TAIWANESE JOURNAL OF MATHEMATICS
Vol. 16, No. 4, pp. 1475-1488, August 2012
This paper is available online at http://journal.taiwanmathsoc.org.tw

SOME CLASSIFICATION RESULTS ON FINITE-TYPE RULED SUBMANIFOLDS IN A LORENTZ-MINKOWSKI SPACE

Dong-Soo Kim[1] and Young Ho Kim[2]

Abstract. Ruled submanifolds of finite type in Lorentz-Minkowski space are studied. We construct a new example of ruled submanifolds with degenerate rulings called a BS-kind ruled submanifold, which is of finite type. Also, it is determined by the restricted minimal polynomial of the shape operator associated with the mean curvature vector field.

1. INTRODUCTION

In the late 1970's the notion of finite type immersion into Euclidean space was introduced by B.-Y. Chen and it was extended to that of submanifolds in pseudo-Euclidean space: A pseudo-Riemannian submanifold M of an m-dimensional pseudo-Euclidean space $\mathbb{E}^m_s$ with signature $(m-s,s)$ is said to be of *finite type* if its position vector field x which is identified with the given isometric immersion can be expressed as a finite sum of eigenvectors of the Laplacian Δ of M, that is, $x = x_0 + \sum_{i=1}^k x_i$, where x_0 is a constant map, $x_1, \cdots, x_k$ non-constant maps such that $\Delta x_i = \lambda_i x_i, \lambda_i \in \mathbb{R}, i = 1, 2, \cdots, k$ ([4, 5]). If $\lambda_1, \lambda_2, \cdots, \lambda_k$ are different, then M is said to be of k-*type*. Similarly, a smooth map ϕ on an n-dimensional pseudo-Riemannian submanifold M of $\mathbb{E}^m_s$ is said to be of *finite type* if ϕ is a finite sum of $\mathbb{E}^m_s$-valued eigenfunctions of Δ. We also similarly define a smooth map of k-type on M as that of immersion x. A very typical interesting smooth map on the submanifold M of Euclidean space or pseudo-Euclidean space is the Gauss map which is useful to examine the geometric character of M.

Received February 25, 2010, accepted September 9, 2011.
Communicated by Bang-Yen Chen.
2010 *Mathematics Subject Classification*: 53B25, 53C40.
Key words and phrases: Ruled submanifold, Finite type, Null scroll kind, B-scroll kind ruled submanifold.
[1] Supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (2010-0022926).
[2] Supported by Kyungpook National University Research Fund, 2009.

Many geometers studied submanifolds with finite type immersion or finite type Gauss map ([1, 2, 4, 5, 6, 9, 10, 11], etc.). Especially, the present authors et al. studied ruled surfaces in Lorentz-Minkowski space $\mathbb{L}^m$ with finite type Gauss maps, in which new examples of ruled surfaces called the generalized B-scrolls were introduced ([9]). In 1992, F. Dillen classified ruled submanifolds in Euclidean space of finite type ([7]).

In this article, we study the ruled submanifolds in Lorentz-Minkowski m-space $\mathbb{L}^m$, and we give the complete classification theorem of ruled submanifolds of finite type, which is a generalization of the authors' results on ruled surfaces in $\mathbb{L}^m$ ([9, 10]).

Throughout this paper, we assume that all objects we are dealing with are smooth and all submanifolds under consideration are connected unless otherwise mentioned.

2. Preliminaries

Let $\mathbb{E}^m_s$ be an m-dimensional pseudo-Euclidean space of signature $(m-s, s)$. In particular, for $m \geq 2$, $\mathbb{E}^m_1$ is called a ***Lorentz-Minkowski m-space*** or simply a ***Lorentzian*** m-space, which will be denoted by $\mathbb{L}^m$. A curve in $\mathbb{E}^m_s$ is said to be ***space-like, time-like*** or ***null*** if its tangent vector field is space-like, time-like or null, respectively.

Let $x : M \longrightarrow \mathbb{E}^m_s$ be an isometric immersion of an n-dimensional pseudo-Riemannian submanifold M into $\mathbb{E}^m_s$. From now on, a submanifold in $\mathbb{E}^m_s$ always means pseudo-Riemannian, that is, each tangent space of the submanifold is non-degenerate.

