TAIWANESE JOURNAL OF MATHEMATICS
Vol. 7, No. 2, pp. 195-205, June 2003
This paper is available online at http://www.math.nthu.edu.tw/tjm/

CONVERGENCE RATE OF GLIMM SCHEME FOR GENERAL SYSTEMS OF HYPERBOLIC CONSERVATION LAWS

Tong Yang

Abstract. In this paper, we consider the convergence rate of the deterministic Glimm scheme for general systems of hyperbolic conservation laws without assuming either genuine nonlinearity or linear degeneracy on the characteristic fields. It is shown that the convergence rate is $o(1)s^{\frac{1}{4}}|\ln s|$ compared to $o(1)s^{\frac{1}{2}}|\ln s|$ obtained in [3] for the case when the charateristic field is either genuinely nonlinear or linear degenerate. Here s is the mesh size in the time direction.

1. Introduction

In this paper, we consider the convergence rate of the Glimm scheme to general hyperbolic conservation laws:

$$u_t + f(u)_x = 0, \tag{1.1}$$

$$u(x,0) = u_0(x), \tag{1.2}$$

here $u = u(x,t) = (u^1(x,t), \cdots, u^n(x,t))$ and $f(u)$ are n-vectors. The analysis depends on the deterministic version of the Glimm scheme introduced in [15] when nonlinear wave propagation can be simplified locally by linear superposition and hence the detailed structure of the solution can be obtained, and the error from the exact solution in L_1 can be analyzed. This problem was solved in [3] when each of the characteristic fields is either genuinely nonlinear or linearly degenerate. Without this assumption on the characteristic fields, we obtain a convergence rate based on

Received January 24, 2003.
Communicated by S. B. Hsu.
2000 *Mathematics Subject Classification*: 35L65, 35L67.
Key words and phrases: Deterministic Glimm scheme, Convergence rate, General systems of conservation laws.
The research was supported by the Strategic Research Grant of City University of Hong Kong, # 7001330.

the estimation of variation of the wave speed times its strength which is bounded by the cubic wave interaction potential and the mesh size to some powers. This estimation was first used in [20] where the consistency of the Glimm scheme for general systems of hyperbolic conservation laws was studied. The approach here for the convergence rate is similar to [3] but with more delicate estimate on L_1 error caused by nonlinear wave interaction. The analysis in [3] depends on the L_1 stability results on the systems of hyerbolic conservation laws with characteristic fields being either genuinely nonlinear or linearly degenerate, [6, 19].

The system (1.1) is assumed to be strictly hyperbolic, that is, the eigenvalues of the $n \times n$ matrix $f'(u)$ are real and distinct:

$$
\begin{aligned}
f'(u)r_i(u) &= \lambda_i(u)r_i(u), \\
l_i(u)f'(u) &= \lambda_i(u)l_i(u), \\
l_i(u)\cdot r_j(u) &= \delta_{ij},\ i,j = 1,2,\dots,n, \\
\lambda_1(u) &< \lambda_2(u) < \cdots < \lambda_n(u).
\end{aligned}
\tag{1.3}
$$

When each characteristic field is either genuinely nonlinear or linearly degenerate, there is the classical existence theory of James Glimm, [10]. An important physical example of such a system is the Euler equations in gas dynamics. Other physical systems, such as those in elasticity and magneto-hydrodynamics, for instance, are not necessarily genuinely nonlinear or linearly degenerate. For such a general system, a characteristic field $\lambda_i(u)$ may have a linearly degenerate manifold $LG_i \equiv \{u : \nabla\lambda_i(u)\cdot r_i(u) = 0\}$ to be neither the empty space, as in the case of genuine nonlinearity, nor the whole space, as in the case of linear degeneracy. When each characteristic field $\lambda_i(u)$ has a linear degeneracy manifold LD_i either is the whole space or consists of a finite number of smooth manifolds of co-dimension one, each transversal to the characteristic vector $r_i(u)$, the global entropy solution was proved in [16, 20] if the initial data (1.2) has sufficiently small total variation. The main improvement is the introduction of a cubic wave interaction potential for interaction of waves of the same family which is the product of the strengths of the two interacting waves and their interacting angle. Since this functional is weaker than the one consists only product of the strengths of the interacting waves of the same family for genuinely nonlinear field, the proof of the consistency of the Glimm scheme relies on a detailed discussion in strong and weak waves, [20]. This technique will also be used in this paper for the study of convergence rate of the Glimm approximate solution to the weak entropy solution.

