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AN INEQUALITY BETWEEN RATIO OF THE EXTENDED LOGARITHMIC MEANS AND RATIO OF THE EXPONENTIAL MEANS

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Abstract. In this article, we prove an inequality between the ratio of the extended logarithmic means and the ratio of the exponential means. The proof is based on an inequality between logarithmic mean and one-parameter mean, which can be deduced from monotonicity of the extended mean values.

1. Introduction

In [1, 14, 39, 42], a double inequalities were proved using the mathematical induction and other techniques, which can be expressed as

$$\frac{n}{n+1} \cdot \left(\frac{1}{n}\sum_{i=1}^{n} i^r \Big/ \frac{1}{n+1}\sum_{i=1}^{n+1} i^r\right)^{1/r} \cdot \frac{\sqrt[n]{n!}}{\sqrt[n+1]{(n+1)!}}, \tag{1}$$

where $r > 0$ and $n \in \mathbb{N}$.

We call the left-hand side of inequality (1) H. Alzer's inequality [1], and the right-hand side of inequality (1) J. S. Martins' inequality [12]. The inequality between two ends of (1) is called Minc-Sathre's inequality [13].

In [18], the first author generalized the Alzer's inequality and obtained

$$\frac{n+k}{n+m+k} < \left(\frac{1}{n}\sum_{i=k+1}^{n+k} i^r \Big/ \frac{1}{n+m}\sum_{i=k+1}^{n+m+k} i^r\right)^{1/r}, \tag{2}$$

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where r is a given positive real number, n and m are natural numbers, and k is a nonnegative integer. The lower bound in (2) is the best possible.

In [3], the Martins' inequality was generalized: Let $\{a_i\}_{i=1}^{\infty}$ be an increasing sequence of positive real numbers and

(i) for any positive integer $\ell > 1$,

$$\frac{a_{\ell+1}}{a_\ell} \cdot \frac{a_\ell}{a_{\ell-1}}; \tag{3}$$

(ii) for any positive integer $\ell > 1$,

$$\left(\frac{a_{\ell+1}}{a_\ell}\right)^{\ell} \geq \left(\frac{a_\ell}{a_{\ell-1}}\right)^{\ell-1}. \tag{4}$$

Then, for any natural numbers n and m, we have

$$\left(\frac{1}{n}\sum_{i=1}^{n} a_i^r \bigg/ \frac{1}{n+m}\sum_{i=1}^{n+m} a_i^r\right)^{1/r} < \frac{\sqrt[n]{a_n!}}{\sqrt[n+m]{a_{n+m}!}}, \tag{5}$$

where $n, m \in \mathbb{N}$ and r is a positive number, $a_i!$ denotes $\prod_{i=1}^{n} a_i$. The upper bound is the best possible.

As a corollary of inequality (5), we have: Let a and b be positive real numbers, k a nonnegative integer, and $m, n \in \mathbb{N}$. Then, for any real number $r > 0$, we have

$$\left(\frac{1}{n}\sum_{i=k+1}^{n+k} (ai+b)^r \bigg/ \frac{1}{n+m}\sum_{i=k+1}^{n+m+k} (ai+b)^r\right)^{1/r} < \frac{\sqrt[n]{\prod_{i=k+1}^{n+k}(ai+b)}}{\sqrt[n+m]{\prod_{i=k+1}^{n+m+k}(ai+b)}}. \tag{6}$$

Inequalities (5) and (6) answer an open problem proposed in [17, 18].

The Alzer's inequality and inequality (2) have been generalized and extended by many mathematicians. For more information, please refer to [2, 4, 5, 10, 17, 25, 24, 29, 38]. The Minc-Sathre's inequality was generalized in [10, 17, 20, 25, 26, 27, 29]. In [20, 26, 27, 29], the following inequalities were obtained:

$$\frac{n+k+1}{n+m+k+1} < \left(\prod_{i=k+1}^{n+k} i\right)^{1/n} \bigg/ \left(\prod_{i=k+1}^{n+m+k} i\right)^{1/(n+m)} \cdot \sqrt{\frac{n+k}{n+m+k}}, \tag{7}$$

where $n, m \in \mathbb{N}$ and k being a nonnegative integer.

