TAIWANESE JOURNAL OF MATHEMATICS
Vol. 16, No. 6, pp. 1959-1970, December 2012
This paper is available online at http://journal.taiwanmathsoc.org.tw

JORDAN HIGHER ALL-DERIVABLE POINTS IN NEST ALGEBRAS

Nannan Zhen and Jun Zhu

Abstract. Let $\mathcal{N}$ be a non-trivial and complete nest on a Hilbert space H. Suppose $d = \{d_n : n \in N\}$ is a group of linear mappings from $Alg\mathcal{N}$ into itself. We say that $d = \{d_n : n \in N\}$ is a Jordan higher derivable mapping at a given point G if $d_n(ST + TS) = \sum\limits_{i+j=n} \{d_i(S)d_j(T) + d_j(T)d_i(S)\}$ for any $S, T \in Alg\mathcal{N}$ with $ST = G$. An element $G \in Alg\mathcal{N}$ is called a Jordan higher all-derivable point if every Jordan higher derivable mapping at G is a higher derivation. In this paper, we mainly prove that any given point G of $Alg\mathcal{N}$ is a Jordan higher all-derivable point. This extends some results in [1] to the case of higher derivations.

1. Introduction and Preliminaries

Let $\mathcal{A}$ be an algebra. A linear mapping δ from $\mathcal{A}$ into itself is called a derivation if $\delta(ST) = \delta(S)T + S\delta(T)$ for any $S, T \in \mathcal{A}$ and is said to be a Jordan derivation if $\delta(ST + TS) = \delta(S)T + S\delta(T) + \delta(T)S + T\delta(S)$ for any $S, T \in \mathcal{A}$. We say that a mapping δ is Jordan derivable at a given point $G \in \mathcal{A}$ if $\delta(ST + TS) = \delta(S)T + S\delta(T) + \delta(T)S + T\delta(S)$ with $ST = G$ and G is called a Jordan all-derivable point of $\mathcal{A}$ if every Jordan derivable mapping at G is a derivation. Suppose that $d = \{d_n : n \in N\}$ is a group of linear mappings from $\mathcal{A}$ into itself and d_0 is the identical mapping. We say that $d = \{d_n : n \in N\}$ is Jordan higher derivable at a given point G if $d_n(ST + TS) = \sum\limits_{i+j=n} \{d_i(S)d_j(T) + d_j(T)d_i(S)\}$ for any $S, T \in \mathcal{A}$ with $ST = G$, and G is called a Jordan higher all-derivable point of $\mathcal{A}$ if every Jordan higher derivable mapping at G is a higher derivation, that is $d_n(ST) = \sum\limits_{i+j=n} \{d_i(S)d_j(T)\}$ for any $S, T \in \mathcal{A}$.

With the development of derivation, higher derivation has attracted much attention of mathematicians as an active subject of research in algebras. Generally speaking, there are two directions in the study of the local actions of derivations of operator algebras. One is the well known local derivation problem. The other is to study

Received December 14, 2011, accepted February 16, 2012.
Communicated by Ngai-Ching Wong.
2010 *Mathematics Subject Classification*: 47L35, 47B47.
Key words and phrases: Nest algebras, Jordan higher all-derivable point, Higher derivation.

conditions under which derivations of operator algebras can be completely determined by the action on some sets of operators. It is obvious that a linear map is a higher derivation if and only if it is higher derivable at all points. It is natural and interesting to ask the question whether or not a linear map is a derivation if it is Jordan higher derivable only at one given point.

We describe some of the results related to ours. In [1] ,Chen proved that any $G \neq 0$ is a Jordan all-derivable point in nest algebras. S. Zhao and J. Zhu pointed that $G = 0$ is a Jordan all-derivable point in nest algebras in [2]. W. Jing, S. J. Lu and P. T. Li [4] showed that every derivable mapping φ at 0 with $\varphi(I) = 0$ on nest algebras is a derivation. In [7], Z.K. Xiao and F. Wei gave the proof of the fact that any Jordan higher derivation on a triangular algebra is a higher derivation.

In this paper, $\mathcal{N}$ is a non-trivial and complete nest on a Hilbert space H. $Alg\mathcal{N} = \{A \in B(H) : AP \subseteq P, \forall P \in \mathcal{N}\}$ is an algebra. Let $N \in \mathcal{N}$ with $0 \subset N \subset H$. Then we can get the orthogonal decomposition $H = N \oplus N^{\perp}$. In this way, we can write $G = \begin{bmatrix} D & E \\ 0 & F \end{bmatrix}$, where $D \in Alg\mathcal{N}_N$, $E \in B(N^{\perp}, N)$ and $F \in Alg\mathcal{N}_{N^{\perp}}$ ($\mathcal{N}_N = \{M \bigcap N : M \in \mathcal{N}\}, \mathcal{N}_{N^{\perp}} = \{M \bigcap N^{\perp} : M \in \mathcal{N}\}$). All the identitical mappings in the proof are represented by I and λ is a positive real number for convenient writing.

2. Jordan Higher All-derivable Points in Nest Algebras

In this section, we assume that $d = \{d_n : n \in N\}$ is a Jordan higher derivable linear mapping at G from $Alg\mathcal{N}$ into itself. We only need to prove that $d = \{d_n : n \in N\}$ is a higher derivation.

