TAIWANESE JOURNAL OF MATHEMATICS
Vol. 12, No. 3, pp. 679-701, June 2008
This paper is available online at http://www.tjm.nsysu.edu.tw/

ON SEMI-LINEAR SECOND ORDER VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS IN HILBERT SPACE

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Abstract. Existence, uniqueness and regularity of weak solutions of semilinear second order Volterra integro-differential equations in Hilbert space are established by the variational method. As an application we give a well-posedness result for the semilinear viscoelastic equations with long memory.

1. Introduction

The dynamics of linear viscoelastic materials with long memory is decribed by the following abstract second order Volterra integro-differential equations of the form

$$y'' + A(t)y + \int_0^t K(t,s)y(s)ds = f, \tag{1.1}$$

where $A(t)$ and $K(t,s)$ are operators corresponding to instantaneous elastic effect and memory effects of the material, respectively and f is the forcing function acting on the material (cf. Dautray and Lions [1; Chapter 1A §3, Chapter XVIII §7]). Under certain physical situations the forcing term f appears as a force depending on the displacement and/or the velocity of the material, i.e., $f = f(t,y,y')$ is considered as a nonlinear perturbation term. In this paper we study the second order Volterra integro-differential equations of perturbed form

$$y'' + A(t)y + \int_0^t K(t,s)y(s)ds = f(t,y,y') \text{ in } (0,T), \tag{1.2}$$

Received March 24, 2006, accepted June 5, 2006.
Communicated by Sze-Bi Hsu.
2000 *Mathematics Subject Classification*: 35L70, 35L10, 35M99 .
Key words and phrases: Volterra integro-differential equation, Weak solution, Energy equality, Viscoelastic equation with long memory.

where $A(t), K(t,s)$ are time varying operators on a Hilbert space V embedded in a pivot Hilbert space H and $f(t,y,y')$ is a nonlinear forcing function. The initial condition attached to (1.2) is given by

$$y(0) = y_0 \in V, \quad y'(0) = y_1 \in H. \tag{1.3}$$

For the linear case where $f(t,y,y') = f(t)$, a *quick proof* of the existence and uniqueness of weak solutions for the Cauchy problem (1.2), (1.3) is given in Dautray and Lions [1; pp. 661-662]. However, in [1] the regularity of solutions is not proved and the detailed analysis on the well-posedness of solutions has not given in spite of the existence of Volterra integral terms. The regularity and the energy equality play essential role in studying optimal control and identification problems for the viscoelastic systems with long memory as in Lions [4]. Also we remark that our analysis is influenced from the excellent books by Lions [3], Lions and Magenes [5], Tanabe [6], Temam [7] and Showalter [8] both for linear and nonlinear equations by the variational approach.

The purposes of this paper is to extend the well-posedness result of Dautray and Lions [1] to our semilinear problem (1.2), (1.3) under the Lipschitz continuity on $f(t,y,y')$ in y and y', and to establish the regularity and the energy equality of solutions. At the same time, this paper is intended to give a complete proof of the regularity of solutions to linear equations.

We now explain the content of this paper. In the first part of Section 2, we give a proof of the regularity and energy equality for the linear problem (1.2), (1.3) with $f(t,y,y') = f(t)$. The proof of regularity is carried over along the line of a proof in Lions and Magenes [5] for our case of Volterra integro-differential equations. In the rest part of Section 2, we state and prove the existence, uniqueness and regularity of weak solutions for the problem (1.2), (1.3). The energy equality for the solution is proved by using the regularization method in [5; p.276]. The key tool of our analysis is the energy inequality and is essentially due to the energy equality for linear equations. We suppose that (V, H, V') is a Gelfand triple. It is assumed in (1.2) that $A(t)$ and $K(t,s)$ are operators defined through bilinear forms on V, and the perturbation term $f(t,y,z)$ is Lipschitz continuous in y and z which maps $V \times H$ into H. We remark that the monotonicity or the compactness of nonlinear term f is not supposed and the Lipschitz continuity is sufficient to derive the pointwise strong convergence of approximate solutions. The strong convergence argument is a refinement of that in Ha and Nakagiri [2] which is based on that of [1; p. 567] for linear equations, and the argument involves new calculations on Volterra integral terms. In section 3 we give an application to viscoelastic equations of material with fading memory with nonlinear perturbations.

2. Volterra Integro-differential Equations

Let H be a real pivot Hilbert space, and the inner product and the norm is denoted by $(\cdot,\cdot)$ and $|\cdot|$, respectively. Let V be a real separable Hilbert space with the norm $\|\cdot\|$. Assume that each pair (V,H) is a Gelfand triple space and that V is continuously embedded in H. We are given a family of symmetric bilinear forms $a(t;\phi,\varphi), t\in[0,T]$ on $V\times V$. We suppose that $a(t;\phi,\varphi)$ satisfies

$$(2.1)\qquad \begin{cases} a(t;\phi,\psi)=a(t;\psi,\phi) \text{and there exists } c_1>0 \text{ such that} \\ |a(t;\phi,\varphi)|\le c_1\|\phi\|\|\varphi\| \ \text{ for all } \ \phi,\psi\in V \ \text{ and } \ t\in[0,T], \\ \text{and there exists } \alpha>0 \text{ such that} \\ a(t;\phi,\phi)\ge\alpha\|\phi\|^2 \ \text{ for all } \ \phi\in V \ \text{ and } \ t\in[0,T], \end{cases}$$

$$(2.2)\qquad \begin{cases} \text{the function } \ t\to a(t;\phi,\varphi) \ \text{ is continuously differentiable in } \ [0,T] \\ \text{and there exists } c_2>0 \ \text{ such that} \\ |a'(t;\phi,\varphi)|\le c_2\|\phi\|\|\varphi\| \ \text{ for all } \ \phi,\psi\in V \ \text{ and } \ t\in[0,T], \end{cases}$$

where $' = \frac{d}{dt}$. Then we can define the operator $A(t),\ A'(t)\in\mathcal{L}(V,V')$ for $t\in[0,T]$ deduced by the relation

$$a(t;\phi,\varphi)=\langle A(t)\phi,\varphi\rangle_{V',V} \text{ for all } \phi,\varphi\in V,$$

$$a'(t;\phi,\varphi)=\langle A'(t)\phi,\varphi\rangle_{V',V} \text{ for all } \phi,\varphi\in V.$$

Next, we consider a family of bilinear forms $k(t,s;\phi,\varphi)$ over $V\times V$ defined over $[0,T]\times[0,T]$ satisfying

$$(2.3)\qquad \begin{cases} \text{there exists } k_0>0 \text{ such that} \\ |k(t,s;\phi,\varphi)|\le k_0\|\phi\|\|\varphi\| \text{ for all } \phi,\varphi\in V \text{ and } (t,s)\in[0,T]\times[0,T], \end{cases}$$

$$(2.4)\qquad \begin{cases} \text{the function } t\to k(t,s;\phi,\varphi) \text{ is partially differentiable for all } \phi,\varphi\in V \\ \text{and } s\in[0,T] \text{ and there exists } k_1>0 \text{ such that} \\ \left|\dfrac{\partial k}{\partial t}(t,s;\phi,\varphi)\right|\le k_1\|\phi\|\|\varphi\| \text{ for all } \phi,\varphi\in V \text{ and} (t,s)\in[0,T]\times[0,T]. \end{cases}$$

This family $k(t,s;\phi,\varphi)$ defines a family of operators $K(t,s)\in\mathcal{L}(V,V')$ by

$$k(t,s;\phi,\varphi)=\langle K(t,s)\phi,\varphi\rangle_{V',V}.$$

Also, the family $\frac{\partial k}{\partial t}(t,s;\phi,\varphi)$ defines a family of operators $K_t(t,s) \in \mathcal{L}(V,V')$ by

$$\frac{\partial k}{\partial t}(t,s;\phi,\varphi) = \langle K_t(t,s)\phi, \varphi\rangle_{V',V}.$$

We impose the following assumptions on the nonlinear term $f = f(t,y,z) : [0,T] \times V \times H \to H$:

(A1) $t \to f(t,y,z)$ is strongly measurable in H for all $(y,z) \in V \times H$;

(A2) there exists a $\beta \in L^2(0,T;\mathbf{R}^+),\ i = 1,2$ such that

$$|f(t,\xi_1,\eta_1) - f(t,\xi_2,\eta_2)| \leq \beta(t)\Big(\|\xi_1 - \xi_2\| + |\eta_1 - \eta_2|\Big) \ \text{ a.e. } \ t \in [0,T]$$
$$\text{for all } \ \xi_i \in V \text{ and } \eta_i \ \in H,\ i = 1,2,$$

(A3) there exists a $\gamma \in L^2(0,T;\mathbf{R}^+)$ such that $|f(t,0,0)| \leq \gamma(t)$ a.e. $t \in [0,T]$.

