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REGULARIZATION FOR HEAT KERNEL IN NONLINEAR PARABOLIC EQUATIONS

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Abstract. We prove existence-uniqueness theorems for some kinds of nonlinear parabolic equations (cf. [2, 3, 15]) with singular initial data and non-Lipschitz's nonlinearities in a framework of Colombeau's algebras using different kinds of regularization for singularities appearing in the equations. We establish the convergence of a family of regularized solutions to the classical solutions (if they exist), when nonlinear term $g(u)$ is of Lipschitz's class and $\varepsilon \to 0$. Moreover, we find solutions not available in classical approach.

1. Introduction

Cauchy problem (1), (2), for nonlinear parabolic equations with singular initial data, existence and uniqueness theorems for local and global solutions are the subject of the papers [2, 3]. Free term $g(u)$ is supposed to be of polynomial growth. If $|g(u)| = u^s$, $s < 1$, (case of sublinear growth), and Lipschitz's condition is satisfied, under some assumptions on s (cf. [2]), Cauchy problem (1), with singular initial data (cf. Section 2), have an unique global solution $u \in C([0,\infty); \mathcal{M}^k(\mathbf{R}^n))$. If singular initial data are smoothed by delta sequences there exists an unique solution $u \in C^{2,1}([0,\infty) \times \mathbf{R}^n) \cap C_0(L^p(\mathbf{R}^n))$, $1 \le p \le \infty$. When nonlinear term has a superlinear growth, $g \in C(\mathbf{R}; \mathbf{R})$ and satisfies

$$|g(u) - g(v)| \le A|u - v|(|u| + |v|)^{s-1}, \;\; u, v \in \mathbf{R},$$

there exists an unique solution $u \in C^2((0,T) \times \mathbf{R}^n)$. The same holds if $\mu(\cdot) \in \mathcal{S}'(\mathbf{R}^n)$. In the article [3] is given the optimal link between the singularities of the nonlinear term and the initial data to have uniqueness. For other classical

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solutions cf. [11, 19]. For Colombeau solution to parabolic equations with nonlinear conservative term cf. [20].

Regularization in evolution equations (w.r.) to the space variable by delta sequences, are introduced in [9] and applied in [20, 12]. Regularization of semigroups given in [14] leads to the uniformly continuous semigroups (cf. [14]) which cover smaller class of the equations than the semigroups with unbounded operators. The attempt of regularizing semigroups (w.r.) to the time variable t is done in [5]. In this paper we give a regularization for the heat kernel in nonlinear parabolic equations (w.r.) to the time variable t to avoid singularities over the diagonal $t=\tau$. In that way we obtain global solutions and the heat semigroup stays unbounded. As a framework we use Colombeau's algebra of generalized functions. In our consideration, the nonlinear term $g(u)$ does not satisfy Lipschitz's condition. We remove it by cut-off. We find a family of nets of regularized solutions which are compatible with classical solutions in a limiting case when $\varepsilon \to 0$. Initial data are strongly singular and regularized with delta sequences. In all cases, we suppose that $g(0)=0$. Note, that many Colombeau's solutions are not available in classical approach.

2. Statement of the Problem

We state the following problems in nonlinear parabolic equations:

1. Cauchy problem (cf. [2])

$$\partial_t u = \triangle u + g(u),\ t>0,\ x\in \mathbf{R}^n, \tag{1}$$

where $g \in L^\infty_{loc}(\mathbf{R}^n)$ is meant to be composed with a real-valued function u on $([0,T)\times \mathbf{R}^n)$, and $g(u)$ is not of Lipschitz's class. The initial data are strongly singular

$$\mu(0,\cdot)=\mu\in \mathcal{M}^k(\mathbf{R}^n)\subset \mathcal{D}'(\mathbf{R}^n),\ k\in Z,$$

where $\mathcal{M}^k(\mathbf{R}^n)=(C^k_b(\mathbf{R}^n))'$ is the strong dual of the Banach space $C^k_b(\mathbf{R}^n)$ of all $C^k(\mathbf{R}^n)$ functions with bounded derivatives up to the order k. $\mathcal{M}^0(\mathbf{R}^n)$ is the space of Radon measure. As an example we consider the delta distribution massed at the point ξ^j and the sum of its derivatives

$$\mu=\sum_{j=1}^{\infty}\sum_{|\alpha|\le k} b_{j\alpha}\partial_x^\alpha \delta(\cdot-\xi^j),\ k\in \mathbf{Z}_+,\ b_{j\alpha}\in\mathbf{R},\ \xi^j\in\mathbf{R}^n,\ j\ge 1,\ \{b_{j\alpha}\}_1^\infty\in l^1.$$

2. Cauchy problem with nonlinear conservative term (cf. [3])

$$\partial_t u - \triangle u + \partial_x\cdot \vec{g}(u)=0,\ t>0,\ x\in\mathbf{R}^n, u(0,\cdot)=D^k\psi(\cdot)\in\mathcal{D}'(\mathbf{R}^n), \tag{2}$$

where $u = (u_1, ..., u_m)^T$, $\vec{g}(u) = (g_1(u), ..., g_n(u))$, where $g_i \in L^\infty_{loc}(\mathbf{R}^n)$, $i = 1, ..., n$, and allow compositions with real-valued functions u, on $([0,T) \times \mathbf{R}^n)$, and $g(u)$ does not satisfy Lipschitz's condition, $\partial_x \cdot \vec{g}(u) = \vec{g}(u)' \cdot \nabla u = \sum_{j=1}^n g_j'(u)\partial_{x_j} u$, $D = (-\triangle)^{1/2}$, $\psi \in L^p(\mathbf{R}^n)$ for some $1 \leq p \leq \infty$. When $p = 1$, $\psi \in \mathcal{M}(\mathbf{R}^n)$ is the space of Radon measure.

3. Equation with Schrӧdinger's operator

$$(\partial_t - \triangle)u + Vu + g(u) = 0, \;\; u(0,\cdot) = \mu(\cdot), \; x \in \mathbf{R}^n, \tag{3}$$

where $V(\cdot)$ and $\mu(\cdot)$ are singular distributions. Suppose that $V(\cdot)$ and $\mu(\cdot)$ are the sums of powers or derivatives of Dirac measure. Without loss of generality suppose that, $V(\cdot) = \delta(\cdot)$, $\mu(\cdot) = \delta(\cdot)$.

3. Basic Spaces

For general theory of Colombeau's generalized functions cf. [6, 7, 1, 16, 10].

We recall construction of the Colombeau's algebras $\mathcal{G}_{p,q}(\Omega)$, ($\Omega$ is an open set), $1 \leq p, q \leq \infty$, from [4].

Let $\Omega \subset \mathbf{R}^n$ be an open set, $m \in \mathbf{Z}$, $1 \leq p \leq \infty$. $W^{\infty,p}(\Omega) = \cap_m W^{m,p}(\Omega)$, $W^{-\infty,p}(\Omega) = \cup_m W^{-m,p}(\Omega)$, where $W^{m,p}(\Omega)$ is usual Sobolev space whose all derivatives up to the order m are finite in corresponding norm. Define

$$\begin{aligned}
\mathcal{E}(\Omega) &= \{u; (0,\infty) \times \Omega \to \mathbf{R}, \text{ s.t. } u_\varepsilon(\cdot) \text{ is } C^\infty \text{ in } x \in \Omega, \; \forall \varepsilon > 0\} \\
\mathcal{E}_p(\Omega) &= \{u \in \mathcal{E}(\Omega); \text{ s.t. } u_\varepsilon \in W^{\infty,p}(\Omega), \; \forall \varepsilon > 0\} \\
\mathcal{E}_{M,p}(\Omega) &= \{u \in \mathcal{E}_p(\Omega); \; \forall \alpha \in \mathbf{N}_0^n \exists N \in \mathbf{N}, \text{ s.t. } ||\partial^\alpha u_\varepsilon(\cdot)||_p = O(\varepsilon^{-N}), \varepsilon \to 0\} \\
\mathcal{N}_{p,q}(\Omega) &= \{u \in \mathcal{E}_{M,p}(\Omega) \cap \mathcal{E}_q(\Omega); \; \forall \alpha \in \mathbf{N}^n \forall M \in \mathbf{N}, \text{ s.t. } ||\partial^\alpha u_\varepsilon(\cdot)||_q \\
&= O(\varepsilon^M), \; \varepsilon \to 0\},
\end{aligned}$$

where $||\cdot||_p$ denotes L^p-norm, and $\partial^\alpha = \partial^{\alpha_1}_{x_1} ... \partial^{\alpha_n}_{x_n}$ for $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbf{N}_0^n$.

Colombeau's space $\mathcal{G}_{p,q}(\Omega)$, $1 \leq p, q \leq \infty$, is the factor set $\mathcal{G}_{p,q}(\Omega) = \mathcal{E}_{M,p}(\Omega)/\mathcal{N}_{p,q}\,(\Omega)$. For structural properties of these spaces cf. [4].

Recall the definition of $\mathcal{G}_{s,g}(\mathbf{R}^n)$ algebras from [8]. Let $\Omega \in \mathbf{R}^n$ be open and $\bar{\Omega}$ be its closure. Let $\mathcal{D}(\Omega)$ be the space of all smooth functions on $\mathbf{R}^n$ with bounded derivatives. Subspace of these functions with compact support in $\bar{\Omega}$ is denoted by $\mathcal{D}(\bar{\Omega})$. $\mathcal{E}_{s,g}(\bar{\Omega})$ is the algebra of all maps from $(0,\infty)$ into $\mathcal{D}_{L^\infty}(\bar{\Omega})$ whose elements are sequences $(u_\varepsilon)_{\varepsilon>0}$ of bounded smooth functions.

$$\begin{aligned}
\mathcal{E}_{M,s,g}(\bar{\Omega}) &= \{(u_\varepsilon)_{\varepsilon>0} \in \mathcal{E}_{s,g}(\bar{\Omega}); \forall \alpha \in \mathbf{N}_0^n \exists p > 0, \text{s.t.} ||\partial^\alpha u_\varepsilon(\cdot)||_{L^\infty(\bar{\Omega})} \\
&= O(\varepsilon^{-p}), \varepsilon \to 0\},
\end{aligned}$$

$$\mathcal{N}_{s,g}(\bar\Omega) = \{(u_\varepsilon)_\varepsilon \in \mathcal{E}_{s,g}(\bar\Omega);\ \forall\alpha \in \mathbf{N}_0^n\ \forall a > 0, \text{ s.t. } ||\partial^\alpha u_\varepsilon(\cdot)||_{L^\infty(\bar\Omega)} = O(\varepsilon^a),\ \varepsilon \to 0\}.$$

The space $\mathcal{G}_{s,g}(\bar\Omega)$ is defined as the factor set $\mathcal{G}_{s,g}(\bar\Omega) = \mathcal{E}_{M,s,g}(\bar\Omega)/\mathcal{N}_{s,g}\ (\bar\Omega)$. The space $\mathcal{D}'_{L^\infty}(\mathbf{R}^n)$, is the space of bounded distributions. The space of finite sums of derivatives of bounded functions can be imbedded into $\mathcal{G}_{s,g}(\mathbf{R}^n)$ by convolution with delta sequence. Let $\phi \in \mathcal{D}(\mathbf{R}^n)$, $\int \phi(\cdot)dx = 1$, $\int x^\alpha \phi(\cdot)dx = 0$, $\forall\alpha \in \mathbf{N}_0^n$, $|\alpha| \geq 1$, and mollifier $\phi_\varepsilon(\cdot) = \varepsilon^{-n}\phi(\cdot/\varepsilon)$. For all $w \in \mathcal{D}'_{L^\infty}(\mathbf{R}^n)$ by $w \to [(\kappa_\varepsilon w * \phi_\varepsilon)_{\varepsilon>0}]$ where κ_ε is the characteristic function of the corresponding set, ($[\cdot]$ denotes the class of equivalence), is obtained an injective map: $\mathcal{D}'_{L^\infty}(\mathbf{R}^n) \to \mathcal{G}_{s,g}(\mathbf{R}^n)$. By Taylor expansion, for every $f \in \mathcal{D}_{L^\infty}(\mathbf{R}^n)$, $(\kappa_\varepsilon f * \phi_\varepsilon - f)_{\varepsilon>0} \in \mathcal{N}_{s,g}(\mathbf{R}^n)$. Thus, $\mathcal{D}_{L^\infty}(\mathbf{R}^n)$ is faithful algebra. The derivatives on $\mathcal{G}_{s,g}(\mathbf{R}^n)$ induce the usual once on $\mathcal{D}'_{L^\infty}(\mathbf{R}^n)$ and $\mathcal{D}_{L^\infty}(\mathbf{R}^n)$.

