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ON THE JENSEN'S EQUATION IN BANACH MODULES

Chun-Gil Park and Won-Gil Park

Abstract. We prove the Hyers-Ulam-Rassias stability of the Jensen's equation in Banach modules over a Banach algebra.

Let E_1 and E_2 be Banach spaces, and $f : E_1 \to E_2$ a mapping such that $f(tx)$ is continuous in $t \in R$ for each fixed $x \in E_1$. Assume that there exist constants $\epsilon \geq 0$ and $p \in [0, 1)$ such that

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon(||x||^p + ||y||^p),$$

for all $x, y \in E_1$. Th.M. Rassias [7] showed that there exists a unique R-linear mapping $T : E_1 \to E_2$ such that

$$\|f(x) - T(x)\| \leq \frac{2\epsilon}{2 - 2^p}||x||^p,$$

for all $x \in E_1$.

The stability problems of functional equations have been investigated in several papers ([2, 3, 4, 5]).

Throughout this paper, let B be a unital Banach algebra with norm $|\cdot|$, R^+ the set of positive real numbers, and B_1 the set of all elements of B having norm 1, and let ${}_B B_1$ and ${}_B B_2$ be left Banach B-modules with norms $||\cdot||$ and $\|\cdot\|$, respectively.

We are going to prove the Hyers-Ulam-Rassias stability of the Jensen's equation in Banach modules over a Banach algebra.

Theorem 1. *Let* $f : {}_B B_1 \to {}_B B_2$ *be a mapping for which there exists a function* $\varphi : {}_B B_1 \setminus \{0\} \times {}_B B_1 \setminus \{0\} \to [0, \infty)$ *such that*

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$$\tilde{\varphi}(x,y) = \sum_{k=0}^{\infty} 3^{-k}\varphi(3^k x, 3^k y) < \infty,$$

$$\left\|2f\left(\frac{ax+ay}{2}\right) - af(x) - af(y)\right\| \le \varphi(x,y),$$

for all $a \in B_1 \cup R^+$ *and all* $x, y \in {}_BB_1 \setminus \{0\}$. *Then there exists a unique* B*-linear mapping* $T : {}_BB_1 \to {}_BB_2$ *such that*

$$\|f(x) - f(0) - T(x)\| \le \frac{1}{3}(\tilde{\varphi}(x,-x) + \tilde{\varphi}(-x,3x)),$$

for all $x \in {}_BB_1 \setminus \{0\}$.

Proof. By [6, Theorem 1], it follows from the inequality of the statement for $a = 1$ that there exists a unique additive mapping $T : {}_BB_1 \to {}_BB_2$ satisfying the condition given in the statement.

By the assumption, for each $a \in B_1 \cup R^+$,

$$\|2f(3^n ax) - af(2 \cdot 3^{n-1}x) - af(4 \cdot 3^{n-1}x)\| \le \varphi(2 \cdot 3^{n-1}x, 4 \cdot 3^{n-1}x),$$

for all $x \in {}_BB_1 \setminus \{0\}$. Using the fact that for each $a \in B$ and each $z \in {}_BB_2$ $\|az\| \le K|a| \cdot \|z\|$ for some $K > 0$,

$$\begin{aligned}
\|f(3^n ax) - af(3^n x)\| &= \|f(3^n ax) - \frac{1}{2}af(2 \cdot 3^{n-1}x) - \frac{1}{2}af(4 \cdot 3^{n-1}x) \\
&\quad + \frac{1}{2}af(2 \cdot 3^{n-1}x) + \frac{1}{2}af(4 \cdot 3^{n-1}x) - af(3^n x)\| \\
&\le \frac{1}{2}\varphi(2 \cdot 3^{n-1}x, 4 \cdot 3^{n-1}x) \\
&\quad + \frac{1}{2}K|a| \cdot \|2f(3^n x) - f(2 \cdot 3^{n-1}x) - f(4 \cdot 3^{n-1}x)\| \\
&\le \frac{1 + K|a|}{2}\varphi(2 \cdot 3^{n-1}x, 4 \cdot 3^{n-1}x),
\end{aligned}$$