We consider the Cauchy problem for semilinear Volterra integro-differential equations

$$\text{(2.5)} \qquad \begin{cases} y'' + A(t)y + \displaystyle\int_0^t K(t,s)y(s)ds = f(t,y,y') \ \text{ in } \ (0,T), \\ y(0) = y_0 \in V, \quad y'(0) = y_1 \in H, \end{cases}$$

where $f(t,y,y')$ satisfies (A1)-(A3). The solution Hilbert space $W(0,T)$ of (2.5) is defined by

$$W(0,T) = \{g | g \in L^2(0,T;V), g' \in L^2(0,T;H), g'' \in L^2(0,T;V')\},$$

endowed with the norm

$$\|g\|_{W(0,T)} = \Big(\|g\|^2_{L^2(0,T;V)} + \|g'\|^2_{L^2(0,T;H)} + \|g''\|^2_{L^2(0,T;V')}\Big)^{\frac{1}{2}}.$$

In what follows we write $L^p(0,T;V),\ L^p(0,T;H)$ and $L^p(0,T;V')$ with $p = 2,\infty$ by $L^p(V),\ L^p(H)$ and $L^p(V')$ for notational simplicity. We denote by $\mathcal{D}'(0,T)$ the space of distributions on $(0,T)$. A function $y = y(\cdot)$ is said to be a weak solution of the problem (2.5) if $y \in W(0,T)$ and y satisfies

$$\text{(2.6)} \qquad \begin{cases} (y''(\cdot),\phi) + a(\cdot;y(\cdot),\phi) + \displaystyle\int_0^{\cdot} k(\cdot,s;y(s),\phi)ds = (f(\cdot,y(\cdot),y'(\cdot)),\phi) \\ \qquad\qquad \text{for all } \ \phi \in V \ \text{ in the sense of } \ \mathcal{D}'(0,T) \\ y(0) = y_0 \in V, \quad y'(0) \in H. \end{cases}$$

First we shall prove the energy equality and the regularity of weak solutions for the linear problem (2.5) with $f(t, y, y') = f(t)$. For this we need the following lemmas. Lemma 2.1 is shown in Lions and Magenes [5].

Lemma 2.1. *Let X, Y be two Banach spaces, $X \subset Y$ with dense, and X being reflexive. Set*

$$C_s([0,T];Y) = \{f \in L^\infty(0,T;Y) | \ \forall \phi \in Y', t \to \langle f, \phi \rangle_{Y,Y'} \text{ is continuous of } [0,T] \to R\},$$

Then

$$L^\infty(0,T;X) \cap C_s([0,T];Y) = C_s([0,T];X).$$

Lemma 2.2. *Assume that y is a weak solution of* (2.5). *Then we can assert* (*after possibly a modification on a set of measure zero*) *that*

$$y \in C_s([0,T];V), \quad y' \in C_s([0,T];H).$$

Proof. The proof is quite similar to that given in [5; pp. 276], in which Lemma 2.1 is used for the case $Y = V$ and $X = H$ (see also Ha and Nakagiri [2; Lemma 4.2]).

Proposition 2.1. *Assume that y is a weak solution of* (2.5) *with* $f(t, y, y') = f(t) \in L^2(H)$. *Then, for each $t \in [0,T]$ we have the energy equality*

$$\begin{aligned} &a(t; y(t), y(t)) + |y'(t)|^2 \\ &= a(0; y_0, y_0) + |y_1|^2 + \int_0^t a'(s; y(s), y(s))ds \\ &\quad + 2\int_0^t (f(s), y'(s))ds + 2\int_0^t k(s, s; y(s), y(s))ds \\ &\quad + 2\int_0^t \Big(\int_0^s \frac{\partial k}{\partial t}(s, \sigma; y(\sigma), y(s))d\sigma - k(t, s; y(s), y(t)) \Big) ds. \end{aligned} \tag{2.7}$$

Proof. By Lemma 2.2 and the uniform boundedness theorem, $y(t) \in V$ and $y'(t) \in H$ for each $t \in [0,T]$. Thus all functions in (2.7) has meaning for all $t \in [0,T]$. We shall show the energy equality (2.7). Let $\delta > 0$ and $t_0 \in (0,T)$ be fixed. We introduce a continuous function

$$\mathcal{O}_\delta(t) = \mathcal{O}(t) = \begin{cases} 1 & \text{in } [\delta, t_0 - \delta], \\ \dfrac{t}{\delta} & \text{in } [0, \delta], \\ \dfrac{1}{\delta}(t_0 - t) & \text{in } [t_0 - \delta, t_0], \\ 0 & \text{otherwise} \end{cases} \tag{2.8}$$

and a step function

$$\mathcal{O}_0(t) = \begin{cases} 1 & \text{in } [0, t_0], \\ 0 & \text{otherwise.} \end{cases} \tag{2.9}$$

Let $\{\rho\}_{n=1}^{\infty}$ be a regularizing sequence of even functions such that $\displaystyle\int_{\mathbf{R}} \rho_n(t)dt = 1$ and supp$\rho_n \subset (-\frac{1}{n}, \frac{1}{n})$ (cf. [5; pp.276-279]). We shall extend $A(t), K(t,s)$ as well as the derivatives $A'(t), K_t(t,s)$ and $f(t)$ for all $t, s \in \mathbf{R}$, with the same properties on $[0,T]$. Especially we can suppose $K(t,s) = 0$ for $(t,s) \in (\mathbf{R} \setminus [0,T]) \times (\mathbf{R} \setminus [0,T])$. In the same way we shall assume that y is defined on $\mathbf{R}$, which is possible by extension by reflection. For the notational simplicity we shall denote by $[\ ,\ ]$ the scalar product of H or the anti duality between V and V', and we shall denote by $(\!(\ ,\ )\!)$ the antiduality between $L^2(\mathbf{R}; V)$ and $L^2(\mathbf{R}; V')$ or the scalar product of $L^2(\mathbf{R}; H)$. We fix n and set $\rho_n = \rho$. Let $\rho * \rho$ be the convolution of ρ and ρ in $L^2(\mathbf{R}; H)$. At first we have

$$\begin{cases} (\!(A'(\rho * (\mathcal{O}_0 y), \rho * (\mathcal{O}_0 y))\!) + 2(\!(\rho * (\mathcal{O}_0 Ay), \rho * (\mathcal{O}_0 y'))\!) \\ \quad + 2(\!(A(\rho * (\mathcal{O}_0 y)) - \rho * (A\mathcal{O}_0 y), \rho' * (\mathcal{O}_0 y))\!) \\ \quad + 2[(\rho * \rho * (\mathcal{O}_0 Ay))(0), y(0)] - 2[(\rho * \rho * (\mathcal{O}_0 Ay))(t_0), y(t_0)] = 0. \end{cases} \tag{2.10}$$

