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ON THE GLIDING HUMPS PROPERTY

Li Ronglu and Min-Hyung Cho

Abstract. We establish two uniform convergence results for duality pairs consisting of vector-sequence spaces with some gliding humps property and the corresponding function-sequence spaces. These uniform convergence results imply some important facts.

Let X, Y be topological vector spaces and $E(X)$ a vector space of X-valued sequences. For $x \in E(X)$, let x_k denote the k^{th} coordinate of x and, hence, if $\{x^n\}$ is a sequence in $E(X)$, then x_k^n is just the k^{th} coordinate of the n^{th} vector-sequence $x^n \in E(X)$.

Let $[Y^{E(X)}]^\beta$ be the family of function-sequences $\{f_k\} \subseteq Y^X$ for which $f_k(0) = 0$ for all k and the series $\sum_{k=1}^{\infty} f_k(x_k)$ converges in Y for each $x = (x_k) \in E(X)$. Especially, let $E(X)^{\beta Y} = \{T = (T_k) \in [Y^{E(X)}]^\beta$: each T_k is linear and continuous$\}$ (see [10, 11, 13, 14, 15, 17]). As usual, for $f = (f_k) \in [Y^{E(X)}]^\beta$ and $x = (x_k) \in E(X)$, we write $f \cdot x = \sum_{k=1}^{\infty} f_k(x_k) = \sum_{k=1}^{\infty} f_k x_k$.

Throughout this paper we assume that $E(X) \supseteq c_{00}(X) = \{x = (x_k) \in X^{\mathbb{N}} : x_k = 0 \text{ eventually}\}$ and $E(X)$ is equipped by some Hausdorff vector topology which is stronger than the topology of coordinatewise convergence and, hence, $E(X)$ is a $K(X)$ space [3]. Let $P_n : E(X) \to E(X)$ be the section map which sends $(x_1, x_2, \ldots)$ to $(x_1, \ldots, x_n, 0, 0, \ldots)$. We say that $E(X)$ has the property SUB if $\{P_n\}_{n=1}^{\infty}$ is uniformly bounded on bounded subsets of the domain space $E(X)$ [13].

Following D. Noll [10], we say that a sequence $\{z^n\}$ of nonzero vectors in $E(X)$ is a block sequence if there is a strictly increasing $\{k_n\} \subseteq \mathbb{N}$ such that $z^n = (0, \ldots, 0, z^n_{k_n+1}, z^n_{k_n+2}, \ldots, z^n_{k_{n+1}}, 0, 0, \ldots)$. We say that $E(X)$ has

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the strong gliding humps property (SGHP) if every bounded block sequence $\{z^n\}$ in $E(X)$ has a subsequence $\{z^{n_i}\}$ such that the coordinatewise sum $\sum_{i=1}^{\infty} z^{n_i} = (0, \ldots, 0, z^{n_1}_{k_{n_1}+1}, \ldots, z^{n_1}_{k_{n_1+1}}, 0, \ldots, 0, z^{n_2}_{k_{n_2}+1}, \ldots, z^{n_2}_{k_{n_2+1}}, 0, \ldots)$ belongs to $E(X)$ [13]. We say that $E(X)$ has the zero gliding humps property (0-GHP) if for $x^n \to 0$ in $E(X)$ and $i_1 \leq j_1 < i_2 \leq j_2 < \cdots$ in $\mathbb{N}$, there is an increasing $\{n_p\} \subseteq \mathbb{N}$ such that $(0, \ldots, 0, x^{n_1}_{i_{n_1}}, \ldots, x^{n_1}_{j_{n_1}}, 0, \ldots, 0, x^{n_2}_{i_{n_2}}, \ldots, x^{n_2}_{j_{n_2}}, 0, \ldots)$ belongs to $E(X)$. This form of the gliding humps property was introduced in Lee Peng Yee [5]; see also [6], [14]. For a discussion of various forms of gliding humps properties and their applications, see [2].

$F \subseteq [Y^{E(X)}]^\beta$ is said to be conditionally $wE(X)$-sequentially compact if every sequence $\{f^n\} \subseteq F$ has a subsequence $\{f^{n_i}\}$ such that $\lim f^{n_i} \cdot x$ exists at each $x \in E(X)$. Finite subsets of $[Y^{E(X)}]^\beta$ are conditionally $wE(X)$-sequentially compact, and if F is sequentially compact under the topology $wE(X)$ of pointwise convergence on $E(X)$, then F is conditionally $wE(X)$-sequentiaiiy compact. In this paper we would like to establish two uniforn convergence results for conditionally $wE(X)$-sequentially compact subfamilies of $[Y^{E(X)}]^\beta$ whenever $E(X)$ has some properties such as SUB, SGHP and 0-GHP, etc. Using these uniform convergence theorems we shall consummate and improve some recent results in [13], [14], [17].