Let $r \in [1,\infty]$ and $g \in L^r_{loc}(\Omega)$. Then $G \in \mathcal{G}_{p,q}(\Omega)$, $1 \leq p,q \leq \infty$, is L^r - associated to g if $||g - G_\varepsilon||_{L^r(\omega)} \to 0$, as $\varepsilon \to 0$, for every $\omega \subset\subset \Omega$ and every representative G_ε of G.

We take $\Omega = ([0,T) \times \mathbf{R}^n)$.

4. Regularization

We shall use three type of regularization to control the singularities: 1. delta sequences for initial data ; 2. the cut-off for nonlinear term; 3. function $k_{\phi,\varepsilon}(t,\tau)$ for the heat kernel.

The initial data

Let $\mu \in \mathcal{D}'(\Omega)$, Ω be an open set in $\mathbf{R}^n$, then we set $\mu_\varepsilon = (\kappa_\varepsilon\mu) * \phi_\varepsilon$ where $\kappa_\varepsilon \in C_0^\infty(\Omega)$ and $\kappa_\varepsilon = \begin{cases} 1 & \text{on } \Omega_{2\varepsilon} \\ 0 & \text{on } \Omega \setminus \Omega_{1\varepsilon} \end{cases}$, where $\Omega_{2\varepsilon} = \{x; d(x, compl.(\Omega)\} > 2\varepsilon\}$.

We use the mollifier $\phi_\varepsilon(\cdot) = h(\varepsilon)^n\phi(\cdot h(\varepsilon))$, $\phi \in C_0^\infty(\mathbf{R}^n)$, $\int\phi(\cdot)dx = 1$ and $\phi(\cdot) \geq 0$, $x \in \mathbf{R}^n$, $h(\varepsilon) \to \infty$, as $\varepsilon \to 0$. We put $h(\varepsilon) = |ln\varepsilon|^a$, $a > 0$. Suppose that $\mu = \delta^{(k)}$, $k \in \mathbf{N}$. Then, $\mu_\varepsilon(\cdot) = |ln\varepsilon|^{an+k}\phi^{(k)}(\cdot|ln\varepsilon|)$ and $||\mu_\varepsilon(\cdot)||_{L^p} \leq C|ln\varepsilon|^{n(1-1/p)+k}$, $k \geq 0$, $1 \leq p \leq \infty$. When $\mu = D^k\psi$, $\psi \in L^p(\mathbf{R}^n)$, $D = (-\triangle)^{1/2}$, we have $\mu_\varepsilon(\cdot) = D^k\psi * \phi_\varepsilon(\cdot) = \psi * D^k\phi_\varepsilon(\cdot) = \psi(\cdot) * |ln\varepsilon|^{an+k/2}\phi^{k/2}(\cdot|ln\varepsilon|)$. In L^p-norm we obtain $||\mu_\varepsilon(\cdot)||_{L^p} \leq C|ln\varepsilon|^{n(a-1/p)+k/2}$, $1 \leq p \leq \infty$, $k \geq 0$. The similar holds for the sums of derivatives of delta functions and its powers. In general, $||\mu_\varepsilon(\cdot)||_{L^p} \leq C|ln\varepsilon|^{\beta n+\gamma}$, $\beta,\gamma > 0$. Without loss of generality suppose

(4) $$\mu_\varepsilon(\cdot) = \delta_\varepsilon(\cdot) = |ln\varepsilon|^{an}\phi(\cdot|ln\varepsilon|),\ \ a > 0,$$

where $\phi(\cdot) > 0,\ \phi(\cdot) \in C_0^\infty(\mathbf{R}^n),\ \int \phi(\cdot)dx = 1.$

The diagonal $t = \tau$.

Due to the estimate (cf. [3]),

$$||t^{k/2+n/2(1-1/r)}\partial_x^\alpha E_n(t,\cdot)||_{L^r} < \infty,\ |\alpha| \le k,\ 1 \le r \le \infty,$$

where $E_n(t,\cdot)$ is the heat kernel, $E_n(t,\cdot) = (4\pi t)^{-n/2}e^{-|\cdot|^2/(4t)}$, and $\partial_x^\alpha = \partial_{x_1}^{\alpha_1}...\partial_{x_n}^{\alpha_n}$, $|\alpha| = \alpha_1 + ... + \alpha_n$, $|\alpha| \ge 0$, we have $||\partial_x^\alpha E_n(t,\cdot)||_{L^r} < Ct^{-(k/2+n/2(1-1/r))}$. The α^{th}-derivative of the heat kernel, where $\alpha \ge 2$ in the equations (11) and (13) and the α^{th}-derivative, $\alpha \ge 1$, in the equation (12) lead to the divergent integrals. To avoid the singularity over the diagonal $t = \tau$ we use the regularization with the function $k_{\phi,\varepsilon}(t,\tau)$, (cf. [18]). We set

$$k_{\phi,\varepsilon}(t,\tau) = 1 - \psi_0(h(\varepsilon)(t-\tau)),\ t,\tau \in \mathbf{R},$$

where $\psi_0 \in C_0^\infty(\mathbf{R}),\ \psi_0(\cdot) \le 1 - \dfrac{1}{ln|ln\varepsilon|}$, when $|\cdot| \le 1/4,\ \psi_0(\cdot) = 0$ when $|\cdot| > 1/2$. Then,

$$k_{\phi,\varepsilon}(t,\tau) = \begin{cases} 1 & |t-\tau| \ge 1/(2h(\varepsilon)) \\ \dfrac{C}{ln|ln\varepsilon|} & |t-\tau| \le 1/(4h(\varepsilon)),\ t,\tau \in \mathbf{R}. \end{cases} \tag{5}$$

We employ the following regularization for the heat kernel

$$\begin{aligned} E_{n\varepsilon}(t,\cdot) &= k_{\phi,\varepsilon}(t,\tau)E_n(t,\cdot) \\ &= \begin{cases} 1 & |t-\tau| \ge 1/(2h(\varepsilon)) \\ \dfrac{C}{ln|ln\varepsilon|} & |t-\tau| \le 1/(4h(\varepsilon)) \end{cases} E_n(t,\cdot),\quad t,\tau \in \mathbf{R}. \end{aligned} \tag{6}$$

Since for $|\alpha| \le k$,

$$||\partial_x^\alpha E_{n\varepsilon}(t,\cdot)||_{L^p} \le Ck_{\phi,\varepsilon}(t,\tau)|t-\tau|^{-(k/2+n/2(1-1/p))},\ \alpha \ge 0,$$

we have

$$||\partial_x^\alpha E_{n\varepsilon}(t,\cdot)||_{L^p} \le \begin{cases} Ch(\varepsilon)^{k/2+n/2(1-1/p)} & |t-\tau| \ge 1/(2h(\varepsilon)) \\ \dfrac{C}{ln|ln\varepsilon|}h(\varepsilon)^{\alpha/2+n/2(1-1/p)} & |t-\tau| \le 1/(4h(\varepsilon)). \end{cases}$$

In particular, in L^1-norm,

$$||\partial_x^\alpha E_{n\varepsilon}(t,\cdot)||_{L^1} \le \begin{cases} \dfrac{C}{ln|ln\varepsilon|}h(\varepsilon)^{\alpha/2} & |t-\tau| \le 1/(4h(\varepsilon)) \\ Ch(\varepsilon)^{\alpha/2} & |t-\tau| \ge 1/(2h(\varepsilon)). \end{cases} \tag{7}$$

We put

(8) $$h(\varepsilon) = O(|ln\varepsilon|^{2/(\alpha+4)}),\ \ \alpha \geq 0,\ \ (\text{resp. O}(ln|ln\varepsilon|)),$$

to handle problem (1) and (2). For the problem (3) we use

(9) $$h(\varepsilon) = O(|ln\varepsilon|^{2/(\alpha+5)}),\ \ \alpha \geq 0,\ \ (\text{resp. O}(ln|ln\varepsilon|)).$$

Cut-off method

Cut-off method is introduced in [8, 9] to compensate the growth of $f \in C^{\infty}(\mathbf{R}^n)$ and its derivatives at infinity. It gives global solutions for equations without Lipschitz's condition for the main term, (cf. [17]). We apply it for nonlinear term $g(u)$ to avoid non-Lipschitz's nonlinearity and obtain global solutions (cf. [18]).

Let $B_{h(\varepsilon)} = \{(t,x), t, x \in w_{h(\varepsilon)}\}$, where $w_{h(\varepsilon)}(s) = \{s \in I, |s| \leq h(\varepsilon), d(s, compl.I) \geq 1/h(\varepsilon)\}$ where I is the n-dimensional interval around zero in a case of x and in a case of t the interval is 1-dimensional; $h(\varepsilon)$ is a scaling function, $h(\varepsilon) \to \infty$, as $\varepsilon \to 0$, and will be determined to follow the singularities of the problem under consideration.

Let

$$\bar{g}_\varepsilon(u) = \begin{cases} g(u),\ \ u \in B_{h(\varepsilon)},\ \ \text{and } |g_\varepsilon(u)| \leq h(\varepsilon) \\ 0 \qquad\qquad\qquad\qquad otherwise \end{cases}$$

for $\varepsilon \in (0,1)$. Set

$$g_\varepsilon(u) = \bar{g}_\varepsilon(\cdot) * (h(\varepsilon)\Theta(h(\varepsilon)\cdot))(u) = h(\varepsilon)^{m+n+1} \int_{B_{h(\varepsilon)}\times\mathbf{R}^m} \bar{g}_\varepsilon(\xi,\eta,\tau)$$
$$\Theta(h(\varepsilon)(u-\xi), h(\varepsilon)(x-\eta), h(\varepsilon)(t-\tau))d\xi d\eta d\tau,\ u \in \mathbf{R}^m,$$

where $\Theta \in C_0^\infty(\mathbf{R}^{m+n+1})$, such that $\Theta = \begin{cases} 1 \text{ on } \{x||x| \leq 1/2\} \\ 0\ \text{ on } \{x||x| \geq 1\} \end{cases}$ and $\int \Theta(\cdot)dx = 1$. We have

$$\begin{aligned}
|\frac{\partial}{\partial u} g_\varepsilon(u)| &= |\bar{g}_\varepsilon(\cdot) * \frac{\partial}{\partial u}(h(\varepsilon)\Theta(h(\varepsilon)\cdot))(u)| \\
&= |\frac{\partial}{\partial u}\int_{B_{h(\varepsilon)}\times\mathbf{R}^m} h(\varepsilon)^{m+n+1}\bar{g}_\varepsilon(\xi,\eta,\tau)\Theta(h(\varepsilon)(u-\xi), \\
&\qquad h(\varepsilon)(x-\eta), h(\varepsilon)(t-\tau))d\xi d\eta d\tau| \\
&= |\int_{\mathbf{R}^m} h(\varepsilon)^{m+n+1}\bar{g}_\varepsilon(\xi,\eta,\tau)\frac{\partial}{\partial u}\Theta(h(\varepsilon)(u-\xi), \\
&\qquad h(\varepsilon)(x-\eta), h(\varepsilon)(t-\tau))d\xi d\eta d\tau| \\
&= |h(\varepsilon)\int_{\mathbf{R}^m} \bar{g}_\varepsilon(u-\xi/h(\varepsilon), x-\eta/h(\varepsilon), t-\tau/h(\varepsilon)) \\
&\qquad \frac{\partial}{\partial u}\Theta(\xi,\eta,\tau)d\xi d\eta d\tau| \leq h(\varepsilon)^2.
\end{aligned}$$

Thus,

$$|g(u)| \leq Ch(\varepsilon), \quad |\nabla g(u)| \leq Ch(\varepsilon)^2. \tag{10}$$

In integral form for the full regularization we use for (1), (2), (3), respectively:

$$\begin{aligned} u_\varepsilon(t,\cdot) &= (E_{n\varepsilon}(t,\cdot) * \mu_\varepsilon(\cdot))(x) \\ &\quad + \int_0^t \int_{\mathbf{R}^n} E_{n\varepsilon}(t-\tau, x-\cdot) g_\varepsilon(u_\varepsilon(\tau,\cdot)) dy d\tau + N_\varepsilon(t,\cdot) \\ u_{0\varepsilon}(0,\cdot) &= \mu_\varepsilon(\cdot) + N_{0\varepsilon}(\cdot), \end{aligned} \tag{11}$$

$$\begin{aligned} u_\varepsilon(t,\cdot) &= (E_{n\varepsilon}(t,\cdot) * \mu_\varepsilon(\cdot))(x) \\ &\quad + \int_0^t \int_{\mathbf{R}^n} \nabla E_{n\varepsilon}(t-\tau, x-\cdot) g_\varepsilon(u_\varepsilon(\tau,\cdot)) dy d\tau + N_\varepsilon(t,\cdot) \\ u_{0\varepsilon}(0,\cdot) &= \mu_\varepsilon(\cdot) + N_{0\varepsilon}(\cdot), \end{aligned} \tag{12}$$

$$\begin{aligned} u_\varepsilon(t,\cdot) &= (E_{n\varepsilon}(t,\cdot) * \mu_\varepsilon(\cdot))(x) \\ &\quad + \int_0^t \int_{\mathbf{R}^n} E_{n\varepsilon}(t-\tau, x-\cdot) V_\varepsilon(\cdot) u_\varepsilon(\tau,\cdot) dy d\tau \\ &\quad + \int_0^t \int_{\mathbf{R}^n} E_{n\varepsilon}(t-\tau, x-\cdot) g_\varepsilon(u_\varepsilon(\tau,\cdot)) dy d\tau, \\ u_{0\varepsilon}(0,\cdot) &= \mu_\varepsilon(\cdot) + N_{0\varepsilon}(\cdot), \end{aligned} \tag{13}$$

where the regularization for the heat kernel, initial data and nonlinear term $g_\varepsilon(u_\varepsilon)$ is given by (6), (4) and (10) respectively. Selection of good mollifiers depends on the problem under consideration.

