TAIWANESE JOURNAL OF MATHEMATICS
Vol. 9, No. 2, pp. 261-280, June 2005
This paper is available online at http://www.math.nthu.edu.tw/tjm/

GEOMETRIC MECHANICS ON A STEP 4 SUBRIEMANNIAN MANIFOLD

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Abstract. In this paper we study the behavior of subRiemannian geodesics on a certain step 4 subRiemannian manifold. We compute the length of the subRiemannian geodesics between the origin and any point on the $t-$axis, where the conjugate locus is. We characterize the number of subRiemannian geodesics between the origin and any other point.

1. Introduction and results

In this paper we shall study geometric problems on $\mathcal{H}_4 = \mathbb{R}^2_{(x_1,x_2)} \times \mathbb{R}_t$ given by the vector fields $X = \{X_1, X_1\}$ where

$$(1.1) \qquad X_1 = \frac{\partial}{\partial x_1} + 4x_2|x|^2\frac{\partial}{\partial t} \ , \qquad X_2 = \frac{\partial}{\partial x_2} - 4x_1|x|^2\frac{\partial}{\partial t},$$

with $|x|^2 = x_1^2 + x_2^2$. We shall give detailed discussion of the geometry induced by the sub-Laplacian $\Delta_X = \frac{1}{2}(X_1^2 + X_2^2)$. As

$$(1.2) \qquad \begin{aligned} [X_1, X_2] &= -16|x|^2\frac{\partial}{\partial t}, \quad [X_1,[X_1,X_2]] = -32x_1\frac{\partial}{\partial t} \\ &\text{and} \quad [X_1,[X_1,[X_1,X_2]]] = -32\tfrac{\partial}{\partial t}, \end{aligned}$$

it is easy to see that $\mathcal{H}_4$ is a step 4 subRiemannian manifold. SubRiemannian geometry starts with Carathéodory's formalization of thermodynamics [10] where

Accepted November 6, 2003.
Communicated by Sze-Bi Hsu.
2000 *Mathematics Subject Classification*: 53C17, 53C22, 35H20.
Key words and phrases: Elliptic functions, SubRiemannian geometry, SubRiemannian geodesics, Conjugate points, Bracket generating condition.
Dedicated to Professor Hwai-Chiuan Wang on the occasion of his retirement.
*Partially supported by a William Fulbright Research grant and a competitive research grant at Georgetown University.

the quasi-static adiabatic processes are related to the integral curves of a Darboux model. It is shown in [10] that any two points in this model can be joined by a piecewise smooth integral curve. Later Chow [11] proves that this fact is true for any two points in a model which involves a distribution with bracket-generating property. This means that the full tangent space at any point can be spanned by iterated Lie brackets of vector fields tangent to the distribution. The integral curves in Chow's connectivity theorem are piecewise smooth. However, we showed that these piecewise smooth curves can be replaced by global curves for this model in [9].

On a subRiemannian manifold the metric is given only on the distribution. Then one may define the Carnot-Carathéodory distance between any two points as the infimum of the length of integral curves joining the points (see [1]). On the other hand these problems have been approached from the sub-elliptic operators point of view. In this case the distribution is defined by a set of linear independent set of vector fields. The subRiemannian metric can be chosen such that the vector fields form an orthonormal system at any point. The sum of the squares of the vector fields is a sub-elliptic operator. The number of the generating brackets $+1$ needed to span the tangent space at any point is called the *step* of the operator.

Using the principal symbol of this sub-elliptic operator one may define a Hamiltonian and associate subRiemannian geodesics to it. The relation between the length of these subRiemannian geodesic and the Carnot-Carathéodory distance has been analyzed by Strichartz [15]. It turns out that the subRiemannian geodesics are integral curves along the distribution and are locally length-minimizing integral curves. The study of subRiemannian geodesics on the Heisenberg group was worked out by Beals, Gaveau and Greiner [5], using Hamiltonian mechanics. They showed that the conjugate points are along the t-axis. They also computed lengths of geodesics and the Carnot-Carathéodory distance between the origin and points on the t-axis.

It has been shown that the Carnot-Carathéodory distance comes into the fundamental solution formula for the Heisenberg Laplacian. It plays the same role as Riemannian distance plays for strongly elliptic operators (see [2-4] and [12]).

This article is one of a series (see [6-9]), whose aim is to give explicit calculation for a step 4 subRiemannian manifold. The tools used are the variational calculus and the theory of elliptic functions which make possible to write explicitly the equations for the subRiemannian geodesics under given boundary conditions. We may state our results as follows. Both theorems take place in the context of the vector fields (1.1).

Theorem 1.1. *The subRiemannian geodesics that join the origin to a point* $(0, 0, t)$ *have lengths* d_1, d_2, $d_3 \dots$, *where*

$$(d_m)^4 = \frac{m^3 K^4}{4Q}|t| \tag{1.3}$$

with

$$K = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}, \qquad k = \frac{\sqrt{2-\sqrt{3}}}{2}, \tag{1.4}$$

$$Q = \frac{1}{4}\frac{\Gamma(1/6)}{\Gamma(2/3)}\sqrt{\pi}. \tag{1.5}$$

For each length d_m, the geodesics of that length are parameterized by the circle S^1.

A similar identity is proved in [5] for the Heisenberg group which is a step 2 subRiemannian manifold. They prove that $(d_m)^2 = m\pi|t|$. One may observe how the step influence the power of the distance. It is shown in [15] that in the case of step 2 models the cut points can be as close as possible to the origin. In the present step 4 example we get something similar, the cut locus being the $t-$axis.

Theorem 1.2. *Let P be a point with the coordinates (x_1, x_2, t).*

(i) *If $|x| = 0$ and $t \neq 0$ there are infinitely many geodesics between the origin and P.*

(ii) *If $0 \leq \dfrac{t}{|x|^4} < \infty$ there are finitely many geodesics between the origin and P. This number increases unbounded as $t/|x|^4 \to \infty$.*

The paper is organized as follows. In section 2, we give some background on elliptic functions. In section 3, we solve the Hamiltonian system in polar coordinates and find all the conjugate points to the origin. We prove Theorems 1.1 and 1.2 in sections 4 and 5 respectively. In section 6, we discuss geodesics between the origin and points away from the t-axis. SubRiemannian geodesics can be realized as trajectories of charged particles in a certain magnetic field. A magnetic field is given by a $2-$form Ω which satisfies $d\Omega = 0$. The trajectories of the charged particles in the magnetic field Ω are given by the spatial component of the corresponding subRiemannian geodesics. This shows the subRiemannian geometry is a good environment for doing Quantum Mechanics. In our case $\Omega = 16|x|^2 dx_1 \wedge dx_2$ and comes from the potential $\omega = dt + 4|x|^2(x_1dx_2 - x_2dx_1)$. We shall give a detailed discussion in section 7.