The equality (2.10) is derived by direct differentiation of the equality

$$\int_{\mathbf{R}} \frac{d}{dt}[A(t)((\rho * (\mathcal{O}y)), \rho * (\mathcal{O}y)]dt = 0$$

and passage $\delta \to 0$ as in Lions and Magenes [5, p. 277]. Next we shall prove

$$\begin{cases} (\!(\rho * (\mathcal{O}_0 K(\cdot,\cdot)y), \rho * (\mathcal{O}_0 y))\!) \\ \quad + (\!(\rho * (\mathcal{O}_0 \int K_t(\cdot, s)y(s)ds), \rho * (\mathcal{O}_0 y))\!) \\ \quad + (\!(\rho * (\mathcal{O}_0 \int K(\cdot, s)y(s)ds), \rho * (\mathcal{O}_0 y'))\!) \\ \quad - [(\mathcal{O}_0 \int K(\cdot, s)y(s)ds)(t_0), \rho * \rho * (\mathcal{O}_0 y)(t_0)] \\ \quad - [\rho * \rho * (\mathcal{O}_0 \int K(\cdot, s)y(s)ds)(t_0), (\mathcal{O}_0 y)(t_0)] = 0, \end{cases} \tag{2.11}$$

where $\int$ means the integration over $\mathbf{R}$ and symbols $K(\cdot,\cdot)y$, $\int K(\cdot, s)y(s)ds$, $\int K_t(\cdot, s)y(s)ds$ denote the functions $K(t,t)y(t)$, $\int_{\mathbf{R}} K(t,s)y(s)ds$, $\int_{\mathbf{R}} K_t(t,s)$ $y(s)ds$, $t \in \mathbf{R}$ belonging to $L^2(\mathbf{R}; V')$, respectively. From the fact

$$\int_{\mathbf{R}} \frac{d}{dt}[\rho * (\mathcal{O} \int K(\cdot, s)y(s)ds), \rho * (\mathcal{O}y)]dt = 0,$$

we obtain by differentiating this directly that

$$(2.12)\qquad \begin{cases} ((\rho * (\mathcal{O}' \int K(\cdot, s)y(s)ds), \rho * (\mathcal{O}y))) \\ \quad +((\rho * (\mathcal{O}K(\cdot, \cdot)y(\cdot)), \rho * (\mathcal{O}y))) \\ \quad + ((\rho * (\mathcal{O} \int K_t(\cdot, s)y(s)ds), \rho * (\mathcal{O}y))) \\ \quad + ((\rho * (\mathcal{O} \int K(\cdot, s)y(s)ds), \rho * (\mathcal{O}'y))) \\ \quad + ((\rho * (\mathcal{O} \int K(\cdot, s)y(s)ds), \rho * (\mathcal{O}y'))) = 0. \end{cases}$$

Now we let $\delta \to 0$ in (2.12). The first term of (2.12) may be written as

$$(2.13)\qquad \begin{aligned} &((\rho * (\mathcal{O}' \int K(\cdot, s)y(s)ds), \rho * ((\mathcal{O} - \mathcal{O}_0)y))) \\ &\quad + ((\rho * (\mathcal{O}' \int K(\cdot, s)y(s)ds), \rho * (\mathcal{O}_0 y))). \end{aligned}$$

Since $(\mathcal{O} - \mathcal{O}_0)y \to 0$ in $L^2(\mathbf{R}; V)$, we see $\rho * ((\mathcal{O} - \mathcal{O}_0)y)) \to 0$ in $L^\infty(\mathbf{R}; V)$. Since $\int_{\mathbf{R}} |\mathcal{O}'|dt = 2$ and $\rho * (\mathcal{O}' \int K(\cdot, s)y(s)ds)$ is bounded in $L^1(\mathbf{R}; V')$, then the first term of (2.13) goes to zero. The second term in (2.13) is equal to

$$(((\mathcal{O}' \int K(\cdot, s)y(s)ds), \rho * \rho * (\mathcal{O}_0 y))),$$

which is also equal to

$$(2.14)\qquad \begin{aligned} &\frac{1}{\delta} \int_0^\delta [\rho * \rho * (\mathcal{O}_0 y)(t), \int_0^t K(t, s)y(s)ds]dt \\ &-\frac{1}{\delta} \int_{t_0-\delta}^{t_0} [\rho * \rho * (\mathcal{O}_0 y)(t), \int_0^t K(t, s)y(s)ds]dt. \end{aligned}$$

Since the map

$$t \;\to\; [\rho * \rho * (\mathcal{O}_0 y)(t), \int_0^t K(t, s)y(s)ds]$$

is continuous, so that the term (2.14) tends to

$$-[(\mathcal{O}_0 \int K(\cdot, s)y(s)ds)(t_0), \rho * \rho * (\mathcal{O}_0 y)(t_0)].$$

Here we note that

$$(\mathcal{O}_0 \int K(\cdot, s)y(s)ds)(0) = 0.$$

Similarily the fourth term in (2.12) tends to

$$-[\rho * \rho * (\mathcal{O}_0 \int K(\cdot, s)y(s)ds)(t_0), (\mathcal{O}_0 y)(t_0)].$$

Also by starting with

$$\int_{\mathbf{R}} \frac{d}{dt}(\rho * (\mathcal{O}y'), \rho * (\mathcal{O}y'))dt = 0,$$

we can prove the following equality

$$\begin{aligned} 2(\!(\rho * (\mathcal{O}_0 y''), \rho * (\mathcal{O}_0 y'))\!) \quad &+ 2[(\rho * \rho * (\mathcal{O}_0 y')(0), y'(0)] \\ -2[(\rho * \rho * (\mathcal{O}_0 y')(t_0), y'(t_0)] = 0, & \end{aligned} \tag{2.15}$$

by letting $\delta \to 0$ as in Lions and Magenes [5, p. 278]. Finally we add (2.10), (2.15) and (2.11) multiplied by 2. Then by taking into account of the equality $y'' = f(t) - A(t)y - \int_0^t K(t,s)y(s)ds$, we obtain

$$\left\{ \begin{aligned} &(\!(A'(\rho * (\mathcal{O}_0 y)), \rho * (\mathcal{O}_0 y))\!) + 2(\!(\rho * (\mathcal{O}_0(f), \rho * (\mathcal{O}_0 y'))\!) \\ &\quad + 2(\!((A(\rho * (\mathcal{O}_0 y)) - \rho * (A\mathcal{O}_0 y))', \rho * \mathcal{O}_0 y)\!) \\ &\quad + 2(\!(\rho * (\mathcal{O}_0 K(\cdot,\cdot)y), \rho * (\mathcal{O}_0 y))\!) \\ &\quad + 2(\!(\rho * (\mathcal{O}_0 \textstyle\int K_t(\cdot,s)y(s)ds), \rho * (\mathcal{O}_0 y)) \\ &\quad - 2[(\mathcal{O}_0 \textstyle\int K(\cdot,s)y(s)ds)(t_0), \rho * \rho * (\mathcal{O}_0 y)(t_0)] \\ &\quad - 2[\rho * \rho * (\mathcal{O}_0 \textstyle\int K(\cdot,s)y(s)ds)(t_0), (\mathcal{O}_0 y)(t_0)] \\ &\quad + 2[\rho * \rho * (\mathcal{O}_0 Ay)(0), y(0)] - 2[\rho * \rho * (\mathcal{O}_0 Ay)(t_0), y(t_0)] \\ &\quad + 2[\rho * \rho * (\mathcal{O}_0 y')(0), y'(0)] - 2[\rho * \rho * (\mathcal{O}_0 y')(t_0), y'(t_0)] = 0. \end{aligned} \right. \tag{2.16}$$

