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THE REAL PART OF AN OUTER FUNCTION AND A HELSON-SZEGÖ WEIGHT

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Dedicated to Professor Kôzô Yabuta on the occasion of his sixtieth birthday

Abstract. Suppose F is a nonzero function in the Hardy space H^1. We study the set $\{f; f$ is outer and $|F| \leq \mathrm{Re}\ f$ a.e. on $\partial\mathbb{D}\}$, where $\partial\mathbb{D}$ is the unit circle. When F is a strongly outer function in H^1 and γ is a positive constant, we describe the set $\{f; f$ is outer, $|F| \leq \gamma\ \mathrm{Re}\ f$ and $|F^{-1}| \leq \gamma\ \mathrm{Re}\ (f^{-1})$ a.e. on $\partial\mathbb{D}\}$. Suppose W is a Helson-Szegö weight. As an application, we parametrize real-valued functions v in $L^\infty(\partial\mathbb{D})$ such that the difference between $\log W$ and the harmonic conjugate function $\tilde{v}$ of v belongs to $L^\infty(\partial\mathbb{D})$ and $\|v\|_\infty$ is strictly less than $\pi/2$ using a contractive function α in H^∞ such that $(1+\alpha)/(1-\alpha)$ is equal to the Herglotz integral of W.

1. Introduction

Let $\mathbb{D}$ be the open unit disc in the complex plane and let $\partial\mathbb{D}$ be the boundary of $\mathbb{D}$. An analytic function f on $\mathbb{D}$ is said to be of class N if the integrals

$$\int_{-\pi}^{\pi} \log^+ |f(re^{i\theta})| d\theta$$

are bounded for $r < 1$. If f is in N, then $f(e^{i\theta})$, which we define to be $\lim_{r\to 1} f(re^{i\theta})$, exists almost everywhere on $\partial\mathbb{D}$. If

$$\lim_{r\to 1} \int_{-\pi}^{\pi} \log^+ |f(re^{i\theta})| d\theta = \int_{-\pi}^{\pi} \log^+ |f(e^{i\theta})| d\theta,$$

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then f is said to be of class N_+. The set of all boundary functions in N or N_+ is denoted by N or N_+, respectively. For $0 < p \leq \infty$, the Hardy space H^p is defined by $N_+ \cap L^p$. Hence any function in H^p has an analytic extension to $\mathbb{D}$.

A function h in N_+ is called *outer* if h is invertible in N_+. A function g in H^1 is called *strongly outer* if the only functions $f \in H^1$ such that $f/g \geq 0$ a.e. on $\partial\mathbb{D}$ are scalar multiples of g. If g is strongly outer then it is outer. Suppose F is a nonzero function in H^1. Define α by

$$\frac{1+\alpha(z)}{1-\alpha(z)} = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}|F(e^{i\theta})|d\theta \quad (z \in \mathbb{D}).$$

The right-hand side is the Herglotz integral of $|F|$. Then α is a contractive function in H^∞. Let $f_0 = (1+\alpha)/(1-\alpha)$. Then Re $f_0(z) > 0$ $(z \in \mathbb{D})$,

$$|F| = \text{Re } f_0 = \frac{1-|\alpha|^2}{|1-\alpha|^2} \quad \text{a.e. on } \partial\mathbb{D},$$

and $f_0 \in \bigcap_{p<1} H^p$ by a theorem of Kolmogorov (c.f. [1, Theorem 4.2]). Since Re $f_0(z) > 0$, $f_0 = c\ e^{\tilde{v}-iv}$, where c is a positive constant, $\|v\|_\infty \leq \frac{\pi}{2}$ and $\tilde{v}$ is a harmonic conjugate function of v satisfying $\tilde{v}(0) = 0$. By a theorem of Kolmogorov, $\tilde{v} - iv \in \bigcap_{p<\infty} H^p$,

$$|F| = e^{u+\tilde{v}} \quad \text{and} \quad e^u = c\ \cos v \quad \text{a.e. on } \partial\mathbb{D},$$

where u is a real-valued function. In Section 2, when F is strongly outer we study an outer function f in N_+ such that $|F| \leq$ Re f a.e. on $\partial\mathbb{D}$. We then show that $|F| \leq \gamma$ Re F if and only if α^2 is a γ-Stolz function, where γ is a positive constant. If β is a contractive function in H^∞ and $|1-\beta| \leq \gamma(1-|\beta|)$ a.e. on $\partial\mathbb{D}$, then we call β a *γ-Stolz function*. Suppose W is a Helson-Szegö weight (cf. [3]). In Section 3, using Theorem 1 in Section 2, we parametrize real-valued functions v such that $\log W - \tilde{v} \in L^\infty$ and $\|v\|_\infty < \pi/2$.