Theorem 2.1. *Let $\mathcal{N}$ be a non-trivial and complete nest on a Hilbert space H. Any element of $Alg\mathcal{N}$ is a Jordan higher all-derivable point.*

Proof. For any $X \in Alg\mathcal{N}_N$, $Y \in B(N^{\perp}, N)$, $Z \in Alg\mathcal{N}_{N^{\perp}}$, we can write

$$d_n\left(\begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix}\right) = \begin{bmatrix} A_{n1}(X)+B_{n1}(Y)+C_{n1}(Z) & A_{n2}(X)+B_{n2}(Y)+C_{n2}(Z) \\ 0 & A_{n3}(X)+B_{n3}(Y)+C_{n3}(Z) \end{bmatrix},$$

Where A_{ij},B_{ij} and C_{ij} are linear mappings on $Alg\mathcal{N}_N$, $B(N^{\perp}, N)$ and $Alg\mathcal{N}_{N^{\perp}}$,respectively. It is clear that $A_{01}(X) = X$, $A_{02}(X) = 0$, $A_{03}(X) = 0$, $B_{01}(Y) = 0$, $B_{02}(Y) = Y$, $B_{03}(Y) = 0$, $C_{01}(Z) = 0$, $C_{02}(Z) = 0$ and $C_{03}(Z) = Z$.

Let $S = \begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix}, T = \begin{bmatrix} U & V \\ 0 & W \end{bmatrix}$ for any $X, U \in Alg\mathcal{N}_N$, $Y, V \in B(N^{\perp}, N)$, $Z, W \in Alg\mathcal{N}_{N^{\perp}}$ with $XU = D$, $XV + YW = E$ and $ZW = F$, then $ST = G$. So we have

$$
\begin{aligned}
&d_n(ST+TS)= \\
&\begin{bmatrix} \begin{array}{c} A_{n1}(D)+A_{n1}(UX)+B_{n1}(E)+ \\ B_{n1}(UY+VZ)+C_{n1}(F)+C_{n1}(WZ) \end{array} & \begin{array}{c} A_{n2}(D)+A_{n2}(UX)+B_{n2}(E)+ \\ B_{n2}(UY+VZ)+C_{n2}(F)+C_{n2}(WZ) \end{array} \\ 0 & \begin{array}{c} A_{n3}(D)+A_{n3}(UX)+B_{n3}(E)+ \\ B_{n3}(UY+VZ)+C_{n3}(F)+C_{n3}(WZ) \end{array} \end{bmatrix} \\
&= \sum_{i+j=n} \{d_i(S)d_j(T)+d_j(T)d_i(S)\} = \sum_{i+j=n} \\
&\left\{ \begin{bmatrix} \begin{array}{c} A_{i1}(X)+B_{i1}(Y) \\ +C_{i1}(Z) \end{array} & \begin{array}{c} A_{i2}(X)+B_{i2}(Y) \\ +C_{i2}(Z) \end{array} \\ 0 & \begin{array}{c} A_{i3}(X)+B_{i3}(Y) \\ +C_{i3}(Z) \end{array} \end{bmatrix} \begin{bmatrix} \begin{array}{c} A_{j1}(U)+B_{j1}(V) \\ +C_{j1}(W) \end{array} & \begin{array}{c} A_{j2}(U)+B_{j2}(V) \\ +C_{j2}(W) \end{array} \\ 0 & \begin{array}{c} A_{j3}(U)+B_{j3}(V) \\ +C_{j3}(W) \end{array} \end{bmatrix} \right. \\
&\left. + \begin{bmatrix} \begin{array}{c} A_{j1}(U)+B_{j1}(V) \\ +C_{j1}(W) \end{array} & \begin{array}{c} A_{j2}(U)+B_{j2}(V) \\ +C_{j2}(W) \end{array} \\ 0 & \begin{array}{c} A_{j3}(U)+B_{j3}(V) \\ +C_{j3}(W) \end{array} \end{bmatrix} \begin{bmatrix} \begin{array}{c} A_{i1}(X)+B_{i1}(Y) \\ +C_{i1}(Z) \end{array} & \begin{array}{c} A_{i2}(X)+B_{i2}(Y) \\ +C_{i2}(Z) \end{array} \\ 0 & \begin{array}{c} A_{i3}(X)+B_{i3}(Y) \\ +C_{i3}(Z) \end{array} \end{bmatrix} \right\}.
\end{aligned}
$$

This implies that

$$
\begin{aligned}
&A_{n1}(D)+A_{n1}(UX)+B_{n1}(E)+B_{n1}(UY+VZ)+C_{n1}(F)+C_{n1}(WZ) \\
&= \sum_{i+j=n} \{A_{i1}(X)A_{j1}(U)+A_{i1}(X)B_{j1}(V)+A_{i1}(X)C_{j1}(W)+B_{i1}(Y)A_{j1}(U) \\
&+B_{i1}(Y)B_{j1}(V)+B_{i1}(Y)C_{j1}(W)+C_{i1}(Z)A_{j1}(U)+C_{i1}(Z)B_{j1}(V) \\
&+C_{i1}(V)C_{j1}(W)+A_{j1}(U)A_{i1}(X)+A_{j1}(U)B_{i1}(Y)+A_{j1}(U)C_{i1}(Z) \\
&+B_{j1}(V)A_{i1}(X)+B_{j1}(V)B_{i1}(Y)+B_{j1}(V)C_{i1}(Z)+C_{j1}(W)A_{i1}(X) \\
&+C_{j1}(W)B_{i1}(Y)+C_{j1}(W)C_{i1}(Z)\},
\end{aligned} \tag{1}
$$