In Colombeau's setting we have, for the equation (11)

$$[u_\varepsilon(t,\cdot)] = [(S_{n\varepsilon}(t,\cdot) * \mu_\varepsilon(\cdot))(x)] + [\int_0^t (S_{n\varepsilon}(t-\tau,\cdot) * g_\varepsilon(u_\varepsilon(\tau,\cdot)))(x) d\tau]$$

where $[\cdot]$ denotes the equivalence class. The similar holds for (12) and (13).

5. Existence-uniqueness Theorems

5.1. The equation (1)

We shall use the following Lemma 1 for the proof of the existence-uniqueness theorem.

Lemma 1. (a) *Let* $u_\varepsilon \in \mathcal{E}_{M,p}([0,\infty)\times\mathbf{R}^n)$. *Then,* $\forall\alpha\in\mathbf{N}_0^n$, $x\in\mathbf{R}^n$, $\varepsilon<\varepsilon_0$, $t\in[0,T)$,

$$[0,\infty)\ni t\mapsto\int_0^t(\partial_x^\alpha E_{n\varepsilon}(t-\tau,\cdot)*g_\varepsilon(u_\varepsilon(\tau,\cdot)))(x)d\tau\in\mathcal{E}_{M,p}([0,\infty)\times\mathbf{R}^n),\ 1\le p\le\infty;$$

(b) *Let* $u_\varepsilon,\tilde u_\varepsilon\in\mathcal{E}_{M,p}([0,T)\times\mathbf{R}^n)$ *such that* $u_\varepsilon-\tilde u_\varepsilon\in\mathcal{N}_{p,q}([0,\infty)\times\mathbf{R}^n)$, $1\le p,q\le\infty$. *Then*

$$[0,\infty)\ni t\mapsto\int_0^t(\partial_x^\alpha E_{n\varepsilon}(t-\tau,\cdot)*(g_\varepsilon(u_\varepsilon(\tau,\cdot))-g_\varepsilon(\tilde u_\varepsilon(\tau,\cdot))))(x)d\tau\in\mathcal{N}_{p,q}([0,\infty)\times\mathbf{R}^n),$$

$1\le p,q\le\infty$.

Proof. Let $\varepsilon<\varepsilon_0$, $D_1^j=\dfrac{\partial^j}{\partial t^j}\partial_x^\alpha E_n(t,\cdot)$ and

$$T_\varepsilon(t)=\int_0^t(\partial_x^\alpha E_{n\varepsilon}(t-\tau,\cdot)*g_\varepsilon(u_\varepsilon(\tau,\cdot)))(x)d\tau,\qquad t\in[0,\infty),\ x\in\mathbf{R}^n,\ \varepsilon<\varepsilon_0. \tag{14}$$

Then $\forall j\in\mathbf{N}_0$, $\varepsilon<\varepsilon_0$ we obtain, since $\partial_x^\alpha E(0,\cdot)=0$, $\forall\alpha\in\mathbf{N}_0^n$, and $k_{\phi,\varepsilon}(t,t)\approx 0$,

$$\frac{d^j}{dt}T_\varepsilon(t)=\int_0^t(D_1^j(\partial_x^\alpha E_{n\varepsilon}(t-\tau,\cdot))*g_\varepsilon(u_\varepsilon(\tau,\cdot)))(x)d\tau.$$

Then, $\exists C>0$, $\exists d_0\in\mathbf{R}$ such that

$$||\frac{d^j}{dt^j}T_\varepsilon(t)||_{L^p}\le\int_0^t||D_1^j(\partial_x^\alpha E_{n\varepsilon}(t-\tau,\cdot))||_{L^1}||g_\varepsilon(u_\varepsilon(\tau,\cdot))||_{L^p}d\tau.$$

By Leibnitz rule

$$\le C\int_0^t||\sum_{k=0}^j\binom{j}{k}k_{\phi,\varepsilon}(t,\tau)^{(k)}\partial_x^\alpha E_n(t-\tau,\cdot)^{(j-k)}||_{L^1}||g_\varepsilon(u_\varepsilon(\tau,\cdot))||_{L^p}d\tau$$
$$\le Ck_{\phi,\varepsilon}(t,\tau)h(\varepsilon)^{j+\alpha/2}\int_0^t||g_\varepsilon(u_\varepsilon(\tau,\cdot))||_{L^p}d\tau,$$

where $k_{\phi,\varepsilon}(t,\tau)$ is given with (5). For $0<\theta<1$, since $g(0)=0$, we have

$$\begin{aligned}||g_\varepsilon(u_\varepsilon(\tau,\cdot))||_{L^p}&=||g_\varepsilon(0)+u_\varepsilon(\tau,\cdot)\nabla_u g_\varepsilon(\theta u_\varepsilon(\tau,\cdot))||_{L^p}\\&=||u_\varepsilon(\tau,\cdot)\cdot\nabla_u g_\varepsilon(\theta u_\varepsilon(\tau,\cdot))||_{L^p}\\&\le h(\varepsilon)^2||u_\varepsilon(\tau,\cdot)||_{L^p}.\end{aligned}$$

Due to $u_\varepsilon \in \mathcal{E}_{M,p}([0,T)\times \mathbf{R}^n)$, we have $||u_\varepsilon(\tau,\cdot)||_{L^p} \le C\varepsilon^{-N}$, $\exists N \in \mathbf{N}$ and

$$||\frac{d^j}{dt^j}T_\varepsilon(t)||_{L^p} \le CTk_{\phi,\varepsilon}(t,\tau)h(\varepsilon)^{j+\alpha/2}\varepsilon^{-N} \le C\varepsilon^{-N},\ \exists N > 0.$$

Note that for $|t-\tau| \le C/(4|ln\varepsilon|)$,

$$||\frac{d^j}{dt^j}T_\varepsilon(t)||_{L^p} \le CT/(ln|ln\varepsilon|)h(\varepsilon)^{j+\alpha/2}\varepsilon^{-N} \le C\varepsilon^{-N},\ \exists N > 0.$$

Thus, $T_\varepsilon(t) \in \mathcal{E}_{M,p}([0,T)\times \mathbf{R}^n)$.

(b) Let $j \in \mathbf{N}$ and $\tilde{T}_\varepsilon(t) = \int_0^t(\partial_x^\alpha E_{n\varepsilon}(t-\tau,\cdot) * g_\varepsilon(\tilde{u}_\varepsilon(\tau,\cdot)))(x)d\tau,\ \varepsilon < \varepsilon_0$, and $A_\varepsilon^j = ||\frac{d^j}{dt^j}(T_\varepsilon(t) - \tilde{T}_\varepsilon(t))||_{L^p}$. Let $B_\varepsilon(t,\cdot) = g_\varepsilon(u_\varepsilon(t,\cdot)) - g_\varepsilon(\tilde{u}_\varepsilon(t,\cdot))$. Then,

$$\begin{aligned} A_\varepsilon^j &\le \int_0^t ||(D_1^j(\partial_x^\alpha E_{n\varepsilon}(t-\tau,\cdot)) * B_\varepsilon(\tau,\cdot))(x)||_{L^p}d\tau \\ &\le \int_0^t ||\sum_{k=0}^j \binom{j}{k} k_{\phi,\varepsilon}(t,\tau)^{(k)}\partial_x^\alpha E_n(t-\tau,\cdot)^{(j-k)}||_{L^1}||B_\varepsilon(\tau,\cdot)||_{L^p}d\tau \\ &\le \int_0^t Ck_{\phi,\varepsilon}(t,\tau)h(\varepsilon)^{j+\alpha/2}||B_\varepsilon(\tau,\cdot)||_{L^p}d\tau. \end{aligned}$$

By mean value theorem we have

$$\begin{aligned} A_\varepsilon^j &\le Ck_{\phi,\varepsilon}(t,\tau)h(\varepsilon)^{j+\alpha/2}\int_0^t ||(u_\varepsilon(\tau,\cdot) - \tilde{u}_\varepsilon(\tau,\cdot))\cdot(\nabla g_\varepsilon(\theta u_\varepsilon(\tau,\cdot) \\ &\quad +(1-\theta)\tilde{u}_\varepsilon(\tau,\cdot))||_{L^p}d\tau \\ &\le Ck_{\phi,\varepsilon}(t,\tau)h(\varepsilon)^{j+\alpha/2+2}||u_\varepsilon(t,\cdot) - \tilde{u}_\varepsilon(t,\cdot)||_{L^p} \le C\varepsilon^a,\ \forall a \in \mathbf{R}. \end{aligned}$$

Since $(u_\varepsilon - \tilde{u}_\varepsilon)(t,\cdot) \in \mathcal{N}_{p,q}([0,\infty)\times\mathbf{R}^n)$, $1 \le p,q \le \infty$, $A_\varepsilon^j \le Ch(\varepsilon)^{j+2}\varepsilon^a$. Thus, $A_\varepsilon^j = O(\varepsilon^a)$ for $\forall a > 0$. ■

Theorem 1. *Let the equation* (1) *where*

(1) $g \in L^\infty_{loc}(\mathbf{R}^n)$ *is meant to be composed with a real-valued function* u *on* $([0,T)\times\mathbf{R}^n)$, $g(u)$ *is not of Lipschitz class;*
(2) $\mu(\cdot) = \delta(\cdot)$, *(resp.* $\mu \in \mathcal{D}'_{L^\infty_{loc}}(\mathbf{R}^n)$*)*,

have the regularized integral form (11) *where the regularization for* ε*-subscript terms are given by* (10), (6), (4) *and* $h(\varepsilon)$ *is from* (8). *Then, there exists an unique solution in the Colombeau's spaces* $[u_\varepsilon] \in \mathcal{G}_{p,q}([0,T)\times\mathbf{R}^n)$, $1 \le p,q \le \infty$ *(resp. in* $\mathcal{G}_{s,g}([0,T)\times\mathbf{R}^n)$*)*.

Proof. We prove the estimate in L^∞-norm. The same holds for L^p-norm where $1 \leq p, q \leq \infty$. Consider the equation (11). By Young's inequality and the first approximation for $g_\varepsilon(u_\varepsilon)$, since $g(0) = 0$,

$$||u_\varepsilon(t,\cdot)||_{L^\infty} \leq ||E_{n\varepsilon}(t,\cdot)||_{L^1}||\mu_\varepsilon(\cdot)||_{L^\infty}$$

$$+\int_0^t ||E_{n\varepsilon}(t-\tau, x-\cdot)||_{L^1}||\nabla g_\varepsilon u_\varepsilon(\tau,\cdot)||_{L^\infty}\, ||u_\varepsilon(\tau,\cdot)||_{L^\infty} d\tau.$$

Since (7) and (4) hold, applying Gronwall inequality we obtain

$$||u(t,\cdot)||_{L^\infty} \leq Ck_{\phi,\varepsilon}(t,\tau)|ln\varepsilon|^{an}\exp\left(CTk_{\phi,\varepsilon}(t,\tau)h(\varepsilon)^2\right) \leq C\varepsilon^{-N},$$

$\exists N > 0, \varepsilon \in (0,1),\ x \in \mathbf{R}^n,\ t \in [0,T)$, where $h(\varepsilon)$ is given by (8), $\alpha \geq 0$. When $|t-\tau| \geq C/(2h(\varepsilon)),\ k_{\phi,\varepsilon}(t,\tau) = 1$ and the moderateness holds. When $|t-\tau| \leq C/(4h(\varepsilon))$, we have

$$||u(t,\cdot)||_{L^\infty} \leq C\frac{|ln\varepsilon|^{an}}{ln|ln\varepsilon|}\exp\left(CT/(ln|ln\varepsilon|)h(\varepsilon)^2\right) \leq C\varepsilon^{-N},$$

$\exists N > 0, \varepsilon \in (0,1),\ x \in \mathbf{R}^n,\ t \in [0,T)$, where $h(\varepsilon)$ is given by (8), $\alpha \geq 0$, $a > 0$.