We set $\rho = \rho_n$ in (2.16) and let $n \to \infty$. Then from the vector Friedrichs' Lemma, the third term of (2.16) tends to 0. In (2.16), the first term tends to

$$(\!(A'\mathcal{O}_0 y, \mathcal{O}_0 y)\!) = \int_0^{t_0} a'(s; y(s), y(s))ds, \tag{2.17}$$

the second term tends to

$$2(\!(\mathcal{O}_0(f), \mathcal{O}_0(y'))\!) = 2\int_0^{t_0} (f(s), y'(s))ds, \tag{2.18}$$

and the fourth and fifth terms tend to

$$\begin{aligned} &2(\!(\mathcal{O}_0 K(\cdot,\cdot)y, \mathcal{O}_0 y)\!) + 2(\!(\mathcal{O}_0 \textstyle\int K_t(\cdot,s)y(s)ds, \mathcal{O}_0 y)\!) \\ &= 2\int_0^{t_0} k(s,s;y(s),y(s))ds + 2\int_0^{t_0}\int_0^s \frac{\partial k}{\partial t}(s,\sigma;y(\sigma),y(s))d\sigma ds. \end{aligned} \tag{2.19}$$

On the remainder part of (2.16), by setting $\gamma_n = \rho_n * \rho_n$, we have

$$\begin{aligned} &-2[(\mathcal{O}_0 \int K(\cdot, s)y(s)ds)(t_0), \gamma_n * (\mathcal{O}_0 y)(t_0)] \\ &\to - \int_0^{t_0} k(t_0, s; y(s), y(t_0))ds, \end{aligned} \tag{2.20}$$

$$-2[\gamma_n * (\mathcal{O}_0 \int K(\cdot, s)y(s)ds)(t_0), y(t_0)] \to - \int_0^{t_0} k(t_0, s; y(s), y(t_0))ds, \tag{2.21}$$

$$2[\gamma_n * (\mathcal{O}_0 Ay)(0), y(0)] \to a(0; y(0), y(0)), \tag{2.22}$$

$$-2[\gamma_n * (\mathcal{O}_0 Ay)(t_0), y(t_0)] \to -a(t_0; y(t_0), y(t_0)), \tag{2.23}$$

$$2[\gamma_n * (\mathcal{O}_0 y')(0), y'(0)] \to |y_1|^2, \tag{2.24}$$

$$-2[\gamma_n * (\mathcal{O}_0 y')(t_0), y'(t_0)] \to -|y'(t_0)|^2. \tag{2.25}$$

For sufficiently large n, we see

$$2[\gamma_n * (\mathcal{O}_0 Ay)(t_0), y(t_0)] = 2 \int_0^{t_0} \gamma_n(t)[Ay(t_0 - t), y(t_0)]dt.$$

Since γ_n is even and

$$\int_0^{t_0} \gamma_n(t)dt = \frac{1}{2},$$

then we have

$$\begin{aligned} &2[\gamma_n * (\mathcal{O}_0 Ay(t_0), y(t_0)] - [Ay(t_0), y(t_0)] \\ = \;\; &2 \int_0^{t_0} \gamma_n(t)[(Ay)(t_0 - t) - (Ay)(t_0), y(t_0)]dt, \end{aligned} \tag{2.26}$$

and the term (2.26) goes to 0 as $n \to \infty$. Combining the above convergences (2.17)-(2.25) we have the energy equality (2.7) with $t = t_0$. The proof of Proposition 2.1 is completed.

The main purpose of this paper is to prove the following theorem on the existence, uniqueness and regularity of a weak solution of (2.5).

Theorem 2.1. *Assume that* a *and* k *satisfy* $(2.1) - (2.2)$ *and* $(2.3) - (2.4)$, *respectively and* f *satisfies* $(A1) - (A3)$. *Then the problem* (2.5) *has a unique weak solution* y *in* $W(0, T)$. *Moreover, the solution* y *has the regularity*

$$y \in C([0, T]; V), \quad y' \in C([0, T]; H) \tag{2.27}$$

and satisfies the energy equality

$$
\begin{aligned}
&a(t; y(t), y(t)) + |y'(t)|^2 \\
= \;& a(0; y_0, y_0) + |y_1|^2 + \int_0^t a'(s; y(s), y(s))ds \\
&+ 2\int_0^t (f(s, y(s), y'(s)), y'(s))ds + 2\int_0^t k(s, s; y(s), y(s))ds \\
&+ 2\int_0^t \Big(\int_0^s \frac{\partial k}{\partial t}(s, \sigma; y(\sigma), y(s))d\sigma - k(t, s; y(s), y(t))\Big)ds.
\end{aligned}
\tag{2.28}
$$

Remark 2.1. Definition 2.4. In Dautray and Lions [1; p.655] the differentiability of $k(t, s; \phi, \varphi)$ with respect to s is assumed to prove the uniqueness of solutions by the Ladyzenskaya's method. However the differentiability is not necessary to prove the uniqueness. We prove the uniqueness by the energy equality as shown in the proof of Theorem 2.1.

Proof of Theorem 2.1. We divide the proof into five steps.

Step 1. Approximate solutions and a priori estimate.

We apply the Faedo-Galerkin procedure as in Dautray and Lions [1]. Since V is separable, there exists a basis $\{w_m\}_{m=1}^{\infty}$ in V such that $\{w_m\}_{m=1}^{\infty}$ is a complete orthonomal system in H, and total and free in V. Let $\mathbf{N}$ be the set of natural numbers. For each $m \in \mathbf{N}$ we define an approximate solution of the equation (2.5) by

$$y_m(t) = \sum_{j=1}^{m} g_{jm}(t) w_j,$$

where $y_m(t)$ satisfies

$$
\begin{cases}
\quad (y''_m(t), w_j) + a(t; y_m(t), w_j) + \displaystyle\int_0^t k(t, s; y_m(s), w_j)ds \\
= (f(t, y_m(t), y'_m(t)), w_j), \;\; t \in [0, T], \;\; 1 \le j \le m, \\
\quad y_m(0) = y_{0m}, \quad y'_m(0) = y_{1m},
\end{cases}
\tag{2.29}
$$

where

$$y_{0m} = \sum_{i=1}^{m} \xi_{im}^0 w_i \to y_0 \;\text{ in }\; V, \tag{2.30}$$

$$y_{1m} = \sum_{i=1}^{m} \xi_{im}^1 w_i \to y_1 \;\text{ in }\; H \tag{2.31}$$

for some real numbers $\xi_{im}^0,\ \xi_{im}^1,\ i=1,\cdots,m$ by the condition on $\{w_m\}$. By standard manipulation the equation (2.29) induces the system of second order Volterra integro-differential equations for $g_{jm}(t)$ with initial conditions $g_{jm}(0)=\xi_{jm}^0,\ g'_{jm}(0) = \xi_{jm}^1,\ j=1,\cdots,m$. By (A1)-(A3) it is verified that the system admits a unique solution $(g_{jm}(t); j=1,\cdots,m)$ over $[0,T]$. Hence we can construct the approximate solution $y_m(t)$ of (2.29). Next we shall derive a priori estimates of $y_m(t)$. Multiply both sides of the equation (2.29) by $g'_{jm}(t)$ and sum over j to have

$$\begin{aligned}(y''_m(t),y'_m(t)) &+ a(t;y_m(t),y'_m(t)) + \int_0^t k(t,s;y_m(s),y'_m(t))ds \\ &= (f(t,y_m(t),y'_m(t)),y'_m(t)).\end{aligned} \tag{2.32}$$