$$
\begin{aligned}
&A_{n2}(D)+A_{n2}(UX)+B_{n2}(E)+B_{n2}(UY+VZ)+C_{n2}(F)+C_{n2}(WZ) \\
&= \sum_{i+j=n} \{A_{i1}(X)A_{j2}(U)+A_{i1}(X)B_{j2}(V)+A_{i1}(X)C_{j2}(W)+B_{i1}(Y)A_{j2}(U) \\
&+B_{i1}(Y)B_{j2}(V)+B_{i1}(Y)C_{j2}(W)+C_{i1}(Z)A_{j2}(U)+C_{i1}(Z)B_{j2}(V) \\
&+C_{i1}(Z)C_{j2}(W)+A_{i2}(X)A_{j3}(U)+A_{i2}(X)B_{j3}(V)+A_{i2}(X)C_{j3}(W) \\
&+B_{i2}(Y)A_{j3}(U)+B_{i2}(Y)B_{j3}(V)+B_{i2}(Y)C_{j3}(W)+C_{i2}(Z)A_{j3}(U) \\
&+C_{i2}(Z)B_{j3}(V)+C_{i2}(Z)C_{j3}(W)+A_{j1}(U)A_{i2}(X)+A_{j1}(U)B_{i2}(Y) \\
&+A_{j1}(U)C_{i2}(Z)+B_{j1}(V)A_{i2}(X)+B_{j1}(V)B_{i2}(Y)+B_{j1}(V)C_{i2}(Z) \\
&+C_{j1}(W)A_{i2}(X)+C_{j1}(W)B_{i2}(Y)+C_{j1}(W)C_{i2}(Z)+A_{j2}(U)A_{i3}(X) \\
&+A_{j2}(U)B_{i3}(Y)+A_{j2}(U)C_{i3}(Z)+B_{j2}(V)A_{i3}(X)+B_{j2}(V)B_{i3}(Y) \\
&+B_{j2}(V)C_{i3}(Z)+C_{j2}(W)A_{i3}(X)+C_{j2}(W)B_{i3}(Y)+C_{j2}(W)C_{i3}(Z)\}
\end{aligned} \tag{2}
$$

and

$$(3)\quad \begin{aligned} &A_{n3}(D)+A_{n3}(UX)+B_{n3}(E)+B_{n3}(UY+VZ)+C_{n3}(F)+C_{n3}(WZ)\\ =&\sum_{i+j=n}\{A_{i3}(X)A_{j3}(U)+A_{i3}(X)B_{j3}(V)+A_{i3}(X)C_{j3}(W)+B_{i3}(Y)A_{j3}(U)\\ &+B_{i3}(Y)B_{j3}(V)+B_{i3}(Y)C_{j3}(W)+C_{i3}(Z)A_{j3}(U)+C_{i3}(Z)B_{j3}(V)\\ &+C_{i3}(Z)C_{j3}(W)+A_{j3}(U)A_{i3}(X)+A_{j3}(U)B_{i3}(Y)+A_{j3}(U)C_{i3}(Z)\\ &+B_{j3}(V)A_{i3}(X)+B_{j3}(V)B_{i3}(Y)+B_{j3}(V)C_{i3}(Z)+C_{j3}(W)A_{i3}(X)\\ &+C_{j3}(W)B_{i3}(Y)+C_{j3}(W)C_{i3}(Z)\}. \end{aligned}$$

Case 1. $G \neq 0$.

Step 1. We show that $C_{i1}(W) = 0$ for any $W \in Alg\mathcal{N}_{N^\perp}$, $i = 0, 1, ..., n$.

Substitute respectively X, Y, Z, U, V and W with $\lambda^{-1}X$, λY, λZ, λU, λV and $\lambda^{-1}W$ with $XU = D$, $XV + YW = E$ and $ZW = F$ in Eq. (1), it follows that

$$(4)\quad \begin{aligned} &A_{n1}(D)+A_{n1}(UX)+B_{n1}(E)+\lambda^2 B_{n1}(UY+VZ)+C_{n1}(F)+C_{n1}(WZ)\\ =&\sum_{i+j=n}\{A_{i1}(X)A_{j1}(U)+A_{i1}(X)B_{j1}(V)+\frac{1}{\lambda^2}A_{i1}(X)C_{j1}(W)\\ &+\lambda^2 B_{i1}(Y)A_{j1}(U)+\lambda^2 B_{i1}(Y)B_{j1}(V)+B_{i1}(Y)C_{j1}(W)+\lambda^2 C_{i1}(Z)A_{j1}(U)\\ &+\lambda^2 C_{i1}(Z)B_{j1}(V)+C_{i1}(Z)C_{j1}(W)+A_{j1}(U)A_{i1}(X)+\lambda^2 A_{j1}(U)B_{i1}(Y)\\ &+\lambda^2 A_{j1}(U)C_{i1}(Z)+B_{j1}(V)A_{i1}(X)+\lambda^2 B_{j1}(V)B_{i1}(Y)\\ &+\lambda^2 B_{j1}(V)C_{i1}(Z)+\frac{1}{\lambda^2}C_{j1}(W)A_{i1}(X)+C_{j1}(W)B_{i1}(Y)+C_{j1}(W)C_{i1}(Z)\}. \end{aligned}$$

Multiplying Eq. (4) by λ^2 and let $\lambda \to 0$, then $\sum\limits_{i+j=n}\{A_{i1}(X)C_{j1}(W)+C_{j1}(W)A_{i1}(X)\}$ =0. It is clearly established when $n = 0$. When $n = 1$, we can get $XC_{11}(W) + C_{11}(W)X = 0$ for any $X \in Alg\mathcal{N}$ with $XU = D$. Substitute X with I, then $C_{11}(W) = 0$. We assume that $C_{m1}(W) = 0$ for all $0 \leq m < n$. In fact, after simplifying the equation, we have $XC_{n1}(W) + C_{n1}(W)X = 0$. Substitute X with I, then $C_{n1}(W) = 0$.

