TAIWANESE JOURNAL OF MATHEMATICS
Vol. 5, No. 3, pp. 519-533, September 2001
This paper is available online at http://www.math.nthu.edu.tw/tjm/

A SPACE OF MEROMORPHIC MAPPINGS AND AN ELIMINATION OF DEFECTS

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Abstract. This is a summary report of my recent articles. Nevanlinna theory asserts that each meromorphic mapping f of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$ has few defects. However, it seems that meromorphic mappings with defects are very few. In this report, we shall show that for any given transcendental meromorphic mapping of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$, there is a small deformation of f which has no Nevanlinna deficient hyperplanes in $\mathbb{P}^n(\mathbb{C})$, and also in the case $m = 1$, there is a small deformation of f which has no Nevanlinna deficient hypersurfaces of degree $\leq d$ for each given positive integer d, or deficient rational moving targets. Furthermore, we shall show that mappings without Nevanlinna defects are dense in a space of transcendental meromorphic mappings.

1. Introduction

Nevanlinna defect relations were established for various cases, for example, holomorphic (or meromorphic) mappings of $\mathbb{C}^m$ into a complex projective space $\mathbb{P}^n(\mathbb{C})$ for constant targets of hyperplanes or moving targets of hyperplanes (arbitrary $m \geq 1$ and $n \geq 1$), or holomorphic mappings of an affine variety A of dimension m into a projective algebraic variety V of dimension n for divisors on V $(m \geq n \geq 1)$, and so on. On the other hand, the size of a set of Valiron deficient hyperplanes or deficient divisors are investigated (e.g., Sadullaev [8], Mori [4]). Nevanlinna theory asserts that for each holomorphic (or meromorphic) mapping, Nevanlinna defects or Valiron defects of the mapping are very few. Until now, there are few results on defects of a family of mappings. Recently, the author [4, 5, 6] proved that for a transcendental meromorphic mapping f of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$, we can eliminate all

Received May 31, 2000; revised November 9, 2000.
Communicated by P. Y. Wu.
2000 *Mathematics Subject Classification*: 30D35, 32H30.
Key words and phrases: Nevanlinna theory, defect, meromorphic mapping.
*This research was partially supported by Grant-in-Aid for Scientific Research (No.10640149), Ministry of Education, Science and Culture of Japan.

deficient hyperplanes ($m \geq 1$), all deficient hypersurfaces of degree at most a given integer d or rational moving targets ($m = 1$) in $\mathbb{P}^n(\mathbb{C})$ by a small deformation of the mapping. The class of meromorphic mappings which does not have a Valiron deficiency is important, because these functions have a counting function $N(r, D) \sim T_f(r)$, $r \to \infty$, for every target D.

We shall now discuss an elimination theorem of defects of a meromorphic mapping or a holomorphic curve by its small deformation, and also discuss a space of meromorphic mappings without defects. Here a small deformation $\tilde{f}$ of f means that their order functions $T_f(r)$ and $T_{\tilde{f}}(r)$ satisfy $|T_f(r) - T_{\tilde{f}}(r)| \leq o(T_f(r))$ as r tends to infinity.

2. Preliminaries

2-1. Notation and Terminology

Let $z = (z_1, ..., z_m)$ be the natural coordinate system in $\mathbb{C}^m$. Set

$$\langle z, \xi \rangle = \sum_{j=1}^{m} z_j \xi_j \text{ for } \xi = (\xi_1, ..., \xi_m),\ \|z\|^2 = \langle z, \bar{z} \rangle,\ B(r) = \{z \in \mathbb{C}^m \mid \|z\| < r\},$$

$$\partial B(r) = \{z \in \mathbb{C}^m \mid \|z\| = r\}, \quad \psi = dd^c \log \|z\|^2 \quad \text{and} \quad \sigma = d^c \log \|z\|^2 \wedge \psi^{m-1},$$

where $d^c = (\sqrt{-1}/4\pi)(\bar{\partial} - \partial)$, and $\psi^k = \psi \wedge \cdots \wedge \psi$ (k-times).

Let f be a meromorphic mapping of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$. Then f has a reduced representation $(f_0 : ... : f_n)$, where $f_0, ..., f_n$ are holomorphic functions on $\mathbb{C}^m$ with $\operatorname{codim}\{z \in \mathbb{C}^m \mid f_0(z) = \cdots = f_n(z) = 0\} \geq 2$. We write $f = (f_0, ..., f_n)$ as the same letter of the meromorphic mapping f. Denote $D^\alpha f = (D^\alpha f_0, ..., D^\alpha f_n)$ for a multi-index α, where $D^\alpha \phi = \partial^{|\alpha|}\phi / \partial z_1^{\alpha_1} \cdots \partial z_m^{\alpha_m}$, $\alpha = (\alpha_1, ..., \alpha_m)$, $|\alpha| = \alpha_1 + \cdots + \alpha_m$ and a function ϕ.

Definition (see Fujimoto [2, §4]). We define the generalized Wronskian of f by

$$W_{\alpha^0,...,\alpha^n}(f) = \det\ (D^{\alpha^k} f \ : \ 0 \leq k \leq n),$$

for $n+1$ multi-indices $\alpha^k = (\alpha_1^k, ..., \alpha_m^k)(0 \leq k \leq n)$.

By Fujimoto [2, §4], for every linearly nondegenerate meromorphic mapping f of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$, there are $n+1$ multi-indices $\alpha^0, ..., \alpha^n$ such that $\{D^{\alpha^0} f, ..., D^{\alpha^n} f\}$ is an admissible basis with $|\alpha^k| \leq n+1$. Then $W_{\alpha^0,...,\alpha^n}(\phi f) = \phi^{n+1} W_{\alpha^0,...,\alpha^n}(f) \not\equiv 0$ holds for any nonzero holomorphic function ϕ on $\mathbb{C}^m$, where $\phi f = (\phi f_0, ..., \phi f_n)$.

Let f be a nonconstant meromorphic mapping f of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$, and let $\mathcal{L} = [H^d]$ be the line bundle over $\mathbb{P}^n(\mathbb{C})$ which is determined by the dth tensor power of the hyperplane bundle $[H]$. A hypersurface D of degree d in $\mathbb{P}^n(\mathbb{C})$ is given by the divisor of a holomorphic section $\delta \in H^0(\mathbb{P}^n(\mathbb{C}), \mathcal{O}(\mathcal{L}))$ which is determined by a homogeneous polynomial $P(w)$ of degree d. A metric $a = \{a_\alpha\}$

on the line bundle $\mathcal{L}$ is given by $a_\alpha = (\sum_{j=0}^n \ |w_j/w_\alpha|^2)^d$ in a neighborhood $U_\alpha = \{w_\alpha \neq 0\}$.