Theorem 1. *Let $E(X)$ be a vector-sequence space having SUB and SGHP. If a subfamily F of $[Y^{E(X)}]^\beta$ is conditionally $wE(X)$-sequentially compact, then for every bounded $B \subseteq E(X)$, the series $\sum_{k=1}^{\infty} f_k x_k$ converges uniformly for $f \in F$ and $x = (x_k) \in B$.*

Proof. If not, there exists a neighborhood U of $0 \in Y$ such that if $n_0 \in \mathbb{N}$ then $\sum_{k=n}^{\infty} f_k x_k \notin U$ for some $n > n_0, f \in F$ and $x \in B$. Thus, there exist $n_1 > 1, f^1 \in F$ and $x^1 \in B$ such that $\sum_{k=n_1}^{\infty} f^1_k x^1_k \notin U$. Pick a neighborhood V of $0 \in Y$ for which $V + V \subseteq U$. Since the series $\sum_{k=1}^{\infty} f^1_k x^1_k$ converges, there is an $m_1 > n_1$ for which $\sum_{k=n_1}^{m_1} f^1_k x^1_k \notin V$. Similarly, there exist integers $m_2 > n_2 > m_1, f^2 \in F$ and $x^2 \in B$ such that $\sum_{k=n_2}^{m_2} f^2_k x^2_k \notin V$. Continuing this construction, we have an integer sequence $n_1 < m_1 < n_2 < m_2 < \cdots, \{f^i\} \subseteq F$ and $\{x^i\} \subseteq B$ such that

$$(*) \qquad \sum_{k=n_i}^{m_i} f^i_k x^i_k \notin V, \ i = 1, 2, 3, \ldots.$$

Let $z^j = (0, \ldots, x^j_{n_j}, \ldots, x^j_{m_j}, 0, 0, \ldots)$. Since $E(X)$ has SUB and B is a bounded subset of $E(X)$, the set $\{P_n x - P_m z : n, m \in \mathbb{N}, x, z \in B\}$ is bounded and, hence, $\{z^j\}$ is a bounded block sequence in $E(X)$.

Now consider the matrix $(f^i \cdot z^j)_{i,j}$. Since F is conditionally $wE(X)$-sequentially compact, by passing to a subsequence if necessary, we may assume that $\lim\limits_i f^i \cdot x$ exists at each $x \in E(X)$ and, hence, $\lim\limits_i f^i \cdot z^j$ exists for all j. Let $j_1 < j_2 < \ldots$ in $\mathbb{N}$. Then there is a subsequence $\{j_{p_r}\} \subseteq \{j_p\}$ such that the sequence $(0, \ldots, 0, x^{j_{p_1}}_{n_{j_{p_1}}}, \ldots, x^{j_{p_1}}_{m_{j_{p_1}}}, 0, \ldots, 0, x^{j_{p_2}}_{n_{j_{p_2}}}, \ldots, x^{j_{p_2}}_{m_{j_{p_2}}}, 0, \ldots, 0, \ldots)$ belongs to $E(X)$ because $E(X)$ has SGHP, i.e., the coordinatewise sum $z = \sum\limits_{r=1}^{\infty} z^{j_{p_r}}$ belongs to $E(X)$. Hence, for each i we have $f^i \cdot z = \sum\limits_{r=1}^{\infty} \sum\limits_{k=n_{j_{p_r}}}^{m_{j_{p_r}}} f^i_k x^{j_{p_r}}_k = \sum\limits_{r=1}^{\infty} f^i . z^{j_{p_r}}$ and $\lim\limits_i \sum\limits_{r=1}^{\infty} f^i \cdot z^{j_{p_r}} = \lim\limits_i f^i \cdot z$ exists. Thus, by the Antosik-Mikusinski matrix theorem [1], [8], $\lim\limits_i \sum\limits_{k=n_i}^{m_i} f^i_k x^i_k = \lim\limits_i f^i \cdot z^i = 0$. This contradicts $(*)$. ∎

Note that, in Theorem 1, each mapping $f = (f_k) \in F$ need not be linear and each coordinate function f_k need not be continuous. In fact, we only require that $f \cdot 0 = 0$, i.e., $f_k(0) = 0$ for all k, while [13], [14], [17] gave results only for continuous linear $f_k : X \to Y$ and $f = (f_k) \in E(X)^{\beta Y}$. A similar improvement to Schur's theorem was given in [9] for abstract function matrices.