2. The Real Part of an Outer Function

In this section, we study the inequality : $|F| \leq \gamma$ Re F a.e. on $\partial\mathbb{D}$ when F is a nonzero function in H^1. The first author [4] studied the inequality : $|F| \leq \gamma$ Re f a.e. on $\partial\mathbb{D}$ when F is strongly outer and f is outer in N_+. We give necessary and sufficient conditions of this inequality. We study two inequalities : $|F| \leq \gamma$ Re f and $|F^{-1}| \leq \gamma$ Re (f^{-1}) a.e. on $\partial\mathbb{D}$ when F is strongly outer and f is in N_+. Results in this section will be used in the later section.

Proposition 1. *Suppose F is a nonzero function in H^1 and γ is a constant satisfying $\gamma \geq 1$. Then the following $(1) \sim (3)$ are equivalent*:

(1) $|F| \leq \gamma$ Re F *a.e. on* $\partial\mathbb{D}$.

(2) $F = (1+\alpha)/(1-\alpha)$ *a.e. on* $\partial\mathbb{D}$ *for a contractive function* α *in* H^∞ *such that* α^2 *is a* γ*-Stolz function.*

(3) $F = c\ e^{\tilde{v}-iv}$ *a.e. on* $\partial\mathbb{D}$, *where* c *is a positive constant and* v *is a real function in* L^∞ *satisfying* $\|v\|_\infty \leq \cos^{-1}(1/\gamma) < \pi/2$.

Proof. (1) $\Leftrightarrow$ (2): Since $F \in H^1$ and Re $F \geq 0$ a.e. on $\partial\mathbb{D}$, it follows that

$$\text{Re } F(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-|z|^2}{|e^{i\theta}-z|^2}\text{Re } F(e^{i\theta})d\theta \geq 0 \quad (z \in \mathbb{D}).$$

Hence $F = (1+\alpha)/(1-\alpha)$ for a contractive function α in H^∞. Since $|F| \leq \gamma$ Re F a.e. on $\partial\mathbb{D}$,

$$\left|\frac{1+\alpha}{1-\alpha}\right| \leq \gamma \text{ Re}\left(\frac{1+\alpha}{1-\alpha}\right) = \gamma\ \frac{1-|\alpha|^2}{|1-\alpha|^2} \quad \text{a.e. on } \partial\mathbb{D}.$$

Hence $|1-\alpha^2| \leq \gamma(1-|\alpha|^2)$ and so α^2 is a γ-Stolz function. The converse is clear.

(2) $\Rightarrow$ (3): Since $\|\alpha\|_\infty \leq 1$, Re $F = \frac{1-|\alpha|^2}{|1-\alpha|^2} \geq 0$ a.e. on $\partial\mathbb{D}$. Since $F \in H^1$, this implies that Re $F(z) \geq 0 \quad (z \in \mathbb{D})$. Hence $F = c\ e^{\tilde{v}-iv}$ and $|v| \leq \pi/2$ a.e. on $\partial\mathbb{D}$. Since α^2 is a γ-Stolz function, it follows that

$$|F| = \left|\frac{1+\alpha}{1-\alpha}\right| = \frac{|1-\alpha^2|}{|1-\alpha|^2} \leq \gamma\frac{1-|\alpha|^2}{|1-\alpha|^2} = \gamma \text{ Re } F \quad \text{a.e. on } \partial\mathbb{D}.$$

Hence $1 \leq \gamma\ \cos\ v$. Since $|v| \leq \pi/2$, this implies that $\|v\|_\infty \leq \cos^{-1}(1/\gamma) < \pi/2$.

(3) $\Rightarrow$ (1): By (3), $|F| = c\ e^{\tilde{v}} \leq \gamma c\ e^{\tilde{v}} \cos\ v = \gamma$ Re F. This implies (1). ∎

By Proposition 1 (3) and [2, Corollary III. 2.6], if $|F| \leq \gamma$ Re F a.e. on $\partial\mathbb{D}$ then both F and F^{-1} belong to H^p for some $p > 1$.