The Nevanlinna order function $T_f(r,\mathcal{L})$ of f for the line bundle $\mathcal{L}$ is given by:

$$T_f(r,\mathcal{L}) := \int_{r_0}^r \frac{dt}{t}\int_{B(t)} f^*\omega\wedge\psi^{m-1},$$

where $\omega = \{\omega_\alpha\} = dd^c \log \sum_{j=0}^n (|w_j/w_\alpha|^2)^d$ in a neighborhood $U_\alpha := \{w_\alpha \neq 0\}$. We say that a meromorphic mapping f is transcendental if

$$\lim_{r\to+\infty} \frac{T_f(r,\mathcal{L})}{\log\ r} = +\infty.$$

A meromorphic mapping f is rational if and only if $T_f(r,\mathcal{L}) = O(\log\ r)$ $(r \to +\infty)$. The norm of a section δ is given by

$$\|\delta\|^2 := \frac{|\delta_\alpha|^2}{a_\alpha} = \frac{|P(w)|^2}{(\sum_{j=0}^n |w_j|^2)^d}.$$

We may assume $\|\delta\| \leq 1$. The proximity function $m_f(r,D)$ of D is defined by

$$m_f(r,D) := \int_{\partial B} \log \frac{1}{\|\delta_f\|}\sigma = \int_{\partial B} \log \frac{\|f\|^d}{|P(f)|}\sigma.$$

The Nevanlinna deficiency $\delta_f(D)$ and the Valiron deficiency $\Delta_f(D)$ of D for f is defined by

$$\delta_f(D) := \liminf_{r\to\infty} \frac{m_f(r,D)}{T_f(r,\mathcal{L})} \text{ and } \delta_f(D) := \limsup_{r\to\infty} \frac{m_f(r,D)}{T_f(r,\mathcal{L})}.$$

In particular, if $\mathcal{L}$ is the hyperplane bundle $[H]$ and D is a hyperplane H which is given by a vector $\mathbf{a} = (a_0, ..., a_n) \in \mathbb{C}^{n+1} \setminus \{0\}$, the proximity function $m_f(r,H)$ and the counting function $N_f(r,H)$ of a hyperplane H in $\mathbb{P}^n(\mathbb{C})$ are given by:

$$m_f(r,H) := \int_{\partial B(r)} \log \frac{\|f\|\,\|\mathbf{a}\|}{|\langle f,\mathbf{a}\rangle|}\sigma \text{ and } N_f(r,H) := \int_{r_0}^r \frac{dt}{t}\int_{(f^*H)\cap B(t)} \psi^{m-1},$$

for some fixed $r_0 > 0$, where $H = \{w = (w_0, ..., w_n) \in \mathbb{C}^{n+1}\setminus\{0\} \mid \sum_{j=0}^n a_j w_j = 0\}$ and f^*H denotes the pullback of H under f. Also, the Nevanlinna order function $T_f(r) \equiv T_f(r,[H])$ of f for the hyperplane bundle $[H]$ is written as:

$$T_f(r) = \int_{\partial B(r)} \log\Big(\sum_{j=0}^n |f_j|^2\Big)^{1/2}\sigma + O(1) = \int_{\partial B(r)} \log \sum_{j=0}^n |f_j|\sigma + O(1)$$

by using Stoke's theorem. We write

$$N(r,(\phi)) := \int_{r_0}^{r} \frac{dt}{t} \int_{(\phi)\cap B(t)} \psi^{m-1},$$

where (ϕ) denotes the divisor determined by a meromorphic function ϕ on $\mathbb{C}^m$.

Let $f : \mathbb{C}^m \to \mathbb{P}^n(\mathbb{C})$ be a meromorphic mapping with a reduced representation $(f_0 : \ldots : f_n)$. Let $\phi : \mathbb{C}^m \to \mathbb{P}^n(\mathbb{C})^*$ be a meromorphic mapping with a reduced representation $(\phi_0 : \ldots : \phi_n)$, which is called a moving target for f. Then the proximity function $m_f(r,\phi)$ and the counting function $N_f(r,\phi)$ of a moving target ϕ into $\mathbb{P}^n(\mathbb{C})^*$ are given by:

$$m_f(r,\phi) := \int_{\partial B} \log \frac{\|f\|\,\|\phi\|}{|\langle f,\phi\rangle|}(re^{i\theta})d\theta \ \text{ and } \ N_f(r,\phi) := \int_{B(r)\cap(A)_0} \psi^{m-1},$$

where $\|f\|^2 = \sum_{j=0}^n |f_j|^2$ and $(A)_0$ denotes the divisor determined by the zeros of $A := \langle f,\phi\rangle = \sum_{j=0}^n \phi_j f_j$. The Nevanlinna deficiency $\delta_f(\phi)$ and the Valiron deficiency $\Delta_f(\phi)$ of a moving target ϕ for f are given by:

$$\delta_f(\phi) := \liminf_{r\to+\infty} \frac{m_f(r,\phi)}{T_f(r)+T_\phi(r)} \ \text{ and } \ \Delta_f(\phi) := \limsup_{r\to+\infty} \frac{m_f(r,\phi)}{T_f(r)+T_\phi(r)}.$$

We now define the projective logarithmic capacity of a set in the projective space $\mathbb{P}^n(\mathbb{C})$. (see, Molzon-Shiffman-Sibony [7, p. 46]). Let E be a compact subset of $\mathbb{P}^n(\mathbb{C})$, and $\mathcal{P}(E)$ denotes the set of probability measures supported on E. We set

$$\begin{aligned} V_\mu(x) &:= \int_{w\in\mathbb{P}^n(\mathbb{C})} \log \frac{\|x\|\,\|w\|}{|\langle x,w\rangle|} d\mu(w) \quad (\mu\in\mathcal{P}(E)) \ \text{ and} \\ V(E) &:= \inf_{\mu\in\mathcal{P}(E)} \sup_{x\in\mathbb{P}^n(\mathbb{C})} V_\mu(x). \end{aligned}$$

Define the projective logarithmic capacity $C(E)$ of E by

$$C(E) := \frac{1}{V(E)}.$$

If $V(E) = +\infty$, we say that the set E is of projective logarithmic capacity zero. For an arbitrary subset K of $\mathbb{P}^n(\mathbb{C})$, we put

$$C(K) = \sup_{E\subset K} C(E),$$

where the supremum is taken over all compact subsets E of K.