Corollary 2. ([13], Th. 4). *Assume that $E(X)$ has properties SUB and SGHP and that X is an $\mathcal{A}$-space. If $F \subseteq E(X)^{\beta Y}$ is pointwise bounded on $E(X)$, then F is uniformly bounded on bounded subsets of $E(X)$.*

Proof. Let $\{x^n\}$ be a bounded sequence in $E(X)$ and $\{f^n\} \subseteq F$. Since $E(X)$ has SUB, for each k, $\{(0, \ldots, 0, x^n_k, 0, 0, \ldots) : n \in \mathbb{N}\}$ is bounded in $E(X)$ and, since the topology on $E(X)$ is stronger than the topology of coordinatewise convergence, $\{x^n_k\}_{n=1}^{\infty}$ is bounded in X and $\lim\limits_n \frac{1}{n} f^n_k x^n_k = 0$ because X is an $\mathcal{A}$-space ([8], Cor. 4). On the other hand, $\{f^n\}$ is pointwise bounded on $E(X)$ and, hence, $\frac{1}{n} f^n \cdot x \to 0$ at each $x \in E(X)$, i.e., the set $\{\frac{1}{n} f^n : n \in \mathbb{N}\}$ is conditionally $wE(X)$-sequentially compact. Now by Theorem 1 the series $\sum\limits_{k=1}^{\infty} \frac{1}{n} f^n_k x^m_k$ converges uniformly for $n, m \in \mathbb{N}$. Thus,

$$\lim_n \frac{1}{n} f^n \cdot x^n = \lim_n \sum_{k=1}^{\infty} \frac{1}{n} f^n_k x^n_k = \sum_{k=1}^{\infty} \lim_n \frac{1}{n} f^n_k x^n_k = 0. \qquad ∎$$

Using 0-GHP instead of SGHP, C. Swartz gave a similar but more clear-cut uniform boundedness result; see [15], Th. 12.5.7.

We say that a map $u : X \to Y$ is boundedly continuous if (x_a) is a bounded net in X such that $x_a \to x_0$ in X, then $u(x_a) \to u(x_0)$. Clearly, a continuous map is boundedly continuous and a boundedly continuous map must be sequentially continuous. Theorem 1 of [13] asserts that if $E(X)$ has some suitable properties, then each $f = (f_k) \in E(X)^{\beta Y}$ is sequentially continuous. Note that if $f = (f_k) \in E(X)^{\beta Y}$, then each coordinate function f_k is continuous and linear. In contrast to Theorem 1 of [13], the following result only requires that coordinate functions be boundedly continuous.

Corollary 3. *Assume that $E(X)$ has SUB and SGHP. If $f = (f_k) \in [Y^{E(X)}]^\beta$ is such that each coordinate function $f_k : X \to Y$ is boundedly continuous, then f is boundedly continuous on $E(X)$ and, hence, f is sequentially continuous. In particular, each $T = (T_k) \in E(X)^{\beta Y}$ is boundedly continuous and, hence, T is sequentially continuous.*

Proof. Let $(x^a)_{a\in I}$ be a bounded net such that $x^a \to x$ in $E(X)$. Then for each k we have $x_k^a \to x_k$ because the topology on $E(X)$ is stronger than the topology of coordinatewise convergence. Since $E(X)$ has SUB, for each k the net $(x_k^a)_{a\in I}$ is bounded in X. Thus, for each k, $\lim_a f_k x_k^a = f_k x_k$ by hypothesis. Since the singleton $\{f\}$ is conditionally $wE(X)$-sequentially compact and $(x^a)_{a\in I}$ is bounded, by Theorem 1 the series $\sum_{k=1}^\infty f_k x_k^a$ converges uniformly for $a \in I$. Therefore,

$$\lim_a f \cdot x^a = \lim_a \sum_{k=1}^\infty f_k x_k^a = \sum_{k=1}^\infty \lim_a f_k x_k^a = \sum_{k=1}^\infty f_k x_k = f \cdot x. \qquad \blacksquare$$

If $E(X)$ has 0-GHP, then for every $T = (T_k) \in E(X)^{\beta Y}$ and $x^n \to 0$ in $E(X)$, the series $\sum_{k=1}^\infty T_k x_k^n$ converges uniformly for $n \in \mathbb{N}$ ([17], Th. 2). We would like to establish a similar result for $F \subseteq [Y^{E(X)}]^\beta$. Recall that $E(X)$ is an AK-space if $(0, \ldots, 0, x_n, x_{n+1}, \ldots) \to 0$ for each $(x_k) \in E(X)$ ([15], p. 128). For example, the sequence spaces $(c_0, \|\cdot\|_\infty)$ and $(1^p, \|\cdot\|_p)$ are AK-spaces having 0-GHP.