Consider α^{th}-derivative, $\alpha \in \mathbf{N}_0^n$, $\alpha \geq 1$,

$$\partial_x^\alpha u_\varepsilon(t,\cdot) = (\partial_x^\alpha E_{n\varepsilon}(t,\cdot) * \mu_\varepsilon(\cdot))(x)$$

$$+\int_0^t\int_{\mathbf{R}^n} \partial_x^\alpha E_{n\varepsilon}(t-\tau, x-\cdot)\nabla g_\varepsilon(\theta u_\varepsilon(\tau,\cdot))u_\varepsilon(\tau,\cdot)dyd\tau.$$

Then,

$$||\partial_x^\alpha u_\varepsilon(t,\cdot)||_{L^\infty} \leq ||\partial_x^\alpha E_{n\varepsilon}(t,\cdot)||_{L^1}||\mu_\varepsilon(\cdot)||_{L^\infty} + \int_0^t ||\partial_x^\alpha E_{n\varepsilon}(t-\tau, x-\cdot)||_{L^1}$$

$$||\nabla g_\varepsilon(\theta u_\varepsilon(\tau,\cdot))||_{L^\infty}||u_\varepsilon(\tau,\cdot)||_{L^\infty}d\tau.$$

We have

$$||\partial_x^\alpha u_\varepsilon(t,\cdot)||_{L^\infty} \leq Ck_{\phi,\varepsilon}(t,\tau)h(\varepsilon)^{\alpha/2} + \int_0^t Ck_{\phi,\varepsilon}(t,\tau)h(\varepsilon)^{\alpha/2+2}||u_\varepsilon(\tau,\cdot)||_{L^\infty}d\tau.$$

By the first step of the induction we obtain

$$||\partial_x^\alpha u_\varepsilon(t,\cdot)||_{L^\infty} \leq Ck_{\phi,\varepsilon}(t,\tau)h(\varepsilon)^{\alpha/2} + CTk_{\phi,\varepsilon}(t,\tau)h(\varepsilon)^{\alpha/2+2}\varepsilon^{-N} \leq C\varepsilon^{-N},$$

since $k_{\phi,\varepsilon}(t,\tau)$ is given with (5), $h(\varepsilon)$ is defined in (8), $\exists N > 0, t \in [0,T), T > 0$, $x \in \mathbf{R}^n$, $\varepsilon < \varepsilon_0,\ \alpha \geq 0,\ a > 0$.

Estimate (w.r.) to t, as well as, the estimate for mixed derivatives we obtain from the equation (1) using the results of Lemma 1. We give the proof for $\alpha\beta^{th}$-derivative $\alpha \in \mathbf{N}_0^n$, $\beta \in \mathbf{N}_0$ (w.r.) to t for the equation (11).

Suppose that $\beta \in \mathbf{N}$, $\alpha \in \mathbf{N}_0^n$. We have proved for $\alpha = \beta = 0$ that u_ε is moderate, by Gronwall inequality. Then, $\forall \beta \in \mathbf{N}_0\ \forall \alpha \in \mathbf{N}_0^n$,

$$\partial_t^\beta \partial_x^\alpha u_\varepsilon(t,\cdot) = (\partial_t^\beta \partial_x^\alpha E_{n\varepsilon}(t,\cdot) * \mu_\varepsilon(\cdot))(x) + \frac{d^\beta}{dt^\beta} T_\varepsilon(t),\ x \in \mathbf{R}^n,\ \varepsilon < \varepsilon_0,\ t \in [0,T),$$

where $T_\varepsilon(t)$ is given by (14). By Lemma 1 we obtain

$$||\partial_t^\beta \partial_x^\alpha u_\varepsilon(t,\cdot)||_{L^\infty} \leq ||\partial_t^\beta \partial_x^\alpha E_{n\varepsilon}(t,\cdot)||_{L^1} ||\mu_\varepsilon(\cdot)||_{L^\infty} + C\varepsilon^{-N}.$$

Then, $\forall \beta \in \mathbf{N}_0 \forall \alpha \in \mathbf{N}_0^n,\ \varepsilon < \varepsilon_0,\ x \in \mathbf{R}^n$, when $|t-\tau| \geq C/(2h(\varepsilon))$,

$$||\partial_t^\beta \partial_x^\alpha u_\varepsilon(t,\cdot)||_{L^\infty} \leq h(\varepsilon)^{\beta/2+\alpha/2} |ln\varepsilon|^{an} + C\varepsilon^{-N} \leq C\varepsilon^{-N},$$

$\exists N > 0$, $t \in [0,T)$, $x \in \mathbf{R}^n$, $\varepsilon < \varepsilon_0$, $h(\varepsilon)$ is from (8), $\alpha \geq 0$.

For $|t-\tau| \leq C/(4|ln\varepsilon|)$ we have

$$||\partial_t^\beta \partial_x^\alpha u_\varepsilon(t,\cdot)||_{L^\infty} \leq h(\varepsilon)^{\beta/2+\alpha/2} \frac{|ln\varepsilon|^{an}}{ln|ln\varepsilon|} + C\varepsilon^{-N} \leq C\varepsilon^{-N},$$

$\exists N > 0$, $t \in [0,T)$, $x \in \mathbf{R}^n$, $\varepsilon < \varepsilon_0$, $h(\varepsilon)$ is from (8), $\alpha \geq 0$.

Follows, $u_\varepsilon \in \mathcal{E}_{M,\infty}([0,T) \times \mathbf{R}^n)$.

Concerning the uniqueness, suppose that $u_{1\varepsilon}$, $u_{2\varepsilon}$ are two solutions to the equation (1). Denote their difference by $w_\varepsilon(t,\cdot) = u_{1\varepsilon}(t,\cdot) - u_{2\varepsilon}(t,\cdot)$. Then, we have in integral form

$$w_\varepsilon(t,\cdot) = (E_{n\varepsilon}(t,\cdot) * N_{0\varepsilon}(\cdot))(x) + \int_0^t \int_{\mathbf{R}^n} E_{n\varepsilon}(t-\tau, x-\cdot) W_\varepsilon(\tau,\cdot) w_\varepsilon(\tau,\cdot)\,dy\,d\tau$$
$$+ \int_0^t \int_{\mathbf{R}^n} E_{n\varepsilon}(t-\tau, x-\cdot) N_\varepsilon(\tau,\cdot)\,dy\,d\tau,$$

where $N_{0\varepsilon}(\cdot) \in \mathcal{N}_{\infty,q}(\mathbf{R}^n)$, $1 \leq q \leq \infty$, $N_\varepsilon(t,\cdot) \in \mathcal{N}_{\infty,q}([0,T) \times \mathbf{R}^n)$ and $W_\varepsilon(t,\cdot) = \int_0^t \nabla g_\varepsilon(\sigma u_{1\varepsilon} + (1-\sigma) u_{2\varepsilon})\,d\sigma$. Then,

$$||w_\varepsilon(t,\cdot)||_{L^q} \leq ||E_{n\varepsilon}(t,\cdot)||_{L^1} ||N_{0\varepsilon}(\cdot)||_{L^q} + \int_0^t ||E_{n\varepsilon}(t-\tau, x-\cdot)||_{L^1} ||W_\varepsilon(\tau,\cdot)||_{L^\infty}$$
$$||w_\varepsilon(\tau,\cdot)||_{L^q} d\tau + \int_0^t ||E_{n\varepsilon}(t-\tau, x-\cdot)||_{L^1} ||N_\varepsilon(\tau,\cdot)||_{L^q} d\tau,$$

and

$$||w_\varepsilon(t,\cdot)||_{L^q} \leq Ck_{\phi,\varepsilon}(t,\tau)\varepsilon^a + \int_0^t Ck_{\phi,\varepsilon}(t,\tau) h(\varepsilon)^2 ||w_\varepsilon(\tau,\cdot)||_{L^q} d\tau + CT\varepsilon^a.$$

By Gronwall inequality we obtain

$$||w_\varepsilon(t,\cdot)||_{L^q} \le Ck_{\phi,\varepsilon}(t,\tau)\varepsilon^a \exp\left(CTk_{\phi,\varepsilon}(t,\tau)h(\varepsilon)^2\right) \le C\varepsilon^a,$$

$\forall a \in \mathbf{R},\ t\in[0,T),\ x\in\mathbf{R}^n,\ \varepsilon<\varepsilon_0$, where $k_{\phi,\varepsilon}(t,\tau)$ is given with (5), $h(\varepsilon)$ with (8), $\alpha \ge 0$. This is sufficient for the negligibility (w.r.) to x (cf. [10]).

Estimate (w.r.) to t we obtain from the equation (1). We use part (b) from Lemma 1 to prove the uniqueness (w.r.) to t for mixed derivatives. We prove that $\forall\beta\in\mathbf{N}_0\forall\alpha\in\mathbf{N}_0^n\forall a\in\mathbf{R}, 1\le q\le\infty$,

$$||\partial_t^\beta\partial_x^\alpha w_\varepsilon(t,\cdot)||_{L^q} \le C\varepsilon^a,\ x\in\mathbf{R}^n,\ t\in[0,T), \varepsilon<\varepsilon_0, \alpha\ge 0.$$

Follows, $w_\varepsilon(t,\cdot)\in\mathcal{N}_{\infty,q}([0,T)\times\mathbf{R}^n)$, $1\le q\le\infty$, i.e. $||\partial_x^\alpha(u_{1\varepsilon}(t,\cdot)-u_{2\varepsilon}(t,\cdot))||_{L^\infty} = O(\varepsilon^a),\ \forall a\in\mathbf{R}$. The same holds for every $1\le p,q\le\infty$. Thus, the solution is unique in the spaces $\mathcal{G}_{p,q}([0,T)\times\mathbf{R}^n)$, $1\le p,q\le\infty$ (resp. for $p=q=\infty$ we have existence-uniqueness result in the space $\mathcal{G}_{s,g}([0,T)\times\mathbf{R}^n)$). ■

5.2. The equation (2)

To handle this problem we use (9) for $h(\varepsilon)$. We prove the first an axillary result useful in the proof of moderatness and uniqueness of the mixed derivatives and derivatives (w.r.) to t.

Lemma 2. (a) *Let* $u_\varepsilon\in\mathcal{E}_{M,p}([0,\infty)\times\mathbf{R}^n)$, $1\le p\le\infty$. *Then,*

$$[0,\infty)\ni t\mapsto\int_0^t(\partial_x^\alpha\nabla E_{n\varepsilon}(t-\tau,\cdot)*g_\varepsilon(u_\varepsilon(\tau,\cdot)))(x)d\tau\in\mathcal{E}_{M,p}([0,\infty)\times\mathbf{R}^n);$$

(b) *Let* $u_\varepsilon,\tilde u_\varepsilon\in\mathcal{N}_{p,q}([0,\infty)\times\mathbf{R}^n)$, $1\le p,q\le\infty$. *Then,*

$$[0,\infty)\ni t\mapsto\int_0^t(\partial_x^\alpha\nabla E_{n\varepsilon}(t-\tau,\cdot)*(g(u(\tau,\cdot))$$
$$-g_\varepsilon(\tilde u_\varepsilon(\tau,\cdot))))(x)d\tau\in\mathcal{N}_{p,q}([0,\infty)\times\mathbf{R}^n).$$

Proof. (a)

$$||\int_0^t(\partial_x^\alpha\nabla E_{n\varepsilon}(t-\tau,\cdot)*g_\varepsilon(u_\varepsilon(\tau,\cdot)))(x)d\tau||_{L^p}$$
$$\le\int_0^t||\partial_x^\alpha\nabla E_{n\varepsilon}(t-\tau,\cdot)||_{L^1}||g_\varepsilon(u_\varepsilon(\tau,\cdot))||_{L^p}d\tau\le C$$
$$\int_0^t|t-\tau|^{-(\alpha+1)/2}||g_\varepsilon(u_\varepsilon(\tau,\cdot))||_{L^p}d\tau\le CTk_{\phi,\varepsilon}(t,\tau)h(\varepsilon)^{(\alpha+1)/2+2}||u_\varepsilon(t,\cdot)||_{L^p}$$
$$\le C\varepsilon^{-N},\exists N>0,x\in\mathbf{R}^n,t\in[0,T),\varepsilon<\varepsilon_0,1\le p\le\infty.$$

We set (7), and for $h(\varepsilon)$ we use (9), $\alpha \geq 0$, and for $k_{\phi,\varepsilon}(t,\tau)$, we use (5). For the derivatives of integral (w.r.) to t cf. Lemma 1. Similarly we prove (b). ∎

Theorem 2. (a) *Let in the equation* (2)

(*1*) $\mu(\cdot) = \delta(\cdot)$;

(*2*) $g \in L^\infty_{loc}(\mathbf{R}^n)$ *is meant to be composed with a real-valued function* u *on* $([0,T) \times \mathbf{R}^n)$, $g(u)$ *is not of Lipschitz class;*

and the equation (12) *stands for its regularized integral form, where the regularization for* ε*-subscript terms are given by* (4), (6) *and* (10). *Then, there exists an unique global solution* $[u_\varepsilon] \in \mathcal{G}_{p,q}([0,T) \times \mathbf{R}^n)$, $1 \leq p,q \leq \infty$.