Let $(x_1, x_2, \cdots, x_n)$ be a local coordinate system of M in $\mathbb{E}^m_s$. For the components g_{ij} of the pseudo-Riemannian metric $\langle \cdot, \cdot \rangle$ on M induced from that of $\mathbb{E}^m_s$, we denote by (g^{ij}) (resp., $\mathcal{G}$) the inverse matrix (resp., the determinant) of the matrix (g_{ij}). Then, the Laplacian Δ on M is given by

$$\Delta = -\frac{1}{\sqrt{|\mathcal{G}|}} \sum_{i,j} \frac{\partial}{\partial x^i}(\sqrt{|\mathcal{G}|} g^{ij} \frac{\partial}{\partial x^j}). \tag{2.1}$$

A non-degenerate $(r+1)$-dimensional submanifold M in $\mathbb{L}^m$ is called a ruled submanifold if there exists a regular curve $\alpha = \alpha(s)$ on M defined on an open interval I along which M is foliated by r-dimesional totally geodesic submanifolds $E(s,r)$ of $\mathbb{L}^m$. Thus, a parametrization of a ruled submanifold M in $\mathbb{L}^m$ can be given by

$$x = x(s, t_1, t_2, \cdots, t_r) = \alpha(s) + \sum_{i=1}^{r} t_i \mathbf{e}_i(s), \quad s \in I, \quad t_i \in I_i,$$

where I_i's are some open intervals for $i = 1, 2, \cdots, r$ and $E(s,r)$ is open in Span$\{\mathbf{e}_1(s), \mathbf{e}_2(s), \cdots, \mathbf{e}_r(s)\}$, which is the linear span of linearly independent vector fields $\mathbf{e}_1(s)$, $\mathbf{e}_2(s), \cdots, \mathbf{e}_r(s)$ along the curve α. Here, we assume $E(s,r)$ are either non-degenerate or degenerate for all s along α. We call $E(s,r)$ the rulings and α the base curve of the ruled submanifold M.

In particular, the ruled submanifold M is said to be *cylindrical* if $E(s, r)$ is parallel along α, or *non-cylindrical* otherwise.

Remark.

(1) If the rulings of M are non-degenerate, then the base curve α can be chosen to be orthogonal to the rulings as follows: Let V be a unit vector field on M which is orthogonal to the rulings, then α can be taken as an integral curve of V.

(2) If the rulings are degenerate, we can choose a null vector field V on M which is not contained in the rulings. An integral curve of V can be the base curve.

By solving a system of ordinary differential equations similarly set up in relation to a frame along a curve in $\mathbb{L}^m$ as given in [3], we have:

Lemma 2.1. ([9]). *Let $V(s)$ be a smooth l-dimensional non-degenerate distribution in a Lorentzian m-space $\mathbb{L}^m$ along a curve $\alpha = \alpha(s)$, where $l \geq 2$ and $m \geq 3$. Then, we can choose orthonormal vector fields $\mathbf{e}_1(s), \cdots, \mathbf{e}_{m-l}(s)$ along α which generate the orthogonal complement $V^\perp(s)$ satisfying $\mathbf{e}_i'(s) \in V(s)$ for $1 \leq i \leq m - l$.*

3. Ruled Submanifolds with Non-degenerate Rulings

Let M be an $(r + 1)$-dimensional ruled submanifold in $\mathbb{L}^m$ generated by non-degenerate rulings. Then, the base curve α can be chosen to be orthogonal to the rulings as was described in Section 2. Without loss of generality, we may assume that α is a unit speed curve, that is, $\langle \alpha'(s), \alpha'(s)\rangle = \varepsilon \ (= \pm 1)$. From now on, the prime $'$ denotes d/ds unless otherwise stated. By Lemma 2.1, we may choose orthonormal vector fields $\mathbf{e}_1(s), \cdots, \mathbf{e}_r(s)$ along α satisfying

$$\langle \alpha'(s), \mathbf{e}_i(s)\rangle = 0, \quad \langle \mathbf{e}_i'(s), \mathbf{e}_j(s)\rangle = 0, \quad i, j \in \{1, 2, \cdots, r\}.$$

If M is cylindrical, its parametrization is given by

$$x(s, t_1, \cdots, t_r) = \alpha(s) + \sum_{i=1}^{r} t_i \mathbf{e}_i(s)$$

for some orthonormal constant vector fields $\mathbf{e}_1(s), \cdots, \mathbf{e}_r(s)$. Since the generators $\mathbf{e}_1(s), \cdots, \mathbf{e}_r(s)$ of $E(s, r)$ are constant vector fields, it is easily seen that M is a cylinder over a finite type curve if it is of finite type. Therefore, we have immediately.