Theorem 4. *Let $E(X)$ be an AK-space with 0-GHP. If F is a conditionally $wE(X)$-sequentially compact subfamily of $[Y^{E(X)}]^\beta$, then for every $x^n \to 0$ in $E(X)$ the series $\sum_{k=1}^\infty f_k x_k^n$ converges uniformly with respect to $f = (f_k) \in F$ and $n \in \mathbb{N}$.*

Proof. Firat we claim that for each $x = (x_k) \in E(X)$ the series $\sum_{k=1}^\infty f_k x_k$ converges uniformly with respect to $f = (f_k) \in F$. If not, there exist a neighborhood V of $0 \in Y, \{f^i\} \subseteq F$ and an integer sequence $n_1 < m_1 < n_2 < m_2 <$

... such that $\sum_{k=n_i}^{m_i} f_k^i x_k \notin V$ for all i. If $z^j = (0,\ldots,0,x_{n_j},\ldots,x_{m_j},0,0,\ldots)$, then $z^j \to 0$ because $E(X)$ is an AK-space. Since F is conditionally $wE(X)$-sequentially compact, by passing to a subsequence if necessary, we may assume that $\lim_i f^i \cdot x$ exists at each $x \in E(X)$. As in the proof of Theorem 1, by 0-GHP, the Antosik-Mikusinski theorem shows that $\sum_{k=n_i}^{m_i} f_k^i x_k \to 0$, but this is a contradiction.

Now suppose the conclusion of Theorem 4 fails. Then there exists a neighborhood V of $0 \in Y$ satisfying

$$(*) \qquad \forall n_0 \in \mathbb{N}\ \exists m > n > n_0,\ f \in F \text{ and } i \in \mathbb{N} \text{ such that } \sum_{k=n}^{m} f_k x_k^i \notin V.$$

Thus, there exist $m_1 > n_1 > 1$, $f^1 \in F$ and $i_1 \in \mathbb{N}$ such that $\sum_{k=n_1}^{m_1} f_k^1 x_k^{i_1} \notin V$. By the first part of this proof, there is an $m_0 \in \mathbb{N}$ such that $\sum_{k=n}^{m} f_k x_k^i \in V$ for all $m \geq n > m_0$, $f = (f_k) \in F$ and $1 \leq i \leq i_1$. Therefore, by $(*)$ again, there exist $m_2 > n_2 > \max(m_0, m_1), f^2 \in F$ and $i_2 > i_1$ such that $\sum_{k=n_2}^{m_2} f_k^2 x_k^{i_2} \notin V$. Continuing this construction we have integer sequences $n_1 < m_1 < n_2 < m_2 < \ldots, i_1 < i_2 < \cdots$ and $\{f^p\} \subseteq F$ such that $\sum_{k=n_p}^{m_p} f_k^p x_k^{i_p} \notin V$ for all $p \in \mathbb{N}$. However, as before the Antosik-Mikusinski theorem, the condition 0-GHP, and $x^{i_p} \to 0$ imply a contradictory fact. ∎

Now we would like to show some applications of Theorems 1 and 4. lt is easy to see that with the norm $\|(t_k)\|_\infty = \sup_k |t_k|$, the space ℓ^∞ of bounded scalar sequences has SUB and SGHP. Observing that ℓ^∞ is the dual of $(\ell^1, \|\cdot\|_1)$, we have the following.

Corollary 5. (Schur lemma [12], [1]) *Every weakly convergent sequence in $(\ell^1, \|\cdot\|_1)$ must be norm convergent.*

Proof. Suppose $r^n = (r_k^n) \in \ell^1$ and $r^n \to 0$ weakly, i.e., for each $t = (t_k) \in \ell^\infty, \lim_n r^n \cdot t = \lim_n \sum_{k=1}^{\infty} t_k r_k^n = 0$. Let $\varepsilon > 0$. By Theorem 1, there is a k_0 such that $\left|\sum_{k=k_0+1}^{\infty} t_k r_k^n\right| < \varepsilon/2$ for all n and (t_k) with $\sup_k |t_k| \leq 1$. On the other hand, for each $k, \lim_n r_k^n = 0$ holds obviously and, hence, there is an n_0

such that $\sum_{k=1}^{k_0} |t_k r_k^n| < \varepsilon/2$ for all $n > n_0$ and (t_k) with $\sup_k |t_k| \leq 1$. Thus, $\left|\sum_{k=1}^{\infty} t_k r_k^n\right| < \varepsilon$ if $n > n_0$ and $\sup_k |t_k| \leq 1$, i.e., $\sum_{k=1}^{\infty} |r_k^n| \leq \varepsilon$ if $n > n_0$. ■