2-2. Some Results

A.Vitter [9] proved the following theorem:

Theorem A (Lemma of the logarithmic derivatives). *Let* $f = (f_0 : f_1)$ *be a reduced representation of a meromorphic mapping* $f : \mathbb{C}^m \to \mathbb{P}^1(\mathbb{C})$*. Set* $F = f_1/f_0$*. Then there exist positive constants* a_1, a_2, a_3 *such that*

$$\int_{\partial B(r)} \log^+ |F_{z_j}/F|\sigma \leq a_1 + a_2 \log\ r + a_3 \log T_f(r), \quad (j = 1, ..., m). \quad //.$$

Here the notation "$A(r) \leq B(r)$ //" means that the inequality $A(r) \leq B(r)$ holds for r outside a Borel set with finite Lebesgue measure.

Molzon-Shiffman-Sibony proved the following result on the projective logarithmic capacity.

Theorem B [7, p. 47]. *Let* $\varphi : [0,\ 1] \to \mathbb{P}^n(\mathbb{C})$ *be a real smooth nondegenerate arc in* $\mathbb{P}^n(\mathbb{C})$*, and* K *a compact subset of the interval [0, 1]* $\subset \mathbb{C}$*. Then the projective logarithmic capacity* $C(\varphi(K))$ *is positive if and only if* K *has a positive logarithmic capacity in* $\mathbb{C}$*.*

Here "smooth nondegenerate arc φ" means that there exists a lift $\tilde{\varphi}\ :\ [0,\ 1] \to \mathbb{C}^{n+1} \setminus \{0\}$ such that the kth derivatives $\{\tilde{\varphi}^{(k)}(t)\}_{k\geq 0}$ of $\tilde{\varphi}(t)$ spans $\mathbb{C}^{n+1}$ for every $t \in [0,\ 1]$.

Theorem C [4]. *Let* f *be a meromorphic mapping of* $\mathbb{C}^m$ *into* $\mathbb{P}^n(\mathbb{C})$ *such that* $\lim_{r\to+\infty} T_f(r) = +\infty$*. Then there exist a sequence* $r_1 < r_2 < \ldots < r_n \to +\infty$ *and sets* $E_n : E_{n+1} \subset E_n$ $(n = 1, 2, ...)$ *in* $\mathbb{P}^n(\mathbb{C})^*$ *with* $V(E_n) \geq 2 \log T_f(r_n)$ *such that, if* H *does not belong to* E_n*, then*

$$m_f(r, H) \leq 4\sqrt{T_f(r)} \log T_f(r)$$

for $r > r_n$*. Hence*

$$\lim_{r\to+\infty} \frac{m_f(r, H)}{T_f(r)} = 0$$

outside a set $E \subset \mathbb{P}^n(\mathbb{C})^*$ *of projective logarithmic capacity zero. Here* $\mathbb{P}^n(\mathbb{C})^*$ *denotes the dual projective space of* $\mathbb{P}^n(\mathbb{C})$*.*

Theorem D [1]. *Set* $\Lambda(r) := \int_{r_0}^{r} \psi(t)/dt\ t$*, where* $\psi(r)$ *is nonnegative, nondecreasing and unbounded. If* $\Lambda(r) < r^K$ *for some* $K > 0$ *and all sufficiently large* r*, then there exists an entire function* $g(z)$ *of finite order such that* $T_g(r) \sim \Lambda(r) (r \to \infty)$*.*

3. Elimination of Defects of Meromorphic Mappings

3-1. Elimination of deficient hyperplanes of a meromorphic mapping

For a transcendental meromorphic mapping g of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$, we can eliminate all deficient hyperplanes by a small deformation of g.

Lemma 1 [4]. *There are monomials $\zeta_1, ..., \zeta_n$ in $z_1, ..., z_m$ such that any n derivatives in $\{D^\alpha\zeta := (D^\alpha\zeta_1, ..., D^\alpha\zeta_n) \mid |\alpha| \leq n+1\}$ are linearly independent over the field M of meromorphic functions on $\mathbb{C}^m$, where $\alpha = (\alpha_1, ..., \alpha_m) \in \mathbb{Z}_{\geq 0}$ is a multi-index.*

Lemma 2 [4]. *Let $h = (h_0 : h_1 : \cdots : h_n)$ be a reduced representation of a meromorphic mapping of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$ and $\zeta_1, ..., \zeta_n$ linearly independent monomiales in $z_1, ..., z_m$ as in Lemma 1. Then there exists $(\tilde{a}_1, ..., \tilde{a}_n)$ such that $\tilde{a}_j = \alpha^{k_j}$ $(j = 1, ..., n)$ with $k_1 = 1, k_m = \sum_{l=1}^{m-1} k_l + 1$ $(m = 2, 3, ..., n)$ $(\alpha \in \mathbb{C})$, and*

$$f := (h_0 : h_1 + \tilde{a}_1\zeta_1 h_0 : h_2 + \tilde{a}_2\zeta_2 h_0 : \cdots : h_n + \tilde{a}_n\zeta_n h_0)$$

is a reduced representation of a linearly nondegenerate meromorphic mapping of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$.

Lemma 3 [4]. *Let $f = (f_0 : \cdots : f_n)$ and $h = (h_0 : \cdots , h_n)$ be as in Lemma 2. Then we have*

$$|T_f(r) - T_h(r)| \leq O(\log\ r) \quad (r \to \infty).$$

Lemma 4 [4]. *The set of vectors*

$$\mathcal{A} := \left\{ \left(1, a_1, ..., \prod_{k=1}^{n} a_k\right) | a_j \in \mathbb{C} \right\}$$

is of positive projective logarithmic capacity in $\mathbb{P}^N(\mathbb{C})$, where $N = 2^n - 1$.

