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MAXIMAL REGULARITY FOR INTEGRO-DIFFERENTIAL EQUATION ON PERIODIC TRIEBEL-LIZORKIN SPACES

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Abstract. We study maximal regularity on Triebel-Lizorkin spaces $F^s_{p,q}(\mathbb{T}, X)$ for the integro-differential equation with infinite delay: (P_2): $u'(t) = Au(t) + \int_{-\infty}^{t} a(t-s)Au(s)ds + f(t)$, $(0 \le t \le 2\pi)$ with the periodic condition $u(0) = u(2\pi)$, where X is a Banach space, $a \in L^1(\mathbb{R}_+)$ and f is an X-valued function. Under a suitable assumption (H3) on the Laplace transform of a, we give a necessary and sufficient condition for (P_2) to have the maximal regularity property on $F^s_{p,q}(\mathbb{T}, X)$.

1. Introduction

In a series of recent publications operator-valued Fourier multipliers on vector-valued function spaces are studied (see e.g. [1-4, 6-12, 14] and [15]). They are useful in the study of the existence and uniqueness of solutions of differential equations on Banach spaces. In [2-4], the authors study the maximal regularity property of the classical abstract non-homogeneous boundary problem (P_1) on L_p spaces, Besov spaces and Triebel-Lizorkin spaces.

$$(P_1) \quad \begin{cases} u'(t) = Au(t) + f(t), & 0 \le t \le 2\pi \\ u(0) = u(2\pi) \end{cases}$$

here X is a Banach space, A is a closed linear operator in X and f is an X-valued function defined on $[0, 2\pi]$. The problem (P_1) has the maximal regularity property on L_p spaces if and only if $i\mathbb{Z} \subset \rho(A)$ and the set $(ikR(ik, A))_{k\in\mathbb{Z}}$ is Rademacher bounded whenever X is a UMD space and $1 < p < \infty$ [2], where

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$R(ik, A)$ is the resolvent $(ik - A)^{-1}$ of A. In the Besov spaces and Triebel-Lizorkin spaces setting, maximal regularity is equivalent to the fact that $i\mathbb{Z} \subset \rho(A)$ and $\sup_{k\in\mathbb{Z}} \|kR(ik, A)\| < \infty$ (see [3,4]).

In this paper, we are interested in a more general evolution equation, namely the integro-differential equation with infinite delay:

$$(P_2) \quad \begin{cases} u'(t) = Au(t) + \int_{-\infty}^{t} a(t-s)Au(s)ds + f(t), \quad 0 \le t \le 2\pi \\ u(0) = u(2\pi) \end{cases}$$

where $a \in \mathrm{L}^1(\mathbb{R}_+)$ is fixed. In [11], Keyantuo and Lizama have considered maximal regularity on periodic Besov spaces $\mathrm{B}^s_{p,q}(\mathbb{T}, X)$ for (P_2). They have shown that if $c_k = \tilde{a}(ik)$ is the Laplace transform of a at ik and if $(b_k)_{k\in\mathbb{Z}}$ is 2-regular and $(c_k)_{k\in\mathbb{Z}}$ satisfies a suitable assumption (H2), then the problem (P_2) has maximal regularity property on $\mathrm{B}^s_{p,q}(\mathbb{T}, X)$ if and only if $(b_k)_{k\in\mathbb{Z}} \subset \rho(A)$ and $\sup_{k\in\mathbb{Z}} \|b_kR(b_k, A)\| < \infty$, where $b_k = \frac{ik}{1+c_k}$. In this paper, we are interested in the maximal regularity property on Triebel-Lizorkin spaces $\mathrm{F}^s_{p,q}(\mathbb{T}, X)$ for the same problem (P_2). We show that if the sequence $(c_k)_{k\in\mathbb{Z}}$ satisfies a similar assumption (H3), then (P_2) has the maximal regularity property on $\mathrm{F}^s_{p,q}(\mathbb{T}, X)$ if and only if $(b_k)_{k\in\mathbb{Z}} \subset \rho(A)$ and $\sup_{k\in\mathbb{Z}} \|b_kR(b_k, A)\| < \infty$. This recovers the known result obtained in [4] when $a = 0$. Here a similar assumption of 2-regularity or 3-regularity on $(b_k)_{k\in\mathbb{Z}}$ is not needed. The main tool in the study of maximal regularity on $\mathrm{B}^s_{p,q}(\mathbb{T}, X)$ for the problem (P_2) is an operator-valued Fourier multiplier theorem on $\mathrm{B}^s_{p,q}(\mathbb{T}, X)$ obtained in [3]. The operator-valued Fourier multiplier theorem on $\mathrm{F}^s_{p,q}(\mathbb{T}, X)$ proved in [4] will be fundamental for us in this paper.

The sufficient condition obtained in [3] for a sequence $(M_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(X)$ to be a $\mathrm{B}^s_{p,q}$-multiplier is a Marcinkiewicz condition of order 2, while in the $\mathrm{F}^s_{p,q}$-multiplier case a stronger Marcinkiewicz condition of order 3 is needed [4]. This is the reason why our assumption (H3) is stronger than the assumption (H2) used in [11] in the Besov space case. It turns out that when $1 < p < \infty$, $1 < q \le \infty$ and $s \in \mathbb{R}$, then a Marcinkiewicz condition of order 2 is already sufficient for a sequence $(M_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(X)$ to be an $\mathrm{F}^s_{p,q}$-multiplier [4], in this case under the weaker assumption (H2) on $(c_k)_{k\in\mathbb{Z}}$, the problem (P_2) has the maximal regularity property on $\mathrm{F}^s_{p,q}(\mathbb{T}, X)$ if and only if $(b_k)_{k\in\mathbb{Z}} \subset \rho(A)$ and $\sup_{k\in\mathbb{Z}} \|b_kR(b_k, A)\| < \infty$.