Since the characteristics $\lambda_i(u)$ depends on the variables u, one needs to consider weak solution to (1.1) because in general there is no global smooth solutions even for smooth and small initial data.

Definition 1.1. A bounded measurable function $u(x,t)$ is a weak solution of

(1.1), (1.2) if and only if

$$(1.4)\qquad \int_0^\infty \int_{-\infty}^\infty [\phi_t u + \phi_x f(u)](x,t)dxdt + \int_{-\infty}^\infty \phi(x,0)u_0(x)dx = 0$$

for any smooth function $\phi(x,t)$ of compact support in $\{(x,t)|(x,t) \in \mathbf{R}^2\}$.

This weak formulation leads to the discuss of a discontinuity (u_-, u_+), i.e. shock, in the weak solution with speed s which satisfies the Rankine-Hugoniot condition

$$(1.5)\qquad s(u_+ - u_-) = f(u_+) - f(u_-),$$

where u_- and u_+ are the left and right states of the discontinuity respectively.

For this, the Hugoniot curves $H(u_0)$ passing through a given state u_0 is introduced as follows:

$$(1.6)\qquad H(u_0) \equiv \{u : \sigma(u_0 - u) = f(u_0) - f(u)\},$$

for some scalar $\sigma = \sigma(u_0, u)$.

The Rankine-Hugoniot condition says that $u_+ \in H(u_-)$ and that $s = \sigma(u_-, u_+)$. It follows easily from the strict hyperbolicity of the system that in a small neighborhood of a given state u_0, the set $H(u_0)$ consists of n smooth curves $H_i(u_0)$, $i = 1, 2, \cdots, n$, through u_0, such that $\sigma_i(u_0, u)$ tends to $\lambda_i(u_0)$ as u moves along $H_i(u_0)$ toward u_0. Here we use the notation $\sigma_i(u_0, u)$ to denote the scalar $\sigma(u_0, u)$ in $H_i(u_0)$. A discontinuity (u_-, u_+), $u_+ \in H_i(u_-)$, is called an i-discontinuity.

In general, weak solutions to the initial value problem (1.1) and (1.2) are not unique. Certain admissibility condition, the entropy condition, needs to be imposed on the weak solution to rule out non-physical discontinuities as follows, [17].

Definition 1.2. A discontinuity (u_-, u_+) is admissible if

$$(1.7)\qquad \sigma(u_-, u_+) \cdot\ \ \sigma(u_-, u),$$

for any state u on the Hugoniot curve $H(u_-)$ between u_- and u_+.

If a characteristic field of the system (1.1) is genuinely nonlinear, [12], in the sense that

$$(1.8)\qquad \nabla\lambda_i(u) \cdot r_i(u) \neq 0. \quad (g.nl.),$$

then the entropy condition is reduced to the Lax's entropy condition

$$(1.9)\qquad \lambda_i(u_+) < \sigma_i(u_-, u_+) < \lambda_i(u_-).$$

Besides the shock wave, there are contact discontinuity and rarefaction wave as basic wave patterns for the hyperbolic system. Since the characteristic field is not genuinely nonlinear, there are also composite waves, i.e. waves consisting rarefaction waves and shocks of the same family starting from a common point in $x-t$ plane. The waves of this type in each family give the wave pattern to the solution of the Riemann problem which has two constant states as initial data, [12, 16].

Recently, there is a great progress on vanishing viscosity limit to hyperbolic conservation laws, [1]. As a consequence of the result obtained in [1], the standard Riemann semigroup S_t generated by (1.1) is unique and Lipschitz continuous in L_1. That is, for two initial data $u_0(x)$ and $v_0(x)$ with small total variations, the semigroup S_t has the following properties:

(1) S_t is Lipschitz continuous in L_1, i.e., there exists a constant L independent of t such that

$$||S_t u_0 - S_t v_0||_{L_1} \cdot \quad L \cdot ||u_0 - v_0||_{L_1}, \quad t \geq 0,$$

(2) If the initial data u_0 is piecewise constant, then for $t > 0$ sufficiently small $S_t u_0$ coincides with the solution of (1.1) and (1.2) which is obtained by piecing together the corresponding Riemann solutions.