Theorem 3.1. *Let M be a cylindrical ruled submanifold of finite type with non-degenerate rulings in Lorentzian m-space $\mathbb{L}^m$. Then, M is a cylinder over a curve of finite type.*

Definition 3.2. A non-cylindrical ruled submanifold with non-degenerate rulings is said to be of *finite rank* k if the base curve α and the generators $\mathbf{e}_{j_1}(s), \cdots, \mathbf{e}_{j_{k-1}}(s)$ of

$E(s,r)$ are of the same finite type with the same eigenvalues and the other generators $\mathbf{e}_{j_k}, \cdots, \mathbf{e}_{j_r}$ are constant vector fields, where $\{j_1, j_2, \cdots, j_r\} = \{1, 2, \cdots, r\}$.

Let M be a non-cylindrical ruled submanifold with the rulings $E(s,r) \subset \mathrm{Span}\{\mathbf{e}_1(s), \cdots, \mathbf{e}_r(s)\}$ over a unit speed base curve $\alpha = \alpha(s)$ so that not all $\mathbf{e}_1'(s), \mathbf{e}_2'(s), \cdots, \mathbf{e}_r'(s)$ are zero and $\mathbf{e}_1(s), \cdots, \mathbf{e}_r(s)$ satisfy $\langle \mathbf{e}_i(s), \mathbf{e}_j(s)\rangle = \epsilon_i \delta_{ij}$ and $\langle \mathbf{e}_i'(s), \mathbf{e}_j(s)\rangle = 0$, where $\epsilon_i = \pm 1$ and $i, j = 1, 2, \cdots, r$. Now, we distinguish two cases:

Case 1. $\mathbf{e}_i'(s)$'s are all non-null for $i = 1, 2, \cdots, r$.

Case 2. There exist some generators $\mathbf{e}_{j_1}(s), \mathbf{e}_{j_2}(s), \cdots, \mathbf{e}_{j_k}(s)$ with $\mathbf{e}_{j_1}'(s), \mathbf{e}_{j_2}'(s), \cdots, \mathbf{e}_{j_k}'(s)$ being null along α for $j_1 < j_2 < \cdots < j_k \in \{1, 2, \cdots, r\}$ $(k \le r)$ and $\mathbf{e}_i'(s)$ are non-null for $i \ne j_1, j_2, \cdots, j_k$.

Let us consider Case 1. Then, an isometric immersion of M is given by

$$x(s, t_1, \cdots, t_r) = \alpha(s) + \sum_{i=1}^{k} t_i \mathbf{e}_i(s) + \sum_{j=k+1}^{r} t_j \mathbf{e}_j(s), \tag{3.1}$$

where $\mathbf{e}_1(s), \cdots, \mathbf{e}_k(s)$ are some constant orthonormal vector fields along α. Then, we have

$$x_s = \alpha'(s) + \sum_{i=k+1}^{r} t_i \mathbf{e}_i'(s), \qquad x_{t_j} = \mathbf{e}_j(s)$$

for $j = 1, 2, \cdots, r$. The straightforward computation with the help of (2.1) implies

$$\Delta = \frac{1}{2P^2}\frac{\partial P}{\partial s}\frac{\partial}{\partial s} - \frac{1}{P}\frac{\partial^2}{\partial s^2} - \frac{1}{2P}\sum_{i=k+1}^{r}\frac{\partial P}{\partial t_i}\frac{\partial}{\partial t_i} - \sum_{i=1}^{r}\frac{\partial^2}{\partial t_i^2},$$

where P is a function of $s, t_{k+1}, \cdots, t_r$ defined by $P = \langle x_s, x_s\rangle$. Then, quite similarly to the case of [7], we have the following

Lemma 3.3. *Let $Q(t)$ be a polynomial in $t = (t_1, t_2, \cdots, t_r)$ with functions of s as the coefficients and deg $Q(t) = d$. Then, we have*

$$\Delta\left(\frac{Q(t)}{P^m}\right) = \frac{\tilde{Q}(t)}{P^{m+3}},$$

where $\tilde{Q}(t)$ is a polynomial in $t = (t_1, t_2, \cdots, t_r)$ with functions in s as the coefficients and deg $\tilde{Q}(t) \le d + 4$, where deg $\tilde{Q}(t)$ denotes the degree of $\tilde{Q}(t)$ in $t = (t_1, t_2, \cdots, t_r)$.