for all $a \in B_1 \cup R^+$ and all $x \in {}_BB_1 \setminus \{0\}$. So $3^{-n}\|f(3^n ax) - af(3^n x)\| \to 0$ as $n \to \infty$ for all $a \in B_1 \cup R^+$ and all $x \in {}_BB_1 \setminus \{0\}$. Hence

$$T(ax) = \lim_{n\to\infty} 3^{-n} f(3^n ax) = \lim_{n\to\infty} 3^{-n} af(3^n x) = aT(x),$$

for all $a \in B_1 \cup R^+$ and all $x \in {}_BB_1 \setminus \{0\}$. Since T is additive, $T(0) = 0$ and $T(a0) = T(0) = 0 = a0 = aT(0)$ for all $a \in B_1 \cup R^+$. So $T(ax) = aT(x)$ for all $a \in B_1 \cup R^+$ and all $x \in {}_BB_1$. Thus

$$T(ax) = T\left(|a| \cdot \frac{a}{|a|}x\right) = |a|T\left(\frac{a}{|a|}x\right) = aT(x),$$

for all $a \in B \setminus \{0\}$ and all $x \in {}_BB_1$. Note that $T(0x) = T(0) = 0 = 0T(x)$ for all $x \in {}_BB_1$. So the unique additive mapping $T : {}_BB_1 \to {}_BB_2$ is a B-linear mapping, as desired. ■

Corollary 1. *Let $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ be a function such that $\frac{\psi(3)}{3} < 1$ and*

$$\psi(ts) \le \psi(t)\psi(s),$$

for all $t, s \in \mathbb{R}^+$. Let $f : {}_BB_1 \to {}_BB_2$ be a mapping such that

$$\|2f\left(\frac{ax + ay}{2}\right) - af(x) - af(y)\| \le \psi(||x||) + \psi(||y||),$$

for all $a \in B_1 \cup \mathbb{R}^+$ and all $x, y \in {}_BB_1 \setminus \{0\}$. Then there exists a unique B-linear mapping $T : {}_BB_1 \to {}_BB_2$ such that

$$\|f(x) - f(0) - T(x)\| \le \frac{3\psi(||x||) + \psi(||3x||)}{3 - \psi(3)},$$

for all $x \in {}_BB_1 \setminus \{0\}$.

Proof. Let $\varphi(x, y) = \psi(||x||) + \psi(||y||)$ for all $x, y \in {}_BB_1 \setminus \{0\}$. Then we get

$$\begin{aligned}
\tilde{\varphi}(x, y) &= \sum_{k=0}^{\infty} 3^{-k} \varphi(3^k x, 3^k y) \\
&= \sum_{k=0}^{\infty} 3^{-k} (\psi(||3^k x||) + \psi(||3^k y||)) \\
&\le \sum_{k=0}^{\infty} \left(\frac{\psi(3)}{3}\right)^k (\psi(||x||) + \psi(||y||)) \\
&= \frac{\psi(||x||) + \psi(||y||)}{1 - \frac{\psi(3)}{3}} < \infty,
\end{aligned}$$

for all $x, y \in {}_BB_1 \setminus \{0\}$. It follows from Theorem 1 that there exists a unique B-linear mapping $T : {}_BB_1 \to {}_BB_2$ such that

$$\|f(x) - f(0) - T(x)\| \le \frac{3\psi(||x||) + \psi(||3x||)}{3 - \psi(3)},$$

for all $x \in {}_BB_1 \setminus \{0\}$. ■

Corollary 2. *Let $p < 1$ and $f : {}_BB_1 \to {}_BB_2$ a mapping such that*

$$\|2f\left(\frac{ax + ay}{2}\right) - af(x) - af(y)\| \le ||x||^p + ||y||^p,$$

for all $a \in B_1 \cup \mathbb{R}^+$ *and all* $x, y \in {}_BB_1 \setminus \{0\}$. *Then there exists a unique* B*-linear mapping* $T : {}_BB_1 \to {}_BB_2$ *such that*

$$\|f(x) - f(0) - T(x)\| \le \frac{3 + 3^p}{3 - 3^p}\|x\|^p,$$

for all $x \in {}_BB_1 \setminus \{0\}$.