Based on the above stability and existence results, we are going to prove the following main theorem in this paper.

Theorem 1.1. *If the total variation of the initial data $u_0(x)$ is sufficiently small and the random sequence in the Glimm scheme is equidistributed and satisfies the condition in Lemma* 2.1, *then for any time* $t \geq 0$, *we have*

$$\lim_{s \to 0} \frac{||u^g(\cdot,t) - u(\cdot,t)||_{L_1}}{s^{\frac{1}{4}}|\ln s|} = 0.$$

here $u^g(x,t)$ and $u(x,t)$ are the approximate solution in the Glimm scheme and the weak solution generated by the semigroup S_t with the same initial data respectively. And $s = \Delta t$ is the mesh size in the time direction.

Notice that here we assume that the number of the linear degenerate manifolds is finite. As for the existence result can be generalized to inifinite number of degenerate manifolds in [14], the same argument can be applied to this case, but we will not consider it here. Furthermore, the convergence rate obtained here may not be optimal and how to obtain the optimal rate is not in the scope of this paper.

The rest of the paper will be organized as follows. In the next section, we will give some prelimenary results on equidistibuted sequence, wave tracing approximation, wave interaction potential and some L_1 error estimates obtained in [3, 20] which are needed for the proof of convergence rate. In Section 3, we will give a proof of the main theorem by considering different errors in the L_1 distance between the Glimm approximate solution and the unique entropy solution by using the intermediate approximate solution with simplified wave patterns through the wave tracing method.

2. Prelimenary

In this section, we will include some estimates obtained in [3, 20] on the equidistibuted random sequence and wave tracing approximation for self-containedness.

The Glimm scheme is a finite difference scheme involving a random sequence $\theta_i,\ i = 0, 1, \cdots,\ 0 < \theta_i < 1$. Let $r = \Delta x,\ s = \Delta t$ be the mesh sizes satisfying the (C-F-L) condition

$$\frac{r}{s} > 2|\lambda_i(u)|, \quad 1 \leq i \leq n, \tag{2.1}$$

for all states u under consideration. The approximate solutions $u(x, t) = u^g(x, t)$ depends on the random sequence $\{\theta_i\}_{i=0}^{\infty}$ and is defined inductively in time as follows:

$$u^g(x, 0) = u_0((h + \theta_0)r), \quad hr < x < (h + 1)r, \tag{2.2}$$

$$\begin{aligned} u^g(x, ks) &= u^g((h + \theta_i)r - 0, ks - 0), \quad hr < x < (h + 1)r, \\ k &= 0, \pm1, \pm2, \cdots. \end{aligned} \tag{2.3}$$

Thus the approximate solution is a step function for each layer $t = ks,\ k = 1, 2, \cdots$. Between the layers it consists of elementary waves by solving the Riemann problems at grid points $x = hr,\ h = 0, \pm1, \cdots$. Due to (C-F-L) condition (2.1) these elementary waves do not interact within the layer. Please refer to [10] for more detailed explanation.

In the Glimm scheme, the random sequence is assumed to be equidistributed in the following sense.

Definition 2.1. A sequence $\{\theta_i\}_{i=0}^{\infty}$ in (0,1) is equidistributed if

$$B(N, I) \equiv |\frac{\Theta(N, I)}{N} - |I|| \to 0, \quad \text{as } N \to \infty,$$

for any subinterval I of $(0, 1)$. Here $\Theta(N, I)$ denotes the number of $i,\ 1 \leq i \leq N$, such that $\theta_i \in I$ and $|I|$ is the length of I.

In particular, we choose a random sequence $\{\theta_i\}_{i=0}^{\infty}$ having the following property, [3].

Lemma 2.1. *For every integer $r \geq 2$, there exists a sequence $\{\theta_i\}_{i=0}^{\infty}$ such that*

$$D_{m,n} \cdot \frac{2r-2}{n-m}\Big[1+\frac{\ln(n-m)}{\ln r}\Big],$$

for any $n > m \geq 1$. Here,

$$D_{m,n} = \sup_{\lambda\in[0,1]} \Big|\lambda - \frac{1}{n-m}\sum_{m\ \ l<n} \chi_{[0,\lambda]}(\theta)\Big|,$$

where χ is the characteristic function and $D_{m,n}$ represents the discrepancy of the random sequence.