This paper is organized as follows: Section 2 collects definitions and basic properties of vector-valued Triebel-Lizorkin spaces and Fourier multipliers. In section 3, we establish the periodic solution for the integro-differential equation (P_2) on Triebel-Lizorkin spaces $\mathrm{F}^s_{p,q}(\mathbb{T}, X)$.

2. Triebel-lizorkin Spaces and the Preliminaries

Let X be a Banach space and let $f \in \mathrm{L}^1(\mathbb{T}, X)$, we denote by

$$\hat{f}(k) = \frac{1}{2\pi}\int_0^{2\pi} e_{-k}(t)f(t)dt$$

the k-th Fourier coefficient of f, where $k \in \mathbb{Z}$, $\mathbb{T} = [0, 2\pi]$ (the points 0 and 2π are identified), and $e_k(t) = e^{ikt}$. For $k \in \mathbb{Z}$ and $x \in X$, we denote by $e_k \otimes x$ the X-valued function defined by $(e_k \otimes x)(t) = e_k(t)x$.

Firstly, we briefly recall the definition of periodic Triebel-Lizorkin spaces in the vector-valued case introduced in [4] (see the monograph [13] for the scalar-valued case). Let $\mathcal{S}(\mathbb{R})$ be the Schwarz space of all rapidly decreasing smooth functions on $\mathbb{R}$. Let $\mathcal{D}(\mathbb{T})$ be the space of all infinitely differentiable functions on $\mathbb{T}$ equipped with the locally convex topology given by the seminorms $\|f\|_\alpha = \sup_{x\in\mathbb{T}} |f^{(\alpha)}(x)|$, where $\alpha \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Let $\mathcal{D}'(\mathbb{T}, X) := \mathcal{L}(\mathcal{D}(\mathbb{T}), X)$ be the space of all bounded linear operators from $\mathcal{D}(\mathbb{T})$ to X. In order to define the periodic Triebel-Lizorkin spaces, we consider the dyadic-like subsets of $\mathbb{R}$:

$$I_0 = \{t \in \mathbb{R} : \ |t| \le 2\}, I_k = \{t \in \mathbb{R} : \ 2^{k-1} < |t| \le 2^{k+1}\}$$

for $k \in \mathbb{N}$. Let $\phi(\mathbb{R})$ be the set of all systems $\phi = (\phi_k)_{k\in\mathbb{N}_0} \subset \mathcal{S}(\mathbb{R})$ satisfying $supp(\phi_k) \subset \bar{I}_k$ for each $k \in \mathbb{N}_0$,

$$\sum_{k\in\mathbb{N}_0} \phi_k(x) = 1 \qquad \text{for } x \in \mathbb{R},$$

and for each $\alpha \in \mathbb{N}_0$

$$\sup_{\substack{x\in\mathbb{R}\\ k\in\mathbb{N}_0}} 2^{k\alpha}|\phi_k^{(\alpha)}(x)| < \infty.$$

Let $\phi = (\phi_k)_{k\in\mathbb{N}_0} \in \phi(\mathbb{R})$ be fixed. For $1 \le p < \infty$, $1 \le q \le \infty$, $s \in \mathbb{R}$, the X-valued periodic Triebel-Lizorkin space is defined by

$$\begin{aligned} &\mathrm{F}^s_{p,q}(\mathbb{T}, X) := \{f \in \mathcal{D}'(\mathbb{T}, X) : \ \|f\|_{\mathrm{F}^s_{p,q}} : \\ &= \|(\sum_{j\ge 0} 2^{sjq}\|\sum_{k\in\mathbb{Z}} e_k \otimes \phi_j(k)\hat{f}(k)\|^q)^{1/q}\|_{L_p} < \infty\} \end{aligned} \tag{2.1}$$

with the usual modification if $q = \infty$. The space $\mathrm{F}^s_{p,q}(\mathbb{T}, X)$ is independent from the choice of ϕ and different choices of ϕ lead to equivalent norms on $\mathrm{F}^s_{p,q}(\mathbb{T}, X)$. $\mathrm{F}^s_{p,q}(\mathbb{T}, X)$ equipped with the norm $\|\cdot\|_{\mathrm{F}^s_{p,q}}$ is a Banach space. See [4, Section 2] for more information about the spaces $\mathrm{F}^s_{p,q}(\mathbb{T}, X)$.

Next, we discuss the Fourier multipliers on Triebel-Lizorkin spaces. Let X and Y be Banach spaces, We denote by $\mathcal{L}(X, Y)$ the space of all bounded linear operators from X to Y. If $X = Y$, we will simply denote it by $\mathcal{L}(X)$. Let $1 \le p < \infty$, $1 \le q \le \infty$, $s \in \mathbb{R}$, and let $(M_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(X, Y)$. We will say that $(M_k)_{k\in\mathbb{Z}}$ is an $\mathrm{F}^s_{p,q}$-multiplier, if for each $f \in \mathrm{F}^s_{p,q}(\mathbb{T}, X)$, there exists $g \in \mathrm{F}^s_{p,q}(\mathbb{T}, Y)$, such that $\hat{g}(k) = M_k \hat{f}(k)$ for all $k \in \mathbb{Z}$. In this case it follows from the Closed Graph Theorem that there exists a constant $C > 0$ such that for $f \in \mathrm{F}^s_{p,q}(\mathbb{T}, X)$, we have $\| \sum_{k\in\mathbb{Z}} e_k \otimes M_k \hat{f}(k)\|_{\mathrm{F}^s_{p,q}} \le C\|f\|_{\mathrm{F}^s_{p,q}}$.