Proposition 2. *Suppose* F *is a strongly outer function in* H^1. *Define* α *by*

$$\frac{1+\alpha(z)}{1-\alpha(z)} = \frac{1}{2\pi}\int_0^{2\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}|F(e^{i\theta})|d\theta \quad (z \in \mathbb{D}).$$

For f *in* N_+, (1) $\sim$ (3) *are equivalent*:

(1) $|F| \leq$ Re f *a.e. on* $\partial\mathbb{D}$ *and* f *is an outer function.*

(2) $f = [(1+\alpha)/(1-\alpha)] + [(1+\beta)/(1-\beta)]$ *a.e. on* $\partial\mathbb{D}$ *for some contractive function* β *in* H^∞.

(3) $|F| = e^{u+\tilde{v}}$, $|v| < \pi/2$, $e^u \leq c\ \cos v$ *and* $f = c\ e^{\tilde{v}-iv}$ *a.e. on* $\partial\mathbb{D}$ *where* c *is a positive constant and* u *and* v *are real functions.*

The following proof is similar to the one of Theorem 6 in the first author's paper [4].

Proof. (1) $\Rightarrow$ (3): Let Arg f denote the argument of f restricted to $-\pi <$ Arg $f \leq \pi$. Let $v = -$Arg f. Then $|v| \leq \pi$ and $f = |f|e^{-iv}$. Since $0 < |F| \leq$ Re f, $|v| < \pi/2$. By the proof of [2, Lemma IV. 5.4], if $|v| \leq \pi/2$ then $e^{\tilde{v}} \cos v \in L^1$. Let $g = e^{iv - \tilde{v}}$. Then $fg = |f|e^{-\tilde{v}} > 0$. Since f is outer, $F/fg \in N_+$. Since

$$\left|\frac{F}{fg}\right| \leq \frac{\text{Re } f}{|fg|} = \frac{\cos v}{|g|} = e^{\tilde{v}} \cos v \in L^1,$$

it follows that $F/fg \in H^1$. Since F is strongly outer, F/fg is a scalar multiple of F. Hence $fg = c$ for some positive constant c. Hence $f = c\ e^{\tilde{v} - iv}$, and hence $|F| \leq c\ e^{\tilde{v}} \cos v$. Define u by $|F| = e^{u + \tilde{v}}$. Then $e^u \leq c\ \cos v$. This implies (3).
(3) $\Rightarrow$ (2): In the following we do not assume that F is strongly outer. We assume that F is a nonzero function in H^1. By (3), $|F| \leq$ Re f and Re $f \in L^1$. Let $(\tilde{v} - iv)(z)$ denote the Poisson transform of $(\tilde{v} - iv)(e^{i\theta})$. Let $g(z) = c\ e^{(\tilde{v} - iv)(z)}$. Then Re $g(z) \geq 0$ $(z \in \mathbb{D})$, $\lim_{r \to 1} g(re^{i\theta}) = f(e^{i\theta})$ a.e. on $\partial\mathbb{D}$, and

$$\sup_{0 \leq r < 1} \frac{1}{2\pi} \int_0^{2\pi} \text{Re } g(re^{i\theta}) d\theta = \text{Re } g(0) < \infty.$$

Hence

$$\begin{aligned} \text{Re } g(z) &\geq \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - |z|^2}{|e^{i\theta} - z|^2} \text{Re } f(e^{i\theta}) d\theta \\ &\geq \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - |z|^2}{|e^{i\theta} - z|^2} |F(e^{i\theta})| d\theta = \text{Re} \left(\frac{1 + \alpha(z)}{1 - \alpha(z)} \right) \quad (z \in \mathbb{D}). \end{aligned}$$

Hence there exists a contractive function β in H^∞ such that

$$g(z) = \frac{1 + \alpha(z)}{1 - \alpha(z)} + \frac{1 + \beta(z)}{1 - \beta(z)} \quad (z \in \mathbb{D}).$$

Since $\lim_{r \to 1} g(re^{i\theta}) = f(e^{i\theta})$ a.e. on $\partial\mathbb{D}$, this implies (2).
(2) $\Rightarrow$ (1): Since $|\beta| \leq 1$, Re $(1 + \beta)/(1 - \beta) \geq 0$. Hence

$$|F| = \text{Re } \frac{1 + \alpha}{1 - \alpha} \leq \text{Re} \left(\frac{1 + \alpha}{1 - \alpha} + \frac{1 + \beta}{1 - \beta} \right) = \text{Re } f \quad \text{a.e. on } \partial\mathbb{D}.$$

This implies (1). ∎

By Proposition 2 (3) and [2, Corollary III. 2.6], if $|F| \le \text{Re } f$ a.e. on $\partial\mathbb{D}$ and f is an outer function then both f and f^{-1} belong to H^p for all $p < 1$.