The paper is based on a lecture given by the second author at the 12th Workshop on Differential Equations and Mathematical Analysis which was held at National Tsing Hua University, Hsinchu, Taiwan, January 9-11, 2004. The author takes great pleasure in expressing his thanks to the organizing committee, especially Professor Sze Bi Hsu for the invitation and the warm hospitality he has received while he visited Taiwan.

2. Elliptic Functions

As mentioned in section 1, the study of step 4 models requires the use of elliptic functions. The Heisenberg group needs only elementary functions while steps greater than 4 require hypergeometric functions. We shall provide in the following the definitions of the elliptic functions used in the next sections. For a detailed description the reader may consult Lawden [14].
The integral

$$z = \int_0^w \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}, \qquad |k| < 1$$

is called an elliptic integral of the first kind. The integral exists if w is real and $|w| < 1$. Using the substitution $t = \sin\theta$ and $w = \sin\phi$

$$z = \int_0^\phi \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}.$$

If $k = 0$, then $z = \sin^{-1} w$ or $w = \sin z$. By analogy, the above integral is denoted by $\mathrm{sn}^{-1}(w; k)$, where $k \neq 0$. k is called the modulus. Thus

$$z = \mathrm{sn}^{-1} w = \int_0^w \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}.$$

The function $w = \mathrm{sn}\, z$ is called a Jacobian elliptic function.

By analogy with the trigonometric functions, it is convenient to define other elliptic functions

$$\mathrm{cn}\, z = \sqrt{1-\mathrm{sn}^2 z}, \qquad \mathrm{dn}\, z = \sqrt{1-k^2\,\mathrm{sn}^2 z}.$$

A few properties of this functions are

$$\mathrm{sn}(0) = 0, \qquad \mathrm{cn}(0) = 1, \qquad \mathrm{dn}(0) = 1,$$

$$\mathrm{sn}(-z) = \mathrm{sn}(z), \qquad \mathrm{cn}(-z) = \mathrm{cn}(z),$$

$$\frac{d}{dz}\mathrm{sn}\, z = \mathrm{cn}\, z\, \mathrm{dn}\, z, \ \frac{d}{dz}\mathrm{cn}\, z = -\mathrm{sn}\, z\, \mathrm{dn}\, z, \ \frac{d}{dz}\mathrm{dn}\, z = -k^2 \mathrm{sn}\, z\, \mathrm{cn}\, z,$$

$$-1 \le \mathrm{cn}\, z \le 1, \quad -1 \le \mathrm{sn}\, z \le 1, \quad 0 \le \mathrm{dn}\, z \le 1$$

Let

$$K = K(k) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}$$

be the complete Jacobi integral. Then, as real functions, the elliptic functions sn and cn are periodic functions of principal period $4K$.

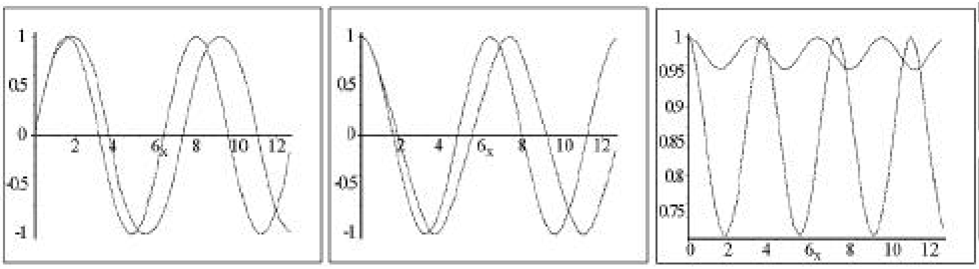

Fig. 1. The graphs of functions $\mathrm{sn}(z,k)$, $\mathrm{cn}(z,k)$ and $\mathrm{dn}(z,k)$ for $k = 0.3$ and 0.7.

3. Hamiltonian System

Let us return to the vector fields defined by (1.1). Obviously, the operator

$$\Delta_X = \frac{1}{2}(X_1^2 + X_2^2)$$

is non-elliptic. From (1.2), it is easy to see that Δ_X is step 4 along the t-axis and step 2 elsewhere. By a well-known result of Hörmander [13], one knows that the sub-Laplacian Δ_X is hypoelliptic. The Hamiltonian H is defined as the principal symbol of Δ_X

$$H(\xi, \theta, x, t) = \frac{1}{2}\Big(\xi_1 + 4x_2|x|^2\theta\Big)^2 + \frac{1}{2}\Big(\xi_2 - 4x_1|x|^2\theta\Big)^2.$$

Definition 3.1. The subRiemannian geodesics between the origin O and the point $P(x,t)$ are the projections on the (x,t)-plane of the solutions of the Hamiltonian system

$$\begin{cases} \dot{x} = \partial H/\partial \xi \\ \dot{t} = \partial H/\partial \theta \\ \dot{\xi} = -\partial H/\partial x \\ \dot{\theta} = -\partial H/\partial t \end{cases}$$

with the boundary conditions

$$x(0) = t(0) = 0, \;\; x(1) = x, \;\; t(1) = t.$$

The Hamiltonian formalism is equivalent to the Lagrangian one; see **[9]**. The associated Lagrangian is given by

$$L(x, \dot{t}, \theta) = \frac{1}{2}(\dot{x}_1^2 + \dot{x}_2^2) + \theta\dot{t} + 4\theta|x|^2(x_1\dot{x}_2 - \dot{x}_1 x_2).$$

• The solution in polar coordinates
In polar coordinates $x_1 = r\cos\phi$, $x_2 = r\sin\phi$ one has

$$L = \frac{1}{2}(\dot r^2 + r^2\dot\phi^2) + \theta\dot t + 4\theta r^4\dot\phi.$$