In [15], the first author proved its integral analogue of inequality (2): Let $b > a > 0$ and $\delta > 0$ be real numbers. Then, for any given positive number $r \in \mathbb{R}$, we have

$$
\begin{aligned}
&\left(\frac{1}{b-a}\int_a^b x^r\,\mathrm{d}x\Big/\frac{1}{b+\delta-a}\int_a^{b+\delta} x^r\,\mathrm{d}x\right)^{1/r}\\
&=\left(\frac{b+\delta-a}{b-a}\cdot\frac{b^{r+1}-a^{r+1}}{(b+\delta)^{r+1}-a^{r+1}}\right)^{1/r} \qquad (8)\\
&>\frac{b}{b+\delta}.
\end{aligned}
$$

The lower bound in (8) is the best possible.

Inequality (8) has been generalized to inequalities for a positive functional in [7].

It is well-known [6] that the extended logarithmic means is defined as

$$
S_p(x,y)=\left(\frac{y^p-x^p}{p(y-x)}\right)^{1/(p-1)}, \quad x\neq y, \quad p\neq 0,1 \qquad (9)
$$

and $S_p(x,x)=x$, which is reduced to $S_0(x,y)=L(x,y)$, the logarithmic mean, and to the identric mean or the exponential mean $I(x,y)$

$$
S_1(x,y)=I(x,y)=e^{-1}\left(\frac{x^x}{y^y}\right)^{1/(x-y)}, \quad x\neq y, \qquad (10)
$$

and $S_1(x,x)=I(x,x)=x$. Please refer to [19] and other literature.

In this paper, by monotonicity of the extended mean values, from which an inequality between the logarithmic mean and one-parameter mean is deduced, we will prove an inequality between ratio of the extended logarithmic means and ratio of the exponential means, from which an open problem proposed in [15] is resolved, as follows:

Thmorem 1. *Let $b>a>0$ and $\delta>0$ be real numbers. Then, for any positive $r\in\mathbb{R}$, we have*

$$
\left(\frac{b+\delta-a}{b-a}\cdot\frac{b^{r+1}-a^{r+1}}{(b+\delta)^{r+1}-a^{r+1}}\right)^{1/r}<\frac{[b^b/a^a]^{1/(b-a)}}{[(b+\delta)^{b+\delta}/a^a]^{1/(b+\delta-a)}}. \qquad (11)
$$

The upper bound in (11) is the best possible.

At last, we will give a new open problem.

2. LEMMAS

In [40], Stolarsky defined the extended mean values $E(r,s;x,y)$ by

$$E(r,s;x,y)=\left(\frac{r}{s}\cdot\frac{y^s-x^s}{y^r-x^r}\right)^{1/(s-r)}, \qquad rs(r-s)(x-y)\neq 0; \tag{12}$$

$$E(r,0;x,y)=\left(\frac{1}{r}\cdot\frac{y^r-x^r}{\ln y-\ln x}\right)^{1/r}, \qquad r(x-y)\neq 0; \tag{13}$$

$$E(r,r;x,y)=e^{-1/r}\left(\frac{x^{x^r}}{y^{y^r}}\right)^{1/(x^r-y^r)}, \qquad r(x-y)\neq 0; \tag{14}$$

$$\begin{aligned} E(0,0;x,y)&=\sqrt{xy}, && x\neq y;\\ E(r,s;x,x)&=x, && x=y \end{aligned} \tag{15}$$

and proved that it is continuous on the domain $\{(r,s;x,y):r,s\in\mathbb{R},x,y>0\}$.