Theorem 1 [4]. *Let $g : \mathbb{C}^m \to \mathbb{P}^n(\mathbb{C})$ be a given transcendental meromorphic mapping. Then there exists a regular matrix $L = (l_{ij})_{0 \leq i,j \leq n}$ of the form $l_{i,j} = c_{ij}\zeta_i + d_{ij}$, $(c_{ij}, d_{ij} \in \mathbb{C} : 0 \leq i, j \leq n)$, such that* $\det L \neq 0$ *and $f := L \cdot g : \mathbb{C}^m \to \mathbb{P}^n(\mathbb{C})$ is a meromorphic mapping without Nevanlinna deficient hyperplanes, where $\zeta_1, ..., \zeta_n$ are some monomials in $z_1, ..., z_m$ which are linearly independent over $\mathbb{C}$.*

Here the mapping $f := L \cdot g : \mathbb{C}^m \to \mathbb{P}^n(C)$ means a product of the matrix $L = (l_{ij})$ and a vector of a reduced representation $\tilde{g} =^t (g_0 : ... : g_n)$ of g which does not depend on a choice of $\tilde{g}$, and also a Nevanlinna deficient hyperplane H for f means a hyperplane with $\delta_f(H) > 0$.

Remark 1. For the mappings as in Theorem 1, the inequality $|T_f(r)-T_g(r)|\leq O(\log\ r)$ $(r\to+\infty)$ holds, and also the mapping g may be linearly degenarate or of infinite order.

Remark 2. A rational mapping g always has a Nevanlinna deficient hyperplane if $m=1$ or there is a regular linear change L_0 such that $L_0\cdot g$ has a reduced representation which consists of polynomials including different degrees. But otherwise g does not have Nevanlinna deficient hyperplanes.

Remark 3. If g is of finite order, we can replace "Nevanlinna deficiency" by "Valiron deficiency" in the conclusion of Theorem 1.

Remark 4. If $m=1$, we can take $\zeta_k=z^k \quad (k=1,...,n)$.

Outline of the proof of Theorem 1 (see [4]).

1st step. There is a regular linear change L_1 of $\mathbb{P}^n(\mathbb{C})$ such that

$$h:=L_1\cdot g=(h_0:\cdots:h_n)\ :\ \mathbb{C}^m\longrightarrow\mathbb{P}^n(\mathbb{C})$$

and a reduced representation of the meromorphic mapping h which satisfies

$$N(r,(h_j))=(1-o(1))T_h(r),\quad (r\to+\infty),\quad (j=0,1,..,n).$$

2nd step. Using Theorems B and C, and Lemmas 1 and 2, there are $f=(h_0:h_1+a_1\zeta_1h_0:...:h_n+a_n\zeta_nh_0)$ and multi-indices $\beta^0,...,\beta^n$ such that f is linearly nondegenerate and its generalized Wronskian satisfies $\mathbf{W}_\beta:=\mathbf{W}_{\beta^0,...,\beta^n}(f)\not\equiv 0$. Note that there are many such $\{a_1,...,a_n\}$. Then it can be written as

$$\mathbf{W}_\beta=h_0^{n+1}\Big(W_0+a_1W_1+\cdots+\prod_{i=1}^n a_iW_N\Big)\not\equiv 0,$$

where W_k is a generalized Wronskian of some of $1,h_1/h_0,a_1\zeta_1,...,h_n/h_0,a_n\zeta_n$ $(0\leq k\leq N=2^n-1)$.

3rd step. Consider the auxiliary meromorphic mapping F of the form

$$F:=(W_0/d\ :\ W_1/d\ :\ \cdots\ :\ W_N/d)\ :\ \mathbb{C}^m\ \longrightarrow\ \mathbb{P}^N(\mathbb{C}),$$

where $d=d(z)$ is a meromorphic function which consists of common factors among $W_0,...,W_N$ such that $W_0/d,...,W_N/d$ are holomorphic functions without common factors up to unit. Then we observe that the meromorphic mapping F is not constant. Therefore, there exists an $\mathbf{a}_0=(1,\tilde{a}_1,...,\tilde{a}_n,\tilde{a}_1\tilde{a}_2,...,\prod_{j=1}^n\tilde{a}_j)$ such that

$$\limsup_{r\to\infty}\frac{m_F(r,H_{\mathbf{a}_0})}{T_F(r)}=0,$$

since the set of Valiron deficient hyperplanes of a nonconstant meromorphic mapping is of projective logarithmic capacity zero in $\mathbb{P}^N(\mathbb{C})^*$.

4th step. Consider the meromorphic mapping given by the following reduced representation by using the vector $\mathbf{a}_0$ in the 3rd step:

$$f := L_2 \cdot h = (f_0 : \cdots : f_n) : \mathbb{C}^m \longrightarrow \mathbb{P}^n(\mathbb{C}),$$

where

$$L_2 = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \tilde{a}_1\,\zeta_1 & 1 & \cdots & 0 \\ \tilde{a}_2\,\zeta_2 & 0 & \cdots & 0 \\ & \cdots & \cdots & \\ \tilde{a}_n\,\zeta_n & 0 & \cdots & 1 \end{pmatrix}, \qquad (\det\, L_2 = 1 \neq 0).$$

Hence $f_0 = h_0$ and $f_k = h_k + \tilde{a}_k \zeta_k h_0$ $(k = 1, ..., n)$. Then we observe that

$$T_f(r) = T_g(r) + O\,(\log\; r) = (1 + o(1))\,T_g(r),\;\; (r \to +\infty),$$

if g is not rational.

Claim 1. Let F and f be as above. Then there exists a positive constant K such that

$$T_F(r) \leq KT_f(r).$$

5th step. Take an arbitrary vector $\mathbf{b} = (b_0, ..., b_n) \in \mathbb{C}^{n+1} \setminus \{0\}$, which determines the hyperplane $H = \{w \in \mathbb{C}^{n+1} \setminus \{0\} |\; \langle w, \mathbf{b}\rangle = 0\}$ in $\mathbb{P}^n(\mathbb{C})$. We may assume that $b_n \neq 0$. Then $f_0, f_1, ..., f_{n-1}, A = \langle f, \mathbf{b}\rangle$ are linearly independent over $\mathbb{C}$. Thus we have