By (1), the set of all functions f satisfying one of the conditions (1) $\sim$ (3) is a convex subset of N_+. If F is a nonzero function in H^1, then (3) $\Rightarrow$ (2) $\Rightarrow$ (1) holds in Proposition 2. But by [4, Theorem 6], (1) $\Rightarrow$ (3) does not hold in general.

Theorem 1. *Suppose F is a strongly outer function in H^1. Define α by*

$$\frac{1+\alpha(z)}{1-\alpha(z)} = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}|F(e^{i\theta})|d\theta \quad (z \in \mathbb{D}).$$

For f in N_+, (1) $\sim$ (4) are equivalent. ($\gamma_1, \ldots, \gamma_5$ are positive appropriate constants.)

(1) $|F| \le \gamma_1 \text{ Re } f$ *and* $|F^{-1}| \le \gamma_1 \text{ Re}(f^{-1})$ *a.e. on* $\partial\mathbb{D}$.

(2) $(1/\gamma_2)\text{Re } f \le |F| \le \gamma_2 \text{ Re } f$ *and* $|f| \le \gamma_2 \text{Re } f$ *a.e. on* $\partial\mathbb{D}$ *and f is in* H^1.

(3) *There exists a contractive function β in H^∞ such that*

$$\gamma_3 f = \frac{1-\alpha\beta}{(1-\alpha)(1-\beta)} \quad \textit{and} \quad \frac{|1-\alpha\beta|}{|1-\alpha|\cdot|1-\beta|} \le \gamma_4 \frac{1-|\alpha|^2}{|1-\alpha|^2} \quad \textit{a.e. on } \partial\mathbb{D}.$$

(4) *There exists a constant $c > 0$ and real functions u, v in L^∞ such that*

$$|F| = e^{u+\tilde{v}}, \quad \|v\|_\infty \le \cos^{-1}\gamma_5 < \frac{\pi}{2} \quad \textit{and} \quad f = c\, e^{\tilde{v}-iv} \quad \textit{a.e. on } \partial\mathbb{D}.$$

Proof. (1) $\Rightarrow$ (2): By (1),

$$(\text{Re } f)^2 \le |f|^2 \le \gamma_1(\text{Re } f)|F| \le \gamma_1^2(\text{Re } f)^2.$$

Hence $|f| \le \gamma_1 \text{ Re } f \le \gamma_1^2|F| \in L^1$. This implies (2) with $\gamma_2 = \gamma_1$.
(2) $\Rightarrow$ (1): By (2),

$$\frac{1}{|F|} \le \gamma_2 \frac{1}{\text{Re } f} \le \gamma_2^3 \frac{\text{Re } f}{|f|^2} = \gamma_2^3 \text{ Re } \frac{1}{f}.$$

This implies (1) with $\gamma_1 = \gamma_2^3$.
(2) $\Rightarrow$ (3): Since $f \in H^1$ and $\text{Re } f \ge 0$ a.e. on $\partial\mathbb{D}$, $\text{Re } f(z) > 0$ $(z \in \mathbb{D})$. Hence f is an outer function. Since $|F| \le \gamma_2 \text{ Re } f$, by Proposition 2,

$$\gamma_2 f = \frac{1+\alpha}{1-\alpha} + \frac{1+\beta}{1-\beta} = \frac{2(1-\alpha\beta)}{(1-\alpha)(1-\beta)}$$

for some contractive function β in H^∞. Since $|f| \le \gamma_2 \text{ Re } f \le \gamma_2^2|F|$,

$$\frac{2|1-\alpha\beta|}{|1-\alpha|\cdot|1-\beta|} = \left|\frac{1+\alpha}{1-\alpha} + \frac{1+\beta}{1-\beta}\right| = \gamma_2|f| \le \gamma_2^3|F| = \gamma_2^3\frac{1-|\alpha|^2}{|1-\alpha|^2}$$

