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CANONICAL FORM RELATED WITH RADIAL SOLUTIONS OF SEMILINEAR ELLIPTIC EQUATIONS AND ITS APPLICATIONS

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Abstract. We explain that boundary value problems which satisfy radial solutions are reduced to a canonical form after a suitable change of variables. We introduce structure theorems to the canonical form to equations with power nonlinearities with the homogeneous Dirichlet boundary condition. By virtue of this fact, we can understand known results systematically, make clear unknown structure of various equations.

As applications, we can investigate the structure of radial solutions including all solutions with singularity at $r = 0$ and $r = \infty$ of Matukuma's equation.

1. Introduction

Radial solutions play a fundamental role in the investigation for the structure of solutions in semilinear elliptic equations.

If we restrict to radial solutions $u = u(|x|)$, then solutions of

$$\Delta u + f(|x|, u) = 0$$

satisfy the ordinary differential equation

$$u_{rr} + \frac{n-1}{r} u_r + f(r, u) = 0.$$

However, even in this simpler situation, it is not easy to show the existence, the nonexistence and the uniqueness of solutions only by known standard techniques.

Traditionally, these problems have been studied separately depending on a slight difference of nonlinear terms or the boundary conditions. However, it becomes

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clear gradually that a class of equations, which are apparently different, has similar structures for their solutions.

We will show that the boundary value problems which radial solutions satisfy can be reduced to a certain canonical form by a suitable change of variables. By virtue of this fact, we can understand known results systematically, make clear unknown structure of various equations, and investigate the structure more precisely.

2. Esamples of Equations

We will investigate the structure of radial solutions of equations with the simplest nonlinearity including $K(r)u^p$. Even for equations with such innocent-looking nonlinearity, the structure of solutions can very sensitively depend on $K(r)$.

The typical examples of equations are as follows:

(2.1) $$\Delta u + u^p = 0$$ Lane-Emden equation (Emden-Fowler equation),

(2.2) $$\Delta u + \frac{1}{1+|x|^2} u^p = 0$$ Matukuma equation,

(2.3) $$\Delta u + \frac{1}{1+|x|^\sigma} u^p = 0$$ Matukuma-type equation,

(2.4) $$\Delta u + \frac{|x|^{\sigma-2}}{(1+|x|^2)^{\sigma/2}} u^p = 0$$ Batt-Faltenbacher-Horst equation,

(2.5) $$\Delta u + K(|x|)u^{(n+2)/(n-2)} = 0$$ scalar curvature equation,

(2.6) $$\Delta u - u + u^p = 0$$ scalar field equation

(2.7) $$\Delta u + \lambda u + u^p = 0$$ Brezis-Nirenberg equation.

For each given equation, for instance,

$$\Delta u + K(|x|)u^p = 0,$$

there are various kinds of problems with various domains such as

$$\begin{aligned}
&\Delta u + K(|x|)u^p = 0 \quad \text{in} \quad \mathbb{R}^n, \\
&\Delta u + K(|x|)u^p = 0 \quad \text{in} \quad \mathbb{R}^n \setminus \{0\}, \\
&\Delta u + K(|x|)u^p = 0 \quad \text{in} \quad B, \\
&\Delta u + K(|x|)u^p = 0 \quad \text{in} \quad B \setminus \{0\}, \\
&\Delta u + K(|x|)u^p = 0 \quad \text{in} \quad \mathbb{R}^n \setminus B, \\
&\Delta u + K(|x|)u^p = 0 \quad \text{in} \quad A,
\end{aligned}$$

and with various boundary conditions, where B is a ball and A is an annulus.

3. Transformation to a Canonical Form

All examples in the previous section are represented by the boundary value problems with various kinds of boundary conditions to

$$u'' + \alpha(x)u' + \beta(x)u + \gamma(x)u^p = 0, \quad -\infty \le a < x < b \le +\infty.$$

It was found by recent studies that the above boundary value problems with the nonlinear term u^p can be reduced to some canonical forms [15, 5, 19, 7].

This means that not only can we treat various problems in a unified way, but also apply results of one equation to other equations through canonical forms. Moreover, the implication of the Kelvin transformation and the Rellich-Pohozaev identity for the original equation can be understood in a more natural manner through a simple canonical form.