Leach and Sholander [11] showed that $E(r,s;x,y)$ are increasing with both r and s, or with both x and y. The monotonicities of E have also been studied by the authors and others in [9, 28] and [30]-[36] using different ideas and simpler methods. In [21], the logarithmic convexity of E was proved.

Most of two variable means are special cases of E, for example

$$E(0,1;x,y)=L(x,y), \tag{16}$$

$$E(r,r+1;x,y)=J_r(x,y), \quad r>0. \tag{17}$$

They are called the logarithmic mean and the one-parameter mean, respectively, and, by monotonicity of E, we have

$$J_r(x,y)>L(x,y), \quad r>0. \tag{18}$$

Recently, the concepts of mean values has been generalized in [16, 19, 22, 23], [30]-[35] and [37].

3. PROOF OF THEOREM 1

Inequality (11) can be rewritten as follows

$$\frac{b^{r+1}-a^{r+1}}{(b-a)(b^b/a^a)^{r/(b-a)}}<\frac{(b+\delta)^{r+1}-a^{r+1}}{(b+\delta-a)((b+\delta)^{b+\delta}/a^a)^{r/(b+\delta-a)}}. \tag{19}$$

Therefore, it suffices to prove the function

$$\phi(x)=\frac{x^{r+1}-a^{r+1}}{(x-a)(x^x/a^a)^{r/(x-a)}} \tag{20}$$

is increasing in $x>a>0$.

Taking logarithm and differentiating yields

$$\begin{aligned}
\frac{\mathrm{d}\ln\phi(x)}{\mathrm{d}x}&=\frac{a^{2+r}+a^{2+r}r-a^{1+r}x-a^{1+r}rx-a^2x^r+ax^{1+r}}{(a-x)^2(a^{1+r}-x^{1+r})}\\
&\quad+\frac{arx^{1+r}-a^2rx^r+ra^{2+r}\ln x+rx^{2+r}\ln x-a^{1+r}rx\ln x}{(a-x)^2(a^{1+r}-x^{1+r})}\\
&\quad+\frac{a^{1+r}r\ln(a^{-a}x^x)-arx^{1+r}\ln x-rx^{1+r}\ln(a^{-a}x^x)}{(a-x)^2(a^{1+r}-x^{1+r})}\\
&=\frac{a[(1+r)(x-a)(x^r-a^r)+r(x^{1+r}-a^{1+r})(\ln a-\ln x)]}{(a-x)^2(a^{1+r}-x^{1+r})}\\
&=\frac{a(1+r)(x^r-a^r)(\ln a-\ln x)\left(\frac{r(x^{1+r}-a^{1+r})}{(r+1)(x^r-a^r)}-\frac{x-a}{\ln x-\ln a}\right)}{(a-x)^2(a^{1+r}-x^{1+r})}\\
&=\frac{a(1+r)(x^r-a^r)(\ln a-\ln x)\left(J_r(a,x)-L(a,x)\right)}{(a-x)^2(a^{1+r}-x^{1+r})}\\
&>0.
\end{aligned}$$

Thus, the function $\ln\phi(x)$, and then $\phi(x)$, is increasing.

Here the L'Hospital rule yields

$$\lim_{r\to0^+}\left(\frac{b+\delta-a}{b-a}\cdot\frac{b^{r+1}-a^{r+1}}{(b+\delta)^{r+1}-a^{r+1}}\right)^{1/r}=\frac{[b^b/a^a]^{1/(b-a)}}{[(b+\delta)^{b+\delta}/a^a]^{1/(b+\delta-a)}}. \tag{21}$$

Hence, the upper bound in inequality (11) is the best possible. The proof is complete.