Proof. Define $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ by $\phi(t) = t^p$ and apply Corollary 1. ∎

Theorem 2. *Let* B *be a unital Banach* $*$*-algebra over* $\mathbb{C}$, *and* B_1^+ *the set of positive elements of* B *having norm 1. Let* $f : {}_BB_1 \to {}_BB_2$ *be a mapping for which there exists a function* $\varphi : {}_BB_1 \setminus \{0\} \times {}_BB_1 \setminus \{0\} \to [0, \infty)$ *such that*

$$\tilde{\varphi}(x, y) = \sum_{k=0}^{\infty} 3^{-k}\varphi(3^k x, 3^k y) < \infty,$$

$$\left\|2f\left(\frac{ax + ay}{2}\right) - af(x) - af(y)\right\| \le \varphi(x, y),$$

for all $a \in B_1^+ \cup \{i\} \cup \mathbb{R}^+$ *and all* $x, y \in {}_BB_1 \setminus \{0\}$. *Then there exists a unique* B*-linear mapping* $T : {}_BB_1 \to {}_BB_2$ *satisfying the condition given in the statement of Theorem* 1.

Proof. By the same reasoning as the proof of Theorem 1, there exists a unique additive mapping $T : {}_BB_1 \to {}_BB_2$ such that

$$\|f(x) - f(0) - T(x)\| \le \frac{1}{3}(\tilde{\varphi}(x, -x) + \tilde{\varphi}(-x, 3x)),$$

for all $x \in {}_BB_1 \setminus \{0\}$. By the same method as the proof of Theorem 1, one can show that

$$T(ax) = \lim_{n\to\infty} 3^{-n}f(3^n ax) = \lim_{n\to\infty} 3^{-n}af(3^n x) = aT(x)$$

for all $a \in B_1^+ \cup \{i\} \cup \mathbb{R}^+$ and all $x \in {}_BB_1 \setminus \{0\}$. So $T(ax) = aT(x)$ for all $a \in (B^+ \setminus \{0\}) \cup \{i\}$ and all $x \in {}_BB_1$. For any element $a \in B$, $a = a_1 + ia_2$, where $a_1 = \frac{a+a^*}{2}$ and $a_2 = \frac{a-a^*}{2i}$ are self-adjoint elements, furthermore, $a = a_1^+ - a_1^- + ia_2^+ - ia_2^-$, where a_1^+, a_1^-, a_2^+, and a_2^- are positive elements (see [1, Lemma 38.8]). Since T is additive, $T(x) = T(x - y + y) = T(x - y) + T(y)$ and $T(x - y) = T(x) - T(y)$ for all $x, y \in {}_BB_1$. So

$$\begin{aligned} T(ax) &= T(a_1^+ x - a_1^- x + ia_2^+ x - ia_2^- x) \\ &= (a_1^+ - a_1^- + ia_2^+ - ia_2^-)T(x) \\ &= aT(x), \end{aligned}$$

for all $a \in B$ and all $x \in {}_BB_1$. Hence there exists a unique B-linear mapping $T : {}_BB_1 \to {}_BB_2$ such that

$$\|f(x) - f(0) - T(x)\| \le \frac{1}{3}(\tilde{\varphi}(x, -x) + \tilde{\varphi}(-x, 3x)),$$

for all $x \in {}_BB_1 \setminus \{0\}$. ■

Corollary 3. *Let E_1 and E_2 be complex Banach spaces. Let $f : E_1 \to E_2$ be a mapping for which there exists a function $\varphi : E_1 \setminus \{0\} \times E_1 \setminus \{0\} \to [0, \infty)$ such that*

$$\tilde{\varphi}(x, y) = \sum_{k=0}^{\infty} 3^{-k} \varphi(3^k x, 3^k y) < \infty,$$

$$\left\|2f\left(\frac{\mu x + \mu y}{2}\right) - \mu f(x) - \mu f(y)\right\| \le \varphi(x, y),$$

for $\mu \in \{i\} \cup \mathbb{R}^+$ and all $x, y \in E_1 \setminus \{0\}$. Then there exists a unique $\mathbb{C}$-linear mapping $T : E_1 \to E_2$ such that

$$\|f(x) - f(0) - T(x)\| \le \frac{1}{3}(\tilde{\varphi}(x, -x) + \tilde{\varphi}(-x, 3x)),$$

for all $x \in E_1 \setminus \{0\}$.