Step 2. We show that $A_{n1}(XU+UX) = \sum\limits_{i+j=n}\{A_{i1}(X)A_{j1}(U)+A_{j1}(U)A_{i1}(X)\}$ for any $X, U \in Alg\mathcal{N}_N$ with $XU = D$ and $B_{i1}(V) = 0$ for any $V \in B(N^\perp, N)$ and all $i = 0, 1, ..., n$.

Substitute respectively X, Y, Z, U, V and W with $\lambda^{-1}X$, λE, λF, λU, 0 and $\lambda^{-1}I$ with $XU = D$ in Eq. (1), then

$$\begin{aligned} &A_{n1}(D) + A_{n1}(UX) + B_{n1}(E) + \lambda^2 B_{n1}(UE)\\ =&\sum_{i+j=n}\{A_{i1}(X)A_{j1}(U) + \lambda^2 B_{i1}(E)A_{j1}(U)+A_{j1}(U)A_{i1}(X)+\lambda^2 A_{j1}(U)B_{i1}(E)\}. \end{aligned}$$

Dividing the above equation by λ^2 and $\lambda \to \infty$, then $B_{n1}(UE) = \sum_{i+j=n} \{B_{i1}(E)A_{j1}(U) + A_{j1}(U)B_{i1}(E)\}$. Substitute respectively X and U with D and I, thus $B_{n1}(E) = \sum_{i+j=n} \{B_{i1}(E)A_{j1}(I) + A_{j1}(I)B_{i1}(E)\}$. By induction, we get $B_{n1}(E) = 0$. It follows that $A_{n1}(XU + UX) = \sum_{i+j=n} \{A_{i1}(X)A_{j1}(U) + A_{j1}(U)A_{i1}(X)\}$ with $XU = D$. Now the simplified Eq. (4) is $\sum_{i+j=n} \{A_{i1}(X)B_{j1}(V) + B_{j1}(V)A_{i1}(X)\} = 0$. Applying mathematical induction, we gain that $B_{i1}(V) = 0$ for any $V \in B(N^\perp, N)$ and all $i = 0, 1, ..., n$.

Step 3. We show that $A_{i3}(X) = 0$ for any $X \in Alg\mathcal{N}_N$ and $B_{i3}(Y) = 0$ for any $Y \in B(N^\perp, N)$ and all $i = 0, 1, ..., n$.

For any $Y \in B(N^\perp, N)$, substitute respectively X, Y, Z, U, V and W with λI, Y, $-\lambda^{-1}F$, $\lambda^{-1}D$, $Y + \lambda^{-1}E$ and $-\lambda I$ in Eq. (3), then

$$\begin{aligned}
&A_{n3}(2D) + B_{n3}(E) + \frac{1}{\lambda}B_{n3}(DY - YF) - \frac{1}{\lambda^2}B_{n3}(EF) + C_{n3}(2F) \\
= &\sum_{i+j=n} \{A_{i3}(I)A_{j3}(D) + \lambda A_{i3}(I)B_{j3}(Y) + A_{i3}(I)B_{j3}(E) - \lambda^2 A_{i3}(I)C_{j3}(I) \\
&+\frac{1}{\lambda}B_{i3}(Y)A_{j3}(D) + B_{i3}(Y)B_{j3}(Y) + \frac{1}{\lambda}B_{i3}(Y)B_{j3}(E) - \lambda B_{i3}(Y)C_{j3}(I) \\
&-\frac{1}{\lambda^2}C_{i3}(F)A_{j3}(D) - \frac{1}{\lambda}C_{i3}(F)B_{j3}(Y) - \frac{1}{\lambda^2}C_{i3}(F)B_{j3}(E) + C_{i3}(F)C_{j3}(I) \\
&+A_{j3}(D)A_{i3}(I) + \lambda B_{j3}(Y)A_{i3}(I) + B_{j3}(E)A_{i3}(I) - \lambda^2 C_{j3}(I)A_{i3}(I) \\
&+\frac{1}{\lambda}A_{j3}(D)B_{i3}(Y) + B_{j3}(Y)B_{i3}(Y) + \frac{1}{\lambda}B_{j3}(E)B_{i3}(Y) - \lambda C_{j3}(I)B_{i3}(Y) \\
&-\frac{1}{\lambda^2}A_{j3}(D)C_{i3}(F) - \frac{1}{\lambda}B_{j3}(Y)C_{i3}(F) - \frac{1}{\lambda^2}B_{j3}(E)C_{i3}(F) + C_{j3}(I)C_{i3}(F)\}.
\end{aligned}$$

With the randomicity of λ, we infer that

$$\begin{aligned}
&A_{n3}(D) + A_{n3}(D) + B_{n3}(E) + C_{n3}(2F) \\
= &\sum_{i+j=n} \{A_{i3}(I)A_{j3}(D) + A_{i3}(I)B_{j3}(E) + B_{i3}(Y)B_{j3}(Y) + C_{i3}(F)C_{j3}(I) \\
&+A_{j3}(D)A_{i3}(I) + B_{j3}(E)A_{i3}(I) + B_{j3}(Y)B_{i3}(Y) + C_{j3}(I)C_{i3}(F)\}
\end{aligned}$$

for any $Y \in B(N^\perp, N)$. The two sides will be, in fact, equal when $Y = 0$. So $\sum_{i+j=n} \{B_{i3}(Y)B_{j3}(Y) + B_{j3}(Y)B_{i3}(Y)\} = 0$. Applying mathematical induction, we claim $B_{i3}(Y) = 0$ for any $Y \in B(N^\perp, N)$ and all $i = 0, 1, .., n$.