The Glimm theory for systems of conservation laws is based on the study of the interactions of elementary waves in the solutions of the Riemann problems solved in [12, 16]. The random choice method, the Glimm scheme, is introduced to construct the general solutions using the Riemann solutions as building blocks. A nonlinear functional, the Glimm functional $F[u]$, is constructed to bound the total variation of the approximate solutions. The functional yields a global measure of the total wave interactions, [11], and allows for the consistency study of the wave tracing method, [16]. This functional is an effective measure of the wave interactions in that the functional decreases only due to the interaction of the waves next to each other and that the decrease is exactly of the same order of the waves produced by the interaction.

The Glimm functional $F(J)$ defined as follows was shown to be non-increasing in [16, 20].

$$F(J) \equiv L(J) + MQ(J),$$

where

$$\begin{aligned}
L(J) &\equiv \sum\{|\alpha| : \ \alpha \text{ any wave crossing } J\},\\
Q_d(J) &\equiv \sum\{|\alpha||\beta| : \ \alpha \text{ and } \beta \text{ interacting waves of distinct}\\
&\qquad\qquad \text{characteristic families crossing } J\},\\
Q_s(J) &\equiv \sum_{i=1}^{n} Q_s^i, \qquad\qquad (2.4)\\
Q_s^i &\equiv \sum\{|\alpha||\beta|(-\min\{\Theta(\alpha,\beta),0\}) : \ \alpha \text{ and } \beta\\
&\qquad\qquad \text{interacting } i-\text{waves crossing } J\},\\
Q(J) &\equiv Q_d(J) + Q_s(J).
\end{aligned}$$

Here M is a sufficiently large constant, J is any space-like curve, and $\Theta(\alpha,\beta)$, called the effective angle between waves α and β of the same family is defined as follows.

$$\Theta(\alpha,\beta) \equiv \theta_\alpha^+ + \theta_\beta^- + \sum \theta_\gamma. \tag{2.5}$$

Here θ_α^+ represents the value of λ_i at the right state of α minus its wave speed. It is negative if α is a shock and is set zero if it is a i-rarefaction wave. Similarly the term θ_β^- denotes the difference between the speed of β and the value of λ_i at its left end state. θ_γ is the value of λ_i at the right state of the wave γ minus that at the left state. It is positive if γ is a rarefaction wave and is negative if it is a shock. The sum $\sum \theta_\gamma$ is over the i-waves γ between α and β. Interacting waves α and β of different families means that the one with larger characteristic speed lies on the left of the one with smaller speed. Hence, they will interact in finite time only once.

It was also shown in [20] that the waves in a fixed time zone can be partitioned so that a simplified wave pattern with linear superposition of nonlinear waves can be used to replace the approximate solutions in the Glimm scheme with controllable error estimates. The error estimate coming from the partition in the wave tracing can be summarized in the following theorem.

Theorem 2.1. *Let ϵ be a constant with $\frac{1}{2} < \epsilon < 1$. The waves in an approximate solution in a given a time zone $\Lambda = \{(x,t) : -\infty < x < \infty, Ms \cdot\ t < (M+N)s\}$, can be partitioned into subwaves of categories I, II or III with the following properties*:

(i) *The subwaves in I are surviving. Given a subwave $\alpha(t)$, $Ms \cdot\ t < (M+N)s$, in I, write $\alpha \equiv \alpha(Ms)$ and denote by $|\alpha(t)|$ its strength at time t, by $[\sigma(\alpha)]$ the variation of its speed and by $[\alpha]$ the variation of the jump of the states across it over the time interval $Ms \cdot\ t < (M+N)s$. Then*

$$\sum_{\alpha \in I}([\alpha] + |\alpha(Ms)|[\sigma(\alpha)]) = O(1)(D(\Lambda)(Ns)^{-\epsilon} + T.V.N^{1+\epsilon}s^\epsilon + s).$$

