TAIWANESE JOURNAL OF MATHEMATICS
Vol. 9, No. 4, pp. 617-627, December 2005
This paper is available online at http://www.math.nthu.edu.tw/tjm/

SOME NEW SEQUENCE SPACES DEFINED BY A SEQUENCE OF ORLICZ FUNCTIONS

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Abstract. In this paper, we introduce some new sequence spaces combining lacunary sequence, invariant means and a sequence of Orlicz functions. We discuss some topological properties and establish some inclusion relations between these spaces. Also we studied connections between lacunary $\sigma-$statistical convergence with these spaces.

1. Introduction

Let σ be a mapping the set of positive integer into itself. A continuous linear functional ϕ on ℓ_∞ ,the space of real bounded sequences, is said to be an invariant mean $\sigma-$ mean if and only if i) $\phi\left(x\right) \geq 0$ when the sequence $x = (x_k)$ has $x_k \geq 0$ for all n, ii) $\phi\left(e\right) = 1$, where $e = (1, 1, 1, ...)$ and , iii) $\phi\left(x_{\sigma(n)}\right) = \phi\left(x\right)$ for all $x \in \ell_{\infty.}$

If $x = (x_n)$, write $Tx = Tx_n = \left(x_{\sigma(n)}\right)$. It can be shown (Schaefer[16]) that

$$V_\sigma = \left\{x \in \ell_\infty \colon \lim_k t_{kn}\left(x\right) = \ell, \text{ uniformly in } n,\, \ell = \sigma - \lim x\right\},$$

where

$$t_{kn}\left(x\right) = \frac{x_n + x_{\sigma^1(n)} + ... + x_{\sigma^k(n)}}{k+1}$$

In the case σ is the translation mapping $n \to n+1$, $\sigma-$mean is often called a Banach limit and V_σ, the set of bounded sequences all of whose invariant means are equal, is the set of almost convergent sequence, (see, Lorentz [8]).

By a lacunary sequence $\theta = (k_r)$ where $k_0 = 0$, we shall mean an increasing sequence of non-negative integers with $k_r - k_{r-1} \to \infty$ as $r \to \infty$. The intervals

Received November 7, 2003; revised December 31, 2003.
Communicated by H. M. Srivastava.
2000 *Mathematics Subject Classification*: 40A05, 40A30.
Key words and phrases: Lacunary sequence, Invariant means, A sequence of Orlicz functions.

determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$. We write $h_r = k_r - k_{r-1}$. The ratio $\frac{k_r}{k_{r-1}}$ will be denoted by q_r. The space of lacunary strongly convergent sequence was defined by Freedman et al. [3].

An Orlicz function is a function $M : [0, \infty) \to [0, \infty)$ which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$ (see,Kamthan and Gupta [5]). If convexity of M is replaced by subadditivity, then this function is called a modulus function, (see, Ruckle [12]).

Let w be the spaces of all real or complex sequence $x = (x_k)$. Lindentrauss and Tzafirir [7] used the idea of Orlicz function to defined the following sequence spaces:

$$\ell_M = \left\{ x : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0 \right\}$$

which is called an Orlicz sequence spaces and ℓ_M is a Banach space with the norm,

$$\|x\| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\}.$$

Parashar and Chaudhary [10] have introduced and examined some properties of four sequence spaces defined by using an Orlicz function M, which generalized the well-known Orlicz sequence spaces $[c, 1, p]_0$, $[c, 1, p]$ and $[c, 1, p]_\infty$. It may be noted here that the spaces of strongly summable sequences were discussed by Maddox [9].

Quite recently, in [15], Savaş and Rhoades introduced some new sequence spaces which were defined by combining the concept of an Orlicz function, invariant mean and lacunary convergence.

The main purpose of this paper is to give a generalization of sequence spaces, defined by [15], using a sequence of Orlicz functions and to examine inclusion relations among these spaces. Also, we shall give the connections between lacunary $\sigma-$statistical convergence with following sequence spaces.