$$\begin{aligned} & m_f(r, H_{\mathbf{b}}) = \int_{\partial B(r)} \log \frac{\|f\|}{|A|}\sigma \\ = \;& \int_{\partial B(r)} \log \frac{|W_{\beta^0,..,\beta^n}(f_0,..,f_n)|}{|A|\,|f_0|\cdots|f_{n-1}|}\sigma + \int_{\partial B(r)} \log \frac{\|f\|\,|f_0|\cdots|f_{n-1}|}{|W_{\beta^0,..,\beta^n}(f_0,..,f_n)|}\sigma \\ \leq \;& \int_{\partial B(r)} \log \frac{|b_n^{-1}|\,|W_{\beta^0,..,\beta^n}(f_0,..,f_{n-1},A)|}{|A||f_0|\cdots|f_{n-1}|}\sigma + \int_{\partial B(r)} \log \frac{\|f\|^{n+1}}{|f_0|^{n+1}}\sigma \\ & + \int_{\partial B(r)} \log \frac{1}{|W_0 + a_1 W_1 + \cdots + \prod_{j=1}^n a_j W_N|}\sigma + O(1), \\ \leq \;& o(T_f(r)) + (n+1) m_f(r, H_{(1,0,..,0)}) \\ & + \int_{\partial B(r)} \log \frac{(|W_0| + |W_1| + \cdots + |W_N|)(1/|d|)}{|W_0 + a_1 W_1 + \cdots + \prod_{j=1}^n a_j W_N|(1/|d|)}\sigma + O(1) \end{aligned}$$

$$= \quad o(T_f(r)) + \int_{\partial B(r)} \log \frac{\|F\|}{|\langle F, \mathbf{a}_0 \rangle|} \sigma = o(T_f(r)) + o(T_F(r)) = o(T_f(r)) \ //.$$

Therefore, we obtain

$$\delta_f(H_{\mathbf{b}}) = \liminf_{r \to +\infty} \frac{m_f(r, H_{\mathbf{b}})}{T_f(r)} = 0,$$

that is, $\delta_f(H) = 0$ for any $H \in \mathbb{P}^n(\mathbb{C})^*$. This proves Theorem 1. ■

Note that we can take the norm $\|\tilde{\boldsymbol{\alpha}}\|$ of a vector $\tilde{\boldsymbol{\alpha}} := (\tilde{a}_1, ..., \tilde{a}_n)$ as small as possible in the proof of Theorem 1.

Problem. Is the conclusion of Theorem 1 true if “Nevanlinna deficiency” is replaced by “Valiron deficiency”?

3-2. Elimination of defect hypersurfaces of a holomorphic mapping of $\mathbb{C}$ into $\mathbb{P}^n(\mathbb{C})$

We shall discuss an elimination theorem on defects of hypersurfaces.

Theorem 2 [5]. *Let g be a given transcendental holomorphic mapping of $\mathbb{C}$ into $\mathbb{P}^n(\mathbb{C})$, and $d \in \mathbb{N}$ be given. Then there exists a regular matrix $L = (l_{ij})$ of the form $l_{ij} = c_{ij} z^{m_j} + d_{ij}$, $(c_{ij}, d_{ij} \in \mathbb{C})$, $|L| \neq 0$ and $f := L \cdot g : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ is a holomorphic mapping without Nevanlinna deficient hypersurfaces of degree $\leq d$, where m_j $(j = 1, ..., n)$ are some integers such that $m_1 < d\, m_1 < m_2 < \cdots < d\, m_{n-1} < m_n$.*

Outline of the proof of Theorem 2. There is a regular linear change L_1 such that the holomorphic mapping $h := L_1 \cdot g = (h_0 : \cdots : h_n) : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ satisfies $N(r, (h_j)) = (1 - o(1)) T_h(r)$, $(r \to \infty)$, $(j = 0, ..., n)$. Consider the Veronese embedding $v_d : \mathbb{P}^n(\mathbb{C}) \to \mathbb{P}^s(\mathbb{C})$, which is defined by homogeneous monomials of degree d in $(w_0 : \cdots : w_n) \in \mathbb{P}^n(\mathbb{C})$. Let $\tilde{h} = (\tilde{h}_0 : \cdots : \tilde{h}_n) := (h_0 : h_1 + a_1 z^{m_1} h_0 : \cdots : h_n + a_n z^{m_n} h_0)$. Consider the composed mapping $\hat{f} := v_d \circ \tilde{h} = (\hat{f}_0 : \cdots : \hat{f}_s) = (\tilde{h}_0^d : \tilde{h}_0^{d-1} \tilde{h}_1 : \cdots : \tilde{h}_0 \tilde{h}_1^{d-1} : \tilde{h}_1^d : \tilde{h}_0^{d-1} \tilde{h}_2 : \cdots : \tilde{h}_n^{d-1} \tilde{h}_{n-1} : \tilde{h}_n^d)$. Here $s = (n+d)!/d!\, n! - 1$. We can prove the following Lemma 5 using a similar method for the proof of Theorem 1.

Lemma 5 [5]. *There is a vector $(a_1, ..., a_n) \in \mathbb{C}^n \setminus \{0\}$ such that $\hat{f}$ is linearly nondegenerate.*

Set

$$\begin{aligned} \mathbf{W} &= \mathbf{W}(\hat{f}_0, ..., \hat{f}_s) = \mathbf{W}(\tilde{h}_0^d, \tilde{h}_0^{d-1} \tilde{h}_1, \tilde{h}_0^{d-2} \tilde{h}_1^2, ..., \tilde{h}_n^d) \\ &= \tilde{h}_0^{d(s+1)} \mathbf{W}(1, \tilde{H}_1, \tilde{H}_1^2, ..., \tilde{H}_1^{k_1} \tilde{H}_2^{k_2} \cdots \tilde{H}_n^{k_n}, ..., \tilde{H}_s^d) \\ &= \tilde{h}_0^{d(s+1)} \Big(\mathbf{W}_0 + a_1 \mathbf{W}_1 + \cdots + \prod_{k=1}^n a_k^d \mathbf{W}_N \Big), \ (\tilde{H}_j = H_j + a_j z^{m_j}), \end{aligned}$$

where $\mathbf{W}_j (j = 0, ..., N)$ are some sums of Wronskians. Consider the auxiliary holomorphic mapping:

$$F := (\mathbf{W}_0/d(z) : \mathbf{W}_1/d(z) : \cdots : \mathbf{W}_N/d(z)) : \mathbb{C} \longrightarrow \mathbb{P}^N(\mathbb{C}).$$

Here $d = d(z)$ is a holomorphic function such that $\mathbf{W}_0/d, ..., \mathbf{W}_N/d$ are holomorphic functions without common zeros. Then there is a vector

$$\mathbf{a} \in \mathcal{A} := \left\{ \left(1, a_1, ..., \prod_{k=1}^{n} a_k^d\right) \mid a_j \in \mathbb{C} \right\}$$

such that $m_F(r, H_{\mathbf{a}}) = o(T_F(r))$, $(r \to \infty)$, since $\mathcal{A}$ has a positive projective logarithmic capacity by Theorem B. Consider the holomorphic mapping given by the following reduced representation which is determined by the vector $(a_1, ..., a_n)$ corresponding to above $\mathbf{a}$:

$$f := L_2 \cdot h \ : \ \mathbb{C} \longrightarrow \mathbb{P}^n(\mathbb{C}),$$

where $L_2 = (s_{ij})$ and $s_{ij} = 1$ $(i = j)$, $s_{ij} = a_i z^{m_i}$ $(j = 1, i \neq 1)$, $s_{ij} = 0$ (otherwise). Then $\det(s_{ij}) \neq 0$.