Using

$$\begin{cases} a(t;y_m(t),y'_m(t)) = \dfrac{1}{2}\dfrac{d}{dt}a(t;y_m(t),y_m(t)) - \dfrac{1}{2}a'(t;y_m(t),y_m(t)), \\ \displaystyle\int_0^t k(t,s;y_m(s),y'_m(t))ds = \frac{d}{dt}\int_0^t k(t,s;y_m(s),y_m(t))ds \\ \quad -k(t,t;y_m(t),y_m(t)) - \displaystyle\int_0^t \frac{\partial k}{\partial t}(t,s;y_m(s),y_m(t))ds, \end{cases} \tag{2.33}$$

(2.32) is rewritten as

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}[a(t;y_m(t),y_m(t)) + |y'_m(t)|^2 \\ &+2\int_0^t k(t,s;y_m(s),y_m(t))ds] - \frac{1}{2}a'(t;y_m(t),y_m(t)) \\ =\ &(f(t,y_m(t),y'_m(t)),y'_m(t)) \\ &+k(t,t;y_m(t),y_m(t)) \\ &+\int_0^t \frac{\partial k}{\partial t}(t,s;y_m(s),y_m(t))ds. \end{aligned} \tag{2.34}$$

We shall integrate the equality (2.34) on $[0,t]$ and estimate it to obtain a priori estimates of $\{y_m\}$. For this we estimate the nonlinear term in (2.34). From (A2) and (A3) we obtain by the Schwartz inequality

$$
\begin{aligned}
&2\Big|\int_0^t (f(s,y_m(s),y_m'(s)),y_m'(s))ds\Big| \\
&= 2\Big|\int_0^t (f(s,y_m(s),y_m'(t)) - f(s,0,0) + f(s,0,0), y_m'(s))ds\Big| \\
&\le 2\int_0^t \beta(s)\Big(\|y_m(s)\| + |y_m'(s)|\Big)|y_m'(s)|ds + 2\int_0^t \gamma(s)|y_m'(s)|ds \\
&\le \|\gamma\|^2_{L^2(0,T;\mathbf{R}^+)} + \int_0^t (1+\beta(s)^2+2\beta(s))\Big(\|y_m(s)\|^2 + |y_m'(s)|^2\Big)ds.
\end{aligned}
\tag{2.35}
$$

And from (2.3) we have

$$
\begin{aligned}
2\Big|\int_0^t k(t,s;y_m(s),y_m(t))ds\Big| &\le 2k_0\|y_m(t)\| \textstyle\int_0^t \|y_m(s)\|ds \\
&\le \epsilon\|y_m(t)\|^2 + c(\epsilon)\int_0^t \|y_m(s)\|^2 ds
\end{aligned}
\tag{2.36}
$$

for any $\epsilon > 0$ and some $c(\epsilon) > 0$. We can deduce from (2.3) and (2.4) that

$$
\left\{
\begin{aligned}
&2\Big|\int_0^t\int_0^s \frac{\partial k}{\partial t}(s,\sigma;y_m(\sigma),y_m(s))d\sigma ds\Big| \le 2k_1\Big(\int_0^t \|y_m(s)\|ds\Big)^2 \\
&2\Big|\int_0^t k(s,s;y_m(s),y_m(s))ds\Big| \le 2k_0\int_0^t \|y_m(s)\|^2 ds.
\end{aligned}
\right.
\tag{2.37}
$$

Therefore by using (2.1)-(2.4) and (2.35)-(2.37), we obtain the following inequality

$$
\begin{aligned}
&|y_m'(t)|^2 + \alpha\|y_m(t)\|^2 \le\ a(0;y_{0m},y_{0m}) + |y_{1m}|^2 + \epsilon\|y_m(t)\|^2 \\
&+\ (c_2 + 2k_0 + c(\epsilon))\int_0^t \|y_m(s)\|^2 ds + 2k_1\Big(\int_0^t \|y_m(s)\|ds\Big)^2 \\
&+\ \|\gamma\|^2_{L^2(0,T;\mathbf{R}^+)} + \int_0^t (1+\beta(s)^2+2\beta(s))\Big(\|y_m(s)\|^2 + |y_m'(s)|^2\Big)ds.
\end{aligned}
\tag{2.38}
$$

Since

$$
\Big(\int_0^T \|y_m(s)\|ds\Big)^2 \le T\int_0^T \|y_m(s)\|^2 ds,
$$

(2.38) imlpies

$$
\begin{aligned}
&|y_m'(t)|^2 + \alpha\|y_m(t)\|^2 \le\ a(0;y_{0m},y_{0m}) + |y_{1m}|^2 + \epsilon\|y_m(t)\|^2 \\
&+ (c_2 + 2k_0 + c(\epsilon) + 2k_1 T)\int_0^t \|y_m(s)\|^2 ds \\
&+ \|\gamma\|^2_{L^2(0,T;\mathbf{R}^+)} + \int_0^t (1+\beta(s)^2+2\beta(s))\Big(\|y_m(s)\|^2 + |y_m'(s)|^2\Big)ds.
\end{aligned}
\tag{2.39}
$$

If we put $\beta_1(s) = 2(c_2 + k_0 + c(\epsilon) + k_1 T + \beta(s)^2 + \beta(s) + 1) \in L^1(0, T; \mathbf{R}^+)$, then we arrive at

$$\begin{aligned} |y_m'(t)|^2 + (\alpha - \epsilon)\|y_m(t)\|^2 &\leq C(\|y_0\|^2 + |y_1|^2 + \|\gamma\|^2_{L^2(0,T;\mathbf{R}^+)}) \\ &+ \int_0^t \beta_1(s)(\|y_m(s)\|^2 + |y_m'(s)|^2)ds \end{aligned} \tag{2.40}$$

for some constant $C > 0$. By choosing $\epsilon = \frac{\alpha}{2}$ and applying the Bellman-Gronwall's inequality to (2.40), we obtain the estimate

$$\|y_m(t)\|^2 + |y_m'(t)|^2 ds \leq C_1(y_0, y_1, \gamma) \exp(\int_0^T \beta_2(t)dt), \tag{2.41}$$

where $\beta_2 = \dfrac{\beta_1}{\min\{1, \frac{\alpha}{2}\}} \in L^1(0, T; \mathbf{R}^+)$ and C_1 is a constant depending only on y_0, y_1 and γ.

Step 2. Weak convergence.

By (2.41) the sequences $\{y_m\}$ and $\{y_m'\}$ remain in a bounded sets of $L^\infty(V)$ and $L^\infty(H)$, respectively. It is readily verified by (A2) and (A3) that $\{f(\cdot, y_m, y_m')\}$ is bounded in $L^2(H)$. Then by the extraction theorem of Rellich's, we can extract a subsequence $\{y_{m_k}\}$ of $\{y_m\}$ and find $z \in L^\infty(V)$ with $z' \in L^\infty(H)$ and $F \in L^2(H)$ such that

$$\text{(i)} \quad y_{m_k} \to z \quad \text{weakly-star in} \quad L^\infty(V), \quad \text{and weakly in } L^2(V), \tag{2.42}$$

$$\text{(ii)} \quad y_{m_k}' \to z' \quad \text{weakly-star in } L^\infty(H) \quad \text{and weakly in } L^2(H), \tag{2.43}$$

$$\text{(iii)} \quad A(\cdot)y_{m_k} \to A(\cdot)z \quad \text{weakly in} \quad L^2(V'), \tag{2.44}$$

$$\text{(iv)} \quad K(t,\cdot)y_{m_k} \to K(t,\cdot)z \quad \text{weakly in} \quad L^2(0,t;V'), \tag{2.45}$$

$$\text{(v)} \quad f(\cdot, y_{m_k}, y_{m_k}') \to F(\cdot) \quad \text{weakly in} \quad L^2(H) \tag{2.46}$$

as $k \to \infty$.