We shall only demonstrate the basic ideas here. By suitable changes of variables [19], each of the following four problems:

(i) Dirichlet problem of Lane-Emden's equation on the unit ball,
(ii) positive entire solutions of Matukuma's equation,
(iii) Dirichlet problem of Brezis-Nirenberg's equation on the unit ball,
(iv) positive entire solutions of the scalar field equation,

can be reduced to the Dirichlet problem for

$$v_{tt}(t) + k(t)v^p = 0, \quad t \in (0,1). \tag{3.1}$$

We note that various information such as the dimension, domain, boundary conditions, spatial inhomogeneity are compressed into $k(t)$.

First, we can show that the equations in the above problems can be rewritten as

$$\frac{1}{g(r)}\{g(r)u_r\}_r + L(r)u^p = 0 \ \text{ in } (a,b), \tag{3.2}$$

where $g(r)$ and $L(r)$ are positive smooth functions defined on (a,b) (see [19]). In fact, we may put $g(r) = r^{n-1}$, $a = 0$, $b = \infty$, $L(r) = K(r)$ for (ii). We can deal with (i) similarly. In the cases of (iii) and (iv), by using solutions of

$$\varphi_{rr} + \frac{n-1}{r}\varphi_r \pm \lambda\varphi = 0, \quad \varphi(0) = 1,$$

we can eliminate the linear term through a transformation $u \mapsto u\varphi$ to reduce the equations to (3.2). For instance, for (iii) with $n = 3$ and $0 < \lambda < \pi^2$, we have

$$\varphi(r) = \sin(\mu r)/(\mu r),\ g(r) = \sin^2(\mu r)/\mu^2,\ a = 0,\ b = 1,$$
$$L(r) = (\sin^2(\mu r)(\cot(\mu r) - \cot(\mu))/(\mu r))^4,\ \mu = \lambda^{1/2}.$$

Further, changing the variables as

$$v(t) := tu(r), \quad t := \left\{1 + \int_b^r \frac{1}{g(s)}\, ds\right\}^{-1}$$

and putting

$$k(t) = t^{-(3+p)} g(r)^2 L(r),$$

we can reduce (3.2) to (3.1).

We note that the function $k(t)$ is smooth on (0,1), but it may not be bounded at boundaries. Hence, we need to introduce the following definition. For any positive solution $v(t)$ of (3.1), it is easy to show that $v(t)/(1-t)$ is decreasing in t, which implies that the behavior of $v(t)$ near $t = 1$ is classified as follows.

(i) $v(t)$ is said to be *regular* at $t = 1$ if $\lim_{t\to 1} v(t)/(1-t)$ exists and is positive.
(ii) $v(t)$ is said to be *singular* at $t = 1$ if $\lim_{t\to 1} v(t)/(1-t) = \infty$.

Let us examine the relation between these definitions and those of the original problems. When $t = 1$ corresponds to the boundary of the unit ball, regular solutions in the above definitions correspond to solutions that satisfy the homogeneous (or, zero) Dirichlet boundary condition in a usual sense. While singular solutions are not Lipshitz continuous at $r = 1$ even if they converge to zero as $r \to 1$, or have positive limit as $r \to 1$, when $t = 1$ corresponds to $r = \infty$, a regular solution corresponds to a rapid-decay solution, and a singular solution corresponds to a slow-decay solution.

The Kelvin transformation $u(r) \mapsto s^{n-2}u(s)$ with $s = 1/r$, which is known to be very useful in the analysis of elliptic equations, exchanges the infinity and the origin. In a canonical form, this transformation corresponds to a simple reflection $v(t) \mapsto v(s)$, $s = 1 - t$ with respect to $t = 1/2$.

4. Structure Theorems to the Canonical Form

Here, we will only explain the structure of positive solutions with the Dirichlet boundary condition at $t = 0$ and $t = 1$. The following results are due to Yanagida and Yotsutani [14, 19].

Let us introduce an auxiliary initial value problem

$$v_{tt}(t) + k(t)v_+^p = 0 \quad \text{in } (0,1), \quad \lim_{t\to -1} \frac{v(t)}{t} = \alpha > 0, \tag{4.1}$$

where $v_+ = \max\{v, 0\}$.

We first state the condition on $k(t)$ for which (4.1) has a unique solution.