(ii) *A subwave $\alpha(t)$ in II has positive initial strength $|\alpha(Ms)| > 0$, but is cancelled in the zone Λ, $|\alpha((M+N)s)| = 0$. Moreover, the total strength and variation of the wave speed satisfy*

$$\sum_{\alpha \in II}([\alpha] + |\alpha(t)|) = O(1)(D(\Lambda) + s), \ Ms \cdot\ t < (M+N)s,$$

$$\sum_{\alpha \in II}([\alpha] + |\alpha(Ms)|[\sigma(\alpha)]) \cdot\ 0(1)(D(\Lambda)(Ns)^{-\epsilon} + T.V.N^{1+\epsilon}s^\epsilon + s).$$

(iii) *A subwave in III has zero initial strength $|\alpha(Ms)| = 0$, and is created in the zone Λ, $|\alpha((M+N)s)$. Moreover, the total strength and variation of the* *wave speed satisfy*

$$\sum_{\alpha\in III}([\alpha]+|\alpha(t)|)=O(1)(D(\Lambda)+s),\ Ms\cdot\quad t<(M+N)s,$$

$$\sum_{\alpha\in III}([\alpha]+|\alpha(Ms)|[\sigma(\alpha)])\cdot\quad 0(1)(D(\Lambda)(Ns)^{-\epsilon}+T.V.N^{1+\epsilon}s^{\epsilon}+s).$$

Here $D(\Lambda)=F(Ms)-F((M+N)s)$, *and* $T.V.=Tot.Var.\{u_0(x)\}$. *And* $F(t)$ *is the Glimm functional on the space-like curve at time* t.

Later in the next section, the approximate solution consisting only the surving waves, i.e. waves in the category I, in each time strip, denoted by $u^l(x,t)$, will be used as the intermediate approximate solution between the one in the Glimm scheme, $u^g(x,t)$, and the unique entropy solution. The advantage of this simplified approximate solution $u^l(x,t)$ is that each wave in the corresponding time strip can be traced and the error from the exact solution can be estimated. That is, the L_1 Lipschitz continuity of the Riemann semigroup can be applied to have the convergence rate in term of s.

3. Proof of Convergence

The estimation on the convergence rate is based on the L_1 Lipschitz continuity of the semigroup S_t and the wave partition estimates in Theorem 2.1. The approach is similar to the one for genuinely nonlinear characteristic fields, cf. [3].

We now come to the proof of the main theorem on the convergence rate. As mentioned at the end of the last section, we will estimate the L_1 distance between $u^g(x,t)$ and $u^l(x,t)$, and the one between $u^l(x,t)$ and the semigroup on $u^l(x,\bar{t})$ with $\bar{t}\cdot\quad t$. Notice that besides the errors from the equidistributed random sequence and the approximate waves, like rarefaction shocks in the wave tracing argument, there are two kind of errors in estimating the L_1 distance. One kind of error comes the new waves created in the wave interaction and the disappearance of waves in wave cancellation; and another comes from the change of the wave speeds through interaction which is different from the associated propagation speeds of the discontinuities in the wave tracing approximation.

Consider the approximate solution up to a time $T=Ms$ without loss of generality. For a positive constant $\delta>>s$ which will be chosen later, we divide the time interval $[0,T]$ into some sub-intervals, $[t_i,t_{i+1}]$, $i=0,1,2,\cdots,N-1$, and rearrange them into two groups E and E^c as in [3]. Here $t_i=n_is$, $i=0,1,2,\cdots,N$ with $t_0=0$ and $t_N=T$. t_i can be defined inductively satisfying one of the following conditions:

1. If $F(t_i)-F(t_{i+1})\cdot\quad\delta$, then let n_{i+1} be the largest integer bounded by N such that $t_{i+1}-t_i\cdot\quad\delta$ and $F(t_i)-F(t_{i+1})\cdot\quad\delta$.

2. If $F(t_i) - F(t_{i+1}) > \delta$, let $n_{i+1} = n_i + 1$.

Denote the set of $0 \cdot\ \ i \cdot\ \ N-1$ satisfying the above conditions by E and E^c respectively. Since the time we consider is bounded by T and the Glimm functional is non-increasing and bounded, we know that the numbers of i in the sets E and E^c are bounded by $\frac{cT}{\delta}$ and $\frac{c}{\delta}$ respectively. Now in each time interval $[t_i, t_{i+1}]$, the L_1 distance between the approximate solution in the Glimm scheme and the one generated by the semigroup can be estimated as follows.