Substitute respectively X, Y, Z, U, V and W with $\lambda^{-1}X$, λE, λF, λU, 0 and

$\lambda^{-1}I$ with $XU = D$ in Eq. (3),then

$$\begin{aligned}
&A_{n3}(D) + A_{n3}(UX) + C_{n3}(F) + C_{n3}(F)\\
= &\sum_{i+j=n} \{A_{i3}(X)A_{j3}(U) + \frac{1}{\lambda^2}A_{i3}(X)C_{j3}(I) + \lambda^2 C_{i3}(F)A_{j3}(U) + C_{i3}(F)C_{j3}(I)\\
&+A_{j3}(U)A_{i3}(X) + \lambda^2 A_{j3}(U)C_{i3}(F) + \frac{1}{\lambda^2}C_{j3}(I)A_{i3}(X) + C_{j3}(I)C_{i3}(F)\}.
\end{aligned}$$

Multiplying the above equation by λ^2 and let $\lambda \to 0$,then $\sum_{i+j=n} \{A_{i3}(X)C_{j3}(I) + C_{j3}(I)A_{i3}(X)\} = 0$. Similarly available $A_{i3}(X) = 0$ for any $X \in Alg\mathcal{N}_N$ and all $i = 0, 1, .., n$.

Step 4. We show that $A_{i1}(I) = 0$ and $C_{i3}(I) = 0$ for all $i = 1, 2, ..., n$.

Case①. D and F, at least one is not 0.

Substitute respectively X, Y, Z, U, V and W with $-X$, Y, Z, $-U$, $-V$ and W with $XU = D$, $XV + YW = E$ and $ZW = F$ in Eq. (2), we get a new Eq. (2).The new one together with the original one yield

$$\begin{aligned}
&A_{n2}(D) + A_{n2}(UX) + B_{n2}(E) + C_{n2}(F) + C_{n2}(WZ)\\
(5) \qquad = &\sum_{i+j=n} \{A_{i1}(X)A_{j2}(U) + A_{i1}(X)B_{j2}(V) + B_{i2}(Y)C_{j3}(W)\\
&+C_{i2}(Z)C_{j3}(W) + A_{j1}(U)A_{i2}(X) + C_{j2}(W)C_{i3}(Z)\}.
\end{aligned}$$

For any $V \in B(N^{\perp}, N)$, substitute respectively X, Y, Z, U, V and W with I, $E-V$, F, D, V and I in Eq. (5), then we can get $\sum_{i+j=n} \{A_{i1}(I)B_{j2}(V) - B_{i2}(V)C_{j3}(I)\} = 0$. Also by mathematical induction, we prove that

$$A_{n1}(I)V = VC_{n3}(I). \tag{6}$$

We can get $A_{n1}(2D) = \sum_{i+j=n} \{A_{i1}(D)A_{j1}(I) + A_{j1}(I)A_{i1}(D)\}$ from step 2 and $C_{n3}(2F) = \sum_{i+j=n} \{C_{i3}(F)C_{j3}(I) + C_{j3}(I)C_{i3}(F)\}$ by substituting respectively Z and W with F and I in Eq. (3). Then, using induction, $DA_{n1}(I) + A_{n1}(I)D = 0$ and $FC_{n3}(I) + C_{n3}(I)F = 0$ are obtained by assuming $C_{m3}(I) = 0, A_{m1}(I) = 0, 0 < m < n$.

By $A_{11}(I) = 0$ and $C_{13}(I) = 0$ from [1,p.56], we get $FC_{23}(I) + C_{23}(I)F = 0, DA_{21}(I) + A_{21}(I)D = 0$. Combining with (6) and the condition of D and F, we have $A_{21}(I) = 0, C_{23}(I) = 0$ by the same method in [1,p.56]. Also, with the help of induction, we point that $A_{i1}(I) = 0, C_{i3}(I) = 0$ for all $i = 1, 2, ..., n$.

Case②. $D = 0, F = 0, E \neq 0$

Substitute respectively X, Y, Z, U, V and W with 0, E, 0, I, V and I in Eq. (2), then we see

$$B_{n2}(E)+B_{n2}(E)=\sum_{i+j=n}\{B_{i2}(E)C_{j3}(I)+A_{j1}(I)B_{i2}(E)\}.$$

By $A_{11}(I)=0$ and $C_{13}(I)=0$ from [1,p.56], the above equation implies $EC_{23}(I)+A_{21}(I)E=0$ when $n=2$. We assume $A_{m1}(I)=0, C_{m3}(I)=0$ for all $0<m<n$. In fact, after simplifying, we get $EC_{n3}(I)+A_{n1}(I)E=0$.

By the same way in [1,p.56], we can prove that $A_{i1}(I)=0, C_{i3}(I)=0$ for all $i=1,2,...,n$.

Step 5. We show that $A_{n2}(X)=-\sum_{i+j=n}A_{i1}(X)C_{j2}(I)$ for any $X\in Alg\mathcal{N}_N$ and $C_{n2}(W)=-\sum_{i+j=n}A_{i2}(I)C_{j3}(W)$ for any $W\in Alg\mathcal{N}_{N^\perp}$.