Now we introduce the following sequence spaces:

Definition 1. Let $M = (M_k)$ be a sequence of Orlicz function and $p = (p_k)$ be any sequence of strictly positive real numbers. We have

$$w_\sigma^0 [M_k, p]_\theta = \left\{ x : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \left[M_k \left(\frac{|t_{kn}(x)|}{\rho} \right) \right]^{p_k} = 0, \rho > 0, \text{ uniformly in } n \right\},$$

$$w_\sigma [M_k, p]_\theta = \left\{ x : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \left[M_k \left(\frac{|t_{kn}(x-\ell)|}{\rho} \right) \right]^{p_k} = 0, \rho > 0, \text{ uniformly in } n \right\},$$

$$w_\sigma^\infty [M_k, p]_\theta = \left\{ x : \sup_{r,n} \frac{1}{h_r} \sum_{k \in I_r} \left[M_k \left(\frac{|t_{kn}(x)|}{\rho} \right) \right]^{p_k} < \infty, \text{ for some } \rho > 0 \right\}.$$

When $(M_k) = M$ for all k, the spaces $w_\sigma^0 [M_k, p]_\theta$, $w_\sigma [M_k, p]_\theta$ and $w_\sigma^\infty [M_k, p]_\theta$ reduce to the spaces $w_\sigma^0 [M, p]_\theta$, $w_\sigma [M, p]_\theta$ and $w_\sigma^\infty [M, p]_\theta$, respectively. When $M_k(x) = x$ for all k, then write respectively the spaces $w_{\sigma\theta}^0 (p)$, $w_{\sigma\theta}(p)$ and $w_\sigma^\infty{}_\theta (p)$ in place of the spaces $w_\sigma^0 [M_k, p]_\theta$, $w_\sigma [M_k, p]_\theta$ and $w_\sigma^\infty [M_k, p]_\theta$. If $p_k = 1$ for all k, $w_\sigma^0 [M_k, p]_\theta$, $w_\sigma [M_k, p]_\theta$ and $w_\sigma^\infty [M_k, p]_\theta$ reduce to $w_\sigma^0 [M_k]_\theta$, $w_\sigma [M_k]_\theta$ and $w_\sigma^\infty [M_k]_\theta$

2. Inclusion Theorems

Theorem 1. *Let $M = (M_k)$ be a sequence of Orlicz functions and a bounded sequence $p = (p_k)$ of strictly positive real numbers. $w_\sigma^0 [M_k, p]_\theta$, $w_\sigma [M_k, p]_\theta$ and $w_\sigma^\infty [M_k, p]_\theta$ linear spaces over the set of complex numbers.*

Proof. We have

$$|x_k + y_k|^{p_k} \leq K \left(|x_k|^{p_k} + |y_k|^{p_k} \right) \tag{2.1}$$

where $K = \max\left(1, 2^{H-1}\right)$, $H = \sup p_k$. We shall prove the result only for $w_\sigma^0 [M_k, p]_\theta$. The others can be treated similarly. Let $x, y \in w_\sigma^0 [M_k, p]_\theta$ and $\alpha, \beta \in C$. In order to prove the result we need to find some ρ_3 such that

$$\lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} \left[M_k \left(\frac{|t_{kn}(\alpha x + \beta y)|}{\rho_3} \right) \right]^{p_k} = 0, \text{ uniformly in } n.$$

Since $x, y \in w_\sigma^0 [M_k, p]_\theta$, there exists positive ρ_1, ρ_2 such that

$$\lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} \left[M_k \left(\frac{|t_{kn}(x)|}{\rho_1} \right) \right]^{p_k} = 0, \text{ uniformly in } n$$

and

$$\lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} \left[M_k \left(\frac{|t_{kn}(y)|}{\rho_2} \right) \right]^{p_k} = 0, \text{uniformly in } n.$$

Define $\rho_3 = \max(2|\alpha| \rho_1, 2|\beta| \rho_2)$. Since (M_k) is non-decreasing and convex

$$\begin{aligned}
\frac{1}{h_r} \sum_{k \in I_r} \left[M_k \left(\frac{|t_{kn}(\alpha x + \beta y)|}{\rho_3} \right) \right]^{p_k} &\leq \frac{1}{h_r} \sum_{k \in I_r} \left[M_k \left(\frac{|t_{kn}(\alpha x)|}{\rho_3} + \frac{|t_{kn}(\beta y)|}{\rho_3} \right) \right]^{p_k} \\
&\leq \frac{1}{h_r} \sum_{k \in I_r} \frac{1}{2^{p_k}} \left[M_k \left(\frac{|t_{kn}(x)|}{\rho_1} \right) + \left(\frac{|t_{kn}(y)|}{\rho_2} \right) \right]^{p_k} \\
&\leq \frac{1}{h_r} \sum_{k \in I_r} \left[M_k \left(\frac{|t_{kn}(x)|}{\rho_1} \right) + \left(\frac{|t_{kn}(y)|}{\rho_2} \right) \right]^{p_k}
\end{aligned}$$