4. Open Problem

Recently, the first author and Mr. N. Towghi, by definition of integral in the sense of Riemann and other techniques, proved the following

Theorem 2 ([41]). *Let $f(x)\not\equiv0$ be a nonnegative integrable function on the closed interval $[a,b+\delta]$, where $b>a$ and $\delta>0$. Then, for any positive parameter*

$r > 0$, *we have*

$$\frac{\sup_{x\in[a,b]} f(x)}{\sup_{x\in[a,b+\delta]} f(x)} \cdot \left(\frac{\frac{1}{b-a}\int_a^b f^r(x)\,\mathrm{d}x}{\frac{1}{b+\delta-a}\int_a^{b+\delta} f^r(x)\,\mathrm{d}x}\right)^{1/r}. \tag{22}$$

Theorem 3 ([33]). *Let $f(x)$ be a positive increasing integrable function on the closed interval $[a, b+\delta]$, where $b > a$ and $\delta > 0$. Then, for any positive parameter $r > 0$, we have*

$$\left(\frac{\frac{1}{b-a}\int_a^b f^r(x)\,\mathrm{d}x}{\frac{1}{b+\delta-a}\int_a^{b+\delta} f^r(x)\,\mathrm{d}x}\right)^{1/r} \cdot \frac{\exp\left(\frac{1}{b-a}\int_a^b \ln f(x)\,\mathrm{d}x\right)}{\exp\left(\frac{1}{b+\delta-a}\int_a^{b+\delta} \ln f(x)\,\mathrm{d}x\right)}. \tag{23}$$

This almost resolved an open problem proposed in [8].

Now it is natural to propose a new open problem as follows.

Open Problem. Let $b > a > 0$ and $\delta > 0$ be real numbers, f a positive integrable function and w a nonnegative weight defined on the interval $[a, b+\delta]$. Then, for any given positive $r \in \mathbb{R}$, we have

$$\frac{\sup_{x\in[a,b]} f(x)}{\sup_{x\in[a,b+\delta]} f(x)} < \left(\frac{\int_a^b w(x)f^r(x)\,\mathrm{d}x}{\int_a^b w(x)\,\mathrm{d}x} \Bigg/ \frac{\int_a^{b+\delta} w(x)f^r(x)\,\mathrm{d}x}{\int_a^{b+\delta} w(x)\,\mathrm{d}x}\right)^{1/r}. \tag{24}$$

Further, if f is increasing, then we have the following

$$\begin{aligned} &\left(\frac{\int_a^b w(x)f^r(x)\,\mathrm{d}x}{\int_a^b w(x)\,\mathrm{d}x} \Bigg/ \frac{\int_a^{b+\delta} w(x)f^r(x)\,\mathrm{d}x}{\int_a^{b+\delta} w(x)\,\mathrm{d}x}\right)^{1/r} \\ &\qquad < \exp\left(\frac{\int_a^b w(x)\ln f(x)\,\mathrm{d}x}{\int_a^b w(x)\,\mathrm{d}x} - \frac{\int_a^{b+\delta} w(x)\ln f(x)\,\mathrm{d}x}{\int_a^{b+\delta} w(x)\,\mathrm{d}x}\right). \end{aligned} \tag{25}$$

The lower and upper bounds in (24) and (25) are best possible.

References

1. H. Alzer, On an inequality of H. Minc and L. Sathre, *J. Math. Anal. Appl.* **179** (1993), 396-402.

2. P. S. Bullen, A Dictionary of Inequalities, *Pitman Monographs and Surveys in Pure and Applied Mathematics* **97**, *Addison Wesley Longman Limited*, 1998.

3. T. H. Chan, P. Gao and F. Qi, On a generalization of Martins' inequality, *RGMIA Res. Rep. Coll.* **4** (2001), no.1, Article 12, 93-101. Available online at http://rgmia.vu.edu.au/v4n1.html.

4. S. S. Dragomir and J. van der Hoek, Some new analytic inequalities and their applications in guessing theory, *J. Math. Anal. Appl.* **225** (1998), 542-556.