Lemma 4.1. *Suppose that* $k(t) \in C((0,1))$, $k(t) \geq 0$ *in* (0,1). *The initial value problem* (4.1) *has a unique solution* $v(t) \in C([0,1)) \cap C^2((0,1))$, *if and only if* $t^p k(t) \in L^1(0, 1/2)$.

Remark 4.1. This kind of lemma is well-known for Lane-Emden type equations (see, e.g., Ni [11], and Propositions 4.1 and 4.2 of [12]).

Thus, the initial value problem (4.1) has a unique solution $v(t) \in C([0,1)) \cap C^1((0,1))$ under the condition

$$t^p k(t) \in C^1((0,1)) \cap L^1(0,1/2), \quad k(t) \geq 0, \not\equiv 0 \text{ in } (0,1), \quad p > 1. \tag{4.2}$$

We will denote the unique solution by $v(t;\alpha)$. Let us discuss the structure of positive solutions to (4.1) under the condition (4.2).

We prepare notation to state a structure theorem. Let $G(t) = G(t;k(\cdot))$ and $H(t) = H(t;k(\cdot))$ be functions defined by

$$G(t) := \frac{1}{p+1} t^{p+2}(1-t)k(t) - \frac{1}{2}\int_0^t \tau^{p+1}k(\tau)d\tau, \tag{4.3}$$

$$H(t) := \frac{1}{p+1} t(1-t)^{p+2}k(t) - \frac{1}{2}\int_t^1 (1-\tau)^{p+1}k(\tau)d\tau. \tag{4.4}$$

We note that $G(t)$ is well-defined under the condition (4.2), $H(t)$ is well-defined provided that $(1-t)^{p+1}k(t) \in L^1(1/2,1)$, and

$$\liminf_{t\downarrow 0} G(t) = 0, \tag{4.5}$$

$$\liminf_{t\uparrow 1} H(t) = 0, \tag{4.6}$$

$$\frac{d}{dt}G(t) := \frac{1}{p+1}\{(t(1-t))^{\frac{p+3}{2}}k(t)\}_t \left(\frac{t}{1-t}\right)^{\frac{p+1}{2}}, \tag{4.7}$$

$$\frac{d}{dt}H(t) := \frac{1}{p+1}\{(t(1-t))^{\frac{p+3}{2}}k(t)\}_t \left(\frac{1-t}{t}\right)^{\frac{p+1}{2}}, \tag{4.8}$$

and

$$H(t;k(\cdot)) = G(1-t;k(1-\cdot)). \tag{4.9}$$

Finally, we define

$$t_G := \inf\{t \in (0,1); G(t) < 0\}, \tag{4.10}$$

$$t_H := \sup\{t \in (0,1); H(t) < 0\}. \tag{4.11}$$

Here we put $t_G = 1$ if $G(t) \geq 0$ on (0,1), and $t_H = 0$ if $H(t) \geq 0$ on (0,1). Thus $0 \leq t_G,\ t_H \leq 1$ by the definition.

Now, we state structure theorems. The first thorem is a consequence of (ii) of Lemma 4.1.

Theorem 4.1. *Let* (4.2) *be satisfied. If* $(1-t)^p k(t) \notin L^1(1/2,1)$, *then the structure of solutions is* Type C : $v(t;\alpha)$ is a crossing solution for all $\alpha > 0$.

Let us consider the case $(1-t)^p k(t) \in L^1(1/2,1)$. The following case is "critical".

Theorem 4.2. *If* $k(t) = C \cdot \{t(1-t)\}^{-(p+3)/2}$ *with a positive constant* C *and* $p > 1$, *then the structure of solutions is* Type R : $v(t;\alpha)$ is a regular solution at $t=1$ for all $\alpha > 0$. More precisely, $v(t;\alpha)$ is explicitly represented by

$$v(t;\alpha) = \alpha\, t \left\{ 1 + \frac{2C\alpha^{p-1}}{p+1} \left(\frac{t}{1-t} \right)^{(p-1)/2} \right\}^{-2/(p-1)}. \tag{4.12}$$

Remark 4.2. Under (4.2), the condition $k(t) = C \cdot \{t(1-t)\}^{-(p+3)/2}$ with a positive constant C is equivalent to $G(t) \equiv 0$ in (0,1).

Let us consider the case $G(t) \not\equiv 0$. Under the condition $t_H \le t_G$, we can completely classify the structure.