$$\leq K\frac{1}{h_r}\sum_{k\in I_r}\left[M_k\left(\frac{|t_{kn}(x)|}{\rho_1}\right)\right]^{p_k}$$
$$+K\frac{1}{h_r}\sum_{k\in I_r}\left[M_k\left(\frac{|t_{kn}(y)|}{\rho_1}\right)\right]^{p_k}$$

$\to 0$ as $r\to\infty$,uniformly in n. So that $\alpha x+\beta y\in w_\sigma^0[M_k,p]_\theta$ .This completes the proof. ∎

Theorem 2. *Let $M=(M_k)$ be a sequence of Orlicz functions and a bounded sequence $p=(p_k)$ of strictly positive real numbers. Then $w_\sigma^0[M_k,p]_\theta$ is a topological linear spaces, totally paranormed*

$$g(x)=\inf\left\{\rho^{\frac{p_r}{H}}:\left(\frac{1}{h_r}\sum_{k\in I_r}\left[M_k\left(\frac{|t_{kn}(x)|}{\rho}\right)\right]^{p_k}\right)^{\frac{1}{H}}\leq 1,\ r=1,2,..,\ n=1,2,...\right\}$$

where $H=\max(1,\sup p_k)$.

The proof of the theorem 1 can be carried out by using similar technical in [15].

Theorem 3. *Let $M=(M_k)$ be e sequence of Orlicz functions. If $\sup_k[M_k(t)]^{p_k}<\infty$ for all $t>0$, then*

$$w_\sigma[M_k,p]_\theta\subset w_\sigma^\infty[M_k,p]_\theta.$$

Proof. Let $x\in w_\sigma[M_k,p]_\theta$. By using (2.1), we have

$$\frac{1}{h_r}\sum_{k\in I_r}\left[M_k\left(\frac{|t_{kn}(x)|}{\rho}\right)\right]^{p_k}\leq\frac{K}{h_r}\sum_{k\in I_r}\left[M_k\left(\frac{|t_{kn}(x-\ell)|}{\rho}\right)\right]^{p_k}+\frac{K}{h_r}\sum_{k\in I_r}\left[M_k\left(\frac{|\ell|}{\rho}\right)\right]^{p_k}.$$

Since $\sup_k[M_k(t)]^{p_k}<\infty$, we can take that $\sup_k[M_k(t)]^{p_k}=T$. Hence we get $x\in w_\sigma^\infty[M_k,p]_\theta$. This completes the proof. ∎

Definition 2. An Orlicz function M is said to satisfy the Δ_2-condition for all values of u, if there exists a constant $L>0$ such that $M(2u)\leq LM(u)$, $u\geq 0$.

It is also easy to see that always $L>2$. The Δ_2-condition equivalent to the satisfaction of inequality $M(lu)\leq LulM(u)$ for all values of u and for all $l>1$, (see, Krasnoselskii and Rutitsky [6]).

Theorem 4. *For a sequence of Orlicz function $M=(M_k)$ which satisfies Δ_2-condition for all k,*

$$w_{\sigma\theta}(p)\subset w_\sigma[M_k,p]_\theta.$$

Proof. Let $x \in w_{\sigma\theta}(p)$. Then we have

$$\tau_r = \frac{1}{h_r} \sum_{k \in I_r} |t_{kn}(x - \ell)|^{p_k} \to 0$$

as $r \to \infty$ uniformly in n, for some ℓ. Let $\varepsilon > 0$ and choose δ with $0 < \delta < 1$ such that $M_k(t) < \varepsilon$ for $0 \le t \le \delta$ and all k. We can write

$$\frac{1}{h_r} \sum_{k \in I_r} \left[M_k \left(\frac{|t_{kn}(x-\ell)|}{\rho} \right) \right]^{p_k} = \frac{1}{h_r} \sum_{\substack{k \in I_r \\ |t_{kn}(x-\ell)| \le \delta}}^{1} \left[M_k \left(\frac{|t_{kn}(x-\ell)|}{\rho} \right) \right]^{p_k} + \frac{1}{h_r} \sum_{\substack{k \in I_r \\ |t_{kn}(x-\ell)| > \delta}}^{2} \left[M_k \left(\frac{|t_{kn}(x-\ell)|}{\rho} \right) \right]^{p_k}.$$