Proof. Since $\mathbb{C}$ is a Banach algebra, the Banach spaces E_1 and E_2 are considered as Banach modules over $\mathbb{C}$. By Theorem 2, there exists a unique $\mathbb{C}$-linear mapping $T : E_1 \to E_2$ satisfying the condition given in the statement. ■

Remark 1. Consider a unital Banach $*$-algebra B over $\mathbb{C}$. In Corollary 1 and Corollary 2, when $a \in B_1 \cup \mathbb{R}^+$ are replaced by $a \in B_1^+ \cup \{i\} \cup \mathbb{R}^+$, the results do also hold.

Theorem 3. *Let $f : {}_BB_1 \to {}_BB_2$ be a mapping for which there exists a function $\varphi : {}_BB_1 \setminus \{0\} \times {}_BB_1 \setminus \{0\} \to [0, \infty)$ such that*

$$\tilde{\varphi}(x, y) = \sum_{k=0}^{\infty} 3^k \varphi(3^{-k} x, 3^{-k} y) < \infty,$$

$$\left\|2f\left(\frac{ax + ay}{2}\right) - af(x) - af(y)\right\| \le \varphi(x, y),$$

for all $a \in B_1 \cup \mathbb{R}^+$ and all $x, y \in {}_BB_1 \setminus \{0\}$. Then there exists a unique B-linear mapping $T : {}_BB_1 \to {}_BB_2$ such that

$$\|f(x) - f(0) - T(x)\| \le \tilde{\varphi}\left(\frac{x}{3}, \frac{-x}{3}\right) + \tilde{\varphi}\left(\frac{-x}{3}, x\right),$$

for all $x \in {}_{B}B_1 \setminus \{0\}$.

Proof. By [6, Theorem 6], it follows from the inequality of the statement for $a = 1$ that there exists a unique additive mapping $T : {}_{B}B_1 \to {}_{B}B_2$ satisfying the condition given in the statement.

By the assumption, for each $a \in B_1 \cup R^+$,

$$\|2f(3^{-n}ax) - af(2 \cdot 3^{-n-1}x) - af(4 \cdot 3^{-n-1}x)\| \le \varphi(2 \cdot 3^{-n-1}x, 4 \cdot 3^{-n-1}x);$$

for all $x \in {}_{B}B_1 \setminus \{0\}$. Using the fact that for each $a \in B$ and each $z \in {}_{B}B_2$ $\|az\| \le K|a| \cdot \|z\|$ for some $K > 0$,

$$\begin{aligned}\|f(3^{-n}ax) - af(3^{-n}x)\| &= \|f(3^{-n}ax) - \frac{1}{2}af(2 \cdot 3^{-n-1}x) - \frac{1}{2}af(4 \cdot 3^{-n-1}x)\\ &\quad + \frac{1}{2}af(2 \cdot 3^{-n-1}x) + \frac{1}{2}af(4 \cdot 3^{-n-1}x) - af(3^{-n}x)\|\\ &\le \frac{1}{2}\varphi(2 \cdot 3^{-n-1}x, 4 \cdot 3^{-n-1}x)\\ &\quad + \frac{1}{2}K|a| \cdot \|2f(3^{-n}x) - f(2 \cdot 3^{-n-1}x) - f(4 \cdot 3^{-n-1}x)\|\\ &\le \frac{1 + K|a|}{2}\varphi(2 \cdot 3^{-n-1}x, 4 \cdot 3^{-n-1}x);\end{aligned}$$

for all $a \in B_1 \cup R^+$ and all $x \in {}_{B}B_1 \setminus \{0\}$. So $3^n\|f(3^{-n}ax) - af(3^{-n}x)\| \to 0$ as $n \to \infty$ for all $a \in B_1 \cup R^+$ and all $x \in {}_{B}B_1 \setminus \{0\}$. Hence

$$T(ax) = \lim_{n \to \infty} 3^n f(3^{-n}ax) = \lim_{n \to \infty} 3^n af(3^{-n}x) = aT(x);$$

for all $a \in B_1 \cup R^+$ and all $x \in {}_{B}B_1 \setminus \{0\}$. By the same reasoning as the proof of Theorem 1, the unique additive mapping $T : {}_{B}B_1 \to {}_{B}B_2$ is a B-linear mapping, as desired. ∎