Claim 2. There is a positive constant K such that $T_F(r) \leq K T_h(r)$, and also $(1 + o(1))T_f(r) = T_g(r) = (1 + o(1))T_h(r)$, $(r \to \infty)$, hold by a similar method in Section 3-1.

Now we take an arbitrary hypersurface $D = D_{\mathbf{b}}$ in $\mathbb{P}^n(\mathbb{C})$ which is determined by a homogeneous polymonial:

$$P(w) := b_0 w_0^d + b_1 w_0^{d-1} w_1 + \cdots + b_k w_0^{j_0} w_1^{j_1} \cdots w_n^{j_n} + \cdots + b_s w_n^d = 0\},$$

$w = (w_0, ..., w_n) \in \mathbb{C}^{n+1} \setminus \{0\}$. Then D corresponds to the vector $\mathbf{b} = (b_0, ..., b_s)$. We may assume that $b_s \neq 0$. We set $\tilde{f} := v_d \circ f$. Consider the function

$$A_{\mathbf{b}} = \sum_{k=0}^{s} b_k f_0^{j_0^k} \cdots f_n^{j_n^k},$$

where $J_k := (j_0^k, ..., j_n^k)$ with $|J_k| := j_0^k + \cdots + j_n^k = d$. Then $\hat{f}_0, ..., \hat{f}_{s-1}$, $A_{\mathbf{b}}$ are linearly independent over $\mathbb{C}$, since $\hat{f} := (\hat{f}_0 : \cdots : \hat{f}_s)$ is linearly nondegenerate. Then, using Theorem A, Claim 2 and the similar method to the proof of Theorem 1, we obtain

$$m_f(r, D_{\mathbf{b}}) = \int_{\partial B(r)} \log \frac{\|f\|^d}{|A_{\mathbf{b}}|} \sigma = o(T_f(r)).$$

Therefore, we obtain

$$\delta_f(D_{\mathbf{b}}) = \liminf_{r\to\infty} \frac{m_f(r, D_{\mathbf{b}})}{d\ T_f(r)} = 0.$$

In case where hypersurfaces of degree $\leq\ d$, for each d' $(\leq d)$, we can take a vector $\mathbf{a} = \mathbf{a}_{d'}$ in a subset of $\mathcal{A}$ of positive projective logarithmic capacity. Hence we can take a common vector $\mathbf{a} \in \mathcal{A}$ for each d'. This proves Theorem 2. ■

Note that Theorem 2 can be extended to the case where meromorphic mappings of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$ by using the similar method to Section 3-1.

3-3. Elimination of defects of holomorphic curves for rational moving targets

For a transcendental holomorphic curve f of $\mathbb{C}$ into $\mathbb{P}^n(\mathbb{C})$, we can eliminate all defects of rational moving targets by a small deformation of f.

Theorem 3 [6]. *Let* $f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ *be a given transcendental holomorphic curve. Then there exists a regular matrix*

$$L = (l_{ij})_{0\leq i,j\leq n} \text{ of the form } l_{i,j} =_{ij} g_j + d_{ij},\ (c_{ij}, d_{ij} \in \mathbb{C} : 0 \leq i, j \leq n),$$

such that $\det L \neq 0$ *and* $\tilde{f} = L \cdot f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ *is a holomorphic curve without Nevanlinna defects of rational moving targets and satisfies*

$$|T_f(r) - T_{\tilde{f}}(r)| = o(T_f(r)), \qquad (r \to \infty),$$

where g_j $(j = 1, ..., n)$ *are some transcendental entire functions satisfying* $T_{g_j}(r) = o(T_{g_{j+1}}(r)),\ (j = 1, ..., n-1)$*, and* $T_{g_n} = o(T_f(r))$ $(r \to \infty)$.

Note that we cannot replace transcendental entire functions g_j by any rational functions.

Problem: Can we extend Theorem 3 to the case of several complex variables?

Outline of the proof of Theorem 3. Let h be a transcendental holomorphic curve and $(h_0, ..., h_n)$ its reduced representation. Then there are indices i, j such that h_j/h_i is transcendental, say $i = 0,\ j = n$. By Theorem D, there are n transcendental entire functions $g_1, ..., g_n$ on $\mathbb{C}$ such that $T_{g_j}(r) = o(T_{g_{j+1}}),\ (j = 1, ..., n-1)$ and $T_{g_n}(r) = o(T_f(r))$ as $r \to \infty$. Then $g_1, ..., g_n$ are linearly independent over $\mathbb{C}$. There is a regular linear change L_1 such that

$$h = L_1 \cdot f = (h_0 : \cdots : h_n) : \mathbb{C} \longrightarrow \mathbb{P}^n(\mathbb{C}),$$

and a reduced representation of the holomorphic curve h satisfying

$$N(r, 0, h_j) \sim T_h(r), \quad (r \to +\infty), \quad (j = 0, ..., n).$$

We put $\bar{h}_k = h_k + a_k g_k h_0$ $(k = 1, ..., n)$ and $\bar{h}_0 = h_0$. Consider the reduced representation of a holomorphic curve

$$\bar{h} := (\bar{h}_0 : \bar{h}_1 : \cdots : \bar{h}_n) : \mathbb{C} \longrightarrow \mathbb{P}^n(\mathbb{C}).$$

Then there exist complex numbers $a_1, ..., a_n$ such that

$$\{\bar{h}_0, z\bar{h}_0, ..., z^m\bar{h}_0, \bar{h}_1, ..., z^m\bar{h}_1, ..., \bar{h}_n, ..., z^m\bar{h}_n\}$$

is linearly independent over $\mathbb{C}$, as in previous theorems.