5. N. Elezović and J. Pecarić, On Alzer's inequality, *J. Math. Anal. Appl.* **223** (1998), 366-369.

6. L. Galvani, Dei limiti a cui tendono alcune media, *Boll. Un. Mat. Ital.* (1927), no. 6, 173-179.

7. B. Gavrea and I. Gavrea, An inequality for linear positive functionals, *J. Inequal. Pure Appl. Math.* **1** (2000), no. 1, Article 5. Available online at http://jipam.vu.edu.au/v1n1/004_99.html.

8. B.-N. Guo and F. Qi, An algebraic inequality, *II*, *RGMIA Res. Rep. Coll.* **4** (2001), no. 1, Article 8, 55-61. Available online at http://rgmia.vu.edu.au/v4n1.html.

9. B.-N. Guo, Sh.-Q. Zhang, and F. Qi, Elementary proofs of monotonicity for extended mean values of some functions with two parameters, *Shuxue de Shijian yu Renshi* (*Mathematics in Practice and Theory*) **29** (1999), no. 2, 169-174. (Chinese)

10. J.-Ch. Kuang, Some extensions and refinements of Minc-Sathre inequality, *Math. Gaz.* **83** (1999), 123-127.

11. E. Leach and M. Sholander, Extended mean values, *Amer. Math. Monthly* **85** (1978), 84-90.

12. J. S. Martins, Arithmetic and geometric means, an applications to Lorentz sequence spaces, *Math Nachr.* **139** (1988), 281-288.

13. H. Minc and L. Sathre, Some inequalities involving $(n!)^{1/r}$, *Proc. Edinburgh Math. Soc.* **14** (1964/65), 41-46.

14. N. Ozeki, On some inequalities, *J. College Arts Sci. Chiba Univ.* **4** (1965), no. 3, 211-214. (Japanese)

15. F. Qi, An algebraic inequality, *J. Inequal. Pure Appl. Math.* **2** (2001), no. 1, Article 13. Available online at http://jipam.vu.edu.au/v2n1/006_00.html. *RGMIA Res. Rep. Coll.* **2** (1999), no. 1, Article 8, 81-83. Available online at http://rgmia.vu.edu.au/v2n1.html.

16. F. Qi, Generalized abstracted mean values, *J. Inequal. Pure Appl. Math.* **1** (2000), no. 1, Article 4. Available online at http://jipam.vu.edu.au/v1n1/013_99.html. *RGMIA Res. Rep. Coll.* **2** (1999), no. 5, Article 4, 633-642. Available online at http://rgmia.vu.edu.au/v2n5.html.

17. F. Qi, Generalizations of Alzer's and Kuang's inequality, *Tamkang J. Math.* **31** (2000), no. 3, 223-227. *RGMIA Res. Rep. Coll.* **2** (1999), no. 6, Article 12, 891-895. Available online at http://rgmia.vu.edu.au/v2n6.html.

18. F. Qi, Generalization of H. Alzer's inequality, *J. Math. Anal. Appl.* **240** (1999), 294-297.

19. F. Qi, Generalized weighted mean values with two parameters, *R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci.* **454** (1998), no. 1978, 2723-2732.

20. F. Qi, Inequalities and monotonicity of sequences involving $\sqrt[n]{(n+k)!/k!}$, *RGMIA Res. Rep. Coll.* **2** (1999), no. 5, Article 8, 685-692. Available online at http://rgmia.vu.edu.au/v2n5.html.

21. F. Qi, Logarithmic convexity of extended mean values, *Proc. Amer. Math. Soc.* **130** (2002), no.6, 1787-1796. *accepted. RGMIA Res. Rep. Coll.* **2** (1999), no. 5, Article 5, 643-652. Available online at http://rgmia.vu.edu.au/v2n5.html.

22. F. Qi, On a two-parameter family of nonhomogeneous mean values, *Tamkang J. Math.* **29** (1998), no. 2, 155-163.

23. F. Qi, Studies on Problems in Topology and Geometry and on Generalized Weighted Abstracted Mean Values, Thesis submitted for the degree of Doctor of Science at University of Science and Technology of China, Hefei City, Anhui Province, China, Winter 1998. (Chinese)

24. F. Qi and L. Debnath, On a new generalization of Alzer's inequality, *Internat. J. Math. Math. Sci.* **23** (2000), no. 12, 815-818.