This implies (3) with $\gamma_3 = \gamma_2/2$ and $\gamma_4 = \gamma_2^3/2$.
(3) $\Rightarrow$ (4): By (3), f is outer, since α and β are contractive. Since

$$|F| = \text{Re}\left(\frac{1+\alpha}{1-\alpha}\right) \leq 2\gamma_3 \text{ Re } f,$$

by Proposition 2, $|F| = e^{u+\tilde{v}}$, $|v| < \pi/2$, $e^u \leq c_0 \ \cos v$ and $2\gamma_3 f = c_0 e^{\tilde{v}-iv}$, where c_0 is a positive constant and u, v are real functions. Hence

$$\begin{aligned} c_0 e^{\tilde{v}} = 2\gamma_3|f| &= \frac{2|1-\alpha\beta|}{|1-\alpha|\cdot|1-\beta|} \\ &\leq 2\gamma_4 \frac{1-|\alpha|^2}{|1-\alpha|^2} = 2\gamma_4|F| = 2\gamma_4 e^{u+\tilde{v}} \\ &\leq 2c_0\gamma_4\, e^{\tilde{v}} \cos v \leq 2c_0\gamma_4\, e^{\tilde{v}}. \end{aligned}$$

Hence $c_0/2\gamma_4 \leq e^u \leq c_0$ and $\cos v \geq 1/2\gamma_4 > 0$. Hence $u, v \in L^\infty$ and $\|v\|_\infty \leq \cos^{-1}(1/2\gamma_4) < \pi/2$. This implies (4) with $c = c_0/2\gamma_3$ and $\gamma_5 = 1/2\gamma_4$.
(4) $\Rightarrow$ (1): Since $\cos v \geq \gamma_5$,

$$|F| = e^{u+\tilde{v}} \leq \frac{1}{\gamma_5} e^{\|u\|_\infty} e^{\tilde{v}} \cos v = \frac{1}{c\gamma_5} e^{\|u\|_\infty} \text{Re}\, f,$$

and

$$\frac{1}{|F|} = e^{-u-\tilde{v}} \leq \frac{c}{\gamma_5} e^{\|u\|_\infty} e^{-\tilde{v}} \cos v = \frac{c}{\gamma_5} e^{\|u\|_\infty} \text{Re} \frac{1}{f}.$$

This implies (1) with $\gamma_1 = (1/\gamma_5)\max(c, c^{-1})e^{\|u\|_\infty}$. ∎

By Theorem 1 (2), the set of all functions f satisfying one of the conditions (1) $\sim$ (4) is a convex subset of H^1.

3. Helson-Szegö Weight

Let W be a positive function in L^1 and $\log W$ be in L^1. For each $\varepsilon > 0$, put

$$\mathcal{E}_{W,\varepsilon} = \left\{ v \in \text{Re } L^\infty; \quad \log W - \tilde{v} \in L^\infty \quad \text{and} \quad \|v\|_\infty \leq \frac{\pi}{2} - \varepsilon \right\}$$

and $\mathcal{E}_W = \bigcup_{\varepsilon>0} \mathcal{E}_{W,\varepsilon}$. $\mathcal{E}_{W,\varepsilon}$ and $\mathcal{E}_W$ are convex subsets of Re L^∞. When $\mathcal{E}_W$ is nonempty, W is called a *Helson-Szegö weight*. Then for each v in $\mathcal{E}_W$ there exists a $u \in$ Re L^∞ such that $\log\ W = u + \tilde{v}$. In this section, we study two problems about a Helson-Szegö weight. In Theorem 2 we describe $\mathcal{E}_W$. Theorem 3 follows from Theorem 2 immediately.

Theorem 2. *Let W be a positive function in L^1. Define α by*

$$\frac{1+\alpha(z)}{1-\alpha(z)} = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} W(e^{i\theta})d\theta \quad (z \in \mathbb{D}).$$

Then v belongs to $\mathcal{E}_W$ if and only if

$$v = -\mathrm{Arg}\frac{1-\alpha\beta}{(1-\alpha)(1-\beta)} \quad a.e. \ on \ \partial\mathbb{D},$$

where β is a contractive function in H^∞ satisfying

$$\frac{|1-\alpha\beta|}{|1-\alpha|\cdot|1-\beta|} \le \gamma \ \frac{1-|\alpha|^2}{|1-\alpha|^2} \quad a.e. \ on \ \partial\mathbb{D}$$

for some constant $\gamma > 0$.