We now consider the Wronskian

$$\mathbf{W} := W(\bar{h}_0, z\bar{h}_0, ..., z^m\bar{h}_0, \bar{h}_1, ..., z^m\bar{h}_1, ..., \bar{h}_n, ..., z^m\bar{h}_n),$$

and we write it as

$$\begin{aligned}\mathbf{W} :=& W_0(h_0, zh_0, ..., z^m h_0, h_1, ..., z^m h_1, ..., h_n, ..., z^m h_n)\\ &+ a_1(W_{11} + \cdots + W_{1s_1}) + \cdots + a_n(W_{n1} + \cdots + W_{ns_n})\\ &+ a_1^2(W_{1^2 1} + \cdots + W_{1^2 s_1^2}) + \cdots + a_1^{m+1}(W_{1^{m+1}1} + \cdots + W_{1^{m+1}s_1^{m+1}})\\ &+ a_1 a_2(W_{1^{12}1} + \cdots + W_{1^{12}s_{12}}) + \cdots\\ &+ \prod_{j=1}^{n} a_j^{m+1} W_N(1, ..., z^m, g_1, ..., z^m g_1, ..., g_n, ..., z^m g_n) \cdot h_0^{(m+1)(n+1)}.\end{aligned}$$

We now rewrite it in an inhomogeneous form as

$$\mathbf{W} = h_0^{(m+1)(n+1)}\Big\{\mathbf{W}_0 + a_1\mathbf{W}_1 + \cdots + \prod_{j=1}^{n} a_j^{m+1}\mathbf{W}_N\Big\},$$

where $\mathbf{W}_k$ $(k = 0, ..., N)$ are sums of some Wronskian determinants, and $N = (m+2)^n - 1$. For any fixed $m \in \mathbb{N}$, we consider an auxiliary holomorphic curve of the form

$$F_m := (\mathbf{W}_0/d : \mathbf{W}_1/d : \cdots : \mathbf{W}_N/d) : \mathbb{C} \longrightarrow \mathbb{P}^N(\mathbb{C}),$$

where $d = d(z)$ is a meromorphic function whose zeros and poles consist of common factors among $\mathbf{W}_0, ..., \mathbf{W}_N$. Then F_m is a reduced representation of nonconstant holomorphic curve in $\mathbb{P}^N(\mathbb{C})$.

Lemma 6 [cf. 6]. *Let*

$$\begin{aligned}\mathcal{A} :=& \Big\{\Big(1, a_1, ..., a_1^{m+1}, a_2, ..., a_1^{i_1} a_2^{i_2} \cdots a_n^{i_n}, ..., \prod_{j=1}^{n} a_j^{m+1}\Big) | a_j \in \mathbb{C},\\ & 0 \le i_1, ..., i_n \le m+1\Big\}.\end{aligned}$$

Then there is a vector $\mathbf{a} = (1, a_1, ..., \prod_{j=1}^{n} a_j^{m+1})$ *such that*

$$\limsup_{r\to\infty} \frac{m_{F_m}(r, H_{\mathbf{a}})}{T_{F_m}(r)} = 0.$$

This holds for any positive integer m, because a countable union of sets of projective logarithmic capacity zero is of projective logarithmic capacity zero. Here

$$H_{\mathbf{a}} = \{\zeta = (\zeta_0, ..., \zeta_N) | \langle \zeta, \mathbf{a}\rangle = 0\} \quad \text{and}$$

$$\langle F, \mathbf{a}\rangle = \Big\{\mathbf{W}_0 + a_1\mathbf{W}_1 + \cdots + \prod_{j=1}^{n} a_j^{m+1}\mathbf{W}_N\Big\}/d.$$

Lemma 7 [6]. *Let* F_m *and* h *be as above. Then there exists a positive constant* K *such that*

$$T_{F_m}(r) \le KT_h(r).$$

For this $(a_1, .., a_n)$, we consider the holomorphic curve given by the following reduced representation:

$$\tilde{f} := L_2 \cdot h \equiv (\tilde{f}_0, ..., \tilde{f}_n) : \mathbb{C} \longrightarrow \mathbb{P}^n(\mathbb{C}),$$

where $L_2 := (s_{ij})$ and $s_{ij} = 1$ $(i = j)$, $s_{ij} = a_i g_i$ $(j = 1, i \ne 1)$, $s_{ij} = 0$ (otherwise). Hence $\tilde{f}_0 = h_0$, $\tilde{f}_k = h_k + a_k g_k h_0$ $(k = 1, ..., n)$, and $\det(s_{ij}) \ne 0$. Then we see

$$T_{\tilde{f}}(r) = T_f(r) + o(T_{\tilde{h}}(r)) = (1 + o(1))T_f(r) \qquad (r \to +\infty).$$

Now we take a given integer m and an arbitrary rational target ϕ of degree m :

$$\phi = (\phi_0(z), ..., \phi_n(z)) : \mathbb{C} \longrightarrow \mathbb{P}^n(\mathbb{C})^*.$$

Then we can choose a reduced representation of ϕ such that each ϕ_j is a polynomial of degree $\le m$ and some ϕ_{i_0} is of degree m. Put $A_m := \langle \tilde{f}, \phi\rangle = \sum_{k=0}^{n} \phi_k \tilde{f}_k$. We may assume that $\phi_n = b_0^n + b_1^n z + \cdots + b_m^n z^m \not\equiv 0$. We note that $\tilde{f}_0, z\tilde{f}_0, ..., z^m\tilde{f}_0, ..., \tilde{f}_{n-1}, z\tilde{f}_{n-1}, ..., z^m\tilde{f}_{n-1}, \tilde{f}_n, ..., z^{m-1}\tilde{f}_n$, A_m are linearly independent over $\mathbb{C}$. Thus we have

$$m_{\tilde{f}}(r, \phi) = \frac{1}{2\pi}\int_0^{2\pi} \log \frac{\|\tilde{f}\|}{|A_m|} d\theta = o(T_{\tilde{f}}(r)), \ //$$

by Lemma 7 and using a similar method to the proof of Theorem 1. Here $s = m(m+1)(n+1)/2$. Therefore, we obtain

$$\delta_{\tilde{f}}(\phi) = \liminf_{r\to+\infty} \frac{m_{\tilde{f}}(r, \phi)}{T_{\tilde{f}}(r)} = 0.$$

We note that $\hat{f} := L_1^{-1} \cdot \tilde{f}$ is also a small deformation of f.