The Euler-Lagrange system of equations verified by $r(s)$ and $\phi(s)$ is

$$\text{(3.6)} \qquad \begin{cases} \ddot r = r\dot\phi(\dot\phi + 16\theta r^2) \\ r^2(\dot\phi + 4\theta r^2) = k(\text{constant}) \\ \theta = \text{constant} \end{cases}$$

with the boundary conditions

$$r(0) = 0\,, \quad r(1) = \rho\,, \quad \phi(1) = \Phi$$

which can be written as two boundary value problems

$$\text{(3.7)} \qquad \begin{cases} \ddot r = -48\theta^2 r^5 \\ \theta = \text{constant} \\ r(0) = 0, \quad r(1) = \rho \end{cases} \qquad \begin{cases} \dot\phi = -4\theta r^2 \\ \theta = \text{constant} \\ \phi(1) = \Phi. \end{cases}$$

The initial argument $\phi(0) = \phi_0$ depends on Φ and ρ.

The law of conservation of energy for the first equation in (3.7) is

$$\text{(3.8)} \qquad \frac{1}{2}\dot r^2 + V(r) = E$$

where E is the constant of the total energy and $V(r) = 8\theta^2 r^6$.

If $\theta = 0$ then $r(s) = s\rho$ and $\phi(s) = \Phi = \phi_0$, $s \in [0,1]$. In the following discussion we assume $\theta > 0$. Then $\phi(s)$ will be decreasing and $\phi_0 > \Phi$. Denoting by r_{max} the solution of the equation $V(r) = E$, one has

$$\text{(3.9)} \qquad r_{max} = \Big(\frac{E}{8\theta^2}\Big)^{1/6}.$$

From (3.8) and second system in (3.7) we can write the simultaneous equations

$$\frac{dr}{ds} = \sqrt{2E - 16\theta^2 r^6} \qquad \text{and} \qquad \frac{d\phi}{ds} = -4\theta r^2,$$

where $r(s)$ is assumed increasing from 0 to r_{max}. By division and integration, the variation of the angle ϕ is obtained

$$\phi - \phi_0 = \int_0^{r(\phi)} \frac{-4\theta x^2}{\sqrt{2E - 16\theta^2 x^6}}\,dx.$$

When $r = r_{max}$ we obtain the angle $w = -\dfrac{\pi}{6}$.

Similarly, when $r(s)$ decreases between $r = r_{max}$ and $r = 0$ the vectorial radius sweeps an angle $\tilde{w} = -\pi/6$. This means the solution $r(\phi)$ is periodic with period $\frac{\pi}{3}$.

Lemma 3.2. *The solution in polar coordinates satisfies*

$$r(\phi)^2 = r_{max}^2 \sin^{2/3}(3(\phi - \phi_0)). \tag{3.10}$$

If $\rho \neq 0$ *then* r_{max} *can be expressed as*

$$r_{max}^2 = \frac{\rho^2}{\sin^{2/3}\Big(3(\Phi - \phi_0)\Big)}. \tag{3.11}$$

Proof. When the radius r increases (decreases) ϕ satisfies the first (second) system bellow.

$$\begin{cases} \dfrac{d\phi}{dr} = \dfrac{-4\theta r^2}{\sqrt{2E - 16\,\theta^2\, r^6}} \\ \phi(r_{max}) = \phi_0 - \pi/6 \end{cases} \qquad \begin{cases} \dfrac{d\tilde{\phi}}{dr} = \dfrac{4\theta r^2}{\sqrt{2E - 16\,\theta^2\, r^6}} \\ \tilde{\phi}(r_{max}) = \phi_0 - \pi/6 \end{cases} \tag{3.12}$$

where $\phi_0 \geq \phi \geq \phi_0 - \frac{\pi}{6} \geq \tilde{\phi} \geq \phi_0 - \frac{\pi}{3}$. The relation between the solutions of the systems (3.12) is

$$\tilde{\phi} = 2(\phi_0 - \frac{\pi}{6}) - \phi \tag{3.13}$$

and then it is enough to solve just the first system. Integrating in the first system in (3.12) between 0 and $r(\phi)$ we get

$$\phi - \phi_0 = -\frac{1}{3} \arcsin\Big(\frac{r(\phi)^3}{r_{max}^3}\Big), \tag{3.14}$$

which leads to (3.10). From (3.13) and (3.14) we obtain

$$r^2(\tilde{\phi}) = r_{max}^2 \sin^{2/3}(3(\tilde{\phi} - \phi_0)).$$

which is also (3.10). Relation (3.11) comes from the boundary condition $r(\phi) = \rho$. ■

• The t-component of the solution

Using one of the Hamiltonian equations, the t-component along the solution is

$$\dot{t} = \frac{\partial H}{\partial \theta} = 4|x|^2(x_2\dot{x}_1 - x_1\dot{x}_2) = -4r^4\dot{\phi}. \tag{3.15}$$

Using Lemma 3.2 and integrating one obtains

$$t(\phi) - t(\phi_0) = -\frac{4}{3} r_{max}^4 \int_0^{3(\phi-\phi_0)} \sin^{4/3} v\, dv. \tag{3.16}$$

Hence $t(\phi) - t(\phi_0)$ depends on the difference $\phi - \phi_0$ and the orbital period

$$T = t(\phi_0 + \pi/3) - t(\phi_0) \tag{3.17}$$

does not depend on ϕ_0. From equation (3.16) one has

$$\begin{aligned} T = t(\phi_0 + \pi/3) - t(\phi_0) &= -\frac{4}{3} r_{max}^4 \int_0^\pi \sin^{4/3} v\, dv \\ &= \frac{4}{3} r_{max}^4 \int_0^1 z^{\frac{1}{6}} (1-z)^{-\frac{1}{2}}\, dz = \frac{4}{3} r_{max}^4 \mathcal{B}\left(\frac{7}{6}, \frac{1}{2}\right) \\ &= \frac{1}{3} r_{max}^4 \frac{\sqrt{\pi}\Gamma(1/6)}{\Gamma(2/3)}. \end{aligned}$$

It follows that

$$T^{\frac{1}{4}} = \Big(\frac{\sqrt{\pi}}{3} \frac{\Gamma(1/6)}{\Gamma(2/3)}\Big)^{\frac{1}{4}} \cdot r_{max}.$$

This is an analogue of the third Kepler's law.