When $i \in E$, by the simplified wave pattern stated in Theorem 2.1, we know that error in L_1 up to the order of s, comes from either the random sequence or the wave interaction. For the wave interaction, part of the the error for surving waves is bounded by the product of the wave strength and the variation of its wave speed and the length of the time interval. And the error for the cancellation and new created waves is bounded by the change of the Glimm functional times the length of the time interval.

Following the partition estimates in Theorem 2.1, we have the following L_1 estimates on the approxiamte solutions $u^g(x,t)$ and $u^l(x,t)$ in each time strip $[t_i, t_{i+1}]$, $0 \cdot\ \ i \cdot\ \ N-1$, [3].

$$
\begin{aligned}
&||u^g(\cdot, t_{i+1}) - u^l(\cdot, t_{i+1})||_{L_1} = 0(1)(F(t_i) - F(t_{i+1}))(t_{i+1} - t_i),\\
&||u^l(\cdot, t_{i+1}) - S_{t_{i+1}-t_i} u^l(\cdot, t_i)||_{L_1} = 0(1)\{(F(t_i) - F(t_{i+1}))^{-\epsilon}\\
&+T.V.(m_{i+1} - m_i)^{1+\epsilon} s^{\epsilon} + \frac{1 + \ln(m_{i+1} - m_i)}{m_{i+1} - m_i} + s\}(t_{i+1} - t_i).
\end{aligned} \tag{3.1}
$$

By Lipschitz continuity of the semigroup S_t, if $i \in E$, then

$$
\begin{aligned}
||u^g(\cdot, t_{i+1}) - S_{t_{i+1}-t_i} u^g(\cdot, t_i)||_{L_1} \cdot\ \ & c\{(F(t_i) - F(t_{i+1})) + (F(t_i)\\
& -F(t_{i+1}))^{-\epsilon} + T.V.(m_{i+1} - m_i)^{1+\epsilon} s^{\epsilon}\\
& + \frac{1 + \ln(m_{i+1} - m_i)}{m_{i+1} - m_i} + s\}(t_{i+1} - t_i),
\end{aligned} \tag{3.2}
$$

where For $i \in E^c$, the error can be bounded by the time length as the total strength of waves in bounded.

$$
||u^g(\cdot, t_{i+1}) - S_{t_{i+1}-t_i} u^g(\cdot, t_i)||_{L_1} \cdot\ \ cs. \tag{3.3}
$$

where the constant c may depend on the time T.

By (3.2) and (3.3), the L_1 distance between $u^g(x,T)$ and $S_T u(x,0)$ can be estimated as follows.

$$\begin{aligned}
&||u^g(\cdot,T)-S_Tu(\cdot,0)||_{L_1}\\
\cdot\ &\sum_{i=0}^{N-1}||S_{T-t_{i+1}}u^g(\cdot,t_{i+1})-S_{T-t_i}u^g(\cdot,t_i)||_{L_1}\\
\cdot\ &L\sum_{i=0}^{N-1}||u^g(\cdot,t_{i+1})-S_{t_{i+1}-t_i}u^g(\cdot,t_i)||_{L_1}\\
\cdot\ &Lc\sum_{i\in E}\Big\{(F(t_i)-F(t_{i+1}))+(F(t_i)-F(t_{i+1}))^{-\epsilon}\\
&+T.V.(m_{i+1}-m_i)^{1+\epsilon}s^\epsilon+\frac{1+\ln(m_{i+1}-m_i)}{m_{i+1}-m_i}\Big\}(t_{i+1}-t_i)+Lc\sum_{i\in E^c}s\\
\cdot\ &c\{\delta^{1-\epsilon}+\tfrac{\delta^{1+\epsilon}}{s}+\frac{s}{\delta}(1+|\ln\frac{\delta}{s}|)+s+\frac{s}{\delta}\}.
\end{aligned}$$

Now we choose $\epsilon=\frac{2}{3}\in(\frac{1}{2},1)$ and $\delta=s^{\frac{3}{4}}\ln|\ln s|$ with $0<s<<1$, we have

$$\begin{aligned}
&||u^g(\cdot,T)-S_Tu(\cdot,0)||_{L_1}\\
\cdot\ &c\{s^{\frac{1}{4}}(\ln|\ln s|)^{\frac{5}{3}}+\frac{s^{\frac{1}{4}}}{\ln|\ln s|}(1+|\ln s|)\}=o(1)s^{\frac{1}{4}}|\ln s|.
\end{aligned}$$

And this completes the proof of the main theorem.