4. Space of Meromorphic Mappings Into $\mathbb{P}^n(\mathbb{C})$

We shall introduce a distance on the space of meromorphic mappings into $\mathbb{P}^n(\mathbb{C})$.

For points $\mathbf{a} = (a_0 : \dots : a_n)$ and $\mathbf{b} = (b_0 : \dots : b_n)$ in $\mathbb{P}^n(\mathbb{C})$, we define the distance $d(\mathbf{a}, \mathbf{b})$ by

$$d_1(\mathbf{a}, \mathbf{b}) := \inf_{\theta} \left\| \frac{\mathbf{a}}{\|\mathbf{a}\|} - e^{i\theta} \frac{\mathbf{b}}{\|\mathbf{a}\|} \right\|.$$

Then $d_1(\mathbf{a}, \mathbf{b})$ satisfies the condition of a distance. Let $f = (f_0 : \dots : f_n)$ and let $g = (g_0 : \dots : g_n)$ be reduced representations of meromorphic mappings of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$. Then the distance $d(f(z), g(z))$ at $z \in \mathbb{C}^m$ is given by

$$d_1(f(z), g(z)) = \inf_{\theta} \left\| \frac{f(z)}{\|f(z)\|} - e^{i\theta} \frac{g(z)}{\|g(z)\|} \right\| \le 2.$$

Define the distance $d(f, g)$ by $d(f, g) := d_1(f, g) + d_2(f, g)$. Here

$$d_1(f, g) := \sum_{n=1}^{\infty} \frac{1}{2^{n+1}} \int_n^{n+1} dt \int_{\partial B(t)} d_1(f(z), g(z))\, \sigma \le 1,$$

which is a distance but does not distinguish rational and transcendental mappings, and

$$\begin{aligned} d_2(f, g) := \liminf_{\alpha \to +1} \limsup_{r \to \infty} \Big\{ & \Big| \frac{T_f(r)}{(\log r)^{1/2} + T_f(r)} - \frac{T_g(r)}{(\log r)^{1/2} + T_g(r)} \Big| \\ & + \Big| \frac{T_f(r)}{(\log r)^{\alpha} + T_f(r)} - \frac{T_g(r)}{(\log r)^{\alpha} + T_g(r)} \Big| \\ & + \sum_{n=1}^{\infty} \Big| \frac{T_f(r)}{r^n + T_f(r)} - \frac{T_g(r)}{r^n + T_g(r)} \Big| \Big\}, \end{aligned}$$

which is a pseudodistance and distinguishes rational and transcendental mappings. Then $d(f, g)$ satisfies the distance conditions on the space of meromorphic mappings into $\mathbb{P}^n(\mathbb{C})$. Here $\partial B(r)$ denotes the boundary of a ball of radius r and σ denotes the normalized surface element as $\int_{\partial B(r)} \sigma = 1$ on $\partial B(r)$.

Note that if f is constant, then $0 \le d_1(f, O) < 1$ and $d_2(f, O) = 0$. Hence $0 \le d(f, O) < 1$. If f is rational, then $0 \le d_1(f, O) < 1$ and $d_2(f, O) = 1$. Hence $1 \le d(f, O) < 2$. If f is transcendental, then $0 \le d_1(f, O) < 1$, while $d_2(f, O) \ge 2$. Hence $d(f, O) \ge 2$. Here O denotes a representation $(1, 0, \dots, 0)$. Therefore we can distinguish constant, rational and transcendental mmappings by this distance.

Now,we consider a space of meromorphic mappings

$$\mathcal{F} := \{f : \mathbb{C}^m \to \mathbb{P}^n(\mathbb{C})| \text{ f is meromorphic}\}.$$

In [4, 5, 6] and this note, a small deformation $\tilde{f} := L_2 \cdot h$ of f is represented as

$$\tilde{f} = (h_0 : h_1 + a_1\zeta_1 h_0 :, ...,: h_n + a_n\zeta_n h_0),$$

where $h = (h_0 : \cdots : h_n) := L_1 \cdot f$. Also, we can choose $(a_1, ..., a_n)$ such that $\|\mathbf{a}\| := |a_1| + \cdots + |a_n|$ is as small as possible. So,we can choose $\hat{f} := L_1^{-1} \cdot \tilde{f}$ which is a small deformation without Nevanlinna defects of f such that $d(\hat{f}, f)$ is as small as possible. Hence transcendental meromorphic mappings without Nevanlinna defects are dense in the space of transcendental meromorphic mappings $\mathcal{F}_0$.

References

1. A. Edrei and W. H. J Fuchs, Entire and meromorphic functions with asymptotically prescribed characteristic, *Canad. J. Math.* **17** (1965), 383-395.
2. H. Fujimoto, Non-integrated defect relation for meromorphic maps of complete Kähler manifolds into $\mathbb{P}^{N_1}(\mathbb{C}) \times \cdots \times \mathbb{P}^{N_k}(\mathbb{C})$, *Japan. J. Math.* **11** (1985), 233-264.
3. W. K. Hayman, *Meromorphic Functions,* Oxford, Clarendon, 1964.
4. S. Mori, Elimination of defects of meromorphic mappings of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$, *Ann. Acad. Sci. Fenn. Math.* **24** (1999), 89-104.
5. S. Mori, Elimination of defects of meromorphic mappings by small deformation, in: *Recent Develpoments in Complex Analisis and Computer Algebra*, Proc. of US-Japan Seminar of the Delaware Conference, Kluwer Acad. Publ., 1999, pp. 247-258.
6. S. Mori, Defects of holomorphic curves into $\mathbb{P}^n(\mathbb{C})$ for rational moving targets and a space of meromorphic mappings, *Complex Variables Theory Appl.* vol. **43** (2000), 363-379.
7. R. E. Molzon, B. Shiffman and N. Sibony, Average growth of estimate for hyperplane section of entire analytic sets, *Math. Ann.* **257** (1987), 43-59.
8. A. Sadullaev, Deficient divisors in Valiron sense, *Math. USSR. Sb.* **36** (1980), 535-547.
9. A. Vitter, The lemma of the logarithmic derivative in several complex variables, *Duke Math. J.* **44** (1977), 89-104.

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