For the first summation above, we immediately write $\sum^{1} \le \varepsilon^H$ by using continuity of M_k for all k. For the second summation, we will make following procedure. We have

$$|t_{kn}(x-\ell)| < 1 + \frac{|t_{kn}(x-\ell)|}{\delta}.$$

Since M_k is non-decreasing and convex for all k, it follows that

$$M_k(|t_{kn}(x-\ell)|) < M_k \left(1 + \frac{|t_{kn}(x-\ell)|}{\delta} \right) \le \frac{1}{2} M_k(2) + \frac{1}{2} M_k \left(2 \frac{|t_{kn}(x-\ell)|}{\delta} \right).$$

Since M_k satisfies Δ_2-condition for all k, we can write

$$\begin{aligned} M_k(|t_{kn}(x-\ell)|) &\le \frac{1}{2} L \frac{|t_{kn}(x-\ell)|}{\delta} M_k(2) + \frac{1}{2} L \frac{|t_{kn}(x-\ell)|}{\delta} M_k(2) \\ &= L \frac{|t_{kn}(x-\ell)|}{\delta} M_k(2). \end{aligned}$$

In this way, we write

$$\frac{1}{h_r} \sum_{k \in I_r} \left[M_k \left(\frac{|t_{kn}(x-\ell)|}{\rho} \right) \right]^{p_k} \le \varepsilon^H + [\max(1, LM_k(2))\,\delta]^H \tau_r.$$

Letting $r \to \infty$, it follows that $x \in w_\sigma [M_k, p]_\theta$ This completes the proof. ∎

After this step of paper, different inclusion relations among these sequence spaces are going to be studied. Now we have

Theorem 5. *Let $M = (M_k)$ be a sequence of Orlicz functions. Then the following statements are equivalent.*

(i) $w_{\sigma\theta}^{\infty}(p) \subset w_{\sigma}^{\infty}[M_k, p]_{\theta}$,

(ii) $w_{\sigma\theta}^{0}(p) \subset w_{\sigma}^{\infty}[M_k, p]_{\theta}$,

(iii) $\sup_r \frac{1}{h_r} \sum\limits_{k\in I_r} [M_k(t)]^{p_k} < \infty$ for all $t > 0$.

Proof. $(i) \Rightarrow (ii)$: Let (i) hold. To verify (ii), it is enough to show $w_{\sigma\theta}^{0}(p) \subset w_{\sigma\theta}^{\infty}(p)$. Let $x \in w_{\sigma\theta}^{0}(p)$. Then there exist $r \geq r_0$, for $\varepsilon > 0$, such that

$$\frac{1}{h_r}\sum_{k\in I_r} |t_{kn}(x)|^{p_k} < \varepsilon.$$

Hence there exists $H > 0$ such that

$$\sup_{r,n} \frac{1}{h_r}\sum_{k\in I_r} |t_{kn}(x)|^{p_k} < H$$

for all n and r. So we get $x \in w_{\sigma\theta}^{\infty}(p)$.

$(ii) \Rightarrow (iii)$: Let (ii) hold. Suppose that (iii) does not hold. Then for some $t > 0$

$$\sup_{r} \frac{1}{h_r}\sum_{k\in I_r} [M_k(t)]^{p_k} = \infty$$

and therefore we can find a subinterval $I_{r(m)}$ of the set of interval I_r such that

$$\frac{1}{h_{r(m)}}\sum_{k\in I_{r(m)}} \left[M_k\left(\frac{1}{m}\right)\right]^{p_k} > m,\ m = 1, 2, ... \tag{2.2}$$

Let us define $x = (x_k)$ as follows. $x_k = \frac{1}{m}$ if $k \in I_{r(m)}$ and $x_k = 0$ if $k \notin I_{r(m)}$.Then $x \in w_{\sigma\theta}^{0}(p)$ but by (2.2), $x \notin w_{\sigma}^{\infty}[M_k, p]_{\theta}$, which contradicts (ii). Hence (iii) must hold.