(b) *If* $\mu \in \mathcal{D}'_{L^\infty_{loc}}(\mathbf{R}^n)$, *the solution to the equation* (2) *is unique in* $[u_\varepsilon] \in \mathcal{G}_{s,g}([0,T) \times \mathbf{R}^n)$.

Proof. (a) We shall give a proof by induction. We have from (12) for every $1 \leq p \leq \infty$,

$$||u_\varepsilon(t,\cdot)||_{L^p} \leq ||E_{n\varepsilon}(t,\cdot)||_{L^1}||\mu_\varepsilon(\cdot)||_{L^p} + \int_0^t ||\nabla E_{n\varepsilon}(t-\tau, x-\cdot)||_{L^1}||g_\varepsilon(u_\varepsilon(\tau,\cdot))||_{L^p} d\tau.$$

By (7), $||\nabla E_{n\varepsilon}(t-\tau,\cdot)||_{L^1} \leq C k_{\phi,\varepsilon}(t,\tau) h(\varepsilon)^{1/2}$, where $k_{\phi,\varepsilon}(t,\tau)$ is defined with (5) and by the first approximation of g, since (10) holds we obtain

$$||u_\varepsilon(t,\cdot)||_{L^p} \leq C k_{\phi,\varepsilon}(t,\tau)||\mu_\varepsilon(\cdot)||_{L^p} + \int_0^t k_{\phi,\varepsilon}(t,\tau) h(\varepsilon)^{1/2+2} ||u_\varepsilon(\tau,\cdot)||_{L^p} d\tau.$$

By Gronwall inequality

$$||u_\varepsilon(t,\cdot)||_{L^p} \leq C k_{\phi,\varepsilon} |ln\varepsilon|^{n(a-1/p)} \exp\left(CT k_{\phi,\varepsilon}(t,\tau) h(\varepsilon)^{1/2+2}\right) \leq C\varepsilon^{-N},$$

$\exists N > 0,\ t \in [0,T),\ T > 0,\ x \in \mathbf{R}^n,\ \varepsilon < \varepsilon_0,\ \ h(\varepsilon)$ is given with (9), $\alpha \geq 0$, $k_{\phi,\varepsilon}(t,\tau)$ is determined in (5).

Suppose that $\alpha \in \mathbf{N}^n_0$, $\alpha \geq 1$. Then,

$$\partial^\alpha_x u_\varepsilon(t,\cdot) = (\partial^\alpha_x E_{n\varepsilon}(t,\cdot) * \mu_\varepsilon(\cdot))(x) + \int_0^t (\partial^\alpha_x \nabla E_{n\varepsilon}(t-\tau,\cdot) * g_\varepsilon(u_\varepsilon(\tau,\cdot)))(x) d\tau.$$

Since from (7), $||\partial^\alpha_x \nabla E_{n\varepsilon}(t-\tau,\cdot)||_{L^1} \leq C k_{\phi,\varepsilon}(t,\tau) h(\varepsilon)^{(\alpha+1)/2}$, we have

$$||\partial^\alpha_x u_\varepsilon(t,\cdot)||_{L^p} \leq C k_{\phi,\varepsilon}(t,\tau) h(\varepsilon)^{\alpha/2} ||\mu_\varepsilon(\cdot)||_{L^p}$$

$$+ \int_0^t k_{\phi,\varepsilon}(t,\tau) h(\varepsilon)^{(\alpha+1)/2} ||\nabla g_\varepsilon(\theta u_\varepsilon(\tau,\cdot))||_{L^\infty} ||u_\varepsilon(\tau,\cdot)||_{L^p} d\tau.$$

Due to moderateness of $u_\varepsilon(t,\cdot)$ and (10), we obtain,

$$||\partial_x^\alpha u_\varepsilon(t,\cdot)||_{L^p} \leq Ck_{\phi,\varepsilon}(t,\tau)h(\varepsilon)^{\alpha/2}|ln\varepsilon|^{n(a-1/p)} + (CTk_{\phi,\varepsilon}(t,\tau)|ln\varepsilon|)\varepsilon^{-N} \leq C\varepsilon^{-N},$$

$\exists N>0$, $t\in[0,T)$, $x\in\mathbf{R}^n$, $\varepsilon<\varepsilon_0$, since $h(\varepsilon)$ is given with (9), $k_{\phi,\varepsilon}(t,\tau)$ is from (5), $\alpha\geq 0$.

The proof for moderateness of derivatives (w.r.) to t and mixed derivatives follows from (2) and Lemma 2.

Thus, $u_\varepsilon\in\mathcal{E}_{M,p}([0,T)\times\mathbf{R}^n)$ when $1\leq p\leq\infty$.

Let us prove the uniqueness. Let $u_\varepsilon,\tilde{u}_\varepsilon$ be two solutions to the equation (12) with different $N_\varepsilon(t,\cdot)$. Denote their difference with $w_\varepsilon(t,\cdot)$. Then, we must solve the equation

$$\begin{aligned} w_\varepsilon(t,\cdot) &= (E_{n\varepsilon}(t,\cdot)*N_{0\varepsilon}(\cdot))(x) \\ &+\int_0^t\int_{\mathbf{R}^n}\nabla E_{n\varepsilon}(t-\tau,x-\cdot)w_\varepsilon(\tau,\cdot)W_\varepsilon(\tau,\cdot)dyd\tau \\ &+\int_0^t\int_{\mathbf{R}^n}\nabla E_{n\varepsilon}(t-\tau,x-\cdot)N_\varepsilon(\tau,\cdot)dy, \end{aligned} \tag{15}$$

where $N_{0\varepsilon}(\cdot)\in\mathcal{N}_{p,q}(\mathbf{R}^n)$, $N_\varepsilon(t,\cdot)\in\mathcal{N}_{p,q}([0,T)\times\mathbf{R}^n)$, $1\leq p,q\leq\infty$, $w_\varepsilon(t,\cdot)=\int_0^1\nabla g_\varepsilon(\sigma u_{1\varepsilon}+(1-\sigma)u_{2\varepsilon})d\sigma$. We have in L^q-norm, $1\leq q\leq\infty$,

$$\begin{aligned} &||w_\varepsilon(t,\cdot)||_{L^q}\leq Ck_{\phi,\varepsilon}(t,\tau)\varepsilon^a \\ &\quad +C\int_0^t k_{\phi,\varepsilon}(t,\tau)h(\varepsilon)^{1/2+2}||w_\varepsilon(\tau,\cdot)||_{L^q}d\tau+Ck_{\phi,\varepsilon}(t,\tau)h(\varepsilon)^{1/2}\varepsilon^a. \end{aligned}$$

By Gronwall inequality

$$||w_\varepsilon(t,\cdot)||_{L^q}\leq Ck_{\phi,\varepsilon}(t,\tau)\varepsilon^a(1+h(\varepsilon)^{1/2})\exp\left(CTk_{\phi,\varepsilon}(t,\tau)h(\varepsilon)^{1/2+2}\right)\leq C\varepsilon^a,$$

$\forall a\in\mathbf{R},\ t\in[0,T),\ T>0,\ \varepsilon<\varepsilon_0,\ h(\varepsilon)$ is from (9). Follows, according to [10] that this is sufficient for the negligibility.

Thus, the solution is unique in $[u_\varepsilon]\in\mathcal{G}_{p,q}([0,T)\times\mathbf{R}^n)$, $1\leq p,q\leq\infty$.

(b) Consider the case $p,q=\infty$. Using (13) we obtain

$$\begin{aligned} &||u_\varepsilon(t,\cdot)||_{L^\infty}\leq||E_{n\varepsilon}(t,\cdot)||_{L^1}||\mu_\varepsilon(\cdot)||_{L^\infty} \\ &\quad+\int_0^t||\nabla E_{n\varepsilon}(t-\tau,x-\cdot)||_{L^1}||\nabla g_\varepsilon(\theta u_\varepsilon(\tau,\cdot))||_{L^\infty}||u_\varepsilon(\tau,\cdot)||_{L^\infty}d\tau. \end{aligned}$$

We have from (7) and (10) that

$$||u_\varepsilon(t,\cdot)||_{L^\infty}\leq Ck_{\phi,\varepsilon}(t,\tau)||\mu_\varepsilon(\cdot)||_{L^\infty}+C\int_0^t k_{\phi,\varepsilon}(t,\tau)h(\varepsilon)^{1/2+2}||u_\varepsilon(\tau,\cdot)||_{L^\infty}d\tau.$$

By Gronwall inequality $||u_\varepsilon(t,\cdot)||_{L^\infty} \leq C\varepsilon^{-N}$, $\exists N > 0$, $t \in [0,T)$, $x \in \mathbf{R}^n$, $\varepsilon < \varepsilon_0$, where $h(\varepsilon)$ is given with (9), $k_{\phi,\varepsilon}(t,\tau)$ with (5).

Suppose that $\alpha \in \mathbf{N}_0^n$, $\alpha \geq 1$. We have for $0 < \theta < 1$,

$$\partial_x^\alpha u_\varepsilon(t,\cdot) = (\partial_x^\alpha E_n(t,\cdot) * \mu_\varepsilon(\cdot))(x) + \int_0^t (\partial_x^\alpha \nabla E_{n\varepsilon}(t-\tau, x-\cdot)$$
$$* \nabla g_\varepsilon(\theta u_\varepsilon(\tau,\cdot))u_\varepsilon(\tau,\cdot))(x)d\tau.$$

Then,

$$||\partial_x^\alpha u_\varepsilon(t,\cdot)||_{L^\infty} \leq Ck_{\phi,\varepsilon}(t,\tau)h(\varepsilon)^{\alpha/2}||\mu_\varepsilon(\cdot)||_{L^\infty}$$
$$+C\int_0^t k_{\phi,\varepsilon}(t,\tau)h(\varepsilon)^{(\alpha+1)/2+2}||u_\varepsilon(\tau,\cdot)||_{L^\infty}d\tau.$$

By the first step we obtain, since $h(\varepsilon)$ is given with (9),

$$||\partial_x^\alpha u_\varepsilon(t,\cdot)||_{L^\infty} \leq Ck_{\phi,\varepsilon}(t,\tau)h(\varepsilon)^{\alpha/2}|ln\varepsilon|^{an} + Ck_{\phi,\varepsilon}(t,\tau)|ln\varepsilon|\varepsilon^{-N} \leq C\varepsilon^{-N},$$

$\exists N \in \mathbf{N}$, $\varepsilon < \varepsilon_0$, $t \in [0,T)$, $x \in \mathbf{R}^n$, $\alpha \geq 0$. Thus, $u_\varepsilon \in \mathcal{E}_{M,\infty}([0,T)\times\mathbf{R}^n)$.