The following $\mathrm{F}^s_{p,q}$-multiplier theorem is due to Bu and Kim [4, Theorem 3.2]:

Theorem 2.1. *Let X and Y be Banach spaces and let $(M_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(X, Y)$.*

(i) *Assume that $(M_k)_{k\in\mathbb{Z}}$ satisfies a Marcinkiewicz estimate of order 3:*

$$\begin{aligned}
&\sup_{k\in\mathbb{Z}} \|M_k\| < \infty\\
&\sup_{k\in\mathbb{Z}} \|k(M_{k+1} - M_k)\| < \infty\\
&\sup_{k\in\mathbb{Z}} \|k^2(M_{k+1} - 2M_k + M_{k-1})\| < \infty\\
&\sup_{k\in\mathbb{Z}} \|k^3(M_{k+1} - 3M_k + 3M_{k-1} - M_{k-2})\| < \infty.
\end{aligned} \tag{2.2}$$

Then $(M_k)_{k\in\mathbb{Z}}$ is an $\mathrm{F}^s_{p,q}$-multiplier whenever $1 \le p < \infty$, $1 \le q \le \infty$, $s \in \mathbb{R}$.

(ii) *If $(M_k)_{k\in\mathbb{Z}}$ satisfies the first three conditions of (2.2), then $(M_k)_{k\in\mathbb{Z}}$ is an $\mathrm{F}^s_{p,q}$-multiplier whenever $1 < p < \infty$, $1 < q \le \infty$ and $s \in \mathbb{R}$.*

Remark 2.2. We notice that even the underlying Banach spaces X, Y are UMD spaces and $1 < p < \infty$, a stronger condition is needed to ensure a sequence $(M_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(X, Y)$ to be an L^p-multiplier: the sets $\{M_k : k \in \mathbb{Z}\}$ and $\{k(M_{k+1} - M_k) : k \in \mathbb{Z}\}$ are Rademacher bounded [2]. Here for Triebel-Lizorkin spaces, we impose conditions on the first three derivatives of $(M_k)_{k\in\mathbb{Z}}$, but the result is true without any conditions on the geometry of the underlying Banach spaces and no assumption of Rademacher boundedness is needed.

Next we give some preliminaries. Given $a \in \mathrm{L}^1(\mathbb{R}_+)$ and $u : [0, 2\pi] \to X$ (extended by periodicity to $\mathbb{R}$), we define

$$F(t) = (a \ast u)(t) := \int_{-\infty}^{t} a(t - s)u(s)ds. \tag{2.3}$$

Let $\tilde{a}(\lambda) = \int_0^{+\infty} e^{-\lambda t} a(t)dt$ be the Laplace transform of a. An easy computation shows that:

$$\hat{F}(k) = \tilde{a}(ik)\hat{u}(k),\ (k \in \mathbb{Z}). \tag{2.4}$$

The notion of 1-regular and 2-regular scalar sequences were introduced in [11] to study the maximal regularity property on periodic Besov spaces for the problem (P_2). We will need the notion of 3-regular scalar sequences in the Triebel-Lizorkin space case: A sequence $(a_k)_{k\in\mathbb{Z}} \subset \mathbb{C}\backslash\{0\}$ is called 1-regular if the sequence $(k(a_{k+1} - a_k)/a_k)_{k\in\mathbb{Z}}$ is bounded; it is called 2-regular if it is 1-regular and the sequence $(k^2(a_{k+1} - 2a_k + a_{k-1})/a_k)_{k\in\mathbb{Z}}$ is bounded; it is called 3-regular if it is 2-regular and the sequence $(k^3(a_{k+1} - 3a_k + 3a_{k-1} - a_{k-2})/a_k)_{k\in\mathbb{Z}}$ is bounded.

The following result is just a direct application of Theorem 2.1.

Theorem 2.3. *Let A be a closed operator in a Banach space X. Let $(b_k)_{k\in\mathbb{Z}} \subset \mathbb{C}\backslash\{0\}$ be a 3-regular sequence such that $(b_k)_{k\in\mathbb{Z}} \subset \rho(A)$. Then the following assertions are equivalent:*

(i) $(b_k R(b_k, A))_{k\in\mathbb{Z}}$ *is an* $\mathrm{F}^s_{p,q}$*-multiplier for* $1 \le p < \infty$, $1 \le q \le \infty$ *and* $s \in \mathbb{R}$.