From now on, for some polynomial $F(t)$ in $t = (t_1, t_2, \cdots, t_r)$, deg $F(t)$ denotes the degree of $F(t)$ in $t = (t_1, t_2, \cdots, t_r)$ unless otherwise stated.

Suppose the ruled submanifold M defined by (3.1) is of finite type. In this case, we easily see that deg $P(t) = 2$. Let x_A be the A^{th}-component function of the immersion x which is identified with the position vector of the point of M in $\mathbb{L}^m$. Since x_A is a linear function in $t = (t_1, t_2, \cdots, t_r)$, we have

$$\Delta x_A = \frac{Q_A(t)}{P^2}$$

for some polynomial $Q_A(t)$ in t with deg $Q_A(t) \leq 3$. By using Lemma 3.3, we get

$$\Delta^j x_A = \frac{Q_{A_j}(t)}{P^{3j-1}}, \quad j = 1, 2, \cdots,$$

where $Q_{A_j}(t)$ is a polynomial in t with functions in s as coefficients with deg $Q_{A_j}(t) \leq 4j - 1$. If j goes up by one, the degree of numerator of $\Delta^j x_A$ goes up by at most 4 while that of the denominator goes up by 6. Thus, for some positive integer i, $\Delta^{i+1}x + \lambda_1 \Delta^i x + \cdots + \lambda_i \Delta x = 0$ never occurs unless $\Delta x = 0$, that is, M is minimal.

Next, we consider Case 2. We may write the parametrization of M as

$$x(s, t_1, \cdots, t_r) = \alpha(s) + \sum_{i \neq j_1, j_2, \cdots, j_k} t_i \mathbf{e}_i(s) + \sum_{i=1}^{k} t_{j_i} \mathbf{e}_{j_i}(s).$$

Then, its Laplacian operator is given by

$$\Delta = \frac{1}{2P^2} \frac{\partial P}{\partial s} \frac{\partial}{\partial s} - \frac{1}{P} \frac{\partial^2}{\partial s^2} - \frac{1}{2P} \sum_{j=1}^{r} \frac{\partial P}{\partial t_j} \frac{\partial}{\partial t_j} - \sum_{j=1}^{r} \frac{\partial^2}{\partial t_j^2},$$

where P is a function of s and $t = (t_1, \cdots, t_r)$ defined by

$$P = P(s,t) = \langle x_s, x_s \rangle = \epsilon + 2 \sum_{i=1}^{r} \langle \alpha'(s), \mathbf{e}'_i(s) \rangle t_i + \sum_{i,j=1}^{r} \langle \mathbf{e}'_i(s), \mathbf{e}'_j(s) \rangle t_i t_j.$$

Subcase 1. Let deg $P(t) = 0$. In this case, $\mathbf{e}_i(s)$'s are all constant vector fields for $i \neq j_1, j_2, \cdots, j_k$, $\mathbf{e}'_{j_i}(s) \wedge \mathbf{e}'_{j_l}(s) = 0$ for $i, l = 1, 2, \cdots, k$ and $\langle \alpha'(s), \mathbf{e}'_j(s) \rangle = 0$ for $j = j_1, j_2, \cdots, j_k$ and the function P is a constant function with value 1. Thus, we have

$$\Delta^l x = -\left(\alpha^{(2l)}(s) + \sum_{i=1}^{k} t_{j_i} \mathbf{e}_{j_i}^{(2l)}(s) \right)$$

for $l = 1, 2, 3, \cdots$. Hence, M is of finite rank $(k+1)$ if M is of finite type. In particular, this case could be obtained only if the base curve α is space-like.

Subcase 2. Let deg $P(t) = 1$. In this case, $\langle \alpha'(s), \mathbf{e}'_i(s) \rangle \neq 0$ for some i $(1 \leq i \leq r)$, $\mathbf{e}'_{j_i}(s) \wedge \mathbf{e}'_{j_l}(s) = 0$ for $i, l = 1, 2, \cdots, k$ and $\mathbf{e}_i(s)$'s are constant vectors for $i \neq j_1, j_2, \cdots, j_k$. Then, the Laplacian Δx_A of the A^{th}-component function of x as

$$\Delta x_A = \frac{Q_A(t)}{P^2}$$

for some polynomial $Q_A(t)$ in $t = (t_1, t_2, \cdots, t_r)$ with deg $Q_A(t) \le 2$. As Case 1, we have

$$\Delta^j x_A = \frac{Q_{Aj}(t)}{P^{3j-1}}, \quad j = 1, 2, 3, \cdots$$

for a polynomial $Q_{Aj}(t)$ in $t = (t_1, t_2, \cdots, t_r)$ with deg $Q_{Aj}(t) \le 2j$. If j goes up by one, the degree of numerator of $\Delta^j x_A$ goes up by at most 2 while that of the denominator goes up by 3. Thus, for some positive integer i, $\Delta^{i+1}x + \lambda_1\Delta^i x + \cdots + \lambda_i\Delta x = 0$ never occurs unless $\Delta x = 0$, that is, M is minimal.