25. F. Qi and B.-N. Guo, Monotonicity of sequences involving convex function and sequence, *RGMIA Res. Rep. Coll.* **3** (2000), no. 2, Article 14, 321-329. Available online at http://rgmia.vu.edu.au/v3n2.html.

26. F. Qi and B.-N. Guo, Some inequalities involving the geometric mean of natural numbers and the ratio of gamma functions, *RGMIA Res. Rep. Coll.* **4** (2001), no. 1, Article 3, 41-48. Available online at http://rgmia.vu.edu.au/v4n1.html.

27. B.-N. Guo and F. Qi, *Inequalities and monotonicity for the ratio of gamma functions*, *Taiwanese J. Math.* **7** (2003), 239-247.

28. F. Qi and Q.-M. Luo, A simple proof of monotonicity for extended mean values, *J. Math. Anal. Appl.* **224** (1998), no. 2, 356-359.

29. F. Qi and Q.-M. Luo, Generalization of H. Minc and J. Sathre's inequality, *Tamkang J. Math.* **31** (2000), no. 2, 145-148. *RGMIA Res. Rep. Coll.* **2** (1999), no. 6, Article 14, 909-912. Available online at http://rgmia.vu.edu.au/v2n6.html.

30. F. Qi and Q.-M. Luo, Refinements and extensions of an inequality, *Mathematics and Informatics Quarterly* **9** (1999), no. 1, 23-25.

31. F. Qi, J.-Q. Mei, Da-F. Xia, and S.-L. Xu, New proofs of weighted power mean inequalities and monotonicity for generalized weighted mean values, *Math. Inequal. Appl.* **3** (2000), no. 3, 377-383.

32. F. Qi, J.-Q. Mei, and S.-L. Xu, Other proofs of monotonicity for generalized weighted mean values, *RGMIA Res. Rep. Coll.* **2** (1999), no. 4, Article 6, 469-472. Available online at http://rgmia.vu.edu.au/v2n4.html.

33. F. Qi and N. Towghi, *Inequalities for the ratios of the mean values of functions*, submitted.

34. F. Qi and S.-L. Xu, Refinements and extensions of an inequality, II, *J. Math. Anal. Appl.* **211** (1997), 616-620.

35. F. Qi and S.-L. Xu, The function $(b^x - a^x)/x$: Inequalities and properties, *Proc. Amer. Math. Soc.* **126** (1998), no. 11, 3355-3359.

36. F. Qi, S.-L. Xu, and L. Debnath, A new proof of monotonicity for extended mean values, *Internat. J. Math. Math. Sci.* **22** (1999), no. 2, 415-420.

37. F. Qi and Sh.-Q. Zhang, Note on monotonicity of generalized weighted mean values, *R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci.* **455** (1999), no. 1989, 3259-3260.

38. J. Sándor, Comments on an inequality for the sum of powers of positive numbers, *RGMIA Res. Rep. Coll.* **2** (1999), no. 2, 259-261. Available online at http://rgmia.vu.edu.au/v2n2.html.

39. J. Sándor, On an inequality of Alzer, *J. Math. Anal. Appl.* **192** (1995), 1034-1035.

40. K. B. Stolarsky, Generalizations of the logarithmic mean, *Math. Mag.* **48** (1975), 87-92.

41. N. Towghi and F. Qi, An inequality for the ratios of the arithmetic means of functions with a positive parameter, *RGMIA Res. Rep. Coll.* **4** (2001), no. 2, Article 15, 305-309. Available online at http://rgmia.vu.edu.au/v4n2.html.

42. J. S. Ume, An elementary proof of H. Alzer's inequality, *Math. Japon.* **44** (1996), no. 3, 521-522.

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