Theorem 4.3. *Let* (4.2), $(1-t)^p k(t) \in L^1(1/2,1)$ *and* $G(t) \not\equiv 0$ *be satisfied. Then the following hold.*

(i) *If* $t_G = 1$, *then the structure is of* Type C.

(ii) *If* $0 = t_H \le t_G < 1$, *then the structure is of* Type S : $v(t;\alpha)$ *is a singular solution for every* $\alpha > 0$.

(iii) *If* $0 < t_H \le t_G < 1$, *then the structure is of* Type M : *There exists a unique positive number* α_r *such that*

$$v(t;\alpha)\ is\ a\ crossing\ solution\ for\ every\ \alpha \in (\alpha_r, \infty),$$

$$v(t;\alpha)\ is\ a\ regular\ solution\ for\ \alpha = \alpha_r,$$

$$v(t;\alpha)\ is\ a\ singular\ solution\ for\ every\ \alpha \in (0, \alpha_r).$$

The next theorem implies that the condition $t_H \le t_G$ is sharp.

Theorem 4.4. *Let* a *and* b *be any given numbers with* $0 \le a < b \le 1$. *Then there exists* $k(t)$ *with* $t_G = a$ *and* $t_H = b$ *such that the structure of solutions to* (4.1) *is not of* Type C, Type S, Type M *nor* Type R.

These theorems are obtained by using Yanagida-Yotsutani [14] combining with the change of variables in reducing to the canonical form [19].

5. Applications

Now, it is possible through the canonical forms to convert results for one problem to that of others. Based on this idea, several results have been obtained by applying Theorems 4.1-4.4 concerning positive entire solutions of the scalar field equation [15], the equation with a gradient term

$$\Delta u + x \cdot \nabla u + \lambda u + u^p = 0$$

(see, e.g., [3, 4]), the Neumann problem and the third boundary problem of Brezis-Nirenberg's equation [1, 6], the structure of radial solutions with possible singularities at $r = 0$ and $r = \infty$ of Matukuma's equation [10]. It seems difficult to obtain these results by previously known methods.

Though systematic applications are now getting started, it is expected that we can understand the relation among known results in an organic way, which in turn contributes to the progress in the study for each equation.

As an concrete example of the application of the canonical form, we show the structure of all positive radial solutions of Matukuma's equation

$$\Delta u + \frac{1}{1+|x|^2} u^p = 0 \quad \text{in} \quad \mathbb{R}^3 \tag{5.1}$$

including both regular and all singular solutions.

There are a lot of results about the structure of positive solutions including all regular solutions or some class of singular solutions. The above problems are reduced to investigating the structure of solutions of one parameter (see, e.g., Ding-Ni [2], Yanagida–Yotsutani [14, 15, 16, 17, 18] and Lin–Takagi [9]).

However, it seems that there is no result about the structure of all positive radial solutions including both regular and all singular solutions.

This problem is reduced to investigating the structure of solutions of two parameters. We need new devices to treat the problem of two parameters.

Since we are interested in the all radial solutions, we investigate the structure of all solutions of

$$u_{rr} + \frac{2}{r} u_r + \frac{1}{1+r^2} u_+^p = 0 \ (r > 0), \ u_r(1) = \mu, \ u(1) = \nu > 0, \tag{5.2}$$

where μ and ν are given real numbers, and $u_+ = \max\{u, 0\}$. We note that the equation (5.2) has the unique solution. We denote it by $u = u(r; \mu, \nu)$.

We can classify each solution of (5.2) according to its behavior as $r \to \infty$. We say that

(i) $u(r; \mu, \nu)$ is a crossing solution in $(1, \infty)$ if $u(r; \mu, \nu)$ has a zero in $(1, \infty)$,

(ii) $u(r; \mu, \nu)$ is a slow-decay solution at $r = \infty$ if $u(r; \mu, \nu) > 0$ on $(1, \infty)$ and $\lim_{r\to\infty} ru(r; \mu, \nu) = \infty$,

(iii) $u(r;\mu,\nu)$ is a rapid-decay solution if $u(r;\mu,\nu) > 0$ on $(1,\infty)$ and $\lim_{r\to\infty} ru(r;\mu,\nu)$ exists and is finite and positive.