• Conjugate points to the origin

Consider an initial geodesic $(r(\phi), t(\phi))$ which starts at origin with the initial argument ϕ_0. From (3.10) and (3.17) we get

$$r(\phi_0 + \frac{\pi}{3}) = 0\,, \qquad t(\phi_0 + \frac{\pi}{3}) = T.$$

A variation of ϕ_0 corresponds to a rotation of the solution around the t-axis. The point $(0, 0, T)$ remains fixed during this variation and belongs to all geodesics which are obtained from the initial one by the variation of the initial angle ϕ_0. There are no conjugate points outside t-axis because under a variation in ϕ_0 the conjugate point will describe a circle and will not be a fixed point any more.

The following result states that the conjugate points to origin along the geodesics is the t-axis.

Theorem 3.3. *Given the point $P = (0, 0, u)$ on the t-axis, $u \neq 0$, there is a geodesic γ which starts at the origin, such that P is the first point conjugate to O along γ.*

Proof. The proof is based on the construction of such a geodesic.
(i) *case* $u < 0$. We choose $\theta < 0$ and require $T = u$. Again, equation (3.16) yields

$$T = t(\phi_0 + \pi/3) - t(\phi_0) = -\frac{4}{3} r_{max}^4 \int_0^\pi \sin^{4/3} v\, dv.$$

It follows that

$$r_{max}^4 = -\frac{3w}{4Q},$$

where $Q = \int_0^\pi \sin^{4/3} v\, dv$. The desired geodesic will be given by

$$\begin{aligned} r(\phi) &= \Big(\frac{3|u|}{4Q}\Big)^{1/4} \sin^{1/3}\big(3(\phi - \phi_0)\big), \\ t(\phi) &= \frac{u}{Q} \int_0^{3(\phi-\phi_0)} \sin^{4/3} v\, dv. \end{aligned}$$

It is easy to see that the point $P = (0, 0, u)$ belongs to the above geodesic. Performing a rotation in the x-plane of the initial velocity, all the geodesics obtained will pass through P, and hence P will be the first conjugate point with the origin.
(ii) *case* $u > 0$. In this case $\theta > 0$ and the rest of the proof is similar to the $\theta < 0$ case. ■

4. Proof of Theorem 1.1

We need to find the Hamiltonian path connecting the origin to $(0, 0, t)$, $t > 0$. The path is considered to be parametrized by $[0, 1]$. The boundary conditions are $r(0) = r(1) = 0$, $t(0) = 0$ and $t(1) = t$. Using the first system in (3.7) one obtains that the constant θ verifies the following boundary value problem

$$\begin{cases} \ddot{r} = -48\theta^2 r^5 \\ r(0) = r(1) = 0. \end{cases} \tag{4.18}$$

Writing $\lambda^2 = 48\theta^2$ then the conservation of energy leads to

$$\frac{1}{2}\dot{r}^2 + \frac{1}{6}\lambda^2 r^6 = E,$$

where E is the constant value of the Hamiltonian along the solutions. By separation and integration we get

$$\int_0^r \frac{dx}{\sqrt{2E - \frac{1}{3}\lambda^2 x^6}} = s,$$

which can be written as

$$\int_0^{\sigma(r)^2} \frac{dt}{\sqrt{t(1-t^3)}} = \frac{2}{3^{1/6}}(2E\lambda)^{1/3} s, \tag{4.19}$$

where

$$\sigma(r) = \Big(\frac{\lambda^2}{6E}\Big)^{1/6} r.$$

The left hand side in (4.19) can be expressed in terms of elliptic functions. This will be done in the following.

Lemma 4.1. *We have*

$$\int_0^{\sigma(r)^2} \frac{dt}{\sqrt{t(1-t^3)}} = \frac{1}{3^{1/4}} sn^{-1}\Big[\sqrt{1 - \frac{g(\sigma)^2}{b^2}}, k\Big],$$

where $b = 2 - \sqrt{3}$, $k = \sqrt{b}/2$ *and* $g(\sigma) = \dfrac{\sqrt{3}}{\sigma^2 + \frac{1+\sqrt{3}}{2}} - 1.$

Proof. We need to evaluate the integral $\int_0^{\sigma^2} \frac{dx}{\sqrt{f(x)}}$ where $f(x) = x(1-x^3)$. Making the substitution $x = \frac{pt+q}{t+1}$ where $p = \frac{-1-\sqrt{3}}{2}$, $q = \frac{-1+\sqrt{3}}{2}$, the function f equals to

$$f = -\frac{1}{(t+1)^4} \cdot \frac{3}{2} \cdot (\frac{3}{2} + \sqrt{3})(t^2 - b^2)(t^2+1)$$

where $b = 2 - \sqrt{3}$. Denoting $g(\sigma) = \frac{\sqrt{3}}{\sigma^2 + \frac{1+\sqrt{3}}{2}} - 1$ the integral becomes

$$\int_0^{\sigma^2} \frac{dx}{\sqrt{f(x)}} = \frac{2}{\sqrt{3+2\sqrt{3}}} \int_{g(\sigma)}^{b} \frac{1}{\sqrt{(b^2-t^2)(t^2+1)}}\, dt.$$

Using the formula (see **[14]** chapter 3)

$$\int_x^b \frac{1}{\sqrt{(a^2+t^2)(b^2-t^2)}}\, dt = \frac{1}{\sqrt{a^2+b^2}}\, sn^{-1}[\frac{\sqrt{b^2-x^2}}{b}\, ,\, \frac{b}{\sqrt{a^2+b^2}}]$$

one obtains

$$\int_0^{\sigma^2} \frac{dx}{\sqrt{f(x)}} = 3^{-1/4}\, sn^{-1}[\sqrt{1 - \frac{g(\sigma)^2}{b^2}}\, ,\, \frac{\sqrt{b}}{2}]. \qquad \blacksquare$$