Let λ be a family of scalar sequences. A sequence $\{x_k\}$ in a topological vector space X is said to be λ-multiplier convergent (λ-mc) if the series $\sum_{k=1}^{\infty} t_k x_k$ converges for each $(t_k) \in \lambda$. Recently, the first author has given the following nice result [7] :

Let X be a Hausdorff locally convex space with the dual X' and $\lambda = c_0$ or ℓ^p $(p \geq 1)$. Then λ-mc is an invariant for all admissible polar topologies, i.e., if $\{x_k\} \subseteq X$, then the series $\sum_{k=1}^{\infty} t_k x_k$ converges for each $(t_k) \in \lambda$ under the strongest admissible topology $\beta(X, X')$ if and only if $\sum_{k=1}^{\infty} t_k x_k$ converges for each $(t_k) \in \lambda$ under the weakest admissible topology $\sigma(X, X')$.

By Theorem 4 we can generalize this result. Recall that if X is a barrelled space with the dual X', then the topology on X is just $\beta(X, X')$; if X is a bornological space and Y is an arbitrary locally convex space, then every bounded linear operator $T : X \to Y$ is continuous and, hence, every sequentially continuous linear operator $T : X \to Y$ must be continuous. Note that a locally convex metric space is bornological and a barrelled bornological space need not be an inductive limit of Banach spaces ([4], p, 39).

Theorem 6. *Let (λ, τ) be a barrelled bornological AK-space of scalar sequences such that τ is stronger than the topology of coordinatewise convergence and (λ, τ) has 0-GHP, e.g., $(\lambda, \tau) = (c_0, \|\cdot\|_\infty)$ or $(\ell^p, \|\cdot\|_p), p \geq 1$. If $\{x_k\}$ is a sequence in a Hausdorff locally convex space X with the dual X' such that for each $(t_k) \in \lambda$ the series $\sum_{k=1}^{\infty} t_k x_k$ converges weakly, then for each $(t_k) \in \lambda$ the series $\sum_{k=1}^{\infty} t_k x_k$ converges under the strongest admissible topology $\beta(X, X')$.*

Proof. Suppose that $t^n = (t_k^n) \to 0$ as $n \to +\infty$ in (λ, τ). Then for each k, $\lim_n t_k^n = 0$ and, hence, $\lim_n t_k^n x_k = 0$. Since $x = (x_k) \in \lambda^{\beta(X, weak)}$ and the singleton $\{x\} = \{(x_k)\}$ is $w\lambda$-sequentially compact, by Theorem 4 the series $\sum_{k=1}^{\infty} t_k^n x_k$ converges in $(X$, weak$)$ uniformly for $n \in \mathbb{N}$. Therefore, in $(X$, weak$)$ we have that $\lim_n \sum_{k=1}^{\infty} t_k^n x_k = \sum_{k=1}^{\infty} \lim_n t_k^n x_k = 0$. Observing that (λ, τ) is bornological and letting $T((t_k)) =$ weak-$\sum_{k=1}^{\infty} t_k x_k$, we obtain that $T : (\lambda, \tau) \to (X$, weak$)$ is continuous and linear. Now let $\lambda' = (\lambda, \tau)'$. By the

Hellinger-Toeplitz theorem ([16] , p. 168, Th. 2), T must be $\beta(\lambda,\lambda')$-$\beta(X,X')$ continuous. But $\beta(\lambda,\lambda')=\tau$ because (λ,τ) is barrelled. Thus T is τ-$\beta(X,X')$ continuous.

Now let $(t_k)\in\lambda$ be arbitrary. Since (λ,τ) is an AK-space, $(t_1, t_2, \ldots, t_n, 0, 0, \ldots) \xrightarrow{\tau} (t_k)$. Therefore,

$$\sum_{k=1}^{n} t_k x_k = T[(t_1,\ldots,t_n,0,0,\ldots)] \xrightarrow{\beta(X,X')} T[(t_k)] = \text{ weak-}\sum_{k=1}^{\infty} t_k x_k,$$

i.e., the series $\sum_{k=1}^{\infty} t_k x_k$ converges in $(X,\beta(X,X'))$. ■

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Li Ronglu
Department of Mathematics, Harbin Institute of Technology
Harbin, China

Min-Hyung Cho
Departmeht of Applied Mathematics, Kum-Oh University of Technology
Kumi, Korea