

and

$$P(M_n = u_t) \leq \min(1, nP(\tau = u_t)) \leq c_1 \min(1, n/u_t^2).$$

Fix n , $u_k < n \leq u_{k+1}$. Notice first that, for $t \leq k - 1$, we have $u_t \leq u_{k-1} \leq u_k^{1/4} \leq n^{1/4}$. Also, $\sum_{j=k+1}^{\infty} 1/u_j \leq c_3/u_{k+1}$ and, thus, splitting the sum into three parts according to t : $t \leq k - 1$, $t = k$ and $t \geq k + 1$, we obtain the bound

$$E[M_n] \leq c_4 \left(n^{1/4} + \frac{n}{u_{k+1}} + u_k \min(1, n/u_k^2) \right).$$

Finally, by the construction of u_n and its relation to a_n , we derive

$$\begin{aligned} \sum_{n=1}^{\infty} a_n J_n / n^{3/2} &\leq c_5 \sum_{n=1}^{\infty} n^{-5/4} + c_6 \sum_{k=1}^{\infty} \frac{1}{k^2} \frac{1}{u_{k+1}} \sum_{n=u_k+1}^{u_{k+1}} n^{-1/2} \\ &\quad + c_7 \sum_{k=1}^{\infty} \frac{1}{k^2} \frac{1}{u_k} \sum_{n=u_k+1}^{u_k^2} n^{-1/2} + c_8 \sum_{k=1}^{\infty} \frac{1}{k^2} u_k \sum_{n=u_k^2+1}^{u_{k+1}} n^{-3/2} \\ &< \infty, \end{aligned}$$

proving (15).

3.3. *Stochastic unboundedness of S_n/\sqrt{n} and the proof of Theorem 2.* We proceed by contradiction; that is, we assume that

$$\{S_n/\sqrt{n}; n \geq 1\} \quad \text{is stochastically bounded}$$

and show that $E\tau^2 < \infty$, which is in contradiction with (14).

Let $\{\tau_j\}$ be independent variables equally distributed as τ . Define

$$T_k = \tau_1 + \dots + \tau_k, \quad \eta_n = \max\{i \geq 1 : T_i \leq n\},$$

$$T(i, n) = T_n - T_i, \quad \eta_n(\xi) = \max\{i \geq 1 : \xi + T(1, i) \leq n\}$$

(where $\max_{i \in \emptyset} a_i = 0$). Then, $S_n = \eta_n(v) - na$, where $a = 1/E[\tau_1] = \pi_0$.

The following proposition will provide a slightly more general result which has interest in itself.

PROPOSITION 3.2. *Assume that, for a nonnegative integer valued variable ξ ,*

$$(16) \quad \left\{ \frac{\eta_n(\xi) - an}{\sqrt{n}}; n \geq 1 \right\} \quad \text{is stochastically bounded.}$$

Then, $E[\tau_1^2] < \infty$.

PROOF. First, let η'_n be a copy of the renewal process $\{\eta_n : n \geq 1\}$ which does not depend on ξ . Then, $\eta_n(\xi)$ is equally distributed as $\eta'_{n-\xi}$ and so, any finite number of renewals do not affect the stochastic boundedness of the normalized renewal processes. As a consequence, condition (16) implies that

$$P([an - \sqrt{n}M] \leq \eta_n < [an + \sqrt{n}M]) \geq 1 - \varepsilon_M,$$

where $\varepsilon_M \rightarrow 0$ as $M \rightarrow \infty$.

Next, we apply the standard relationship $\{\eta_n \geq k\} = \{T_k \leq n\}$, yielding

$$P([an - \sqrt{n}M] \leq \eta_n < [an + \sqrt{n}M]) = P(T_{[an-\sqrt{n}M]} \leq n, T_{[an+\sqrt{n}M]} > n)$$

$$\equiv P(T_L \leq n, T_R > n) = I \geq 1 - \varepsilon_M,$$

where

$$L = L[n, M] = [an - \sqrt{n}M], \quad R = R[n, M] = [an + \sqrt{n}M].$$

Now, we take $k = R - L$. Since $T(i, n) = T_n - T_i$ is equally distributed as T_{n-i} , we can write

$$I = P(T_L \leq n, T_L + T(L, R) > n)$$

$$= P(T_L \leq n - kN, T_L + T(L, R) > n)$$

$$+ P(n - kN < T_L \leq n, T_L + T(L, R) > n)$$

$$\leq P(T(L, R) > kN) + P(n - kN < T_L \leq n)$$

$$= P(T_k > kN) + P(n - kN < T_L \leq n).$$

By the law of the large numbers,

$$P(T_k/k > N) \leq \delta_N,$$

where $\delta_N \rightarrow 0$ as $N \rightarrow \infty$. Thus,

$$P(n - kN < T_L \leq n) \geq 1 - \varepsilon_M - \delta_N.$$

Since $2\sqrt{n}M - 1 \leq k \leq 2\sqrt{n}M + 1$, we derive

$$P(|T_L - n|/\sqrt{n} \leq (2M + 1)N) \geq 1 - \varepsilon_M - \delta_N.$$

Now we use the symmetrization argument. We consider an independent copy of $\{\tau_j\}$, namely, $\{\tau'_j\}$ and denote by $T'_k = \tau'_1 + \dots + \tau'_k$, $T^s_k = T_k - T'_k$. Clearly,

$$P(|T^s_L|/\sqrt{n} \geq 2(2M + 1)N) \leq \varepsilon_M + \delta_N.$$

Here

$$\lim_{n \rightarrow \infty} L(n, M)/n = a.$$

By standards arguments involving an application of the Lévy maximal inequality for sums of symmetric independent random variables, we easily derive that the sequence $\{T^s_n/\sqrt{n}\}$ is stochastically bounded. By Theorem 3 in Esseen and Janson (1985), the fact that $\{T^s_n/\sqrt{n}\}$ is stochastically bounded implies $E(\tau_1 - \tau'_1)^2 < \infty$. Thus, $E\tau_1^2 < \infty$. \square

PROOF OF THEOREM 1.2. By combining Proposition 3.1 with the bound (15), we obtain the first part of (3). To prove the second part, we proceed by absurd and notice that if $\{S_n/\sqrt{n}\}$ is stochastically bounded, then by Proposition 3.2, $E[\tau^2] < \infty$, which is in contradiction with (14).

The proof of Theorem 1.2 is complete. \square

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