For any invertible X, substitute respectively X, Y, Z, U, V and W with X, λE, λF, $X^{-1}D$, 0 and $\lambda^{-1}I$ in Eq. (2), it follows that

$$\begin{aligned}
&A_{n2}(X^{-1}DX+D)+B_{n2}(E)+\lambda B_{n2}(X^{-1}DE)+C_{n2}(2F)\\
&=\sum_{i+j=n}\{A_{i1}(X)A_{j2}(X^{-1}D)+\frac{1}{\lambda}A_{i1}(X)C_{j2}(I)+\frac{1}{\lambda}A_{i2}(X)C_{j3}(I)\\
&\quad+\lambda A_{j1}(X^{-1}D)C_{i2}(F)+\lambda A_{j2}(X^{-1}D)C_{i3}(F)+\lambda A_{j1}(X^{-1}D)B_{i2}(E)\\
&\quad+A_{j1}(X^{-1}D)A_{i2}(X)+B_{i2}(E)C_{j3}(I)+C_{i2}(F)C_{j3}(I)+C_{j2}(I)C_{i3}(F)\}.
\end{aligned}$$

Multiplying the above equation by λ and let $\lambda\to 0$, then $\sum_{i+j=n}\{A_{i1}(X)C_{j2}(I)+A_{i2}(X)C_{j3}(I)\}=0$. In fact, from step 4, we can get $A_{n2}(X)=-\sum_{i+j=n}A_{i1}(X)C_{j2}(I)$. So, for any invertible operator $X\in Alg\mathcal{N}_N$, $A_{n2}(X)=-\sum_{i+j=n}A_{i1}(X)C_{j2}(I)$. We can get the fact that $A_{n2}(X)=-\sum_{i+j=n}A_{i1}(X)C_{j2}(I)$ for any $X\in Alg\mathcal{N}_N$ from [1, Lemma 4.1].

For any invertible $W\in Alg\mathcal{N}_{N^\perp}$, substitute respectively X, Y, Z, U, V and W with I, 0, $\lambda^{-1}FW^{-1}$, D, E and λW in Eq. (2), that is

$$\begin{aligned}
&A_{n2}(2D)+B_{n2}(E)+\frac{1}{\lambda}B_{n2}(EFW^{-1})+C_{n2}(F)+C_{n2}(WFW^{-1})\\
&=\sum_{i+j=n}\{A_{i1}(I)A_{j2}(D)+A_{i1}(I)B_{j2}(E)+\lambda A_{i1}(I)C_{j2}(W)+\lambda A_{i2}(I)C_{j3}(W)\\
&\quad+C_{i2}(FW^{-1})C_{j3}(W)+A_{j1}(D)A_{i2}(I)+\frac{1}{\lambda}A_{j1}(D)C_{i2}(FW^{-1})\\
&\quad+\frac{1}{\lambda}A_{j2}(D)C_{i3}(FW^{-1})+\frac{1}{\lambda}B_{j2}(E)C_{i3}(FW^{-1})+C_{j2}(W)C_{i3}(FW^{-1})\}.
\end{aligned}$$

Dividing the above equation by λ and let $\lambda \to \infty$, then $\sum\limits_{i+j=n} \{A_{i1}(I)C_{j2}(W) + A_{i2}(I)C_{j3}(W)\} = 0$. By step 4, we have $C_{n2}(W) = -\sum\limits_{i+j=n} A_{i2}(I)C_{j3}(W)$ for any invertible $W \in Alg\mathcal{N}_{N^\perp}$. By [1, Lemma 4.1], for any $W \in Alg\mathcal{N}_{N^\perp}$, $C_{n2}(W) = -\sum\limits_{i+j=n} A_{i2}(I)C_{j3}(W)$.

Step 6. We show that $C_{n2}(ZW+WZ) = \sum\limits_{i+j=n} \{C_{i2}(Z)C_{j3}(W)+C_{j2}(W)C_{i3}(Z)\}$ with $ZW = F$ and $\sum\limits_{i+j=n} \{A_{i1}(I)C_{j2}(W) + A_{i2}(I)C_{j3}(W)\} = 0$ for any $X \in Alg\mathcal{N}_N$ and $W \in Alg\mathcal{N}_{N^\perp}$.

We have $\sum\limits_{i+j=n} \{A_{i1}(I)C_{j2}(W) + A_{i2}(I)C_{j3}(W)\} = 0$ by step 5.

Substitute respectively X, Y, Z, U, V and W with I, 0, Z, D, E and W with $ZW = F$ in Eq. (5), then

$$\begin{aligned} &A_{n2}(D) + A_{n2}(D) + B_{n2}(E) + C_{n2}(F) + C_{n2}(WZ) \\ = &\sum_{i+j=n} \{A_{i1}(I)A_{j2}(D) + A_{i1}(I)B_{j2}(E) \\ &+C_{i2}(Z)C_{j3}(W) + A_{j1}(D)A_{i2}(I) + C_{j2}(W)C_{i3}(Z)\}. \end{aligned}$$

We can easily get $C_{n2}(ZW + WZ) = \sum\limits_{i+j=n} \{C_{i2}(Z)C_{j3}(W) + C_{j2}(W)C_{i3}(Z)\}$ with $ZW = F$ after simplifying the above equation.

Step 7. We show that both $A_{n1}(\cdot)$ and $C_{n3}(\cdot)$ are derivations.

For any invertible $W \in Alg\mathcal{N}_{N^\perp}$, substitute respectively X, Y, Z, U, V and W with I, Y, FW^{-1}, D, $E - YW$ and W in Eq. (5), then we get

$$B_{n2}(YW) = \sum_{i+j=n} \{B_{i2}(Y)C_{j3}(W)\}.$$

By [1, Lemma 4.1], we know that $B_{n2}(YW) = \sum\limits_{i+j=n} \{B_{i2}(Y)C_{j3}(W)\}$ for any $W \in Alg\mathcal{N}_{N^\perp}$. For any $W_1, W_2 \in Alg\mathcal{N}_{N^\perp}$, we have $B_{n2}(YW_1W_2) = \sum\limits_{i+j=n} \{B_{i2}(Y)C_{j3}(W_1W_2)\}$. On the other hand, $B_{n2}(YW_1W_2) = \sum\limits_{i+j=n} \{B_{i2}(YW_1)C_{j3}(W_2)\} = \sum\limits_{i+j+k=n} \{B_{i2}(Y)C_{k3}(W_1)C_{j3}(W_2)\}$. The above equations imply that $C_{n3}(W_1W_2) = \sum\limits_{i+j=n} \{C_{i3}(W_1)C_{j3}(W_2)\}$. Hence $C_{i3}(\cdot)$ is a derivation.