Corollary 4. *Let* $\psi : R^+ \to R^+$ *be a function such that* $\frac{\psi(3)}{3} > 1$ *and*

$$\psi(ts) \ge \psi(t)\psi(s);$$

for all $t, s \in R^+$. *Let* $f : {}_{B}B_1 \to {}_{B}B_2$ *be a mapping such that*

$$\|2f\left(\frac{ax + ay}{2}\right) - af(x) - af(y)\| \le \psi(||x||) + \psi(||y||);$$

for all $a \in B_1 \cup R^+$ *and all* $x, y \in {}_{B}B_1 \setminus \{0\}$. *Then there exists a unique* B*-linear mapping* $T : {}_{B}B_1 \to {}_{B}B_2$ *such that*

$$\|f(x) - f(0) - T(x)\| \le \frac{3\psi(||\frac{x}{3}||) + \psi(||x||)}{1 - \frac{3}{\psi(3)}};$$

for all $x \in {}_B B_1 \setminus \{0\}$.

Proof. Let $\varphi(x, y) = \psi(||x||) + \psi(||y||)$ for all $x, y \in {}_B B_1 \setminus \{0\}$. Then we get

$$\begin{aligned}
\tilde{\varphi}(x, y) &= \sum_{k=0}^{\infty} 3^k \varphi(3^{-k}x, 3^{-k}y) \\
&= \sum_{k=0}^{\infty} 3^k(\psi(||3^{-k}x||) + \psi(||3^{-k}y||)) \\
&\leq \sum_{k=0}^{\infty} \left(\frac{3}{\psi(3)}\right)^k (\psi(||x||) + \psi(||y||)) \\
&= \frac{\psi(||x||) + \psi(||y||)}{1 - \frac{3}{\psi(3)}} < \infty,
\end{aligned}$$

for all $x, y \in {}_B B_1 \setminus \{0\}$. It follows from Theorem 3 that there exists a unique B-linear mapping $T : {}_B B_1 \to {}_B B_2$ such that

$$\|f(x) - f(0) - T(x)\| \leq \frac{3\psi(||\frac{x}{3}||) + \psi(||x||)}{1 - \frac{3}{\psi(3)}},$$

for all $x \in {}_B B_1 \setminus \{0\}$. ■

Corollary 5. *Let* $p > 1$ *and* $f : {}_B B_1 \to {}_B B_2$ *a mapping such that*

$$\left\|2f\left(\frac{ax + ay}{2}\right) - af(x) - af(y)\right\| \leq ||x||^p + ||y||^p,$$

for all $a \in B_1 \cup R^+$ *and all* $x, y \in {}_B B_1 \setminus \{0\}$. *Then there exists a unique* B*-linear mapping* $T : {}_B B_1 \to {}_B B_2$ *such that*

$$\|f(x) - f(0) - T(x)\| \leq \frac{3^p + 3}{3^p - 3}||x||^p,$$

for all $x \in {}_B B_1$.

Proof. Define $\psi : R^+ \to R^+$ by $\psi(t) = t^p$ and apply Corollary 4. ■

Theorem 4. *Let* B *be a unital Banach* $*$*-algebra over* C. *Let* $f : {}_B B_1 \to {}_B B_2$ *be a mapping for which there exists a mapping* $\varphi : {}_B B_1 \setminus \{0\} \times {}_B B_1 \setminus \{0\} \to [0, \infty)$ *such that*

$$\tilde{\varphi}(x, y) = \sum_{k=0}^{\infty} 3^k \varphi(3^{-k}x, 3^{-k}y) < \infty,$$

$$\left\|2f\left(\frac{ax + ay}{2}\right) - af(x) - af(y)\right\| \leq \varphi(x, y),$$

for all $a \in B_1^+ \cup \{i\} \cup \mathbb{R}^+$ *and all* $x, y \in {}_B B_1 \setminus \{0\}$. *Then there exists a unique* B*-linear mapping* $T : {}_B B_1 \to {}_B B_2$ *satisfying the condition given in the statement of Theorem* 3.