Similarly, we can classify each solution of (5.2) according to its behavior as $r \to 0$. We say that

(i) $u(r;\mu,\nu)$ is a crossing solution in (0,1) if $u(r;\mu,\nu)$ has a zero in (0,1),

(ii) $u(r;\mu,\nu)$ is a singular solution at $r = 0$ if $u(r;\mu,\nu) > 0$ on (0,1) and $\lim_{r\to 0} u(r;\mu,\nu) = \infty$,

(iii) $u(r;\mu,\nu)$ is a regular solution if $u(r;\mu,\nu) > 0$ on (0,1) and $\lim_{r\to 0} u(r;\mu,\nu)$ exists and is finite and positive.

Now we state the main theorems due to Morishita–Yanagida –Yotsutani [10].

Theorem 5.1. *Let $p > 1$ be fixed. There exists a continuous function $R(\theta;p) \in C([0,3\pi/4])$ with $R(\theta;p) > 0$ for $\theta \in [0,3\pi/4)$ and $R(3\pi/4;p) = 0$ such that*

(i) *if $(\mu,\nu) \in R_{out}$ then $u(r;\mu,\nu)$ is a crossing solution in $(1,\infty)$;*

(ii) *if $(\mu,\nu) \in R_{on}$ then $u(r;\mu,\nu)$ is a rapid-decay solution at $r = \infty$;*

(iii) *if $(\mu,\nu) \in R_{in}$ then $u(r;\mu,\nu)$ is a slow-decay solution at $r = \infty$,*

where

$$\begin{aligned}
R_{out} &:= \{(\rho\cos\theta, \rho\sin\theta) : \rho > R(\theta;p), 0 < \theta < 3\pi/4\},\\
R_{on} &:= \{(R(\theta;p)\cos\theta, R(\theta;p)\sin\theta) : 0 < \theta < 3\pi/4\},\\
R_{in} &:= \{(\rho\cos\theta, \rho\sin\theta) : 0 < \rho < R(\theta;p), 0 < \theta < 3\pi/4\}.
\end{aligned}$$

Theorem 5.2. *Let p with $1 < p \le 5$ be fixed. There exists a continuous function $L(\theta;p) \in C([\pi/2,\pi])$ with $L(\pi/2;p) = 0$, $L(\theta;p) > 0$ for $\theta \in (\pi/2,\pi)$, $L(\pi;p) > 0$ $(0 < p < 5)$, $L(\pi;p) = 0$ $(p = 5)$ such that*

(i) *if $(\mu,\nu) \in L_{out}$ then $u(r;\mu,\nu)$ is a crossing solution in $(0,1)$;*

(ii) *if $(\mu,\nu) \in L_{on}$ then $u(r;\mu,\nu)$ is a regular solution at $r = 0$;*

(iii) *if $(\mu,\nu) \in L_{in}$ then $u(r;\mu,\nu)$ is a singular solution at $r = 0$,*

where

$$\begin{aligned}
L_{out} &:= \{(\rho\cos\theta, \rho\sin\theta) : \rho > L(\theta;p), \pi/2 < \theta < \pi\},\\
L_{on} &:= \{(L(\theta;p)\cos\theta, L(\theta;p)\sin\theta) : \pi/2 < \theta < \pi\},\\
L_{in} &:= \{(\rho\cos\theta, \rho\sin\theta) : 0 < \rho < L(\theta;p), \pi/2 < \theta < \pi\}.
\end{aligned}$$

Theorem 5.3. *The following relations hold* :

(i) *If $1 < p < 5$, then $L_{on} \cap R_{on} = \{one\ point\}$.*

(ii) *If* $p = 5$, *then* $L_{in} \cup L_{on} \subset R_{in}$.

6. Outline of the Proof of Theorems 5.1-5.3

It is difficult to treat (5.2) directly. Instead of (5.2), we introduce the following initial value problem satisfying the third boundary condition

$$u_{rr} + \frac{2}{r}u_r + K(r)u_+^p = 0 \ (r > 0), \ u_r(1) = \ell \, u(1), \ u(1) = \nu > 0, \tag{6.1}$$

where $K(r) = (1+r^2)^{-1}$, ℓ is a fixed real number, and positive number ν is moved. The unique solution of (6.1) is denoted by $u(r; \ell\nu, \nu)$.

The following proposition is the first step.

Proposition 6.1. *Let* $p > 1$. *The following properties hold.*

(i) *If* $\ell \leq -1$, *then* $u(r; \ell\nu, \nu)$ *is a crossing solution in* $(1, \infty)$ *for all* $\nu > 0$.