For any invertible $X \in Alg\mathcal{N}_N$, substitute respectively X, Y, Z, U, V and W with X, $E - XV$, F, $X^{-1}D$, V and I in Eq. (5), it follows that $B_{n2}(XV) = \sum\limits_{i+j=n} \{A_{i1}(X)B_{j2}(V)\}$. Similarly, we can prove that $A_{n1}(\cdot)$ is a derivation.

Step 8. We show that $d_n(\cdot)$ is a higher derivation.

For arbitrary $S = \begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix}$ and $T = \begin{bmatrix} U & V \\ 0 & W \end{bmatrix}$ in $Alg\mathcal{N}$, we only need to prove that $d_n(ST) = \sum\limits_{i+j=n} d_i(S)d_j(T)$. By the above steps, we calculate

$$\begin{aligned}
&\sum_{i+j=n} d_i(S)d_j(T)\\
&= \sum_{i+j=n}\left\{\begin{bmatrix} A_{i1}(X) & A_{i2}(X)+B_{i2}(Y) \\ & +C_{i2}(Z) \\ 0 & C_{i3}(Z)\end{bmatrix}\begin{bmatrix} A_{j1}(U) & A_{j2}(U)+B_{j2}(V) \\ & +C_{j2}(W) \\ 0 & C_{j3}(W)\end{bmatrix}\right\}\\
&= \begin{bmatrix} A_{n1}(XU) & \sum\limits_{i+j=n}\{A_{i1}(X)A_{j2}(U)+A_{i1}(X)B_{j2}(V) \\ & +B_{i2}(Y)C_{j3}(W)+C_{i2}(Z)C_{j3}(W)\} \\ 0 & C_{n3}(ZW)\end{bmatrix}\\
&= \begin{bmatrix} A_{n1}(XU) & B_{n2}(XV)+B_{n2}(YW)+\sum\limits_{i+j=n}\{A_{i1}(X)A_{j2}(U)+C_{i2}(Z)C_{j3}(W)\} \\ 0 & C_{n3}(ZW)\end{bmatrix}\\
&= \begin{bmatrix} A_{n1}(XU) & B_{n2}(XV)+B_{n2}(YW)+\sum\limits_{i+j+K=n}\{-A_{i1}(X)A_{m1}(U)C_{k2}(I) \\ & -A_{i2}(I)C_{m3}(Z)C_{j3}(W)\} \\ 0 & C_{n3}(ZW)\end{bmatrix}\\
&= \begin{bmatrix} A_{i1}(XU) & B_{n2}(XV)+B_{n2}(YW) \\ & +\sum\limits_{i+j+K=n}\{-A_{i1}(XU)C_{k2}(I)-A_{i2}(I)C_{m+j3}(ZW)\} \\ 0 & C_{n3}(ZW)\end{bmatrix}\\
&= \begin{bmatrix} A_{i1}(XU) & B_{n2}(XV)+B_{n2}(YW)+A_{n2}(XU)+C_{n2}(ZW)\} \\ 0 & C_{n3}(ZW)\end{bmatrix} = d_n(ST).
\end{aligned}$$

This completes the proof of case 1.

Case 2. $G = 0$.

Step 1. We show that $B_{i1}(V) = 0$ and $B_{i3}(V) = 0$ for any $V \in B(N^{\perp}, N)$ and all $i = 0, 1, ..., n$.

Substitute respectively X, Y, Z, U, V and W with 0, V, 0, X, 0 and 0 in Eq. (1), Eq. (2) and Eq. (3) respectively, it follows that

$$B_{n1}(XV) = \sum_{i+j=n}\{B_{i1}(V)A_{j1}(X)+A_{j1}(X)B_{i1}(V)\}, \tag{7}$$

$$\begin{aligned} B_{n2}(XV) &= \sum_{i+j=n}\{B_{i1}(V)A_{j2}(X)+B_{i2}(V)A_{j3}(X) \\ &+ \quad A_{j2}(X)B_{i3}(V)+A_{j1}(X)B_{i2}(V)\}, \end{aligned} \tag{8}$$

$$B_{n3}(XV) = \sum_{i+j=n} \{B_{i3}(V)A_{j3}(X) + A_{j3}(X)B_{i3}(V)\}. \tag{9}$$

Subtitute X with I in Eq. (7) and Eq. (9), by mathematical induction, we prove $B_{i1}(V) = 0$ and $B_{i3}(V) = 0$ for all $i = 0, 1, ..., n$, respectively. The Eq. (8) can be simplified to

$$B_{n2}(XV) = \sum_{i+j=n} \{B_{i2}(V)A_{j3}(X) + A_{j1}(X)B_{i2}(V)\}. \tag{10}$$

Step 2. We show that $A_{i3}(X) = 0$ for any $X \in Alg\mathcal{N}_N$ and $C_{i1}(W) = 0$ for any $W \in Alg\mathcal{N}_{N^\perp}$ and all $i = 0, 1, ..., n$.