Let $\phi \in \mathcal{D}(0, T)$ and $v \in V$ be fixed. Then, there exists a sequence $\{v_m\}_{m \in \mathbf{N}}, v_m \in V_m$ for all m such that $v_m \to v$ strongly in V, where V_m is the linear space spanned by $\{w_1, \cdots, w_m\}$. We introduce the product functions

$$\begin{cases} \psi_m = \phi \otimes v_m, \\ \psi = \phi \otimes v. \end{cases} \tag{2.47}$$

Then we particularly have

$$
\left\{ \begin{array}{ll} \psi_m \to \psi & \text{strongly in } \ L^2(V), \\ \psi'_m \to \psi' & \text{strongly in } \ L^2(H). \end{array} \right. \tag{2.48}
$$

From (2.32) we deduce

$$
\begin{aligned} &-\int_0^T (y'_m(t), \psi'_m(t))dt + \int_0^T \Big[a(t; y_m(t), \psi_m(t)) \\ &\quad + \int_0^t k(t,s; y_m(s), \psi_m(t))ds\Big]dt = \int_0^T (f(t, y_m, y'_m), \psi_m(t))dt. \end{aligned} \tag{2.49}
$$

By letting $m = m_k \to \infty$ in (2.49) and using (2.42)-(2.48) we have

$$
\begin{aligned} &-\int_0^T (z'(t), \psi'(t))dt + \int_0^T \Big[a(t; z(t), \psi(t)) \\ &\quad + \int_0^t k(t,s; z(s), \psi(t))ds\Big]dt = \int_0^T (F(t), \psi(t))dt. \end{aligned} \tag{2.50}
$$

Hence

$$
\begin{aligned} &-\int_0^T (z'(t), v)\phi'(t)dt + \int_0^T \Big[\Big(a(t; z(t), v) \\ &\quad + \int_0^t k(t,s; z(s), v)ds\Big)\phi(t)\Big]dt = \int_0^T (F(t), v)\phi(t)dt. \end{aligned}
$$

This shows that z satisfies for all $v \in V$

$$
\langle z''(\cdot), v\rangle_{V',V} + a(\cdot; z(\cdot), v) + \int_0^{\cdot} k(\cdot; z(s), v)ds = (F(\cdot), v), \tag{2.51}
$$

in the sense of $\mathcal{D}'(0,T)$. We already know that $z \in L^2(V), z' \in L^2(H)$. The mapping

$$
\begin{aligned} \psi \to \langle z'', \psi\rangle &= -\int_0^T (z'(t), \psi'(t))dt \\ &= \int_0^T \Big\langle F(t) - A(t)z(t) - \int_0^t K(t,s)z(s)ds, \psi(t)\Big\rangle_{V',V} dt \end{aligned}
$$

is continuous over $\mathcal{D}((0,T);V)$ equipped with the topology of $L^2(V)$. Therefore, over $L^2(V)$ by density, we can know that $z'' \in L^2(V')$ and therefore $z \in W(0,T)$. Let $\phi \in C^1[0,T]$ such that $\phi(0) \neq 0$ and $\phi = 0$ in a neighborhood of T. For such a ϕ we set

$$
\left\{ \begin{array}{l} \psi_m = \phi \otimes v_m \\ \psi = \phi \otimes v, \end{array} \right. \tag{2.52}
$$

where $v_m \in V_m$ is such that $v_m \to v$ strongly in V. From (2.29), we have

$$\begin{aligned}
&-\int_0^T (y'_m(t), \psi'_m(t))dt + \int_0^T a(t; y_m(t), \psi_m(t))dt \\
&+\int_0^T \int_0^t k(t,s; y_m(s), \psi_m(t))dsdt \\
&= (y_{1m}, \psi_m(0)) + \int_0^T (f(t, y_m(t), y'_m(t)), \psi_m(t))dt,
\end{aligned}$$

and by the passage of $m = m_k \to \infty$, we obtain

$$\begin{aligned}
&-\int_0^T (z'(t), \psi'(t))dt + \int_0^T a(t; z(t), \psi(t))dt \\
&+\int_0^T \int_0^t k(t,s; z(s), \psi(t))dsdt \\
&= (y_1, \psi(0)) + \int_0^T (F(t), \psi(t))dt.
\end{aligned} \tag{2.53}$$

Moreover from (2.51) we can deduce

$$\begin{aligned}
&-\int_0^T (z'(t), \psi'(t))dt + \int_0^T a(t; z(t), \psi(t))dt \\
&+\int_0^T \int_0^t k(t,s; z(s), \psi(t))dsdt \\
&= (z'(0), \psi(0)) + \int_0^T (F(t), \psi(t))dt,
\end{aligned} \tag{2.54}$$

From (2.53), (2.54) and $\psi(0) = v$, we have

$$(z'(0), v) = (y_1, v) \quad \text{for all } v \in V,$$

from which $z'(0) = y_1$ follows. And we may have that

$$\int_0^T (y'_m(t), \psi_m(t))dt = -(y_{0m}, \psi_m(0)) - \int_0^T (y_m(t), \psi'_m(t))dt, \tag{2.55}$$

from which by tending $m = m_k \to \infty$, it follows that

$$\int_0^T (z'(t), \psi(t))dt = -(y_0, \psi(0)) - \int_0^T (z(t), \psi'(t))dt. \tag{2.56}$$

Since

$$\int_0^T (z'(t),\psi(t))dt = -(z(0),\psi(0)) - \int_0^T (z(t),\psi'(t))dt, \tag{2.57}$$

then by the comparison of (2.56) with (2.57) we can see

$$(z(0),v) = (y_0,v) \quad \text{for all } v \in V,$$

which implies $z(0) = y_0$. This proves that z is a weak solution of (2.5) in which $f(t,y,y')$ is replaced by $F(t)$. Hence z satisfies the energy equality (2.7) with $f(t) = F(t)$.

Step 3. Strong covergence.

In order to prove that z is a weak solution of (2.5) we have to prove $F(\cdot) = f(\cdot, z(\cdot), z'(\cdot))$. The main difficulty exists in showing this equality. In the following we set $y_{m_k} = y_m$ again for simplicity. To overcome this difficulty we shall show the strong convergences of $\{y_m\}$ and $\{y_m'\}$. That is for all $t \in [0,T]$ we shall show $y_m(t) \to z(t)$ strongly in V and $y_m'(t) \to z'(t)$ strongly in H. Let $t \in [0,T]$ be fixed. Integrating (2.32) on $[0,t]$, we obtain

$$\begin{aligned}
& a(t;y_m(t),y_m(t)) + |y_m'(t)|^2 \\
={}& a(0;y_{0m},y_{0m}) + |y_{1m}|^2 + \int_0^t a'(s;y_m(s),y_m(s))ds \\
& - 2\int_0^t k(t,s;y_m(s),y_m(t))ds + 2\int_0^t k(s,s;y_m(s),y_m(s))ds \\
& + 2\int_0^t \int_0^s \frac{\partial k}{\partial t}(s,\sigma;y_m(\sigma),y_m(s))d\sigma ds \\
& + 2\int_0^t (f(s,y_m(s),y_m'(s)),y_m'(s))ds.
\end{aligned} \tag{2.58}$$

Since z is a weak solution of (2.5) with $f(t,y,y') = F(t)$, we can verify by Proposition 2.1 the following energy equality

$$\begin{aligned}
& a(t;z(t),z(t)) + |z'(t)|^2 \\
={}& a(0;y_0,y_0) + |y_1|^2 + \int_0^t a'(s;z(s),z(s))ds \\
& - 2\int_0^t k(t,s;z(s),z(t))ds + 2\int_0^t k(s,s;z(s),z(s))ds \\
& + 2\int_0^t \int_0^s \frac{\partial k}{\partial t}(s,\sigma;z(\sigma),z(s))d\sigma ds + 2\int_0^t (F(s),z'(s))ds.
\end{aligned} \tag{2.59}$$