Using Lemma 4.1 the relation (4.19) becomes

$$\sqrt{1 - \frac{g(\sigma)^2}{b^2}} = sn\,[2^{4/3} \cdot 3^{1/12}(E\lambda)^{1/3} s\ ,\ k]. \tag{4.20}$$

Using the boundary values from (4.18) one may verify that for $s = 0, 1$ the left hand side in (4.20) is zero and hence

$$2^{4/3} \cdot 3^{1/12} (E\lambda)^{1/3} = 2mK\,, \qquad m = 1, 2, 3 \dots \tag{4.21}$$

where $k = \sqrt{b}/2$ and

$$K = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}.$$

From (4.21) one has

$$\theta = \Big(\frac{mK}{2 \cdot 3^{1/4}}\Big)^3 / E,$$

and hence

$$\frac{E}{\theta^2} = (2 \cdot 3^{1/4})^6 \Big(\frac{E}{m^2K^2}\Big)^3. \tag{4.22}$$

Then (3.16) becomes

$$t = \frac{m}{3}\Big(\frac{E}{\theta^2}\Big)^{2/3} Q, \tag{4.23}$$

with $Q = \int_0^{\pi} \sin^{4/3} u\, du$. We used a decreasing ϕ because θ was assumed positive, according to the formula $\dot{\phi} = -4\theta r^2$. Using (4.22) and (4.23) one obtains

$$(2E)^2 = \frac{m^3 K^4}{4Q} t\,, \qquad m = 1, 2, 3... \tag{4.24}$$

The physical interpretation of the above relation is the fact that the energy is discrete and depends on t. Geometrically this means the subRiemannian geodesics joining the origin O to $P(0,0,t)$ have discrete lengths. Let $\gamma : [0,1] \to \mathbb{R}^3$ be a geodesic between O and P. As $|\dot{\gamma}|$ is constant in the subRiemannian metric, we have identity in the Cauchy's inequality

$$l(\gamma) = \int_0^1 |\dot{\gamma}(s)|\, ds = \Big(\int_0^1 ds\Big)^{1/2} \Big(\int_0^1 |\dot{\gamma}(s)|^2\, ds\Big)^{1/2} = \sqrt{2E}. \tag{4.25}$$

Using (4.24) and (4.25) one obtains that the lengths of the solution γ are quantified, namely, $l(\gamma) = d_m$ where

$$(d_m)^4 = \frac{m^3 K^4}{4Q} t\,, \qquad m = 1, 2, 3... \tag{4.26}$$

The constant Q can be expressed in terms of gamma functions as follows.

Lemma 4.2.
$$Q = \frac{1}{4} \frac{\Gamma(1/6)}{\Gamma(2/3)} \sqrt{\pi}.$$

Proof. Making the substitution $t = \sin u$ one gets

$$Q = \int_0^{\pi} \sin^{4/3} u\, du = 2 \int_0^{\frac{\pi}{2}} \sin^{4/3} u\, du = 2 \int_0^1 t^{4/3}(1-t^2)^{-1/2}\, dt.$$

With the substitution $t^2 = x$ the integral becomes a beta function which can be expressed in terms of gamma functions

$$Q = \int_0^1 x^{1/6}(1-x)^{-1/2}\, dx = B(7/6, 1/2) = \frac{\Gamma(7/6)\,\Gamma(1/2)}{\Gamma(7/6 + 1/2)}.$$

Using $\Gamma(1/2) = \sqrt{\pi}$, $\Gamma(7/6) = \frac{1}{6}\Gamma(1/6)$ and $\Gamma(7/6 + 1/2) = \frac{2}{3}\Gamma(2/3)$ one obtains

$$Q = \frac{1}{4} \frac{\Gamma(1/6)}{\Gamma(2/3)} \sqrt{\pi}.$$ ∎

Remark 4.3. These lengths do not depend on the angle ϕ_0. When this angle is varied the solution is rotating around t-axis having the same length all the time.

Remark 4.4. The shortest solution corresponds to $m = 1$ which is the Carnot-Carathéodory distance between O and $P(0,0,t)$. One has

$$(d_1)^4 = \frac{K^4}{4Q}|t|.$$

5. Proof of Theorem 1.2

Let P be a point with polar coordinates ρ, Φ and t-component t and consider a solution which joins the origin and P. The t-component is computed by integration in (3.15) between ϕ_0 and Φ

$$t = -\frac{4}{3} r_{max}^4 \int_0^{3(\Phi-\phi_0)} \sin^{4/3} v \, dv.$$

Applying the relation for r_{max} given in Lemma 3.2 we get

$$\frac{3}{4} \frac{t}{\rho^4} \sin^{4/3} \Big(3(\Phi - \phi_0) \Big) = - \int_0^{3(\Phi-\phi_0)} \sin^{4/3} v \, dv. \tag{5.27}$$

Define

$$F(x) = \int_0^x \sin^{4/3} u \, du,$$

which is an unbounded, increasing and odd function. Denoting $\mu = 3(\phi_0 - \Phi)$ the relation (5.27) can be written as

$$\frac{3}{4} \frac{t}{\rho^4} \sin^{4/3} \mu = F(\mu).$$

As the left hand side is a periodic and bounded function in μ, the above equation has only a finite numbers of roots μ. This means that for $\rho \neq 0$ there are only a finite number of subRiemannian geodesics joining the origin O and the point P. This number increases unbounded as the quotient t/ρ^4 increases.

Remark 5.1. For $t = 0$ the only solution is the segment line OP.

6. Geodesics Between the Origin and Points Away from the t-Axis

We have seen that if $|x| = 0$ and $t \neq 0$, there are infinitely many geodesics between the origin and P. In this section we shall study the $|x| \neq 0$ case. Let us start with an easy case.