(ii) $(b_k R(b_k, A))_{k\in\mathbb{Z}}$ *is bounded.*

Proof. Let $M_k = b_k R(b_k, A)$ for $k \in \mathbb{Z}$. Assume that (i) is valid, it follows from the Closed Graph Theorem that there exists $C > 0$ such that for $f \in \mathrm{F}^s_{p,q}(\mathbb{T}, X)$, we have $\| \sum_{k\in\mathbb{Z}} e_k \otimes M_k \hat{f}(k)\|_{\mathrm{F}^s_{p,q}} \le C\|f\|_{\mathrm{F}^s_{p,q}}$. Let $x \in X$ and $n \in \mathbb{Z}$, we let $f = e_n \otimes x$. The above inequality implies that $\|e_n\|_{\mathrm{F}^s_{p,q}} \|M_n x\| = \|e_n M_n x\| \le C\|e_n\|_{\mathrm{F}^s_{p,q}} \|x\|$. Hence $\sup_{n\in\mathbb{Z}} \|M_n\| \le C$. This proves the implication (i) $\Rightarrow$ (ii). To prove the implication (ii) $\Rightarrow$ (i), we assume that the sequence $(M_k)_{k\in\mathbb{Z}}$ is bounded. From the proof of [11, Proposition 3.4], we have

$$\sup_{k\in\mathbb{Z}} \|k(M_{k+1} - M_k)\| < +\infty$$
$$\sup_{k\in\mathbb{Z}} \|k^2(M_{k+1} - 2M_k + M_{k-1})\| < +\infty$$

as $(b_k)_{k\in\mathbb{Z}}$ is 2-regular. In order to verify the fourth Marcinkiewicz condition of (2.2), we observe that for $\lambda, \mu, \nu, \xi \in \rho(A)$, we have the identity:

$$\begin{aligned} &3\lambda R(\lambda, A) - 3\mu R(\mu, A) + \nu R(\nu, A) - \xi R(\xi, A) \\ = &-(\nu - 3\mu + 3\lambda - \xi) AR(\mu, A)R(\lambda, A) \\ &+(\nu - \xi)(\nu - 2\mu + \lambda) AR(\mu, A)R(\lambda, A)R(\nu, A) \\ &+(\nu - \xi)(\mu - 2\lambda + \xi) AR(\mu, A)R(\nu, A)R(\xi, A) \\ &+2(\mu - \lambda)(\xi - \lambda)(\nu - \xi) AR(\mu, A)R(\lambda, A)R(\nu, A)R(\xi, A). \end{aligned}$$

Substituting $\nu = b_{k+1}, \mu = b_k, \lambda = b_{k-1}, \xi = b_{k-2}$, we obtain:

$$\begin{aligned} &k^3(M_{k+1} - 3M_k + 3M_{k-1} - M_{k-2}) \\ = &-\frac{k^3(b_{k+1} - 3b_k + 3b_{k-1} - b_{k-2})}{b_k} M_k(M_{k-1} - I) \\ &+\frac{k^2(b_{k+1} - 2b_k + b_{k-1})}{b_k} \frac{k(b_{k+1} - b_{k-2})}{b_{k+1}} M_k M_{k+1}(M_{k-1} - I) \end{aligned}$$

$$+\frac{k^2(b_k-2b_{k-1}+b_{k-2})}{b_k}\frac{k(b_{k+1}-b_{k-2})}{b_{k+1}}M_kM_{k+1}(M_{k-2}-I)$$
$$+\frac{2k(b_{k-2}-b_{k-1})}{b_{k-1}}\frac{k(b_k-b_{k-1})}{b_k}\frac{k(b_{k+1}-b_{k-2})}{b_{k+1}}M_{k-1}M_kM_{k+1}(M_{k-2}-I).$$

Since $(b_k)_{k\in\mathbb{Z}}$ is 1-regular, $|k(b_{k+1}-b_k)/b_k| \le D$ for some constant $D > 0$ independent from k. From this, we deduce that $|b_{k+1}/b_k - 1| \le D/|k|$ and thus $b_{k+1}/b_k \to 1$ as $k \to \infty$. We have

$$\begin{aligned}
\frac{k(b_{k-2}-b_{k-1})}{b_{k-1}} &= \frac{-(k-2)(b_{k-1}-b_{k-2})}{b_{k-2}}\frac{b_{k-2}}{b_{k-1}}\frac{k}{k-2}\\
\frac{k(b_k-b_{k-1})}{b_k} &= \frac{k(b_k-b_{k-1})}{b_{k-1}}\frac{b_{k-1}}{b_k}\\
\frac{k(b_{k+1}-b_{k-2})}{b_{k+1}} &= \frac{k(b_{k+1}-b_k)}{b_k}\frac{b_k}{b_{k+1}}+\frac{k(b_k-b_{k-1})}{b_{k-1}}\frac{b_{k-1}}{b_k}\frac{b_k}{b_{k+1}}\\
&\quad+\frac{k(b_{k-1}-b_{k-2})}{b_{k-2}}\frac{b_{k-2}}{b_{k-1}}\frac{b_{k-1}}{b_k}\frac{b_k}{b_{k+1}}.
\end{aligned}$$

Then $k(b_{k-2}-b_{k-1})/b_{k-1}, k(b_k-b_{k-1})/b_k, k(b_{k+1}-b_{k-2})/b_{k+1}$ are bounded. Since $(b_k)_{k\in\mathbb{Z}}$ is 2-regular, $k^2(b_{k+1}-2b_k+b_{k-1})/b_k$ and $k^2(b_k-2b_{k-1}+b_{k-2})/b_k$ are bounded. Since $(b_k)_{k\in\mathbb{Z}}$ is 3-regular, $k^3(b_{k+1}-3b_k+3b_{k-1}-b_{k-2})/b_k$ is bounded. Hence, $\sup_{k\in\mathbb{Z}}\|k^3(M_{k+1}-3M_k+3M_{k-1}-M_{k-2})\| < \infty$, and the result follows from Theorem 2.1. ■

3. Maximal Regularity on Triebel-lizorkin Space

We will consider the problem

$$(P_2)\qquad \begin{cases} u'(t) = Au(t) + \int_{-\infty}^t a(t-s)Au(s)ds + f(t) & 0 \le t \le 2\pi\\ u(0) = u(2\pi) \end{cases}$$

where $a \in \mathrm{L}^1(\mathbb{R}_+)$, A is a closed operator in X and f is an X-valued function defined on $[0, 2\pi]$.