$(iii) \Rightarrow (i)$:Let (iii) hold and that $x \in w_{\sigma\theta}^{\infty}(p)$. Suppose that $x \notin w_{\sigma}^{\infty}[M_k, p]_{\theta}$.Then for $x \in w_{\sigma\theta}^{\infty}(p)$,we can write

$$\sup_{r,n} \frac{1}{h_r}\sum_{k\in I_r} \left[M_k\left(\frac{|t_{kn}(x)|}{\rho}\right)\right]^{p_k} = \infty. \tag{2.3}$$

Let $t = |t_{kn}(x)|$ for each k and fixed n.Then, by(2.3)

$$\sup_{r} \frac{1}{h_r}\sum_{k\in I_r} [M_k(t)]^{p_k} = \infty$$

which contradicts (iii). Hence (i) must hold. ∎

Theorem 6. *Let $M = (M_k)$ be a sequence of Orlicz functions. Then the following statements are equivalent.*

(i) $w_{\sigma}^{0}[M_k,p]_{\theta} \subset w_{\sigma\theta}^{0}(p)$,

(ii) $w_{\sigma}^{0}[M_k,p]_{\theta} \subset w_{\sigma\theta}^{\infty}(p)$,

(iii) *inf*$_r \frac{1}{h_r}\sum\limits_{k\in I_r}[M_k(t)]^{p_k} > 0$ *for all* $t>0$.

Proof. $(i) \Rightarrow (ii)$: It is obvious.

$(ii) \Rightarrow (iii)$: Let (ii) hold. Suppose that (iii) does not hold. Then

$$inf_r \frac{1}{h_r}\sum_{k\in I_r}[M_k(t)]^{p_k} = 0 \text{ for some } t>0,$$

and we can find a subinterval $I_{r(m)}$ of the set of interval I_r such that

$$\frac{1}{h_{r(m)}}\sum_{k\in I_{r(m)}}[M_k(m)]^{p_k} < \frac{1}{m},\ m=1,2,... \tag{2.4}$$

Let us define $x_k = m$ if $k \in I_{r(m)}$ and $x_k = 0$ if $k \notin I_{r(m)}$. Thus, by (2.4), $x \in w_{\sigma}^{0}[M_k,p]_{\theta}$ but $x \notin w_{\sigma\theta}^{\infty}(p)$ which contradicts (ii). Hence (iii) must hold.

$(iii) \Rightarrow (i)$: Let (iii) hold. Suppose that $x \in w_{\sigma}^{0}[M_k,p]_{\theta}$. Therefore

$$\frac{1}{h_r}\sum_{k\in I_r}\left[M_k\left(\frac{|t_{kn}(x)|}{\rho}\right)\right]^{p_k} \to 0 \tag{2.5}$$

as $r\to\infty$ uniformly in n. Again suppose that $x \notin w_{\sigma\theta}^{0}(p)$ for some number $\varepsilon>0$ and a subinterval $I_{r(m)}$ of the set of interval I_r,we have $|t_{kn}(x)|^{p_k} \geq \varepsilon$ for all n and for all $I_{r(m)}$. Then, from properties of the Orlicz function, we have

$$\left[M_k\left(\frac{\varepsilon}{\rho}\right)\right]^{p_k} \leq \left[M_k\left(\frac{|t_{kn}(x)|}{\rho}\right)\right]^{p_k}.$$

Consequently, by (2.5) ,we have

$$\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}\left[M_k\left(\frac{\varepsilon}{\rho}\right)\right]^{p_k} = 0$$

uniformly in n which contradicts (iii). Hence (i) must hold. ■

Theorem 7. *Let* $M=(M_k)$ *be a sequence of Orlicz functions.* $w_{\sigma}^{\infty}[M_k,p]_{\theta} \subset w_{\sigma\theta}^{0}(p)$ *if and only if*

$$\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}[M_k(t)]^{p_k} = \infty \tag{2.6}$$