Theorem 6.1. *Given a point $P(x, 0) \in \mathbb{R}^3$, there is a unique geodesic between the origin and P. It is a straight line in the x-plane of length $|x|$.*

Proof. In this case $\theta = 0$, the Hamiltonian is $H(\xi) = \frac{1}{2}\xi_1^2 + \frac{1}{2}\xi_2^2$. From Hamiltonian equation $\dot{t} = \partial H/\partial\theta = 0$, $t(s)$ is constant. As $t(0) = 0$, it follows that $t(s) = 0$ for $s \in [0, 1]$ and the solution belongs to the x-plane. Using the equations $\ddot{r} = 0$ and $\dot{\phi} = 0$, it follows the solution is a straight line. ■

Theorem 6.2. *Let $P(x, t) \in \mathbb{R}^3$ be a point such that $x \neq 0$ and $t > 0$. There are finitely many geodesics between the origin and P. This number increases unbounded as $t/|x|^4 \to \infty$.*

Proof. Let ρ and Φ be the polar coordinates for the end point (x_1, x_2). Consider a geodesic which joins the origin and P. The t-component is computed by integration between ϕ_0 and Φ as in Section 2:

$$t = -\frac{4}{3} r_{max}^4 \int_0^{3(\Phi-\phi_0)} \sin^{4/3} v \, dv. \tag{6.28}$$

Writing $\rho = r(\Phi)$ in Proposition 3.2, yields

$$\rho^2 = r_{max}^2 \sin^{2/3}\Big(3(\Phi - \phi_0)\Big). \tag{6.29}$$

Eliminating r_{max} from equations (6.28) and (6.29) yields

$$\frac{3}{4}\frac{t}{\rho^4} \sin^{4/3}\Big(3(\Phi - \phi_0)\Big) = -\int_0^{3(\Phi-\phi_0)} \sin^{4/3} v \, dv. \tag{6.30}$$

Denote

$$F(x) = \int_0^x \sin^{4/3} u \, du.$$

It is easy to see that $F(x)$ is an increasing, unbounded and odd function. Let $\psi = 3(\phi_0 - \Phi)$. The relation (6.30) becomes

$$\frac{3}{4}\frac{t}{\rho^4} \sin^{4/3} \psi = F(\psi). \tag{6.31}$$

As the left hand side is a periodic and bounded function of variable ψ, the above equation has only a finite numbers of solutions; see Figure 1. This means that for $\rho \neq 0$ there are only a finite number of subRiemannian geodesics joining the origin O and the point P. This number increases unbounded as the quotient t/ρ^4 increases. ■

A detailed analysis of the number of solutions of equation (6.31) is done in the following lemmas.

Let $\lambda(\psi) = \frac{3}{4}\frac{t}{\rho^4} \sin^{4/3} \psi$. The functions $\lambda(\psi)$ and $F(\psi)$ start from the origin and their graphs are tangent at $\psi = 0$. The following lemma shows that the graph of $F(\psi)$ is below the graph of $\lambda(\psi)$ on a small neighborhood.

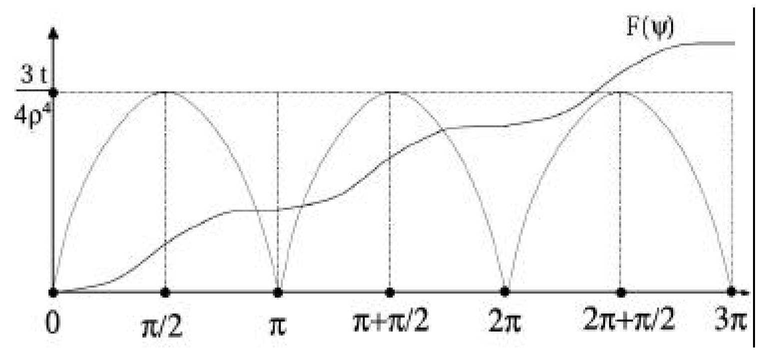

Fig. 2. The functions $\lambda(\psi) = \frac{3}{4}\frac{t}{\rho^4} \sin^{4/3} \psi$ and $F(\psi)$.

Lemma 6.3. *Let $t > 0$ and $\rho \neq 0$. There is $\epsilon > 0$ such that*

$$0 < \lambda(\psi) < F(\psi), \quad \textit{for} \quad 0 < \psi < \epsilon.$$

Hence, there are no solutions for the equation (6.31) *on the interval* $(0, \epsilon)$.

Proof. The derivatives of λ at $\psi = 0$ blow up faster than the derivatives of F, as the following computation shows

$$\begin{aligned} F'(\psi) &= \sin^{4/3}\psi \Longrightarrow F'(0) = 0, \\ F''(\psi) &= \frac{4}{3}\sin^{1/3}\psi\,\cos\psi \Longrightarrow F''(0) = 0, \\ F'''(\psi) &= \frac{4}{9}\cdot\frac{4\cos^2\psi - 3}{\sin^{2/3}\psi} \Longrightarrow F'''(0) = +\infty, \\ \lambda'(0) &= 0, \qquad \lambda''(0) \;=\; +\infty. \end{aligned}$$

■

Lemma 6.4. *Let $t > 0$ and $\rho \neq 0$.*

(i) *Let $n \in \mathbb{N}^*$ be an integer such that*

$$(n - \frac{1}{2})Q < \frac{3}{4}\frac{t}{\rho^4} < (n + \frac{1}{2})Q, \tag{6.32}$$

where

$$Q = \frac{1}{4}\frac{\Gamma(1/6)}{\Gamma(2/3)}\sqrt{\pi} \approx 1.82.$$

Then equation (6.31) *has no more than $2n + 1$ solutions and no less than $2n - 1$. All solutions belong to the interval $\Big(0, (n + \frac{1}{2})\pi\Big)$.*

(ii) *If there is $n \in \mathbb{N}^*$ such that*

$$\frac{3}{4}\frac{t}{\rho^4} = (n + \frac{1}{2})Q,$$

there are exactly $2n+1$ nonzero solutions for equation (6.31)*, and they belong to $\Big(0, (n + \frac{1}{2})\pi\Big]$. The largest solution is $(n + \frac{1}{2})\pi$.*

(iii) *If there is $n \in \mathbb{N}^*$ such that*

$$\frac{3}{4}\frac{t}{\rho^4} > (n + \frac{1}{2})Q,$$

then in the interval $\Big(0, (n + \frac{1}{2})\pi\Big]$ there are $2n$ distinct solutions for equation (6.31). However, equation (6.31) may have other solutions outside of the above interval.