Let $1 \le p < \infty$, $1 \le q \le \infty$, $s > 0$, and let $f \in \mathrm{F}^s_{p,q}(\mathbb{T}, X)$. A function $u \in \mathrm{F}^{s+1}_{p,q}(\mathbb{T}, X)$ is called a strong $\mathrm{F}^s_{p,q}$-solution of (P_2), if $u(t) \in D(A)$, (P_2) holds true for a.e. $t \in [0, 2\pi]$ and $Au \in \mathrm{F}^s_{p,q}(\mathbb{T}, X)$.

We remark that by [4, Proposition 2.3], if $u \in \mathrm{F}^{s+1}_{p,q}(\mathbb{T}, X)$, then u is differentiable a.e. and $u' \in \mathrm{F}^s_{p,q}(\mathbb{T}, X)$. This implies that if u is a strong $\mathrm{F}^s_{p,q}$-solution of (P_2), then every term in (P_2) is in $\mathrm{F}^s_{p,q}(\mathbb{T}, X)$.

For convenience, we introduce the following notations

$$c_k = \tilde{a}(ik)$$

(3.1)
$$b_k = \frac{ik}{1+\tilde{a}(ik)} \quad \text{for all} \quad k \in \mathbb{Z}\backslash\{0\},\ b_0 = 0$$

In order to give our result, the following hypotheses are fundamental.

(**(H1)**) $(c_k)_{k\in\mathbb{Z}}$, $(k(c_{k+1}-c_k))_{k\in\mathbb{Z}}$ and $(1/(1+c_k))_{k\in\mathbb{Z}}$ are bounded sequences.

(**(H2)**) $(kc_k)_{k\in\mathbb{Z}}$, $(k^2(c_{k+1}-2c_k+c_{k-1}))_{k\in\mathbb{Z}}$ are bounded sequences.

(**(H3)**) $(kc_k)_{k\in\mathbb{Z}}$, $(k^2(c_{k+1}-2c_k+c_{k-1}))_{k\in\mathbb{Z}}$ and $(k^3(c_{k+1}-3c_k+3c_{k-1}-c_{k-2}))_{k\in\mathbb{Z}}$ are bounded sequences.

Now we are ready to state the main result of this paper:

Theorem 3.1. *Let $1 \le p < \infty$, $1 \le q \le \infty$ and $s > 0$. Let A be a closed operator in a Banach space X. Let $a \in L^1(\mathbb{R}_+)$ be such that the condition (H3) is satisfied. Then the following assertions are equivalent:*

(i) $(b_k)_{k\in\mathbb{Z}} \subset \rho(A)$ *and* $\sup_{k\in\mathbb{Z}} \|b_k R(b_k, A)\| < \infty$.

(ii) *For every $f \in \mathrm{F}^s_{p,q}(\mathbb{T}, X)$, there exists a unique strong $\mathrm{F}^s_{p,q}$-solution of (P_2).*

Before proving our main result, we first discuss the relations between the assumptions (H2), (H3) and the conditions of 2-regularity and 3-regularity of the sequence $(b_k)_{k\in\mathbb{Z}}$.

Lemma 3.2.

(i) *If $(c_k)_{k\in\mathbb{Z}}$ satisfies the condition (H2), then $(b_k)_{k\in\mathbb{Z}}$ is 2-regular.*

(ii) *If $(c_k)_{k\in\mathbb{Z}}$ satisfies the condition (H3), then $(b_k)_{k\in\mathbb{Z}}$ is 3-regular.*

Proof. First we assume that $(c_k)_{k\in\mathbb{Z}}$ satisfies the condition (H2). From the assumption $(kc_k)_{k\in\mathbb{Z}}$ is bounded, we deduce that $\lim_{k\to\infty} c_k = 0$ and thus $(c_k)_{k\in\mathbb{Z}}$ is bounded. We have

$$\frac{k(b_{k+1}-b_k)}{b_k} = \frac{1+c_k+kc_k-kc_{k+1}}{1+c_{k+1}}$$

$$\frac{k^2(b_{k+1}-2b_k+b_{k-1})}{b_k} = \frac{k(c_{k-1}-c_{k+1})}{(1+c_{k-1})(1+c_{k+1})} - \frac{k^2(c_{k+1}-2c_k+c_{k-1})}{(1+c_{k-1})(1+c_{k+1})} + \frac{k(k+1)c_{k-1}c_k + k(k-1)c_kc_{k+1} - 2k^2c_{k-1}c_{k+1}}{(1+c_{k-1})(1+c_{k+1})}.$$