Proof. Let $w_{\sigma}^{\infty}\left[M_k,p\right]_{\theta} \subset w_{\sigma\theta}^{0}\left(p\right)$. Suppose that (2.6) does not hold. Therefore there is a subinterval $I_{r(m)}$ of the set of interval I_r and a number $t_0 > 0$, where $t_0 = \left|t_{kn}\left(x\right)\right|$ for all k and n,such that

$$\frac{1}{h_{r(m)}}\sum_{k\in I_{r(m)}}\left[M_k\left(t_0\right)\right]^{p_k} \leq M < \infty, m = 1,2,... \tag{2.7}$$

Let us define $x_k = t_0$ if $k \in I_{r(m)}$ and $x_k = 0$ if $k \notin I_{r(m)}$. Then, by (2.7), $x \in w_{\sigma}^{\infty}\left[M_k,p\right]_{\theta}$. But $x \notin w_{\sigma\theta}^{0}\left(p\right)$. Hence (2.6) must hold.

Conversely, suppose that (2.6) hold and that $x \in w_{\sigma}^{\infty}\left[M_k,p\right]_{\theta}$. Then for each r and n

$$\frac{1}{h_r}\sum_{k\in I_r}\left[M_k\left(\frac{\left|t_{kn}\left(x\right)\right|}{\rho}\right)\right]^{p_k} \leq M < \infty. \tag{2.8}$$

Now suppose that $x \notin w_{\sigma\theta}^{0}\left(p\right)$. Then for some number $\varepsilon > 0$ and for a subinterval I_{r_i} of the set of interval I_r, there is k_0 such that $\left|t_{kn}\left(x\right)\right|^{p_k} > \varepsilon$ for $k \geq k_0$. From the properties of sequence of Orlicz functions, we obtain

$$\left[M_k\left(\frac{\varepsilon}{\rho}\right)\right]^{p_k} \leq \left[M_k\left(\frac{\left|t_{kn}\left(x\right)\right|}{\rho}\right)\right]^{p_k}$$

which contradict (2.7), by using (2.8). This completes the proof. ∎

3. Some Connections Between $St_{\theta\sigma}$ and $w_{\sigma}\left[M_k\right]_{\theta}$

In this section, we introduce natural relationship between lacunary strong convergence with respect to a sequence Orlicz functions and lacunary $\sigma-$statistical convergence.

In [2], Fast introduced the idea of statistical convergence. These idea were studied later in Connor [1], Freedman and Sember [4], Salat [13], Savas and Nuray [14], Pehlivan and Fisher [11], and other authors independently.

A complex number sequence $x = (x_k)$ is said to be statistically convergent to the number ℓ if for every $\varepsilon > 0$

$$\lim_{n\to\infty}\frac{1}{n}\left|\left\{k \leq n\colon\ \left|x_k - \ell\right| \geq \varepsilon\right\}\right| = 0$$

The set of statistically convergent sequences is denoted by s.

Definition 3. Let $\theta = (k_r)$ be lacunary sequence. A sequence $x = (x_k)$ is said to be lacunary strong $\sigma-$ statistically convergent if for every $\varepsilon > 0$

$$\lim_{r\to\infty}\frac{1}{h_r}\left|\left\{k \in I_r\colon\ \left|t_{kn}\left(x - \ell\right)\right| \geq \varepsilon\right\}\right| = 0,\ \text{uniformly in } n.$$

The set of all lacunary strong $\sigma-$ statistically convergent sequences is denoted by $St_{\theta\sigma}$. If $\ell = 0$, write $St^0_{\theta\sigma}$ in place of $St_{\theta\sigma}$.

We now establish inclusion relations between $St_{\theta\sigma}$ and $w_\sigma [M_k]_\theta$.

Theorem 8. *Let $M = (M_k)$ be a sequence of Orlicz functions. Then $w^0_\sigma [M_k]_\theta \subset St^0_{\theta\sigma}$ if and only if*

$$\inf_k M_k (|t_{kn} (x)|) > 0 \textit{ for all } n.$$

Proof. If $\inf_k M_k (|t_{kn} (x)|) > 0$, then there exists a number $\beta > 0$ such that $M_k (|t_{kn} (x)|) > \beta$ for all n and k. Let $x \in w^0_\sigma [M_k]_\theta$. For given $\varepsilon > 0$

$$\begin{aligned} \frac{1}{h_r} \sum_{k \in I_r} M_k (|t_{kn} (x)|) &= \frac{1}{h_r} \sum_{\substack{k \in I_r \\ |t_{kn}(x)| \geq \varepsilon}} M_k (|t_{kn} (x)|) + \frac{1}{h_r} \sum_{\substack{k \in I_r \\ |t_{kn}(x)| < \varepsilon}} M_k (|t_{kn} (x)|) \\ &\geq \frac{1}{h_r} \sum_{\substack{k \in I_r \\ |t_{kn}(x)| \geq \varepsilon}} M_k (|t_{kn} (x)|) \\ &> \beta \frac{1}{h_r} |\{k \in I_r \colon \ |t_{kn} (x)| \geq \varepsilon\}| . \end{aligned}$$