Proof. (i) The function $x \longrightarrow \sin^{4/3} x$ is positive and periodic with the principal period π. Let $n \in \mathbb{N}^*$ be the smallest positive integer such that

$$\frac{3}{4}\frac{t}{\rho^4} < F\Big(n\pi + \frac{\pi}{2}\Big).$$

Then equation (6.31) has no solutions on the interval $(n\pi + \frac{\pi}{2}, +\infty)$.
Hence, if

$$F\Big((n-1)\pi + \frac{\pi}{2}\Big) < \frac{3}{4}\frac{t}{\rho^4} < F\Big(n\pi + \frac{\pi}{2}\Big), \tag{6.33}$$

there are three cases:

(1) If the graph of $F(\psi)$ is above the graph of $\lambda(\psi)$ on the interval $(n\pi, n + \frac{1}{2}\pi)$, there are exactly $2n - 1$ nonzero solutions. See Figure 2.
(2) If the graph of $F(\psi)$ is tangent to the graph of $\lambda(\psi)$ on the interval $(n\pi, n + \frac{1}{2}\pi)$, there are $2n$ nonzero solutions. The largest solution is a double solution. See Figure $3(a)$.
(3) If the graph of $F(\psi)$ intersects the graph of $\lambda(\psi)$ twice on the interval $(n\pi, n + \frac{1}{2}\pi)$, there are exactly $2n + 1$ nonzero solutions. See Figure $3(b)$.

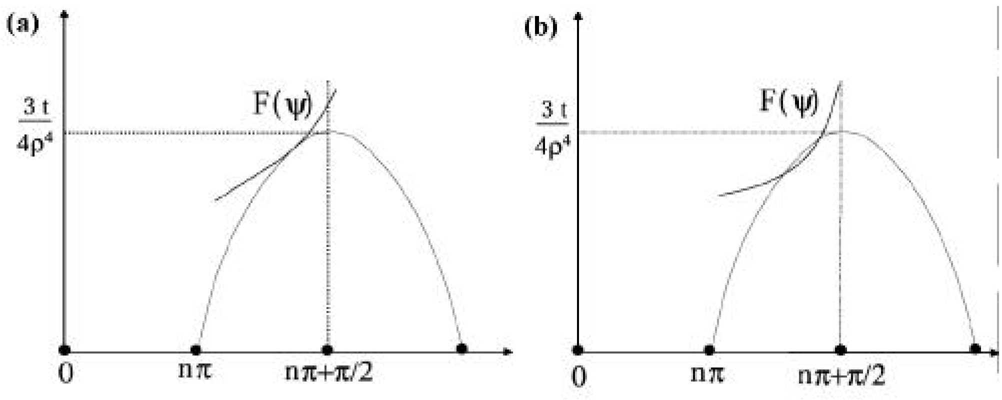

Fig. 3.

All solutions belong to the interval $\big(0, (n + \frac{1}{2})\pi\big)$. In the following we shall show that

$$F\Big(n\pi + \frac{\pi}{2}\Big) = \Big(n + \frac{1}{2}\Big)Q. \tag{6.34}$$

Using the properties of the integral of a periodic function,

$$\begin{aligned} F\Big(n\pi + \frac{\pi}{2}\Big) &= \int_0^{n\pi + \frac{\pi}{2}} \sin^{\frac{4}{3}} x\, dx \\ &= (2n+1)\int_0^{\frac{\pi}{2}} \sin^{\frac{4}{3}} x\, dx \\ &= (2n+1)\frac{Q}{2} = \Big(n + \frac{1}{2}\Big)Q. \end{aligned}$$

Replacing n by $n-1$, yields

$$F\Big((n-1)\pi+\frac{\pi}{2}\Big)=\Big(n-1+\frac{1}{2}\Big)Q=(n-\frac{1}{2})Q. \tag{6.35}$$

Substituting (6.34) and (6.35) in (6.33) yields (6.32).
(ii) If there is an integer $n\in\mathbb{N}^*$ such that

$$F\Big(n\pi+\frac{\pi}{2}\Big)=\frac{3}{4}\frac{t}{\rho^4},$$

then

$$(n+\frac{1}{2})Q=\frac{3}{4}\frac{t}{\rho^4},$$

and there is one more solution, equal to $n\pi+\frac{\pi}{2}$.
(iii) The condition $\frac{3}{4}\frac{t}{\rho^4}>(n+\frac{1}{2})Q=F\big(n\pi+\frac{\pi}{2}\big)$ implies that the graph of F is below the line $y=\frac{3t}{4\rho^4}$ on the interval $[0,n\pi+\frac{\pi}{2}]$. The graph of F intersects the graph of λ right before and after each turning point such as $\pi,\dots,n\pi$. Hence are $n+n$ intersections, which corresponds to $2n$ solutions in the interval $[0,n\pi+\pi/2]$. See Figure 2. ■
As a conclusion, one may state the following theorem.

Theorem 6.5. *Let $P(x,t)$ be a point of $\mathbb{R}^3$, away from the t-axis and x-plane. There are not less than $2n-1$ and not more than $2n+1$ subRiemannian geodesics between the origin and the point P, where the integer n is defined by*

$$(n-\frac{1}{2})Q<\frac{3}{4}\frac{|t|}{|x|^4}\leq(n+\frac{1}{2})Q. \tag{6.36}$$

Proof. The number of subRiemannian geodesics is equal to the number of solutions of equation (6.31), and apply Lemma 6.4. ■

• *The limit case* $|x|\to 0$
One may see from the Figure 2 that the solutions of equation (6.31) become

$$0<\psi_1=\psi_2=\pi<\psi_3=\psi_4=2\pi<\psi_5=\psi_6=3\pi<\dots,$$

and hence we have an infinite number of geodesics joining the origin and $P(x,t)$.