Proof. If $v \in \mathcal{E}_W$, then $v \in \mathcal{E}_{W,\varepsilon}$ for some constant $\varepsilon > 0$. Hence $W = e^{u+\tilde{v}}$ where $u \in L^\infty$ and $\|v\|_\infty \le (\pi/2) - \varepsilon$. Hence there exists a constant $\gamma > 0$ such that

$$W \le \gamma \ e^{\tilde{v}} \cos v \quad \text{and} \quad W^{-1} \le \gamma \ e^{-\tilde{v}} \cos v,$$

where $e^{\|u\|_\infty} \le \gamma \cos v$. If $f = e^{\tilde{v}-iv}$, then $W \le \gamma \ \mathrm{Re} \ f$, $W^{-1} \le \gamma \ \mathrm{Re} \ (f^{-1})$ and $f \in H^1$. Since $W, W^{-1} \in L^1$, there exists an outer function F such that $|F| = W$ and $F, F^{-1} \in H^1$. Hence F is strongly outer. By Theorem 1, there exist constants $\gamma_3, \gamma_4 > 0$ and a contractive function $\beta \in H^\infty$ such that

$$\gamma_3 f = \frac{1-\alpha\beta}{(1-\alpha)(1-\beta)} \quad \text{and} \quad \frac{|1-\alpha\beta|}{|1-\alpha|\cdot|1-\beta|} \le \gamma_4 \frac{1-|\alpha|^2}{|1-\alpha|^2} \quad \text{a.e. on } \partial\mathbb{D}.$$

Hence

$$v = -\mathrm{Arg} \ f = -\mathrm{Arg}\frac{1-\alpha\beta}{(1-\alpha)(1-\beta)} \quad \text{a.e. on } \partial\mathbb{D}.$$

This implies the 'only if' part. Conversely, suppose v satisfies the condition. Define f by

$$f = \frac{1-\alpha\beta}{(1-\alpha)(1-\beta)}.$$

Then

$$v = -\mathrm{Arg} \ f \quad \text{and} \quad |f| \le \gamma\frac{1-|\alpha|^2}{|1-\alpha|^2} \quad \text{a.e. on } \partial\mathbb{D}$$

for some constant $\gamma > 0$. Then f satisfies (3) of Theorem 1 and

$$W = \frac{1-|\alpha|^2}{|1-\alpha|^2} \le \frac{1-|\alpha|^2}{|1-\alpha|^2} + \frac{1-|\beta|^2}{|1-\beta|^2} = 2 \ \mathrm{Re} f \le 2|f| \le 2\gamma\frac{1-|\alpha|^2}{|1-\alpha|^2} = 2\gamma W.$$

Since W is a positive function in L^1, Re $f \geq 0$ a.e. on $\partial\mathbb{D}$ and $f \in H^1$. Hence f is strongly outer. Since $\log W \in L^1$, there exists an outer function $F \in H^1$ such that $|F| = W$. Let k be any function satisfying $k \in H^1$ and $k/F \geq 0$ a.e. on $\partial\mathbb{D}$. Since $f/F \in H^\infty$, $kf/F \in H^1$. Since f is strongly outer, $kf/F = cf$ for some constant c. Hence $k = cF$. Therefore F is strongly outer. By Theorem 1, there exists a constant $c > 0$ and real functions $u, v_0 \in L^\infty$ such that $\|v_0\|_\infty < \pi/2$, $W = e^{u+\tilde{v}_0}$ and $f = c\, e^{\tilde{v}_0 - iv_0}$ a.e. on $\partial\mathbb{D}$. Hence

$$v_0 = -\text{Arg } f = -\text{Arg}\frac{1-\alpha\beta}{(1-\alpha)(1-\beta)} = v.$$

Hence $W = e^{u+\tilde{v}}$ a.e. on ∂D and $\|v\|_\infty < \pi/2$. Hence v belongs to $\mathcal{E}_W$. ■

By Theorem 2, if $W = 1$ then $\alpha = 0$ and hence

$$\begin{aligned}\mathcal{E}_1 &= \left\{ v \in \text{Re } L^\infty; \quad \|v\|_\infty < \frac{\pi}{2} \quad \text{and} \quad \tilde{v} \in L^\infty \right\} \\ &= \left\{ -\text{Arg}\frac{1}{1-\beta};\ \beta \in H^\infty, \quad \|\beta\| \leq 1 \quad \text{and} \quad \frac{1}{1-\beta} \in L^\infty \right\}.\end{aligned}$$