Moreover the following equalities hold true:

$$\begin{aligned}
&a(t;y_m,y_m)+a(t;z,z)=a(t;y_m-z,y_m-z)+2a(t;y_m,z);\\
&a(0;y_{0m},y_{0m})+a(0;y_0,y_0)=a(0;y_{0m}-y_0,y_{0m}-y_0)+2a(0;y_{0m},y_0);\\
&|y_{1m}|^2+|y_1|^2=(y_{1m}-y_1,y_{m1}-y_1)+2(y_{1m},y_1);\\
&|y_m'(t)|^2+|z'(t)|^2=(y_m'(t)-z'(t),y_m'(t)-z'(t))+2(y_m'(t),z'(t));\\
&a'(s;y_m,y_m)+a'(s;z,z)=a'(s;y_m-z,y_m-z)+2a'(s;y_m,z);\\
&k(t,s;y_m(s),y_m(t))+k(t,s;z(s),z(t))\\
=\ &k(t,s;y_m(s)-z(s),y_m(t)-z(t))\\
&+\ k(t,s;y_m(s),z(t))+k(t,s;z(s),y_m(t));\\
&k(t,s;y_m(s),y_m(s))+k(t,s;z(s),z(s))\\
&k(t,s;y_m(s)-z(s),y_m(s)-z(s))\\
&+\ k(t,s;y_m(s),z(s))+k(t,s;z(s),y_m(s));\\
&\frac{\partial k}{\partial t}(s,\sigma;y_m(\sigma),y_m(s))+\frac{\partial k}{\partial t}(s,\sigma;z(\sigma),z(s))\\
&=\ \frac{\partial k}{\partial t}(s,\sigma;y_m(\sigma)-z(\sigma),y_m(s)-z(s))\\
&+\frac{\partial k}{\partial t}(s,\sigma;y_m(\sigma),z(s))+\frac{\partial k}{\partial t}(s,\sigma;y_m(s),z(\sigma));\\
&(f(t,y_m,y_m'),y_m')+(F(t),z')=(f(t,y_m,y_m')-f(t,z,z'),y_m'-z')\\
&+\ (f(t,z,z')-F(t),y_m'-z')+(f(t,y_m,y_m'),z')\\
&+\ (F(t),y_m').
\end{aligned}$$

Adding (2.58) to (2.59) and using the above equalities, we have

$$\begin{aligned}
&a(t;y_m(t)-z(t),y_m(t)-z(t))+|y_m'(t)-z'(t)|^2\\
=\ &\sum_{i=1}^{7}\Phi_m^i(t)+a(0;y_{0m}-y_0,y_{0m}-y_0)+|y_{1m}-y_1|^2\\
&+\int_0^t a'(s;y_m(s)-z(s),y_m(s)-z(s))ds\\
&-2\int_0^t k(t,s;y_m(s)-z(s),y_m(t)-z(t))ds\\
&+2\int_0^t k(s,s;y_m(s)-z(s),y_m(s)-z(s))ds\\
&+2\int_0^t\int_0^s\frac{\partial k}{\partial s}(s,\sigma;y_m(\sigma)-z(\sigma),y_m(s)-z(s))d\sigma ds\\
&+2\int_0^t(f(s,y_m(s),y_m'(s))-f(s,z(s),z'(s)),y_m'(s)-z'(s))ds,
\end{aligned}\tag{2.60}$$

where

$$\Phi_m^1(t) \equiv \Phi_m^1 = 2a(0; y_{0m}, y_0) + 2(y_{1m}, y_1),$$
$$\Phi_m^2(t) = -2a(t; y_m(t), z(t)) - 2(y_m'(t), z'(t)),$$
$$\Phi_m^3(t) = 2\int_0^t a'(s; y_m, z)ds,$$
$$\Phi_m^4(t) = -2\int_0^t k(t, s; y_m(s), z(t))ds - 2\int_0^t k(t, s; z(s), y_m(t))ds,$$
$$\Phi_m^5(t) = 2\int_0^t k(t, s; y_m(s), z(s))ds + 2\int_0^t k(t, s; z(s), y_m(s))ds,$$
$$\Phi_m^6(t) = 2\int_0^t \int_0^s (\frac{\partial k}{\partial t}(s, \sigma; y_m(\sigma), z(s)) + \frac{\partial k}{\partial t}(s, \sigma; z(\sigma), y_m(s)))d\sigma ds,$$
$$\Phi_m^7(t) = 2\int_0^t (f(s, y_m(s), y_m'(s)), z'(s)) + (F(s), y_m'(s)))ds$$
$$+ 2\int_0^t (f(s, z(s), z'(s)) - F(s), y_m'(s) - z'(s))ds.$$

For simplicity we set

$$\Phi_m(t) = \sum_{i=1}^7 \Phi_m^i(t).$$

By (2.42), (2.43) we have $y_m(t) \to z(t)$ weakly in V and $y_m'(t) \to z'(t)$ weakly in H, so that

$$\Phi_m^2(t) \;\to\; -2a(t; z(t), z(t)) - 2|z'(t)|^2. \tag{2.61}$$

It is clear from (2.30) and (2.31) that

$$\Phi_m^1 \;\to\; 2a(0; y_0, y_0) + 2|y_1|^2. \tag{2.62}$$

From (2.42)-(2.46) it follows immediately that

$$\Phi_m^3(t) \;\to\; 2\int_0^t a'(s; z, z)ds, \tag{2.63}$$

$$\Phi_m^4(t) \;\to\; -4\int_0^t k(t, s; z(s), z(t))ds, \tag{2.64}$$

$$\Phi_m^5(t) \ \rightarrow \ 4\int_0^t k(t,s;z(s),z(s))ds, \tag{2.65}$$

$$\Phi_m^6(t) \ \rightarrow \ 4\int_0^t \int_0^s \frac{\partial k}{\partial t}(s,\sigma;y_m(\sigma),z(s))d\sigma ds, \tag{2.66}$$

$$\Phi_m^7(t) \ \rightarrow \ 4\int_0^t (F(s),z'(s))ds. \tag{2.67}$$

Hence by (2.61)-(2.67) and the energy equality (2.28) for z, we have

$$\Phi_m(t) \ \rightarrow \ 0 \quad \text{as} \ \ m \ \rightarrow \ \infty. \tag{2.68}$$

From (2.60) we can derive the following estimation

$$\begin{aligned} &(\alpha-\epsilon)\|y_m(t)-z(t)\|^2+|y_m'(t)-z'(t)|^2 \\ \leq\ &\Phi_m(t)+c_0\|y_{0m}-y_0\|^2+|y_{1m}-y_1|^2 \\ &+\int_0^t h(s)(\|y_m(s)-z(s)\|^2+|y_m'(s)-z'(s)|^2)ds, \end{aligned} \tag{2.69}$$

where $h(s)=2(c_2+k_0+k_1T+\beta(s)^2+\beta(s)+1+\frac{k_0^2T}{\epsilon}) \ \in \ L^1(0,T;\mathbb{R}^+)$. We divide (2.69) by $\alpha_1=\min\{1,\alpha-\epsilon\}>0$ and if we set

$$M_m(t)=\|y_m(t)-z(t)\|^2+|y_m'(t)-z'(t)|^2, \tag{2.70}$$

$$\Psi_m(t)=\frac{1}{\alpha_1}\Phi_m(t), \tag{2.71}$$

$$Z_m=\frac{c_0}{\alpha_1}\|y_{m0}-y_0\|^2+\frac{1}{\alpha_1}|y_{m1}-y_1|^2, \tag{2.72}$$

$$h_1(s)=\frac{1}{\alpha_1}h(s), \tag{2.73}$$

then we can have

$$M_m(t)\leq Z_m+|\Psi_m(t)|+\int_0^t h_1(s)M_m(s)ds. \tag{2.74}$$

Applying the Bellmann Gronwall's inequality to (2.74), we obtain

$$\begin{aligned} M_m(t)\leq Z_m+|\Psi_m(t)|+\exp\Big(\int_0^T h_1(s)ds\Big) \\ \exp\Big(\int_0^T h_1(s)(Z_m+|\Psi_m(s)|)ds\Big). \end{aligned} \tag{2.75}$$