Subcase 3. Let deg $P(t) = 2$. In this case, we get the Laplacian Δx_A of the A^{th}-component function of x as

$$\Delta x_A = \frac{R_A(t)}{P^2}$$

for a polynomial $R_A(t)$ in $t = (t_1, t_2, \cdots, t_r)$ with deg $R_A(t) \le 3$. As Case 1, we get

$$\Delta^j x_A = \frac{R_{Aj}(t)}{P^{3j-1}}, \quad j = 1, 2, 3, \cdots$$

for some polynomial $R_{Aj}(t)$ in $t = (t_1, t_2, \cdots, t_r)$ with deg $R_{Aj}(t) \le 4j - 1$. Using the similar argument as Case 1, $\Delta x = 0$ provided M is of finite type.

Hence, if M is of finite type, M is either minimal or of finite rank k for some k $(2 \le k \le r + 1)$.

Combining the results of Cases 1 and 2, we have

Theorem 3.4. *Let M be an $(r+1)$-dimensional non-cylindrical ruled submanifold with non-degenerate rulings of finite type in Lorentzian m-space $\mathbb{L}^m$. Then, it is minimal or of finite rank k for some k $(2 \le k \le r + 1)$.*

4. Ruled Submanifolds with Degenerate Rulings

Let M be an $(r+1)$-dimensional ruled submanifold in $\mathbb{L}^m$ with degenerate rulings $E(s, r)$ along a regular curve and let its parametrization be given by $\tilde{x}(s, t)$ where $t = (t_1, t_2, \cdots, t_r)$. Since $E(s, r)$ is degenerate, it can be spanned by a degenerate frame $\{B(s) = \mathbf{e}_1(s), \mathbf{e}_2(s), \cdots, \mathbf{e}_r(s)\}$ such that

$$\langle B(s), B(s)\rangle = \langle B(s), \mathbf{e}_i(s)\rangle = 0, \langle \mathbf{e}_i(s), \mathbf{e}_j(s)\rangle = \delta_{ij}, \ i, j = 2, 3, \cdots, r.$$

Without loss of generality as before, we may assume that

$$\langle \mathbf{e}_i'(s), \mathbf{e}_j(s)\rangle = 0, \ i, j = 2, 3, \cdots, r.$$

Since the tangent space of M at $\tilde{x}(s, t)$ is a Lorentzian $(r + 1)$-space which contains the degenerate ruling $E(s, r)$, there exists a tangent vector field A to M which satisfies

$$\langle A(s, t), A(s, t)\rangle = 0, \ \langle A(s, t), B(s)\rangle = -1, \ \langle A(s, t), \mathbf{e}_i(s)\rangle = 0, \ i = 2, 3, \cdots, r$$

at $\tilde{x}(s,t)$.

Let $\alpha(s)$ be an integral curve of the vector field A on M. Then we can define another parametrization x of M as follows:

$$x(s, t_1, \cdots, t_r) = \alpha(s) + \sum_{i=1}^{r} t_i \mathbf{e}_i(s), \tag{4.1}$$

where $\alpha'(s) = A(s)$.

Lemma 4.1. *We may assume that* $\langle A(s), B'(s)\rangle = 0$ *for all* s.

Proof. Let $\bar{B}(s) = f(s)B(s), \bar{A}(s) = \frac{1}{f(s)}A(s)$. We consider an integral curve $\bar{\alpha}(s)$ of $\bar{A}(s)$ through $\alpha(0)$, which is a reparametrization of $\alpha(s)$. Note that

$$\langle \bar{A}(s), \bar{B}'(s)\rangle = \frac{-f'(s)}{f(s)} + \langle A(s), B'(s)\rangle.$$

Hence, if we take $f(s)$ as $e^{\int \langle A(s), B'(s)\rangle ds}$, $\{\bar{A}, \bar{B}\}$ satisfies the desired property. ∎

We will denote $\bar{\alpha}(s), \bar{A}(s)$, and $\bar{B}(s)$ as $\alpha(s), A(s)$, and $B(s)$, respectively.