The uniqueness holds as follows. Suppose that $u_{1\varepsilon}, u_{2\varepsilon}$ are two solutions to the equation (12). Then, we should solve the equation (15), where $w_\varepsilon(t,\cdot) = u_{1\varepsilon}(t,\cdot) - u_{2\varepsilon}(t,\cdot)$, $N_{0\varepsilon}(\cdot) \in \mathcal{N}_{\infty,q}(\mathbf{R}^n)$, $N_\varepsilon(t,\cdot) \in \mathcal{N}_{\infty,q}([0,T)\times\mathbf{R}^n)$, $1 \leq q \leq \infty$, $W_\varepsilon(t,\cdot) = \int_0^1 \nabla g_\varepsilon(\sigma u_{1\varepsilon} + (1-\sigma)u_{2\varepsilon})d\sigma$. Then, we have, by Gronwall inequality

$$||w_\varepsilon(t,\cdot)||_{L^q} \leq Ck_{\phi,\varepsilon}(t,\tau)\varepsilon^a \exp\left(CTk_{\phi,\varepsilon}(t,\tau)h(\varepsilon)^{1/2+2}\right) \leq C\varepsilon^a,$$

$\forall a \in \mathbf{R}$, $t \in [0,T)$, $x \in \mathbf{R}^n$, $\varepsilon < \varepsilon_0$, $h(\varepsilon)$ is given with (9), $k_{\phi,\varepsilon}(t,\tau)$ with (5), $\alpha \geq 0$.

Thus, $w_\varepsilon(t,\cdot) \in \mathcal{N}_{\infty,q}([0,T)\times\mathbf{R}^n)$. The solution is unique in $\mathcal{G}_{\infty,q}([0,T)\times\mathbf{R}^n)$, $1 \leq q \leq \infty$. When $q = \infty$ we deal with the space $\mathcal{G}_{s,g}([0,T)\times\mathbf{R}^n)$. ■

5.3. The equation (3)

We set in the equation (13): $V_\varepsilon(\cdot) = \delta_\varepsilon(\cdot) = h(\varepsilon)^2\phi(\cdot h(\varepsilon)^2)$, where $h(\varepsilon)$ is given with (8), $\mu_\varepsilon(\cdot)$ is from (4), $g \in L^\infty_{loc}(\mathbf{R}^n)$ is meant to be composed with a real-valued function u, $g(u)$ is non-Lipschitz's and regularized by cut-off such that (10) holds.

Theorem 3. *Let the equation* (3)*, where*

(1) $V(\cdot) = \delta(\cdot)$, $\mu(\cdot) = \delta(\cdot)$;

(2) $L^\infty_{loc}(\mathbf{R}^n)$ *is meant to be composed with a real-valued function* u *on* $([0,T)\times\mathbf{R}^n)$, $g(u)$ *is not of Lipschitz class;*

have the regularized integral form (13) *where the regularization with* ε*-subscript terms are given by* (6), (4) *and* (10). *Then, there exists an unique solution* $[u_{\,\varepsilon}] \in \mathcal{G}_{p,q}([0,T) \times \mathbf{R}^n)$, $1 \le p,q \le \infty$.

Proof. From (13) we have for $1 \le p \le \infty$, due to $||E_{n\varepsilon}(t,\cdot)||_{L^1} \le C$ and (10) holds, that

$$||u_\varepsilon(t,\cdot)||_{L^p} \le Ck_{\phi,\varepsilon}(t,\tau)|ln\varepsilon|^{n(a-1/p)} + C\int_0^t k_{\phi,\varepsilon}(t,\tau)h(\varepsilon)^2||u_\varepsilon(\tau,\cdot)||_{L^p}d\tau$$
$$+\int_0^t Ck_{\phi,\varepsilon}(t,\tau)h(\varepsilon)^2||u_\varepsilon(\tau,\cdot)||_{L^p}d\tau.$$

By Gronwall inequality

$$||u_\varepsilon(t,\cdot)||_{L^p} \le Ck_{\phi,\varepsilon}(t,\tau)|ln\varepsilon|^{n(a-1/p)}\exp\left(CTk_{\phi,\varepsilon}(t,\tau)(h(\varepsilon)^2 + h(\varepsilon)^2)\right) \le C\varepsilon^{-N},$$

$\exists N > 0$, $x \in \mathbf{R}^n$, $t \in [0,T)$, $\varepsilon < \varepsilon_0$, $h(\varepsilon)$ is given with (8) and $k_{\phi,\varepsilon}(t,\tau)$ is given with (5). It can be seen that the singularities of the potential and nonlinearity of $g(u)$ should be at the same level.

Suppose that $\alpha \in \mathbf{N}_0^n$, $\alpha \ge 1$. We have

$$\begin{aligned}\partial_x^\alpha u_\varepsilon(t,\cdot) &= (\partial_x^\alpha E_{n\varepsilon}(t,\cdot) * \mu_\varepsilon(\cdot))(x)\\ &+ \int_0^t\int_{\mathbf{R}^n}\partial_x^\alpha E_{n\varepsilon}(t-\tau,x-\cdot)V_\varepsilon(\cdot)u_\varepsilon(\tau,\cdot)dyd\tau\\ &+ \int_0^t\int_{\mathbf{R}^n}\partial_x^\alpha E_{n\varepsilon}(t-\tau,x-\cdot)g_\varepsilon(u_\varepsilon(\tau,\cdot))dyd\tau.\end{aligned}$$

Due to (7) and (10)

$$\begin{aligned}||\partial_x^\alpha u_\varepsilon(t,\cdot)||_{L^p} &\le Ck_{\phi,\varepsilon}(t,\tau)h(\varepsilon)^{\alpha/2}|ln\varepsilon|^{n(a-1/p)}\\ &+ \int_0^t Ck_{\phi,\varepsilon}(t,\tau)h(\varepsilon)^{\alpha/2+2}||u_\varepsilon(\tau,\cdot)||_{L^p}d\tau\\ &+ \int_0^t Ck_{\phi,\varepsilon}(t,\tau)h(\varepsilon)^{\alpha/2+2}||u_\varepsilon(\tau,\cdot)||_{L^p}d\tau.\end{aligned}$$

By Gronwall inequality

$$||\partial_x^\alpha u_\varepsilon(t,\cdot)||_{L^p} \le Ck_{\phi,\varepsilon}(t,\tau)h(\varepsilon)^{\alpha/2}|ln\varepsilon|^{n(a-1/p)}$$
$$\exp\left(CTk_{\phi,\varepsilon}(t,\tau)(h(\varepsilon)^{\alpha/2+2} + h(\varepsilon)^{\alpha/2+2})\right) \le C\varepsilon^{-N},$$

$\exists N > 0,\ x \in \mathbf{R}^n,\ t \in [0,T),\ \varepsilon < \varepsilon_0$. For $h(\varepsilon)$ and $k_{\phi,\varepsilon}(t,\tau)$, we use (8) and (5) respectively.

Let us see the uniqueness. Suppose that $u_{1\varepsilon}(t,\cdot)$ and $u_{2\varepsilon}(t,\cdot)$ are two solutions to the equation (13) and denote their difference with $w_\varepsilon(t,\cdot)$. Then, we must solve the equation

$$\begin{aligned}
w_\varepsilon(t,\cdot) = {} & (E_{n\varepsilon}(t,\cdot) * N_{0\varepsilon}(\cdot))(x) \\
& + \int_0^t \int_{\mathbf{R}^n} E_{n\varepsilon}(t-\tau, x-\cdot) V_\varepsilon(\cdot) w_\varepsilon(\tau,\cdot) dy d\tau \\
& + \int_0^t \int_{\mathbf{R}^n} E_{n\varepsilon}(t-\tau, x-\cdot) W_\varepsilon(\tau,\cdot) w_\varepsilon(\tau,\cdot) dy d\tau \\
& + \int_0^t \int_{\mathbf{R}^n} E_{n\varepsilon}(t-\tau, x-\cdot) N_\varepsilon(\tau,\cdot) dy d\tau,
\end{aligned}$$

where $W_\varepsilon(t,\cdot) = \int_0^1 \nabla g_\varepsilon(t, \theta u_{1\varepsilon} + (1-\theta) u_{2\varepsilon}) d\theta$, $w_\varepsilon(0,\cdot) = N_{0\varepsilon}(\cdot) \in \mathcal{N}_{p,q}(\mathbf{R}^n)$, $N_\varepsilon(t,\cdot) \in \mathcal{N}_{p,q}([0,T) \times \mathbf{R}^n)$, $1 \le p, q \le \infty$. We have

$$\begin{aligned}
||w_\varepsilon(t,\cdot)||_{L^q} \le {} & C k_{\phi,\varepsilon}(t,\tau) \varepsilon^a \\
& + \int_0^t C k_{\phi,\varepsilon}(t,\tau) h(\varepsilon)^2 ||w_\varepsilon(\tau,\cdot)||_{L^q} d\tau \\
& + \int_0^t C k_{\phi,\varepsilon}(t,\tau) h(\varepsilon)^2 ||w_\varepsilon(\tau,\cdot)||_{L^q} d\tau \\
& + \int_0^t C k_{\phi,\varepsilon}(t,\tau) \varepsilon^a d\tau.
\end{aligned}$$

By Gronwall inequality

$$||w_\varepsilon(t,\cdot)||_{L^q} \le C k_{\phi,\varepsilon}(t,\tau) \varepsilon^a \exp\left(CT k_{\phi,\varepsilon}(t,\tau) h(\varepsilon)^2\right) \le C\varepsilon^a,$$

$\forall a > 0,\ \ t \in [0,T),\ \ x \in \mathbf{R}^n,\ \ \varepsilon < \varepsilon_0$, $h(\varepsilon)$ and $k_{\phi,\varepsilon}(t,\tau)$ is given with (8) and (5) respectively.

Follows, $w_\varepsilon(t,\cdot) \in \mathcal{N}_{L^p,L^q}([0,T) \times \mathbf{R}^n)$, $1 \le p, q \le \infty$. Thus, the solution is unique in the spaces $[u_\varepsilon] \in \mathcal{G}_{p,q}([0,T) \times \mathbf{R}^n)$, $1 \le p, q \le \infty$. ∎

6. Consistency with Classical Results

We shall give proofs when $|t-\tau| \ge 1/(2h(\varepsilon))$. When $|t-\tau| \le 1/(4h(\varepsilon))$we consider $k_{\phi,\varepsilon}(t,\tau)$ as zero in limiting case when $\varepsilon \to 0$. We have $k_{\phi,\varepsilon}(t,\tau) \approx 0$, i.e. $k_{\phi,\varepsilon}(t,\tau)$ is associated to zero due to the definition (5).

Proposition 1. (a) *Let* u *be the classical solution to the equation* (1), *where* $g \in L^\infty_{loc}(\mathbf{R}^n)$ *is meant to be composed with a real-valued function* u *on* $([0,T) \times \mathbf{R}^n)$,

$\mu \in L^p(\mathbf{R}^n)$. *Then,* u *is* L^p*-associated to the solution to the equation* (11)*, where regularization for* $g(u)$ *is given with* (10)*, heat kernel is regularized with* (6) *and* μ *and* μ_ε *are* L^p*-associated.*

(b) *If* $\mu \in \mathcal{D}'_{L^\infty}(\mathbf{R}^n)$*, the solutions are* L^∞*-associated in* $\mathcal{G}_{s,g}([0,T) \times \mathbf{R}^n)$ *space.*

Proof. (a) Subtracting integral forms for the classical equation (1) and regularized one (11), we obtain

$$u(t,\cdot) - u_\varepsilon(t,\cdot) = (E_n(t,\cdot) * \mu(\cdot) - E_{n\varepsilon}(t,\cdot) * \mu_\varepsilon(\cdot))(x)$$
$$+ \int_0^t (E_n(t-\tau,\cdot) * g(u(\tau,\cdot)) - E_{n\varepsilon}(t-\tau,\cdot) * g_\varepsilon(u_\varepsilon(\tau,\cdot)))(x) d\tau.$$

By adding $\pm(E_n(t,\cdot) * \mu_\varepsilon(\cdot))(x)$ to the first row of the above expression we obtain $(E_n(t,\cdot) * (\mu_\varepsilon(\cdot) - \mu(\cdot)))(x) + \mu_\varepsilon(E_n(t,\cdot) - E_{n\varepsilon}(t,\cdot))$. Since $(1 - k_{\phi,\varepsilon}(t,\tau)) = 0$ when $|t-\tau| \geq 1/(2h(\varepsilon))$ it remains to estimate $(E_n(t,\cdot) * (\mu_\varepsilon(\cdot) - \mu(\cdot)))(x)$.