It follows that $x \in St^0_{\theta\sigma}$.

The reverse of this theorem will be shown by an example function. Let $M_k (x) = x^{p_k} |\log x|^{\alpha_k}$ where $1 < p_k < \infty$ and $1 < \alpha_k < \infty$. Therefore it is a sequence of Orlicz functions. Now let us define sequence $x = (x_k)$ as following $x_k = 1$ if $k \in I_r$ and $x_k = 0$ if $k \notin I_r$. Then there exists $|t_{kn} (x)| \geq \varepsilon > 0$ for all k and n, such that $M_k (|t_{kn} (x)|) = 0$ for all every $k \in I_r$. From here, it can be seen that $x \in w^0_\sigma [M_k]_\theta$ but $x \notin St^0_{\theta\sigma}$. ∎

Theorem 9. $St^0_{\theta\sigma} \subset w^0_\sigma [M_k]_\theta$ *if and only if*

$$\sup_k M_k (|t_{kn} (x)|) < \infty \textit{ for all } n.$$

Proof. We suppose that $R = \sup_k \sup_n M_k (|t_{kn} (x)|)$ and $x \in St^0_{\theta\sigma}$. Since $M_k (|t_{kn} (x)|) \leq R$ for all k and n, we

$$\begin{aligned} \frac{1}{h_r} \sum_{k \in I_r} M_k (|t_{kn} (x)|) &= \frac{1}{h_r} \sum_{\substack{k \in I_r \\ |t_{kn}(x)| \geq \varepsilon}} M_k (|t_{kn} (x)|) + \frac{1}{h_r} \sum_{\substack{k \in I_r \\ |t_{kn}(x)| < \varepsilon}} M_k (|t_{kn} (x)|) \\ &\leq 2R \frac{1}{h_r} |\{k \in I_r \colon \ |t_{kn} (x)| \geq \varepsilon\}| . \end{aligned}$$

Taking the limit as $r \to \infty$ uniformly in n, it follows that $x \in w_\sigma^0 [M_k]_\theta$.

Conversely suppose that $\sup_k \sup_n M_k (|t_{kn}(x)|) = \infty$. Then we have $0 < t_{1n} < t_{2n} < ... < t_{kn} < ...$ such that $M_{k_r}\left(\left|t_{k_r,n}(x)\right|\right) \geq h_r$ for $r \geq 1$. We define the sequence $x = (x_k)$ by $x_i = 1$ if $i = k \in I_r$ for some $r = 1, 2, 3, ...$ and $x_i = 0$ for $i \neq k \in I_r$. Then we have

$$\frac{1}{h_r} \sup_n |\{k \in I_r \colon\ |t_{kn}(x)| \geq \varepsilon\}| = \frac{|k_r|}{|h_r|} \left| \frac{1}{k_r + 1} \sum_{i=0}^{k_r} \sup_n x_{\sigma^i(n)} \right|.$$

Since $\sup_n x_{\sigma^i(n)} = 1$ for $i = k_r$ for some $r = 1, 2, 3, ...$, we have

$$\lim_{r \to \infty} \frac{|k_r|}{|h_r| \, |k_r + 1|} = 0.$$

So, it follows that $x \in St_{\theta\sigma}^0$. Now , since $M_{k_r}\left(\left|t_{k_r,n}(x)\right|\right) \geq h_r$, we have

$$\frac{1}{h_r} \sum_{k \in I_r} M_k \left(\frac{|t_{kn}(x)|}{\rho} \right) \geq h_{r.} \tag{3.1}$$

Since (3.1) tends infinity as $r \to \infty$, we get that $x \notin w_\sigma^0 [M_k]_\theta$. This completes the proof. ∎

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