Let $P = \langle x_s, x_s\rangle$ and $Q = -\langle x_s, x_{t_1}\rangle$. Then, we have

$$P(s,t) = 2\sum_{i=1}^{r} U_i(s)t_i + \sum_{i,j=1}^{r} V_{ij}(s)t_i t_j,$$
$$Q(s,t) = 1 + \sum_{i=2}^{r} w_i(s)t_i,$$

where

$$w_i(s) = \langle B'(s), \mathbf{e}_i(s)\rangle, U_i(s) = \langle A(s), \mathbf{e}_i'(s)\rangle, V_{ij}(s) = \langle \mathbf{e}_i'(s), \mathbf{e}_j'(s)\rangle,\ i, j = 1, 2, \cdots, r.$$

It follows from Lemma 4.1 that $U_1(s) = 0$. Note that P and Q are polynomials in $t = (t_1, \cdots, t_r)$ with functions in s as coefficients. Then the Laplacian Δ of M can be expressed as follows:

$$\begin{aligned}\Delta = \frac{1}{Q^2}\{&\frac{\partial \bar{P}}{\partial t_1}\frac{\partial}{\partial t_1} - 2Q\sum_{i=2}^{r} w_i \frac{\partial}{\partial t_i} + 2Q\frac{\partial^2}{\partial s \partial t_1} + \bar{P}\frac{\partial^2}{\partial t_1^2}\\ &- t_1 Q\sum_{i=2}^{r}\frac{\partial^2}{\partial t_1 \partial t_i} - Q\sum_{i=2}^{r}(t_1 w_i + Q)\frac{\partial^2}{\partial t_i^2}\},\end{aligned} \tag{4.2}$$

where $\bar{P} = P + t_1^2\sum_{i=2}^{r} w_i^2$. Hence, by a straightforward computation, we obtain

$$\Delta x = \frac{S_1(t)}{Q^2(t)}, \tag{4.3}$$

where $S_1(t)$ is a polynomial in $t = (t_1, t_2, \cdots, t_r)$ with functions in s as the coefficients defined by

$$S_1(t) = \frac{\partial \bar{P}}{\partial t_1}\mathbf{e}_1(s) - 2Q\sum_{i=2}^{r} w_i\mathbf{e}_i(s) + 2Q\mathbf{e}_1'(s). \tag{4.4}$$

Quite similarly to the case of Lemma 3.3, we have the following:

Lemma 4.2. *Let S be a polynomial in $t = (t_1, t_2, \cdots, t_r)$ with functions of s as the coefficients and deg $S(t) = d$. Then, we have*

$$\Delta(\frac{S(t)}{Q^m}) = \frac{\tilde{S}(t)}{Q^{m+3}},$$

where $\tilde{S}$ is a polynomial in $t = (t_1, t_2, \cdots, t_r)$ with functions in s as the coefficients and deg $\tilde{S}(t) \leq d + 1$.

Suppose the ruled submanifold M defined by (4.1) is of finite type. Then, for some positive integer l, we have

$$\Delta^{l+1}x + \lambda_1\Delta^l x + \cdots + \lambda_l\Delta x = 0. \tag{4.5}$$

According to the degree of Q, we divide it into two cases.

Case 1. Suppose that deg $Q(t) = 1$. Since $x(s, t)$ is a linear function in $t = (t_1, t_2, \cdots, t_r)$, Lemma 4.2 implies for $k = 1, 2, 3, \cdots$

$$\Delta^k x = \frac{S_k(t)}{Q^{3k-1}},$$

where $S_k(t)$ is a polynomial in t with functions in s as the coefficients with deg $S_k(t) \leq k$. Therefore, if k goes up by one, the degree of numerator of $\Delta^k x$ goes up by at most 1 while that of the denominator goes up by 3. Thus, for some positive integer l, $\Delta^{l+1}x + \lambda_1\Delta^l x + \cdots + \lambda_l\Delta x = 0$ occurs only when $\Delta x = 0$, that is, M is minimal.