By the energy equality (2.7), we see $\Psi_m(t) \to 0$ and $\Psi_m(t)$ is uniformly bounded. And also we know that $y_{0m} \to y_0$ strongly in V, $y_{1m} \to y_1$ strongly in H from (2.30), (2.31). Therefore

$$\lim_{m\to\infty} M_m(t) = 0. \tag{2.76}$$

Hence from (2.74) and (2.76) we can deduce that

$$y_m(t) \to z(t) \text{ strongly in } V, \tag{2.77}$$

$$y_m'(t) \to z'(t) \text{ strongly in } H. \tag{2.78}$$

Moreover with the condition (A2) and (2.46) it follows readily that

$$F(\cdot) = f(\cdot, z(\cdot), z'(\cdot)). \tag{2.79}$$

Therefore we have proved the existence of a weak solution and strong convergence of approximate solutions.

Step 4. Energy equality and regularity.

Since $f(\cdot, z(\cdot), z'(\cdot)) \in L^2(H)$, we have by Proposition 2.1 that $y = z$ satisfies the energy equality (2.28). It is verified from the assumptions (A1)-(A3) and (2.1)-(2.4) on a and k that the right hand side of (2.28) is continuous in t. Hence we have that

$$t \to a(t; y(t), y(t)) + |y(t)|^2$$

is continuous on $[0, T]$. Therefore as in the proof of Lions and Magenes [5, p. 279] we have the regularity

$$y \in C([0,T]; V) \quad \text{and} \quad y' \in C([0,T]; H).$$

Step 5. Uniqueness.

The uniqueness of weak solutions follows directly from the energy equality (2.28). Indeed, let y_1 and y_2 be the solutions of (2.5). Then by (2.1)-(2.4) and (A1)-(A3), we can obtain the following inequality for $z = y_1 - y_2$

$$\begin{aligned} &\|z(t)\|^2 + |z'(t)|^2 \\ &\le C_1 \int_0^t \|z(s)\|^2 ds + C_2 \int_0^t \Big|(f(s, y_1(s), y_1'(s)) \\ &\quad - f(s, y_2(s), y_2'(s)), z'(s))\Big| ds \\ &\le C_1 \int_0^t \|z(s)\|^2 ds + C_2 \int_0^t \beta(s)\Big(\|z(s)\| + |z'(s)|\Big)|z'(s)| ds \\ &\le \int_0^t h(s)(\|z(s)\|^2 + |z'(s)|^2) ds, \end{aligned} \tag{2.80}$$

where C_1 and C_2 are some constants and $h(s) = C_1 + C_2 + C_2\beta(s)^2 + C_2\beta(s) \in L^1(0,T;\mathbf{R}^+)$. Finally, applying the Gronwall's inequality to (2.80), we have $z = 0$. This completes the proof of Theorem 2.1.

3. An Application to Viscoelastic Equations

Let Ω be an open bounded set of $\mathbf{R}^n$ with a smooth boundary $\Gamma = \partial\Omega$. Let $Q = (0,T)\times\Omega$ and $\Sigma = (0,T)\times\Gamma$. We consider the following Dirichlet boundary value problem for nonlinear perturbed viscoelastic equations with long memory.

$$(3.1) \quad \begin{cases} \dfrac{\partial^2 y}{\partial t^2} - \displaystyle\sum_{i,j=1}^n \frac{\partial}{\partial x_i}(a_{ij}(t,x)\frac{\partial y}{\partial x_j}) + \sum_{i,j=1}^n \int^t \frac{\partial}{\partial x_i}(b_{ij}(t-s,x)\frac{\partial y(s,x)}{\partial x_j})ds \\ \qquad = f(t,x,y,\nabla y,\dfrac{\partial y}{\partial t}) \quad \text{in } Q, \\ y = 0 \quad \text{on } \Sigma, \\ y(0,x) = y_0(x), \quad \dfrac{\partial y}{\partial t}(0,x) = y_1(x) \quad \text{in } \Omega, \end{cases}$$

where y denotes the displacement of the material, $a_{ij}(t,x)$ are instantaneous elastic coefficients and $b_{ij}(t,x)$ are elastic coefficients taking into account of the effects of memory of the material. We assume

$$(3.2) \quad \begin{cases} a_{ij} = a_{ji} \quad \forall i,j = 1,\cdots,n, \\ \displaystyle\sum_{i,j=1}^n a_{ij}\xi_i\xi_j \geq \alpha|\xi|^2 \quad \exists\alpha > 0, \quad \forall \xi_i,\xi_j \in \mathbf{R}, \\ a_{ij} \in W^{1,\infty}(0,T;L^\infty(\Omega)) \quad \forall i,j = 1,\cdots,n, \\ b_{ij} \in W^{1,\infty}(0,T;L^\infty(\Omega)) \quad \forall i,j = 1,\cdots,n. \end{cases}$$

We take $V = H_0^1(\Omega)$ and $H = L^2(\Omega)$ and introduce two bilinear forms

$$a(t;\phi,\varphi) = \sum_{i,j=1}^n \int_\Omega a_{ij}(t,x)\frac{\partial\phi}{\partial x_i}\frac{\partial\varphi}{\partial x_j}dx, \quad \forall\phi,\varphi \in H_0^1(\Omega)$$

and

$$k(t;\phi,\varphi) = -\sum_{i,j=1}^n \int_\Omega b_{ij}(t,x)\frac{\partial\phi}{\partial x_i}\frac{\partial\varphi}{\partial x_j}dx, \quad \forall\phi,\varphi \in H_0^1(\Omega).$$

The nonlinear forcing function $f : [0,T]\times\mathbf{R}\times\mathbf{R}^n\times\mathbf{R} \to \mathbf{R}$ is assumed to satisfy the following conditions

(i) $f(\cdot,x,\xi,\eta,\zeta)$ is measurable on $[0,T]$ for all $\xi,\zeta \in R$ and $\eta \in \mathbf{R}^n$,

(ii) there is a $\beta \in L^2(0,T;L^\infty(\Omega))$ such that for each $t \in [0,T]$

$$\begin{aligned} &|f(\cdot,x,\xi_1,\eta_1,\zeta_1) - f(\cdot,x,\xi_2,\eta_2,\zeta_2)| \\ \leq\ &\beta(t,x)(|\xi_1-\xi_2| + |\eta_1-\eta_2| + |\zeta_1-\zeta_2|), \\ &\forall\zeta_1,\zeta_2 \text{ and } \forall\xi_1,\xi_2 \in \mathbf{R},\ \forall\eta_1,\eta_2 \in \mathbf{R}^n, \end{aligned}$$

(iii) there is a $\gamma \in L^2(0,T;L^\infty(\Omega))$ such that

$$|f(t,x,0,0,0)| \leq \gamma(t,x), \quad \text{a.e. } t \in [0,T].$$

It is verified that the nonlinear term f in (3.1) satisfies the assumption (A1)-(A3) and $a(t;\phi,\varphi), k(t,s;\phi,\varphi) \equiv k(t-s;\phi,\varphi)$ satisfy (2.1)-(2.4). Hence by Theorem 2.1, for $y_0 \in V = H_0^1(\Omega)$ and $y_1 \in H = L^2(\Omega)$, there exists a unique y satisfying (3.1) in the weak sense and $y \in C([0,T];V) \cap C^1([0,T];H)$.

Acknowledgment

The supports by Grant-in-Aid for Scientific Research (C) (2) 19540216 of the second author is gratefully acknowledged.

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