Proof. The proof is similar to the proof of Theorem 2. ■

Corollary 6. *Let* E_1 *and* E_2 *be complex Banach spaces. Let* $f : E_1 \to E_2$ *be a mapping for which there exists a function* $\varphi : E_1 \setminus \{0\} \times E_1 \setminus \{0\} \to [0, \infty)$ *such that*

$$\tilde{\varphi}(x, y) = \sum_{k=0}^{\infty} 3^k \varphi(3^{-k}x, 3^{-k}y) < \infty,$$
$$\left\|2f\left(\frac{\lambda x + \lambda y}{2}\right) - \lambda f(x) - \lambda f(y)\right\| \le \varphi(x, y)$$

for $\lambda \in \{i\} \cup \mathbb{R}^+$ *and all* $x, y \in E_1 \setminus \{0\}$. *Then there exists a unique* $\mathbb{C}$*-linear mapping* $T : E_1 \to E_2$ *such that*

$$\|f(x) - f(0) - T(x)\| \le \tilde{\varphi}\left(\frac{x}{3}, \frac{-x}{3}\right) + \tilde{\varphi}\left(\frac{-x}{3}, x\right)$$

for all $x \in E_1 \setminus \{0\}$.

Proof. The proof is similar to the proof of Corollary 3. ■

Remark 2. Consider a unital Banach $*$-algebra B over $\mathbb{C}$. In Corollary 4 and Corollary 5, when $a \in B_1 \cup \mathbb{R}^+$ are replaced by $a \in B_1^+ \cup \{i\} \cup \mathbb{R}^+$, the results do also hold.

Remark 3. If the inequalities

$$\left\|2f\left(\frac{ax + ay}{2}\right) - af(x) - af(y)\right\| \le \varphi(x, y)$$

in the statements are replaced by

$$\left\|2f\left(\frac{ax + y}{2}\right) - af(x) - f(y)\right\| \le \varphi(x, y),$$

then

$$\left\|2f\left(\frac{ax + ay}{2}\right) - af(x) - f(ay)\right\| \le \varphi(x, ay),$$
$$\left\|2f\left(\frac{ax + ay}{2}\right) - f(ax) - af(y)\right\| \le \varphi(ax, y),$$
$$\left\|2f\left(\frac{ax + ay}{2}\right) - f(ax) - f(ay)\right\| \le \varphi(ax, ay).$$

So

$$\|2f\left(\frac{ax+ay}{2}\right) - af(x) - af(y)\| \leq \varphi(x, ay) + \varphi(ax, y) + \varphi(ax, ay);$$

hence the results do also hold.

Remark 4. When the inequalities

$$\|2f\left(\frac{ax+ay}{2}\right) - af(x) - af(y)\| \leq \varphi(x, y)$$

in the statements of Theorem 1 and Theorem 3 are replaced by

$$\|2f\left(\frac{a^m x + a^m y}{2}\right) - a^d f(x) - a^d f(y)\| \leq \varphi(x, y)$$

for nonnegative integers m and d, by similar methods to the proofs of Theorem 1 and Theorem 3, one can show that there exist unique additive mappings $T : {}_B B_1 \to {}_B B_2$, satisfying the conditions given in the statements of Theorem 1 and Theorem 3, such that

$$T(a^m x) = a^d T(x)$$

for all $a \in B_1 \cup R^+$ and all $x \in {}_B B_1$.

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Chun-Gil Park and Won-Gil Park
Department of Mathematics, Chungnam National University
TaeJon 305-764, South Korea
E-mail: cgpark@math.chungnam.ac.kr & wgpark@math.chungnam.ac.kr