Theorem 3. *Let W be a positive function in L^1. Define α by*

$$\frac{1+\alpha(z)}{1-\alpha(z)} = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} W(e^{i\theta})d\theta \quad (z \in \mathbb{D}).$$

(1) *W is a Helson-Szegö weight, that is, $\mathcal{E}_W \neq \emptyset$ if and only if there exists a constant $\gamma > 0$ and a contractive function β in H^∞ such that*

$$\frac{|1-\alpha\beta|}{|1-\alpha|\cdot|1-\beta|} \leq \gamma\frac{1-|\alpha|^2}{|1-\alpha|^2} \quad a.e.\ on\ \partial\mathbb{D}.$$

(2) *If α is a Stolz function, then W is a Helson-Szegö weight, and W^{-1} belongs to L^∞.*

Proof. By Theorem 2, (1) follows immediately. By Theorem 2 with $\beta = 0$, if α is a Stolz function, then $v = -\text{Arg}\,(1-\alpha)^{-1}$ belongs to $\mathcal{E}_W$, and hence $\mathcal{E}_W \neq \emptyset$. By (1), W is a Helson-Szegö weight. Since $W = (1-|\alpha|^2)/|1-\alpha|^2 = [(1+|\alpha|)/|1-\alpha|][(1-|\alpha|)/|1-\alpha|]$ a.e. on $\partial\mathbb{D}$ and α is a Stolz function, it follows that $W^{-1} \in L^\infty$. ■

Note that if α is a Stolz function, then α^2 is also a Stolz function. In fact, if α is a γ-Stolz function, then $|\alpha| \leq 1$ and

$$|1-\alpha^2| \leq |1-\alpha| + |\alpha(1-\alpha)| \leq 2|1-\alpha| \leq 2\gamma(1-|\alpha|) \leq 2\gamma(1-|\alpha|^2).$$

Let W be a positive function in L^1. By Proposition 1, $W = c\, e^{\tilde{v}}$ for a constant $c > 0$ and a real function v with $\|v\|_\infty < \pi/2$ if and only if there exists an $\alpha \in H^\infty$ such that α^2 is a Stolz function and $W = |1+\alpha|/|1-\alpha|$. Then there exists a $u \in$ Re L^∞ such that

$$W = \frac{|1-\alpha^2|}{1-|\alpha|^2}\,\frac{1-|\alpha|^2}{|1-\alpha|^2} = e^u \frac{1-|\alpha|^2}{|1-\alpha|^2} = e^u \,\mathrm{Re}\ F,$$

where $F = \frac{1+\alpha}{1-\alpha}$.

4. Remark

Put $B_r = \{\beta \in H^\infty; \|\beta\|_\infty \le r\}$ and put

$$B^\alpha = \left\{\beta \in B_1; \frac{|1-\alpha\beta|}{|1-\alpha|\cdot|1-\beta|} \le \gamma \frac{1-|\alpha|^2}{|1-\alpha|^2} \text{ a.e. on } \partial\mathbb{D} \quad \text{for some constant } \gamma > 0\right\},$$

where α is a contractive function in H^∞. The set B^α was important in Theorems 1, 2 and 3. Let W be a Helson-Szegö weight. Define α by

$$\frac{1+\alpha(z)}{1-\alpha(z)} = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} W(e^{i\theta})d\theta.$$

Then by Theorem 2,

$$\mathcal{E}_W = \left\{v = -\mathrm{Arg}\frac{1-\alpha\beta}{(1-\alpha)(1-\beta)};\ \beta \in B^\alpha\right\}.$$

If $W = 1$ then $\alpha = 0$ and

$$\mathcal{E}_1 = \left\{-\mathrm{Arg}\frac{1}{1-\beta};\ \beta \in B^0\right\}.$$

In this section, we study such a set B^α. α is a Stolz function if and only if $0 \in B^\alpha$. α^2 is a Stolz function if and only if $\alpha \in B^\alpha$. Hence if $0 \in B^\alpha$ then $\alpha \in B^\alpha$. If α is a Stolz function and $\beta \in B_r$, $r < 1$, then for some constant $\gamma > 0$

$$\frac{|1-\alpha\beta|}{|1-\alpha|\cdot|1-\beta|} \le \frac{2}{(1-r)|1-\alpha|} \le \frac{2\gamma(1-|\alpha|^2)}{(1-r)|1-\alpha|^2} \quad \text{a.e. on } \partial\mathbb{D},$$

and hence $\beta \in B^\alpha$. Hence if α is a Stolz function, then $B_r \subset B^\alpha \quad (r < 1)$.