Case 2. Suppose that deg $Q(t) = 0$. Then we have $Q = 1$ and $w_i = 0$ for $i = 2, 3, \cdots, r$. Hence, it follows from (4.2), (4.3) and (4.4) that for $k = 1, 2, \cdots$

$$\Delta^k x = 2^k\{(\sum_{j=1}^{r} t_j V_{1j})V_{11}^{k-1}\mathbf{e}_1(s) + (V_{11}^{k-1}\mathbf{e}_1)'(s)\}. \tag{4.6}$$

Since the coefficient of t_1 of the polynomial in the left hand side of (4.5) vanishes, we have

$$V_{11}\{(2V_{11})^l + \lambda_1(2V_{11})^{l-1} + \cdots + \lambda_l(2V_{11}) + \lambda_l\} = 0,$$

which shows that V_{11} is a constant since it is a root of an algebraic equation.

Suppose that $V_{11} = 0$. From the causal character of the vector field $\mathbf{e}_1(s)$, $\mathbf{e}_1'(s)$ must be space-like. Thus, we have $\mathbf{e}_1'(s) = 0$, and hence $V_{1j} = 0$ for all $j = 1, 2, \cdots, r$. Therefore (4.6) with $k = 1$ implies that M is minimal.

Now, suppose that $V_{11} = a^2$ $(a > 0)$. Then $B'(s)$ is orthogonal to $\{A(s), B(s), \mathbf{e}_2(s), \cdots, \mathbf{e}_r(s)\}$, which has constant length a. Since the mean curvature vector field H is derived from (4.6) with $k = 1$ as

$$H = \frac{-2}{r+1}\{a^2 t_1 \mathbf{e}_1(s) + \left(\sum_{j=2}^{r} V_{1j} t_j\right) \mathbf{e}_1(s) + \mathbf{e}_1'(s)\},$$

we get

$$\Delta H = 2a^2 H. \tag{4.7}$$

Using the results of ([5], p. 57), we have the following:

Theorem 4.3. *Let M be a ruled submanifold in $\mathbb{L}^m$ with degenerate rulings. Then, M is of finite type if and only if it is of either 1-type or null 2-type.*

Now we establish an existence theorem for ruled submanifolds with degenerate rulings in $\mathbb{L}^m$ satisfying (4.7). For a null curve $\tilde{\alpha}(s)$ in $\mathbb{L}^m$, we consider a null frame $\{A(s), B(s) = \mathbf{e}_1(s), \mathbf{e}_2(s), \cdots, \mathbf{e}_{m-1}(s)\}$ along $\tilde{\alpha}(s)$ satisfying

$$\begin{aligned} &\langle A(s), A(s)\rangle = \langle B(s), B(s)\rangle = \langle A(s), \mathbf{e}_i(s)\rangle = \langle B(s), \mathbf{e}_i(s)\rangle = 0, \\ &\langle A(s), B(s)\rangle = -1, \langle \mathbf{e}_i(s), \mathbf{e}_j(s)\rangle = \delta_{ij}, \tilde{\alpha}'(s) = A(s) \end{aligned}$$

for $i, j = 2, 3, \cdots, m-1$. Let $X(s)$ be the matrix $(A(s)\ B(s)\ \mathbf{e}_2(s)\ \ldots\ \mathbf{e}_{m-1}(s))$ consisting of column vectors of $A(s), B(s), \mathbf{e}_2(s), \cdots, \mathbf{e}_{m-1}(s)$ with respect to the standard coordinate system in $\mathbb{L}^m$. We then have

$$X^t(s) E X(s) = T,$$

where $X^t(s)$ denotes the transpose of $X(s)$, $E = \mathrm{diag}(-1, 1, \cdots, 1, 1)$ and

$$T = \begin{pmatrix} 0 & -1 & 0 & & \\ -1 & 0 & 0 & \mathbf{O} & \\ 0 & 0 & 1 & & \\ & \mathbf{O} & & \ddots & \\ & & & & 1 \end{pmatrix}.$$

Here, we assume that

$$\mathbf{e}_{r+1}(s) = \frac{1}{a} B'(s), \langle \mathbf{e}_i'(s), \mathbf{e}_j(s)\rangle = \langle \mathbf{e}_\alpha'(s), \mathbf{e}_\beta(s)\rangle = 0 \tag{4.8}$$

for $2 \leq i, j \leq r$ and $r+2 \leq \alpha, \beta \leq m-1$, where a is a non-zero constant.