We add in integrand: $\pm(E_n(t-\tau,\cdot) * (g_\varepsilon(u_\varepsilon(\tau,\cdot))))(x)$. We have

$$(E_n(t-\tau,\cdot) * g(u(\tau,\cdot)) - g_\varepsilon(u_\varepsilon(\tau,\cdot))))(x) + (1-k_{\phi,\varepsilon}(t,\tau))(E_n(t-\tau,\cdot) * g_\varepsilon(u_\varepsilon(\tau,\cdot))))(x).$$

Since $(k_{\phi,\varepsilon}(t,\tau) - 1) = 0$ when $|t-\tau| \geq 1/(2h(\varepsilon))$ we shall estimate only the first part of the last expression. We have

$$||u(t,\cdot) - u_\varepsilon(t,\cdot)||_{L^p} \leq ||E_n(t,\cdot)||_{L^1} ||(\mu - \mu_\varepsilon)(\cdot)||_{L^p}$$
$$+ \int_0^t ||E_n(t-\tau,\cdot)||_{L^1} ||(g(u) - g_\varepsilon(u))(\tau,\cdot)||_{L^p} d\tau \leq C||(\mu - \mu_\varepsilon)(\cdot)||_{L^p}$$
$$+ C \int_0^t ||(g(u) - g_\varepsilon(u))(\tau,\cdot)||_{L^p} d\tau.$$

Denote by $I_1 = \int_0^t ||(g(u) - g_\varepsilon(u_\varepsilon))(\tau,\cdot)||_{L^p} d\tau$. We add $\pm g_\varepsilon(u)$. By Minkowsky inequality

$$I_1 \leq \int_0^t (||g(u) - g_\varepsilon(u)||_{L^p} + ||g_\varepsilon(u) - g_\varepsilon(u_\varepsilon)||_{L^p}) d\tau.$$

Due to the regularization $||g(u) - g_\varepsilon(u)||_{L^p} = O(\varepsilon^a)$, $\forall a \in \mathbf{R}$. Thus, we have

$$||u(t,\cdot) - u_\varepsilon(t,\cdot)||_{L^p} \leq C||(\mu - \mu_\varepsilon)(\cdot)||_{L^p} + CT\varepsilon^{a+1}$$
$$+ \int_0^t ||\nabla g_\varepsilon(\theta u + (1-\theta) u_\varepsilon)||_{L^\infty} ||u(\tau,\cdot) - u_\varepsilon(\tau,\cdot)||_{L^p} d\tau.$$

By Gronwall inequality, due to (10), we obtain

$$||u(t,\cdot) - u_\varepsilon(t,\cdot)||_{L^p} \leq (C||(\mu - \mu_\varepsilon)(\cdot)||_{L^p} + C\varepsilon^{a+1}) \exp{(CTh(\varepsilon)^2)},\ \alpha \geq 0,$$

$h(\varepsilon)$ is given with (8). Since $||(\mu-\mu_\varepsilon)(\cdot)||_{L^p} \le C\varepsilon^a$ for every $a \in \mathbf{R}$, we obtain $||(u(t,\cdot)-u_\varepsilon(t,\cdot))(x)||_{L^p} \le C\varepsilon^a,\ \forall a>0$, i.e. $u(t,\cdot)$ and $u_\varepsilon(t,\cdot)$ are L^p-associated (resp. L^∞-associated when $p=\infty$ in case (b)). ∎

Proposition 2. *(a) Let in (1), $g \in C^1(\mathbf{R})$ and allows composition with a real-valued function $u(t,\cdot)$ on $(I\times\Omega)$, $I\subset[0,T)$, $\Omega\subset\mathbf{R}^n$ and $\mu\in C(\Omega)$. Then, $\exists T>0$ such that the solution $[u_\varepsilon]$ to the equation (11) is L^∞-associated with the classical solution u in $C(I\times\Omega)$ to the equation (1).*

(b) Let $g\in C^1(\mathbf{R})$, and there exists a composition with u on $(I\times\Omega)$, $I\subset[0,T)$, $\Omega\subset\mathbf{R}^n$ such that

$$\sup_{\substack{t\in[0,T)\\ x\in\mathbf{R}^n}}\{|g(u)|\}<\infty,\quad \sup_{\substack{t\in[0,\infty)\\ x\in\mathbf{R}^n}}\{|\nabla_u g(u)|\}<\infty$$

and $\mu\in L^p(\Omega), 1\le p\le\infty$. Then, $\exists T>0$ such that the solution $[u_\varepsilon]$ to regularized equation (11) is L^p-associated with the classical solution u to the equation (1).

Proof. (a) $\exists C>0$, such that $||((\kappa_\varepsilon\mu)*\phi_\varepsilon)(\cdot)||_{L^\infty(\Omega)}\le C,\ \varepsilon\in(0,1),\ x\in\Omega,\ \Omega\subset\mathbf{R}^n$. There exists, by classical theory, $T>0$ and a family of smooth functions on $[0,T)$ to the equation

$$u_\varepsilon(t,\cdot)=((\kappa_\varepsilon\mu)*\phi_\varepsilon)(x)+\int_0^t (E_n(t-\tau,\cdot)*g(u_\varepsilon(\tau,\cdot)))(x)d\tau,\ x\in\Omega,\ t\in[0,T).$$

Let U_ε be the family of regularized solutions to (11). Since $g\in C^1$ by fixed point theorem $\exists T>0$, such that $\{U_\varepsilon(t,\cdot);t\in[0,T),\varepsilon\in(0,1)\}$ is bounded. For $x\in\Omega$

$$|U_\varepsilon(t,\cdot)-u(t,\cdot)|\le|U_\varepsilon(t,\cdot)-u_\varepsilon(t,\cdot)|+|u_\varepsilon(t,\cdot)-u(t,\cdot)|.$$

Since $g\in C^1$, $|u_\varepsilon(t,\cdot)-u(t,\cdot)|\to0$, as $\varepsilon\to0$, we obtain

$$U_\varepsilon(t,\cdot)-u_\varepsilon(t,\cdot)=\int_0^t((E_{n\varepsilon}(t-\tau,\cdot)*g_\varepsilon(U_\varepsilon(\tau,\cdot)))-(E_n(t-\tau,\cdot)*g(u_\varepsilon(\tau,\cdot))))(x)d\tau.$$

We add : $\pm(E_n(t-\tau,\cdot)*g_\varepsilon(U_\varepsilon(\tau,\cdot)))(x)$. We have

$$U_\varepsilon(t,\cdot)-u_\varepsilon(t,\cdot)=\int_0^t(((k_{\phi,\varepsilon}(t,\tau)-1)E_n(t-\tau,\cdot)*g_\varepsilon(U_\varepsilon(\tau,\cdot))+E_n(t-\tau,\cdot)$$

$$*(g(u_\varepsilon(\tau,\cdot)-g_\varepsilon(U_\varepsilon(\tau,\cdot))))(x)d\tau\le\int_0^t(E_n(t-\tau,\cdot)*(g(u_\varepsilon(\tau,\cdot)-g_\varepsilon(U_\varepsilon(\tau,\cdot)))(x)d\tau,$$

since $(k_{\phi,\varepsilon}(t,\tau)-1)=0$ when $|t-\tau|\ge1/(2h(\varepsilon))$. Then, we should estimate only the last term. We have

$$||U_\varepsilon(t,\cdot)-u_\varepsilon(t,\cdot)||_{L^\infty}\le\int_0^t||(E_n(t-\tau,\cdot)*(g_\varepsilon(U_\varepsilon(\tau,\cdot))-g(u_\varepsilon(\tau,\cdot)))||_{L^\infty})(x)d\tau.$$

By adding $\pm g(U_\varepsilon)$ we obtain

$$||U_\varepsilon(t,\cdot)-u_\varepsilon(t,\cdot)||_{L^\infty}\le C||g_\varepsilon(U_\varepsilon(\tau,\cdot))-g(U_\varepsilon(\tau,\cdot))||_{L^\infty}$$

$$+||g(U_\varepsilon(\tau,\cdot))-g(u_\varepsilon(\tau,\cdot))||_{L^\infty}$$

$$\le CTO(1/h(\varepsilon))+\int_0^t ||U_\varepsilon(\tau,\cdot)-u_\varepsilon(\tau,\cdot)||_{L^\infty}||\nabla g(\theta U_\varepsilon+(1-\theta)u_\varepsilon)||_{L^\infty}d\tau.$$

Since g is of Lipschitz's class, $||\nabla g||_{L^\infty}\le C$. Thus,

$$||U_\varepsilon(t,\cdot)-u_\varepsilon(t,\cdot)||_{L^\infty}\le CO(1/h(\varepsilon))+C\int_0^t ||U_\varepsilon(\tau,\cdot)-u_\varepsilon(\tau,\cdot)||_{L^\infty}d\tau.$$

By Gronwall inequality

$$||U_\varepsilon(t,\cdot)-u_\varepsilon(t,\cdot)||_{L^\infty}\le CO(1/h(\varepsilon))\exp CT\to 0,\ \text{as } \varepsilon\to 0.$$

(b) $\exists T>0$ and the unique solution in $L^p(I\times\Omega)$, $I\subset[0,T)$, $\Omega\subset\mathbf{R}^n$ by classical results (cf. [2]). Let $U_\varepsilon(t,\cdot)$ be the regularized solution, $t\in[0,T)$, $x\in\mathbf{R}^n$, $\varepsilon\in(0,1)$. Then,

$$||U_\varepsilon(t,\cdot)-u(t,\cdot)||_{L^p}\le||(\mu_\varepsilon-\mu)(\cdot)||_{L^p}$$

$$+\int_0^t ||(E_n(t-\tau,\cdot)*(g_\varepsilon(U_\varepsilon(\tau,\cdot)-g(u(\tau,\cdot))))(x)||_{L^p}d\tau$$

$$\le||(\mu_\varepsilon-\mu)(\cdot)||_{L^p}+\int_0^t C||g_\varepsilon(U_\varepsilon(\tau,\cdot))-g(u(\tau,\cdot))||_{L^p}d\tau.$$

We add $\pm g(U_\varepsilon)$ and denote by

$$A=\int_0^t ||g_\varepsilon(U_\varepsilon)-g(U_\varepsilon)||_{L^p}d\tau,\ B=\int_0^t ||g(U_\varepsilon)-g(u)||_{L^p}d\tau.$$

We have

$$B=\int_0^t ||U_\varepsilon-u||_{L^p}||\nabla g(\theta U_\varepsilon+(1-\theta)u)||_{L^\infty}d\tau\le C\int_0^t ||U_\varepsilon-u||_{L^\infty}d\tau$$

since g is of Lipschitz's class. By mean value theorem and boundeddness of ∇g

$$A=\int_0^t ||g_\varepsilon(U)-g(U_\varepsilon)||_{L^p}d\tau=\int_0^t\int_{\mathbf{R}^n}(g(U-\frac{\xi}{h(\varepsilon)})-g(U_\varepsilon))\theta(\xi)d\xi d\tau\le C/h(\varepsilon).$$

Thus,

$$||U_\varepsilon(t,\cdot)-u(t,\cdot)||_{L^p}\le||(\mu_\varepsilon-\mu)(\cdot)||_{L^p}+C(1/h(\varepsilon))+C\int_0^t ||U_\varepsilon(\tau,\cdot)-u(\tau,\cdot)||_{L^p}d\tau.$$

Gronwall inequality implies

$$||U_\varepsilon(t,\cdot)-u(t,\cdot)||_{L^p}\le (||(\mu_\varepsilon-\mu)(\cdot)||_{L^p}+O(1/h(\varepsilon)))\exp C\to 0,\ \text{as } \varepsilon\to 0,$$

what proves the assertion. ■

Proposition 3. *Assume that $g\in C^1(\mathbf{R})$ and allows composition with a real valued function u on $(I\times D)$, $I\subset[0,T),\Omega\subset\mathbf{R}^n$, such that $\sup\{|g(u)|,|\nabla_u g(u)|\}<\infty$ and $\mu\in L^p(\Omega)$, $1\le p\le\infty$, $\Omega\subset\mathbf{R}^n$. Then, there exists $T>0$, such that the unique classical solution u to the equation (2) is L^p-associated with the solution to the equation (12) i.e. $||u_\varepsilon(t,\cdot)-u(t,\cdot)||_{L^p(0,T)}\to 0,$ as $\varepsilon\to 0$.*

Proof. The existence of the classical solution under above conditions for ε fixed follows by results of [3]. Let u_ε be the regularized solution and u be the classical one. Let $\varepsilon<\varepsilon_0$. We have,

$$\begin{aligned}&||u_\varepsilon(t,\cdot)-u(t,\cdot)||_{L^p}\le ||(E_{n\varepsilon}(t,\cdot)*\mu_\varepsilon-E_n(t,\cdot)*\mu)(x)||_{L^p}\\&\quad+\int_0^t||(\nabla E_{n\varepsilon}(t-\tau,\cdot)*g_\varepsilon(u_\varepsilon(\tau,\cdot))\\&\quad-\nabla E_n(t-\tau,\cdot)*g(u(\tau,\cdot)))(x)||_{L^p}d\tau,\end{aligned}$$

and since $(k_{\phi,\varepsilon}(t,\tau)-1)=0$ when $|t-\tau|\ge 1/(2h(\varepsilon))$, then

$$\begin{aligned}&||u_\varepsilon(t,\cdot)-u(t,\cdot)||_{L^p}\le C||(\mu_\varepsilon-\mu)(\cdot)||_{L^p}\\&\quad+\int_0^t||(\nabla E_{n\varepsilon}(t-\tau,\cdot))*g_\varepsilon(u_\varepsilon(\tau,\cdot)))(x)||_{L^p}\\&\quad-||(\nabla E_n(t-\tau,\cdot)*g(u(\tau,\cdot)))(x)||_{L^p}d\tau.\end{aligned}\tag{16}$$