REFERENCES

1. S. Bianchini and A. Bressan, Vanishing viscosity solutions of nonlinear hyperbolic systems. (preprint, 2001)
2. A. Bressan, Hyperbolic systems of conservation laws. The one-dimensional Cauchy problem. *Oxford Lecture Series in Mathematics and its Applications*, **20** (2000).
3. A. Bressan, A. Marson, Error bounds for a deterministic version of the Glimm scheme, *Arch. Rational Mech. Anal.* **142** (1998), no. 2, 155-176.
4. A. Bressan and R. M. Colombo, The semigroup generated by 2×2 conservation laws, *Arch. Rational Mech. Anal.* **133** (1995), 1-75.
5. A. Bressan, G. Crasta, and B. Piccoli, Well posedness of the Cauchy problem for $n\times n$ systems of conservation laws, *Memoir Amer. math. Soc.* **694** (2000).
6. A. Bressan, T.-P. Liu, and T. Yang, L_1 stability estimates for $n\times n$ conservation laws, *Arch. Ration. Mech. Anal.* **149** (1999), no. 1, 1-22.
7. I.-L. Chern, Local and global interaction for nongenuinely nonlinear hyperbolic conservation laws, *Indiana Univ. Math. J.* **49** (2000), no. 3, 1199-1227.

8. R. Courant and K. O. Friedrichs, "*Supersonic flow and shock waves*", Springer-Verlag, 1948.

9. C. M. Dafermos, Polygonal approximations of solutions of the initial value problem for a conservation law, *J. Math. Anal. Appl.* **38** (1972), 33-41.

10. J. Glimm, Solutions in the large for nonlinear hyperbolic systems of equations, *Comm. Pure Appl. Math.* **18** (1965), 697-715.

11. J. Glimm and P. D. Lax, Decay of solutions of systems of hyperbolic conservation laws, *Memoirs Amer. Math. Soc.* **101** (1970).

12. P. D. Lax, Hyperbolic systems of conservation laws II, *Comm. Pure Appl. Math.* **10** (1957), 537-566.

13. P. D. Lax, Shock waves and entropy, Contribution to Nonlinear Functional Analysis, ed. E. Zarantonello, *Academic Press, N.Y.* (1971), 603-634.

14. P. LeFloch and T. Iguchi, Existence theory for hyperbolic systems of conservation laws with general flux-functions, to appear in the Archive for Rational Mechanics and Analysis.

15. T.-P. Liu, The deterministic version of the Glimm scheme, *Comm. Math. Phys.* **57** (1977), 135-148.

16. T.-P. Liu, Admissible solutions of hyperbolic conservation laws, *Memoirs of Amer. Meth. Soc.*, Vol 30, **240** (1981).

17. T.-P. Liu, The entropy condition and the admissibility of shocks, *J. Math. Anal. Appl.* **53** (1976), 78-88.

18. T.-P. Liu and T. Yang, L_1 stability for 2×2 systems of hyperbolic conservation laws, *J. Amer. Math. Soc.* **12** (1999), no. 3, 729-774.

19. T.-P. Liu and T. Yang, Well-posedness theory for hyperbolic conservation laws, *Comm. Pure Appl. Math.* **52** (1999), no. 12, 1553-1586.

20. T.-P. Liu and T. Yang, Weak solutions of general systems of hyperbolic conservation laws, *Commun. Math. Phys.* **230** (2002), 289-327.

21. B. J. Lucier, Error bounds for the motheds of Glimm, Godunov and LeVeque, *SIAM J. Numer. Anal.* **22** (1985), 1074-1081.

22. O. Oleinik, Uniqueness and stability of the generalized solution of the Cauchy problem for a quasilinear equation, *Usp. Mat. Nauk* **14** (1959), 165-170, *Amer. Math. Soc. Transl.* Ser. 2, **33** (1964), 285-290.

23. H. L. Keng and W. Yuan, Applications of number theory to numerical analysis, Springer-Verlag, 1981.

Tong Yang
Department of Mathematics,
City University of Hong Kong,
Hong Kong