For two positive functions f and g on $\partial\mathbb{D}$, if there exists a constant $\gamma > 0$ such that $(1/\gamma)g \le f \le \gamma g$ a.e. on $\partial\mathbb{D}$, then we write $f \sim g$.

Lemma. *Suppose α and β are contractive functions in H^∞. Then the following* $(1) \sim (5)$ *are equivalent*:

(1) $\|(\alpha-\bar\beta)/(1-\alpha\beta)\|_\infty < 1$.

(2) $|1-\alpha\beta|^2 \le \gamma_2(1-|\alpha|^2)(1-|\beta|^2)$ *a.e. on* $\partial\mathbb{D}$ *for some constant* $\gamma_2 > 0$.

(3) *There exists a constant* $\gamma_3 > 0$ *such that for any function* $t > 0$

$$\frac{|1-\alpha\beta|}{|1-\alpha|\cdot|1-\beta|} \le \gamma_3 \left\{ t\frac{1-|\alpha|^2}{|1-\alpha|^2} + \frac{1}{t}\,\frac{1-|\beta|^2}{|1-\beta|^2} \right\} \quad a.e.\ on\ \partial\mathbb{D}.$$

(4) *There exists a constant* $\gamma_4 > 0$ *such that*

$$\frac{|1-\alpha\beta|}{|1-\alpha|\cdot|1-\beta|} \le \gamma_4\,\frac{1-|\alpha|^2}{|1-\alpha|^2} \quad a.e.\ on\ \partial\mathbb{D}$$

and

$$\frac{|1-\alpha\beta|}{|1-\alpha|\cdot|1-\beta|} \le \gamma_4\,\frac{1-|\beta|^2}{|1-\beta|^2} \quad a.e.\ on\ \partial\mathbb{D}.$$

(5) $|1-\alpha| \sim |1-\beta|$ and $1-|\alpha| \sim 1-|\beta| \sim |1-\alpha\beta|$.

Proof. (1) and (2) are equivalent because

$$1-\left|\frac{\alpha-\bar\beta}{1-\alpha\beta}\right|^2 = \frac{(1-|\alpha|^2)(1-|\beta|^2)}{|1-\alpha\beta|^2}.$$

(cf. [5, p. 58]). (2) and (3) are equivalent because if $a, b > 0$ then $2\sqrt{ab} \le a+b$ and the equality holds when $a = b$. (1) $\Rightarrow$ (5): Let $f = (\bar\alpha-\beta)/(1-\alpha\beta)$. Then $\|f\|_\infty < 1$, $\beta = (\bar\alpha - f)/(1-\alpha f)$ and

$$|1-\beta| = \frac{|(1-\bar\alpha)+f(1-\alpha)|}{|1-\alpha f|} \ge \frac{|1-\alpha|-|f|\cdot|1-\alpha|}{2} \ge \frac{1-\|f\|_\infty}{2}|1-\alpha|.$$

Let $g = (\alpha-\bar\beta)/(1-\alpha\beta)$. Then $\|g\|_\infty = \|f\|_\infty < 1$, $\alpha = (g+\bar\beta)/(1+g\beta)$ and

$$|1-\alpha| = \frac{|(1-\bar\beta)-g(1-\beta)|}{|1+g\beta|} \ge \frac{|1-\beta|-|g|\cdot|1-\beta|}{2} \ge \frac{1-\|g\|_\infty}{2}|1-\beta|.$$

Hence $|1-\alpha| \sim |1-\beta|$. Since $0 < 1-\|f\|_\infty \le |1-\alpha f| \le 2$ and

$$1-|\beta|^2 = \frac{(1-|\alpha|^2)(1-|f|^2)}{|1-\alpha f|^2},$$

$1-|\alpha| \sim 1-|\beta|$. Since $|1-\alpha f| = (1-|\alpha|^2)/|1-\alpha\beta|$, $|1-\alpha\beta| \sim 1-|\alpha|$. It is clear that (5) implies (4). If we multiply both sides of the two inequalities in (4), then (2) follows. ∎

By the above lemma, Proposition 3 follows immediately.

Proposition 3. *If* $\alpha \in B_1$, *then*

$$B^\alpha \supset \left\{ \beta \in B_1; \quad \left\| \frac{\alpha - \bar{\beta}}{1 - \alpha\beta} \right\|_\infty < 1 \right\}.$$

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