The two sequences are bounded as $(c_k)_{k\in\mathbb{Z}}$ satisfies the assumption (H2). Hence $(b_k)_{k\in\mathbb{Z}}$ is 2-regular. Next we assume that $(c_k)_{k\in\mathbb{Z}}$ satisfies the condition (H3), then $(b_k)_{k\in\mathbb{Z}}$ is 2-regular by (i) . We have

$$\frac{k^3(b_{k+1}-3b_k+3b_{k-1}-b_{k-2})}{b_k} = -\frac{k^3(c_{k+1}-3c_k+3c_{k-1}-c_{k-2})}{(1+c_{k-2})(1+c_{k-1})(1+c_{k+1})} - \frac{2kc_{k-2}k^2(c_{k+1}-2c_k+c_{k-1})}{(1+c_{k-2})(1+c_{k-1})(1+c_{k+1})} + \frac{2kc_{k+1}k^2(c_k-2c_{k-1}+c_{k-2})}{(1+c_{k-2})(1+c_{k-1})(1+c_{k+1})}$$

$$-\frac{k^2(c_{k+1}-2c_k+c_{k-1})}{(1+c_{k-2})(1+c_{k-1})(1+c_{k+1})}-\frac{2k^2(c_k-2c_{k-1}+c_{k-2})}{(1+c_{k-2})(1+c_{k-1})(1+c_{k+1})}$$
$$+\frac{k^2(c_{k-2}c_{k-1}-2c_{k-2}c_k-3c_{k-2}c_{k+1}+3c_{k-1}c_k+2c_{k-1}c_{k+1}-c_kc_{k+1})}{(1+c_{k-2})(1+c_{k-1})(1+c_{k+1})}$$
$$+\frac{k^2(k+1)c_{k-2}c_{k-1}c_k-3k^3c_{k-2}c_{k-1}c_{k+1}+(3k-3)k^2c_kc_{k-2}c_{k+1}-(k-2)k^2c_{k-1}c_kc_{k+1}}{(1+c_{k-2})(1+c_{k-1})(1+c_{k+1})}$$

which is bounded by the assumption (H3). This finishes the proof. ■

We notice that the assumption (H2) and the notion of 2-regular sequences were introduced in [11] to study the maximal regularity property on Besov spaces $\mathrm{B}^s_{p,q}(\mathbb{T},X)$ for the problem (P_2). In the proof of our main result, we will use the following result.

Proposition 3.3. *Let X be a Banach space. Under the assumption $(H3)$, the sequences $((1+c_k)I)_{k\in\mathbb{Z}}$, $((1+c_k)I/(ik))_{k\in\mathbb{Z}}$ and $(I/(1+c_k))_{k\in\mathbb{Z}}$ are $\mathrm{F}^s_{p,q}$-multipliers for $1\le p<\infty$, $1\le q\le\infty$, $s\in\mathbb{R}$.*

Proof. Since $(kc_k)_{k\in\mathbb{Z}}$ is bounded, we have $\lim_{k\to\infty}c_k=0$ and thus $(c_k)_{k\in\mathbb{Z}}$ is bounded. To show that $((1+c_k)I)_{k\in\mathbb{Z}}$ is an $\mathrm{F}^s_{p,q}$-multiplier, it will suffice to show that the sequence $(c_kI)_{k\in\mathbb{Z}}$ is an $\mathrm{F}^s_{p,q}$-multiplier, this follows directly from the assumption (H3) and Theorem 2.1. Since the product of two $\mathrm{F}^s_{p,q}$-multipliers is still an $\mathrm{F}^s_{p,q}$-multiplier, to show that $((1+c_k)I/ik)_{k\in\mathbb{Z}}$ is an $\mathrm{F}^s_{p,q}$-multiplier, it will suffice to show that $(I/ik)_{k\in\mathbb{Z}}$ is an $\mathrm{F}^s_{p,q}$-multiplier. This is a direct consequence of Theorem 2.1.

Finally, we show that $u_k=I/(c_k+1)$ is an $\mathrm{F}^s_{p,q}$-multiplier. We have

$$\begin{aligned}
k(u_{k+1}-u_k) &= \frac{k(c_k-c_{k+1})}{(1+c_k)(1+c_{k+1})}\\
k^2(u_{k+1}-2u_k+u_{k-1}) &= \frac{-k^2(c_{k+1}-2c_k+c_{k-1})}{(1+c_{k+1})(1+c_k)(1+c_{k-1})}\\
&\quad -\frac{kc_{k-1}k(c_{k+1}-c_k)}{(1+c_{k+1})(1+c_k)(1+c_{k-1})}\\
&= +\frac{kc_{k+1}k(c_k-c_{k-1})}{(1+c_{k+1})(1+c_k)(1+c_{k-1})}\\
k^3(u_{k+1}-3u_k+3u_{k-1}-u_{k-2}) &= \frac{-k^3(c_{k+1}-3c_k+3c_{k-1}-c_{k-2})}{(1+c_{k+1})(1+c_k)(1+c_{k-1})(1+c_{k-2})}\\
&= +\frac{2kc_{k+1}k^2(c_k-2c_{k-1}+c_{k-2})}{(1+c_{k+1})(1+c_k)(1+c_{k-1})(1+c_{k-2})}\\
&\quad -\frac{2kc_{k-2}k^2(c_{k-1}-2c_k+c_{k+1})}{(1+c_{k+1})(1+c_k)(1+c_{k-1})(1+c_{k-2})}
\end{aligned}$$

$$= +\frac{k^3 c_k c_{k-1} c_{k-2} - 3k^3 c_{k+1} c_{k-1} c_{k-2} + 3k^3 c_k c_{k+1} c_{k-2} - k^3 c_{k+1} c_k c_{k-1}}{(1+c_{k+1})(1+c_k)(1+c_{k-1})(1+c_{k-2})}$$

are bounded sequences. Hence $(u_k)_{k\in\mathbb{Z}}$ is also an $\mathrm{F}^s_{p,q}$-multiplier by Theorem 2.1. ■