Substitute respectively X, Y, Z, U, V and W with 0, V, W, X, 0 and 0 in Eq. (1), Eq. (2) and Eq. (3) respectively, hence the following three equations hold

$$0 = \sum_{i+j=n} \{C_{i1}(W)A_{j1}(X) + A_{j1}(X)C_{i1}(W)\}, \tag{11}$$

$$\begin{aligned} B_{n2}(XV) = \sum_{i+j=n} \{&C_{i1}(W)A_{j2}(X) + C_{i2}(W)A_{j3}(X) + B_{i2}(V)A_{j3}(X) \\ &+A_{j1}(X)C_{i2}(W) + A_{j1}(X)B_{i2}(V) + A_{j2}(X)C_{i3}(W)\}, \end{aligned} \tag{12}$$

$$0 = \sum_{i+j=n} \{C_{i3}(W)A_{j3}(X) + A_{j3}(X)C_{i3}(W)\}. \tag{13}$$

Substituting respectively X and W with I and I in Eq. (11) and Eq. (13) respectively, applying mathematical induction , we can prove $C_{i1}(W) = 0$ and $A_{i3}(X) = 0$ for all $i = 0, 1, ..., n$. Eq. (10) together with Eq. (12) yield

$$B_{n2}(XV) = \sum_{i+j=n} A_{j1}(X)B_{i2}(V) \tag{14}$$

and

$$\sum_{i+j=n} \{A_{j1}(X)C_{i2}(W) + A_{j2}(X)C_{i3}(W)\} = 0. \tag{15}$$

Step 3. We show that $C_{n2}(W) = \sum\limits_{i+j=n} C_{i2}(I)C_{j3}(W)$ for any $W \in Alg\mathcal{N}_{N^\perp}$, $A_{n1}(\cdot)$ is is a derivation and $A_{i1}(I) = 0$ for all $i = 1, 2, ..., n$.

Similarly, from step 7 in case 1, we can prove $A_{n1}(\cdot)$ is a derivation by Eq. (14). Substituting X with I in Eq. (14), by mathematical induction, we get $A_{i1}(I) = 0$ for all $i = 1, 2, ..., n$. Substitute X with I in Eq. (15), then we have $C_{n2}(W) = \sum\limits_{i+j=n} C_{i2}(I)C_{j3}(W)$.

Step 4. We show that $A_{n2}(X) = -\sum\limits_{i+j=n} A_{i1}(X)C_{j2}(I)$ for any $X \in Alg\mathcal{N}_N$, $C_{n3}(\cdot)$ is a derivation and $C_{i3}(I) = 0$ for all $i = 1, 2, ..., n$.

Substitute respectively X, Y, Z, U, V and W with λI, $-V$, 0, 0, V and λI in Eq. (2), then

$$\sum_{i+j=n} \{\lambda A_{i1}(I)B_{j2}(V) + \lambda^2 A_{i1}(I)C_{j2}(I) + \lambda^2 A_{i2}(I)C_{j3}(I) - \lambda B_{i2}(V)C_{j3}(I)\} = 0.$$

Dividing the above equation by λ and let $\lambda \to 0$, then we get $\sum\limits_{i+j=n} \{A_{i1}(I)B_{j2}(V) - B_{i2}(V)C_{j3}(I)\} = 0$. In fact, by step 3, $B_{n2}(V) = \sum\limits_{i+j=n} \{B_{i2}(V)C_{j3}(I)\}$. Using mathematical induction, we prove $C_{i3}(I) = 0$ for all $i = 1, 2, ..., n$. Substituting W with I in Eq. (15), that is $A_{n2}(X) = -\sum\limits_{i+j=n} A_{i1}(X)C_{j2}(I)$.

Substitute respectively X, Y, Z, U, V and W with 0, $-X^{-1}VW$, W, X, V and 0 in Eq. (2), it follows that

$$\sum_{i+j=n} \{-A_{j1}(X)B_{i2}(VW)+A_{j1}(X)C_{i2}(W)+A_{j2}(X)C_{i3}(W)+B_{j2}(V)C_{i3}(W)\} = 0.$$

Substituting X with I, we get $B_{n2}(VW) = \sum\limits_{i+j=n} B_{j2}(V)C_{i3}(W)$ with the help of Eq. (15). Similarly, we can prove that $C_{n3}(\cdot)$ is a derivation.

Similarly available $d = \{d_n : n \in N\}$ is a higher derivation.

References

1. Y. H. Chen, *The depicting of mappings in operator algebra*, the doctoral dissertation of ECUST, 2011.
2. S. Zhao and J. Zhu, Jordan all-derivable points in the algebra of all upper triangular matrices, *J. Linear Algebra Appl.*, **433(11-12)** (2010), 1922-1938.
3. J. Zhu, C. P. Xiong and L. Zhang, All-derivable points in matrix algebras, *J. Linear Algebra Appl.*, **430(8-9)** (2009), 2070-2079.
4. W. Jing, S. J. Lu and P. T. Li, Characterizations of derivations on some operator algebras, *Bull. Austral. Math. Soc.*, **66** (2002), 227-232, MR 1932346 (2003f:47059).
5. J. C. Hou and X. F. Qi, Characterizations of derivations of Banach space nest algebras: all-derivable point, *Linear Algebra Appl.*, **432** (2010), 3183-3200.
6. M. Gong and J. Zhu, Jordan multiplicative mappings at some points on matrix algebras, *Journal of Advanced Research in Pure Mathematics*, **2(4)** (2010), 84-93.
7. Z. K. Xiao and F. Wei, Higher derivations of triangular algebras and its generations, *Linear Algebra Appl.*, **432** (2010), 2615-2622.

Nannan Zhen and Jun Zhu
Institute of Mathematics
Hangzhou Dianzi University
Hangzhou 310018
P. R. China
E-mail: nannanzhen66@sina.com
zhu_gjun@yahoo.com.cn