Denote the integrand by

$$I=(\nabla E_{n\varepsilon}(t-\tau,\cdot)*g_\varepsilon(u_\varepsilon(\tau,\cdot)))(x)-(\nabla E_n(t-\tau,\cdot)*g(u(\tau,\cdot)))(x).$$

We add $\pm((\nabla E_{n\varepsilon}(t-\tau,\cdot))*g(u(\tau,\cdot)))(x)$. Then, we have

$$\begin{aligned}I&=(\nabla E_{n\varepsilon}(t-\tau,\cdot)*g(u_\varepsilon(\tau,\cdot)))(x)-(\nabla E_{n\varepsilon}(t-\tau,\cdot)*g(u(\tau,\cdot)))(x)\\&\quad+(\nabla E_{n\varepsilon}(t-\tau,\cdot)*g(u(\tau,\cdot)))(x)-(\nabla E_n(t-\tau,\cdot)*g(u(\tau,\cdot)))(x)\\&=(\nabla E_{n\varepsilon}(t-\tau,\cdot)*(g(u_\varepsilon(\tau,\cdot))-g(u(\tau,\cdot))))(x)\\&\quad+(k_{\phi,\varepsilon}(t,\tau)-1)(\nabla E_n(t-\tau,\cdot)*g(u(\tau,\cdot)))(x)=I_1+I_2.\end{aligned}$$

Since $(k_{\phi,\varepsilon}(t,\tau)-1)=0$ when $|t-\tau|\geq 1/(2h(\varepsilon))$ then I_2 equals zero. We obtain by Young's inequality and mean value theorem that

$$||I||_{L^p}\leq ||\nabla E_{n\varepsilon}(t-\tau,\cdot)||_{L^1}||(u_\varepsilon(\tau,\cdot)-u(\tau,\cdot))\nabla g(\theta u_\varepsilon+(1-\theta)u)||_{L^p}$$
$$\leq Ch(\varepsilon)^{(\alpha+1)/2+2}||u_\varepsilon(\tau,\cdot)-u(\tau,\cdot)||_{L^p}.$$

Putting this in (16) we obtain

$$||u_\varepsilon(t,\cdot)-u(t,\cdot)||_{L^p}\leq C||(\mu_\varepsilon-\mu)(\cdot)||_{L^p}$$
$$+C\int_0^t h(\varepsilon)^{(\alpha+1)/2+2}||u_\varepsilon(\tau,\cdot)-u(\tau,\cdot)||_{L^p}d\tau.$$

By Gronwall inequality we have

$$||u_\varepsilon(t,\cdot)-u(t,\cdot)||_{L^p}\leq C||(\mu_\varepsilon-\mu)(\cdot)||_{L^p}\exp\left(Ch(\varepsilon)^{(\alpha+1)/2+2}\right).$$

Since $h(\varepsilon)$ is given with (9) we obtain

$$||u_\varepsilon(t,\cdot)-u(t,\cdot)||_{L^p}\leq C||(\mu_\varepsilon-\mu)(\cdot)||_{L^p}\exp\left(CT|\log\varepsilon|\right)\leq C(||(\mu_\varepsilon-\mu)(\cdot)||_{L^p}\varepsilon^{-N}).$$

Because μ_ε and μ are L^p-associated, i.e. $||(\mu_\varepsilon-\mu)(\cdot)||_{L^p}\to 0$, as $\varepsilon\to 0$, the same holds for u_ε and u. Thus, $||u_\varepsilon(t,\cdot)-u(t,\cdot)||_{L^p(0,T)}\to 0,$ as $\varepsilon\to 0$. ∎

Proposition 4. *Let u be the solution to the equation (2), where $g\in C^1(\mathbf{R})$ and allows composition with a real valued function u on $(I\times D)$, $I\subset[0,T)$, $\Omega\subset\mathbf{R}^n$, $\mu(\cdot)\in C(\Omega)$. Assume that for every compact set $\Omega\subset\subset D$, $D\subset\subset\mathbf{R}^n$,*

$$\sup_{\substack{t\in[0,\infty)\\ x\in D}}\{|\nabla g(u(t,\cdot))|\}<\infty. \tag{17}$$

Then, there exists $T>0$, such that the unique solution $[u_\varepsilon]$ to the equation (12) is $C(0,T)$-associated to u, i.e. $||u_\varepsilon(t,\cdot)-u(t,\cdot)||_{C(0,T)}\to 0,$ as $\varepsilon\to 0$.

Proof. Let the initial data $\mu\in C(\Omega)$. Then $||(\kappa_\varepsilon\mu*\phi_\varepsilon)(\cdot)||_{L^\infty}\leq C$, $\varepsilon\in(0,1)$ where $\kappa_\varepsilon\in C_0^\infty(I)$. Since g satisfies (17), $\exists T>0$ such that

$$u_\varepsilon(t,\cdot)=((\kappa_\varepsilon\mu*\phi_\varepsilon)*E_n(t,\cdot))(x)+\int_0^t(\nabla E_n(t-\tau,\cdot)*g(u_\varepsilon(\tau,\cdot)))(x)d\tau,$$

$t\in[0,T)$, $\varepsilon\in(0,1)$, has a family of solutions which are bounded and unique. Let $U_\varepsilon(t,\cdot)$ be a family of unique solutions to regularized equation (12). This family is bounded in $C([0,T)\times\Omega)$ and by regularization $U_\varepsilon=u_\varepsilon$, $\varepsilon<\varepsilon_0$, since $g(u_\varepsilon)=g_\varepsilon(U_\varepsilon)$ on bounded set $\varepsilon<\varepsilon_0$, and $(k_{\phi,\varepsilon}-1)=0$ when $|t-\tau|\geq 1/(2h(\varepsilon))$.

For $x \in \Omega \subset \mathbf{R}^n$, $\varepsilon \in (0,1)$ and u is a classical solution, we have

$$||U_\varepsilon(t,\cdot) - u(t,\cdot)||_{L^\infty} \leq ||U_\varepsilon(t,\cdot) - u_\varepsilon(t,\cdot)||_{L^\infty} + ||u_\varepsilon(t,\cdot) - u(t,\cdot)||_{L^\infty}. \tag{18}$$

Let us see the first part of the above inequality.

$$\begin{aligned}
&||U_\varepsilon(t,\cdot) - u_\varepsilon(t,\cdot)||_{L^\infty} \\
&\leq \int_0^t ||(\nabla E_{n\varepsilon}(t-\tau,\cdot) * g(U_\varepsilon(\tau,\cdot)) - \nabla E_n(t-\tau,\cdot) * g(u_\varepsilon(\tau,\cdot)))(x)||_{L^\infty} d\tau \\
&+ \int_0^t ||(\nabla E_n(t-\tau,\cdot) * g(u_\varepsilon(\tau,\cdot)) - \nabla E_n(t-\tau,\cdot) * g(u(\tau,\cdot)))(x)||_{L^\infty} d\tau = A + B.
\end{aligned}$$

Consider the part A. We add $\pm(\nabla E_{n\varepsilon}(t-\tau,\cdot) * g(u_\varepsilon(\tau,\cdot)))(x)$. We have

$$\begin{aligned}
A \leq \int_0^t &||\nabla E_{n\varepsilon}(t-\tau,\cdot)||_{L^1} ||g_\varepsilon(U_\varepsilon(\tau,\cdot)) - g(u_\varepsilon(\tau,\cdot))||_{L^\infty} \\
&+ (k_{\phi,\varepsilon}(t,\tau) - 1)||\nabla E_n(t-\tau,\cdot)||_{L^1} ||g(u_\varepsilon(\tau,\cdot))||_{L^\infty} d\tau.
\end{aligned}$$

Since $(k_{\phi,\varepsilon}(t,\tau) - 1) = 0$ when $|t-\tau| \geq 1/(2h(\varepsilon))$, and by regularization, we obtain that part A is negligible. Part B is negligible due to cut-off. We must estimate only the second part in (18). We have

$$\begin{aligned}
||u_\varepsilon(t,\cdot) - u(t,\cdot)||_{L^\infty} &\leq ||(E_n(t,\cdot) * (\kappa_\varepsilon \mu(\cdot) * \phi_\varepsilon(\cdot) - \mu(\cdot)))(x)||_{L^\infty} \\
&+ \int_0^t ||(\nabla E_n(t-\tau,\cdot) * (g(u_\varepsilon(\tau,\cdot)) - g(u(\tau,\cdot))))(x)||_{L^\infty} d\tau, \\
&\leq C||(\mu_\varepsilon - \mu)(\cdot)||_{L^\infty} + C\int_0^t \frac{1}{\sqrt{t-\tau}} ||g(u_\varepsilon(\tau,\cdot)) - g(u(\tau,\cdot))||_{L^\infty} d\tau.
\end{aligned}$$

Because of $g \in C^1(\Omega)$, and (17) holds, we obtain by Gronwall inequality

$$||u_\varepsilon(t,\cdot) - u(t,\cdot)||_{L^\infty} \leq ||(\mu_\varepsilon - \mu)(\cdot)||_{L^\infty} \exp(CT).$$

Since $||(\mu_\varepsilon - \mu)(\cdot)||_{L^\infty} \to 0$ as $\varepsilon \to 0$, the same holds for $||u_\varepsilon(t,\cdot) - u(t,\cdot)||_{L^\infty} \to 0$ as $\varepsilon \to 0$. Setting this in (18) we obtain $||U_\varepsilon(t,\cdot) - u(t,\cdot)||_{L^\infty} \to 0,$ as $\varepsilon \to 0$, i.e. the solutions of regularized and classical equations are L^∞-associated. ∎

Proposition 5. *Let $\mu \in C(\Omega)$, $\Omega \subset \mathbf{R}^n$, $g \in C^1(\mathbf{R})$ and allows composition with a real valued function u on $(I \times D)$, $I \subset [0,T)$, $D \subset \mathbf{R}^n$ and satisfies Lipschitz's condition. Assume that (2) is globally L^∞-well-posed. Then, the solution $[u_\varepsilon]$ to regularized equation (12) is L^∞-associated with continuous solution u to (2) on each $[0,T]$, $T > 0$.*

Proof. Let u be a continuous solution to the equation (12), where $\mu \in C(\Omega),\ \Omega \subset \mathbf{R}^n$, is an open set. Let $\tilde{U} \in C^\infty(\Omega)$ be the solution to integral form of the equation (2) on $[0, T),\ T > 0,\ \mu_\varepsilon = \mu * \phi_\varepsilon$. Due to (2) is well-posed, $\tilde{U}_\varepsilon \to u$, as $\varepsilon \to 0$. Follows, (cf. [13]), $\exists C_{\tilde{U}} > 0$ such that $||\tilde{U}(t, \cdot)||_{L^\infty(0,T)} \leq C_{\tilde{U}},\ \varepsilon < \varepsilon_0$. Let $\{\varepsilon < \min(\varepsilon_{i_0}, \varepsilon_0)\}$, then $\{\tilde{U} | t \in [0, T),\ |\tilde{U}| \leq C_{\tilde{U}}\} \subset B_{i_0}$. Because of the cut-off we have $g(\tilde{U}_\varepsilon) = g_\varepsilon(\tilde{U}_\varepsilon)$, $\tilde{U}_\varepsilon$ is also the solution to

$$\tilde{U}_\varepsilon(t, \cdot) = \mu_\varepsilon(\cdot) + \int_0^t (\nabla E_n(t - \tau, \cdot) * g_\varepsilon(\tilde{U}_\varepsilon(\tau, \cdot)))(x) d\tau$$

in $[0, T)$ and follows $\tilde{U}_\varepsilon \approx U_\varepsilon$, where U_ε is the solution to regularized equation

$$U_\varepsilon(t, \cdot) = \mu_\varepsilon(\cdot) + \int_0^t (\nabla E_{n\varepsilon}(t - \tau, \cdot) * g_\varepsilon(U_\varepsilon(\tau, \cdot)))(x) d\tau.$$

Consequently, $U_\varepsilon \approx u$. ■

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