Corollary 3.4. *Let A be a closed operator in a Banach space X. Let $(c_k)_{k\in\mathbb{Z}} \subset \mathbb{C}\backslash\{0\}$ be a sequence satisfying the assumption (H3), such that $(b_k)_{k\in\mathbb{Z}} \subset \rho(A)$. If $(b_k R(b_k, A))_{k\in\mathbb{Z}}$ is bounded, then $(R(b_k, A))_{k\in\mathbb{Z}}$ is an $\mathrm{F}^s_{p,q}$-multiplier for $1 \le p < \infty$, $1 \le q \le \infty$ and $s \in \mathbb{R}$.*

This is a direct consequence of the following observations: when $(c_k)_{k\in\mathbb{Z}}$ satisfies the assumption (H3), $(b_k)_{k\in\mathbb{Z}}$ is 3-regular by Lemma 3.2. By the boundedness of $(b_k R(b_k, A))_{k\in\mathbb{Z}}$ and Theorem 2.3, $(b_k R(b_k, A))_{k\in\mathbb{Z}}$ is an $\mathrm{F}^s_{p,q}$-multiplier for $1 \le p < \infty$, $1 \le q \le \infty$ and $s \in \mathbb{R}$. Then the result follows from Proposition 3.3 and the fact that the product of two $\mathrm{F}^s_{p,q}$-multipliers is still an $\mathrm{F}^s_{p,q}$-multiplier.

Now we are ready to give the proof of our main result.

Proof of Theorem 3.1. (ii)⇒(i): Let $y \in X$ and $k \in \mathbb{Z}$ be fixed. We let $f = e_k \otimes y$. Note that $f \in \mathrm{F}^s_{p,q}(\mathbb{T}, X)$. Hence there exists $u \in \mathrm{F}^{s+1}_{p,q}(\mathbb{T}, X)$ such that $u(t) \in D(A)$, $u'(t) = Au(t) + a{*}Au(t) + f(t)$ holds for a.e. $t \in [0, 2\pi]$ and $Au \in \mathrm{F}^s_{p,q}(\mathbb{T}, X)$ (see (2.3) for the definition of $a{*}Au$). Taking Fourier series on both sides, we obtain $\hat{u}(k) \in D(A)$ by [2, Lemma 3.1] and

$$ik\hat{u}(k) = A\hat{u}(k) + \tilde{a}(ik)A\hat{u}(k) + \hat{f}(k) = A\hat{u}(k) + \tilde{a}(ik)A\hat{u}(k) + y$$

by (2.4). Thus $[ik-(1+\tilde{a}(ik))A]\hat{u}(k) = y$. We have shown that $ik-(1+\tilde{a}(ik))A$ is surjective. To show that the operator $ik-(1+\tilde{a}(ik))A$ is also injective, we take $x \in D(A)$ be such that $[ik-(1+\tilde{a}(ik))A]x = 0$, then $Ax = b_k x$. This implies that $u = e_k \otimes x$ defines a periodic solution of $u'(t) = Au(t) + \int_{-\infty}^t a(t-s)Au(s)ds$. Indeed,

$$Au(t) + \int_{-\infty}^t a(t-s)Au(s)ds = e^{ikt}Ax + \int_{-\infty}^t a(t-s)e^{iks}Axds$$
$$= e^{ikt}Ax + e^{ikt}\tilde{a}(ik)Ax = e^{ikt}(1+\tilde{a}(ik))Ax = ike^{ikt}x = u'(t)$$

By the assumption of uniqueness, we must have $x = 0$. We have shown that $ik-(1+\tilde{a}(ik))A$ is bijective. Since A is closed, we conclude that $b_k \in \rho(A)$.

Next, we show that $\sup_{k\in\mathbb{Z}} \|b_k R(b_k, A)\| < \infty$. We consider $f = e_k \otimes x$ for some fixed $k \in \mathbb{Z}$ and $x \in X$, we let u be the unique solution in $\mathrm{F}^{1+s}_{p,q}(\mathbb{T}, X)$ of (P_2). Taking Fourier series, we have $[ik-(1+\tilde{a}(ik))A]\hat{u}(k) = x$. Hence

$$ik\ \hat{u}(k) = b_k R(b_k, A)x$$
$$in\ \hat{u}(n) = 0,\ (n \ne k).$$

This implies that the solution u satisfies $u' = b_k R(b_k, A) e_k \otimes x$. By hypothesis and using the Closed Graph Theorem, we can find $C > 0$ independent from k and x such that

$$\|u'\|_{\mathrm{F}^s_{p,q}} + \|Au\|_{\mathrm{F}^s_{p,q}} + \|a\dot{*}Au\|_{\mathrm{F}^s_{p,q}} \le C\|f\|_{\mathrm{F}^s_{p,q}}.$$

This implies that $\|b_k R(b_k, A)x\| \le C\|x\|$ for all $k \in \mathbb{Z}$. Hence $\sup_{k\in\mathbb{Z}} \|b_k R(b_k, A)\| < \infty$. We have proved (i).