(ii) *If* $\ell \geq 0$, *then* $u(r; \ell\nu, \nu)$ *is a crossing solution in* $(0, 1)$ *for all* $\nu > 0$.

Let us consider (6.1) in $(1, \infty)$ by fixing the parameter $\ell > -1$.

Proposition 6.2. *Let* $p > 1$ *and* $\ell > -1$. *There exists a unique* $\nu^* = \nu^*(\ell; p)$ *such that* $\nu^*(\ell; p)$ *is continuous with respect to* $\ell \in (-1, \infty)$, $\nu^*(\ell; p) \to 0$ *as* $\ell \to -1$, $\nu^*(\ell; p) \to \nu^*(\infty; p)$ *as* $\ell \to \infty$ *for some* $\nu^*(\infty; p) > 0$, *and*

(i) $u(r; \ell\nu, \nu)$ *is a crossing solution in* $(1, \infty)$ *for* $\nu \in (\nu^*, \infty)$,

(ii) $u(r; \ell\nu^*, \nu^*)$ *is a rapid-decay solution*,

(iii) $u(r; \ell\nu, \nu)$ *is a slow-decay solution in* $(1, \infty)$ for $\nu \in (0, \nu^*)$.

Let us consider (6.1) in the interval (0,1) by fixing the parameter $\ell < 0$.

Proposition 6.3. *Let* $1 < p \leq 5$ *and* $\ell < 0$. *There exists a unique* $\nu_* = \nu_*(\ell; p)$ *such that* $\nu_*(\ell; p)$ *is continuous with respect to* $\ell \in (-\infty, 0)$, $\nu_*(\ell; p) \to 0$ *as* $\ell \to 0$, $\nu_*(\ell; p) \to \nu_*(-\infty; p)$ *as* $\ell \to -\infty$ *for some* $\nu_*(-\infty; p) > 0$, *and*

(i) $u(r; \ell\nu, \nu)$ *is a crossing solution in* (0,1) *for* $\nu \in (\nu_*, \infty)$,

(ii) $u(r; \ell\nu_*, \nu_*)$ *is a regular solution at* $r = 0$,

(iii) $u(r; \ell\nu, \nu)$ *is a singular solution at* $r = 0$ *for* $\nu \in (0, \nu_*)$.

We can prove Theorem 5.1 by using Propositions 6.1 and 6.2. Similarly, we can prove Theorem 5.2 by using Propositions 6.1 and 6.3. For the proof of Theorem 5.3, we combine Theorems 5.1, 5.2 and the following facts (see, e.g., [12, 8, 13]).

Proposition 6.4. *The following facts hold* :

(i) *If* $1 < p < 5$, *then there exists a unique positive solution of* (5.2), *which is regular at* $r = 0$ *and rapid-decay at* $r = \infty$.

(ii) *If* $p \geq 5$, *then there exists no positive solutions of* (5.2) *which are regular at* $r = 0$ *and rapid-decay at* $r = \infty$.

We explain the idea of the proof of the propositions briefly. We transform the equation (6.1) to

$$v_{tt} + k(t)v_+^p = 0 \ (-1 < t < 1), \ v_t(0) = m\ v(0), \ v(0) = \nu > 0, \tag{6.2}$$

where

$$\begin{aligned} v(t) &:= (1+t)u(r), \ r := (1+t)/(1-t), \\ k(t) &:= 4(1+t)^{1-p}(1-t)^{-4}K((1+t)/(1-t)) \\ &= 4(1+t)^{1-p}(1-t)^{-2}/\{(1+t)^2 + (1-t)^2\}, \\ m &:= 2\ell + 1. \end{aligned}$$

We denote the unique solution of (6.2) by $v = v(t;\nu)$. We may investigate the behavior of solutions of (6.2). For instance, Proposition 6.1 is equivalent to the following lemma.

Lemma 6.1. *Let* $p > 1$. *The following properties hold.*

(i) *If* $m \leq -1$, *then* $v(t;\nu)$ *has a zero in* (0,1) *for all* $\nu > 0$.

(ii) *If* $m \geq 1$, *then* $v(t;\nu)$ *has a zero in* $(-1, 0)$ *for all* $\nu > 0$.

We can prove Propositions 6.2 and 6.3 by applying Theorems 4.1 – 4.3, and their modifications to the third boundary condition [7].

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