7. Physical Interpretation

A magnetic field on $\mathbb{R}^3=\{(x,t)=(x_1,x_2,x_3)\}$ is given by a two form Ω which satisfies the Maxwell equation:

$$d\Omega=0.$$

The case of concern is when the magnetic field comes from a potential

$$\Omega=d\omega,$$

where $\omega = \sum_{j=1}^{3} \omega_j \, dx_j$ and

$$\Omega = \Omega_1 \, dx_2 \wedge dx_3 + \Omega_2 \, dx_1 \wedge dx_3 + \Omega_3 \, dx_1 \wedge dx_2.$$

The equation of motion for a particle with charge e, speed c and mass m is given by the Lorentz equation

$$m\frac{d\vec{v}}{dt} = \frac{e}{c}\vec{v} \times \vec{\Omega}, \tag{7.37}$$

where $\vec{\Omega} = (\Omega_1, \Omega_2, \Omega_3)$ is the magnetic vector field. Equation (7.37) is the Euler-Lagrange equation for the following Lagrangian

$$L(x, \dot{x}) = \frac{m}{2}\dot{x}^\alpha \dot{x}^\beta \delta_{\alpha\beta} - e\dot{x}^i \omega_i(x) = \frac{1}{2}m|\dot{x}|^2 - e\,\omega(\dot{x}).$$

The associate Hamiltonian is

$$H(x; \xi, \theta) = \frac{1}{2m}\sum_\alpha (\xi_\alpha + \omega_\alpha(x)\theta)^2.$$

In our case, the vector fields are given by (1.1). It follows that $span\{X_1, X_2\} = ker\,\omega$ where $\omega = dt + 4|x|^2(x_1\,dx_2 - x_2\,dx_1)$. The magnetic field is given by the 2-form $\Omega = d\omega = 16|x|^2 dx_1 \wedge dx_2$. The corresponding Lagrangian is

$$L = \frac{1}{2}(\dot{x}_1^2 + \dot{x}_2^2) + \theta\big(\dot{t} + 4|x|^2(x_1\dot{x}_2 - x_2\dot{x}_1)\big). \tag{7.38}$$

The Euler-Lagrange equations for the Lagrangian (7.38) are

$$\ddot{x}_1 = 16\,\theta|x|^2\dot{x}_2, \quad \ddot{x}_2 = -16\,\theta|x|^2\dot{x}_1, \quad \theta = \text{constant}. \tag{7.39}$$

Let $\phi(s) = (x_1(s), x_2(s))$. Then (7.39) becomes

$$\frac{d\dot{\phi}(s)}{ds} = \theta\dot{\phi}(s) \times \vec{\Omega},$$

which is the Lorentz equation (7.37) with mass $m = 1$, charge $e = \theta$ and magnetic field oriented along the t-axis $\vec{\Omega} = \big(0, 0, 16|x|^2\big)$. This magnetic vector field vanishes at the origin and it has spherical symmetry. The trajectories for the charged particles depend whether the particle passes through the origin. Theorem 1.1 states the energy levels for a charged particle which bounces back in the time t are quantified by formula (1.3), taking into account (4.25). The first part of Theorem 1.2 states the particle can bounce back in any time $t > 0$. The second part deals with some finite number of energy levels if certain boundary conditions are satisfied. Figure 4 represents the trajectory of a charged particle which passes through the origin. The trajectory is a union of loops of angle $\pi/3$ at the origin (see Lemma 3.2).

The particle remains all the time between two circles. Using a result of [7], the particle will never leave the circular crown.

If the trajectory is not passing through the origin, it looks like in Figures 5 and 6.

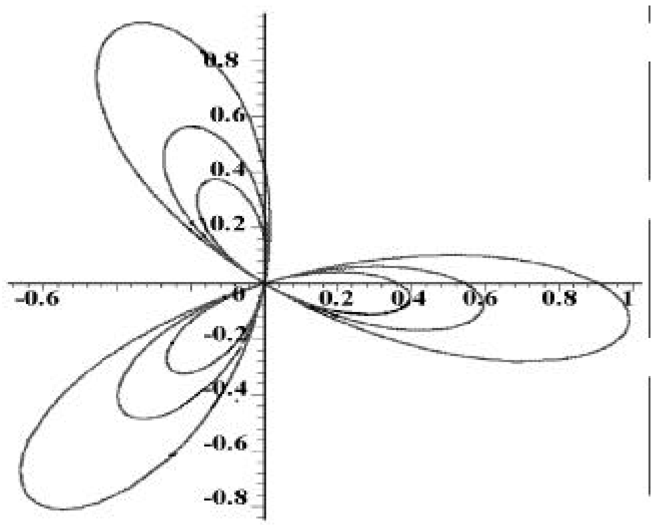

Fig. 4. Trajectories through the origin.

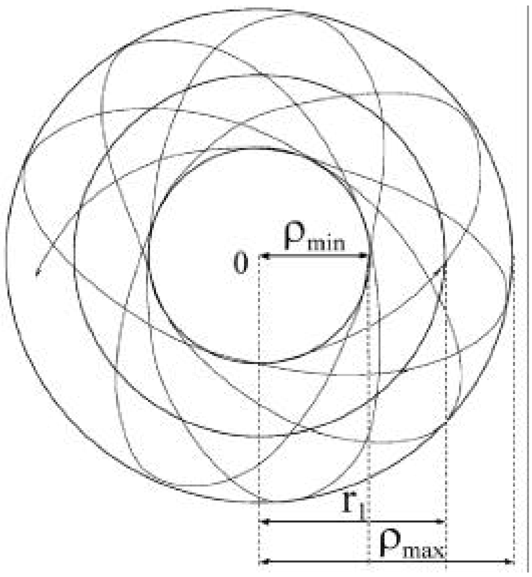

Fig. 5. Trajectories outside of the origin.

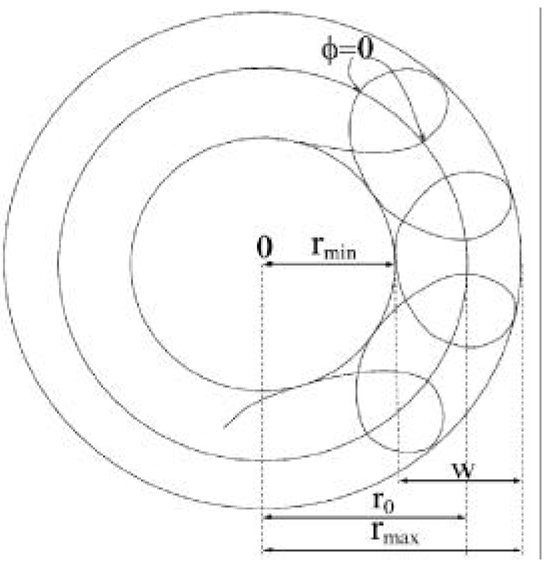

Fig. 6. Trajectories outside of the origin.

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