(i)⇒(ii): Let $f \in \mathrm{F}^s_{p,q}(\mathbb{T}, X)$. Since $(I/(1 + c_k))_{k\in\mathbb{Z}}$ is an $\mathrm{F}^s_{p,q}$-multiplier by Proposition 3.3, there exists $g \in \mathrm{F}^s_{p,q}(\mathbb{T}, X)$, such that $\hat{g}(k) = \hat{f}(k)/(1+c_k)$ for all $k \in \mathbb{Z}$. Since $(b_k R(b_k, A))_{k\in\mathbb{Z}}$ is bounded by assumption and $(b_k)_{k\in\mathbb{Z}}$ is 3-regular as the condition (H3) is satisfied by Lemma 3.2, the sequence $(b_k R(b_k, A))_{k\in\mathbb{Z}}$ defines an $\mathrm{F}^s_{p,q}$-multiplier by Theorem 2.3. By Proposition 3.3, $(1 + c_k)_{k\in\mathbb{Z}}$ is also an $\mathrm{F}^s_{p,q}$-multiplier. We deduce that $(ikR(b_k, A))_{k\in\mathbb{Z}}$ defines an $\mathrm{F}^s_{p,q}$-multiplier. There exists $v \in \mathrm{F}^s_{p,q}(\mathbb{T}, X)$, such that $\hat{v}(k) = ikR(b_k, A)\hat{g}(k)$ for $k \in \mathbb{Z}$. By Corollary 3.4, $(R(b_k, A))_{k\in\mathbb{Z}}$ is also an $\mathrm{F}^s_{p,q}$-multiplier, there exists $u \in \mathrm{F}^s_{p,q}(\mathbb{T}, X)$ such that $\hat{u}(k) = R(b_k, A)\hat{g}(k)$. Hence we have $\hat{v}(k) = ik\hat{u}(k)$ for $k \in \mathbb{Z}$. By [2, Lemma 2.1], u is differentiable a.e. and $u' = v, u(0) = u(2\pi)$. By [4, Proposition 2.3], this implies that $u \in \mathrm{F}^{s+1}_{p,q}(\mathbb{T}, X)$. By $\hat{u}(k) = R(b_k, A)\hat{g}(k)$ and [2, Lemma 3.1], $u(t) \in D(A)$ for a.e. $t \in [0, 2\pi]$. On the other hand $A\hat{u}(k) = AR(b_k, A)\hat{g}(k)$, we deduce that $Au \in \mathrm{F}^s_{p,q}(\mathbb{T}, X)$ as $(AR(b_k, A))_{k\in\mathbb{Z}}$ is an $\mathrm{F}^s_{p,q}$-multiplier by (i).

From $(b_k I - A)\hat{u}(k) = \hat{g}(k)$, we have

$$ik\,\hat{u}(k) = (1 + \tilde{a}(ik))A\hat{u}(k) + (1 + \tilde{a}(ik))\hat{g}(k) = A\hat{u}(k) + \tilde{a}(ik)A\hat{u}(k) + \hat{f}(k)$$

for all $k \in \mathbb{Z}$. From the uniqueness theorem of Fourier coefficient, we deduce that (P_2) holds true for almost $t \in [0, 2\pi]$. This shows existence.

To show the uniqueness, let $u \in \mathrm{F}^{s+1}_{p,q}(\mathbb{T}, X) \cap \mathrm{F}^s_{p,q}(\mathbb{T}, D(A))$ be such that $u'(t) - Au(t) - \int_{-\infty}^{t} a(t-s)Au(s)ds = 0$. Then $\hat{u}(k) \in D(A)$ by[2, Lemma 3.1] and $[ikI - (1 + \tilde{a}(ik))A]\hat{u}(k) = 0$ by taking the Fourier series. Since $(ik/(1 + \tilde{a}(ik)) \subset \rho(A)$, this implies that $\hat{u}(k) = 0$ for all $k \in \mathbb{Z}$. Thus $u = 0$ and the proof is finished.

Remark 3.5.

(i) When $1 < p < \infty,\ 1 < q \le \infty,\ s \in \mathbb{R}$, the first three conditions in (2.2) are already sufficient for a sequence $(M_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(X)$ to be an $\mathrm{F}^s_{p,q}$-multiplier [4, Theorem 3.2]. This fact together with the argument used in [11] shows that under the weaker assumption (H2) on $(c_k)_{k\in\mathbb{Z}}$, the problem (P_2) has the $\mathrm{F}^s_{p,q}$-maximal regularity if and only if $(b_k)_{k\in\mathbb{Z}} \subset \rho(A)$ and $\sup_{k\in\mathbb{Z}} \|b_k R(b_k, A)\| < \infty$ whenever $1 < p < \infty,\ 1 < q \le \infty$ and $s > 0$.

(ii) When the underlying Banach space X has a non trivial Fourier type and $1 \le p,\ q \le \infty,\ s \in \mathbb{R}$, the first two conditions in (2.2) are already sufficient for a sequence $(M_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(X)$ to be a $\mathrm{B}^s_{p,q}$-multiplier [3, Theorem 4.5]. This fact together with the argument used in [11] shows that under the weaker assumption (H1) on $(c_k)_{k\in\mathbb{Z}}$, the problem (P_2) has the $\mathrm{B}^s_{p,q}$-maximal regularity if and only if $(b_k)_{k\in\mathbb{Z}} \subset \rho(A)$ and $\sup_{k\in\mathbb{Z}} \|b_k R(b_k, A)\| < \infty$ whenever $1 \leq p, q \leq \infty$ and $s > 0$.

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