Then, $X(s)$ satisfies the matrix differential equation

$$X'(s) = X(s)M(s), \tag{4.9}$$

where

$$M(s)=\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & a & 0 & \cdots & 0 \\ 0 & 0 & -U_2 & \cdots & -U_r & -U_{r+1} & -U_{r+2} & \cdots & -U_{m-1} \\ -U_2 & 0 & 0 & \cdots & 0 & V_2 & z_{r+2,2} & \cdots & z_{m-1,2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -U_r & 0 & 0 & \cdots & 0 & V_r & z_{r+2,r} & \cdots & z_{m-1,r} \\ -U_{r+1} & a & -V_2 & \cdots & -V_r & 0 & -V_{r+2} & \cdots & -V_{m-1} \\ -U_{r+2} & 0 & -z_{r+2,2} & \cdots & -z_{r+2,r} & V_{r+2} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -U_{m-1} & 0 & -z_{m-1,2} & \cdots & -z_{m-1,r} & V_{m-1} & 0 & \cdots & 0 \end{pmatrix}, \tag{4.10}$$

$V_i (2 \leq i \leq r), U_j (2 \leq j \leq m-1)$ and $z_{b,j} (r+2 \leq b \leq m-1, 2 \leq j \leq r)$ are some smooth functions of s.

For a given initial condition $X(0) = (A(0)\ B(0)\ \mathbf{e}_2(0)\ \cdots\ \mathbf{e}_{m-1}(0))$ satisfying $X^t(0)EX(0) = T$, there is a unique solution to equation (4.9) on the whole domain I of $\tilde{\alpha}(s)$ containing 0. Since T is symmetric and MT is skew-symmetric, $\frac{d}{ds}(X^t(s)EX(s)) = 0$ and hence we have

$$X^t(s)EX(s) = T$$

for all $s \in I$. Therefore, $A(s), B(s), \mathbf{e}_2(s), \cdots, \mathbf{e}_{m-1}(s)$ form a null frame along a null curve $\tilde{\alpha}(s)$ in $\mathbb{L}^m$ on I. Let $\alpha(s) = \int_0^s A(u)du$.

Thus, we can define a parametrization for a ruled submanifold M by

$$x(s, t_1, t_2, \cdots, t_r) = \alpha(s) + t_1 B(s) + \sum_{i=2}^{r} t_i \mathbf{e}_i(s). \tag{4.11}$$

Then, the proof of Theorem 4.3 shows that M is an $(r+1)$-dimensional ruled submanifold in $\mathbb{L}^m$ with degenerate rulings which satisfies (4.7). Thus, it is of either 1-type or null 2-type.

Definition 4.4. A ruled submanifold in Lorentzian m-space $\mathbb{L}^m$ with degenerate rulings satisfying (4.8) and (4.9) is called a *ruled submanifold of the B-scroll kind* or simply a *BS-kind ruled submanifold*.

Remark. If $r = 1$ and $m = 3$, then a BS-kind ruled surface is just an ordinary B-scroll ([8]). If $m \geq 4$, either an extended B-scroll or a generalized B-scroll in $\mathbb{L}^m$

is a BS-kind ruled surface ([9]). Also, a time-like ruled surface over a null curve with degenerate rulings in $\mathbb{L}^m$ is called a null scroll ([9, 10]). Thus, we have

Theorem 4.5. *Let M be a ruled submanifold in Lorentzian m- space $\mathbb{L}^m$ with degenerate rulings. Then, M is of finite type if and only if it is a BS-kind ruled submanifold defined by* (4.11)*.*

Corollary 4.6. ([10]). *Let M be a null scroll of finite type in Lorentzian m-space $\mathbb{L}^m$. Then, M is minimal or open part of a generalized B-scroll.*

5. BS-kind Ruled Submanifolds and the Restricted Minimal Polynomials

In this section, we characterize a non-minimal BS-kind ruled submanifold in terms of some minimal polynomial of the restricted shape operator $\tilde{A}_H$ associated with the mean curvature vector field H.

Let M be a ruled submanifold with degenerate rulings parametrized by

$$x(s, t_1, t_2, \cdots, t_r) = \alpha(s) + t_1 B(s) + \sum_{i=2}^{r} t_i \mathbf{e}_i(s) \tag{5.1}$$

over a null curve $\alpha = \alpha(s)$ with $A(s) = \alpha'(s)$, $\mathbf{e}_1(s) = B(s)$, $\langle B(s), B(s)\rangle = 0$, $\langle A(s), B(s)\rangle = -1$, $\langle A(s), B'(s)\rangle = \langle A'(s), B(s)\rangle = 0$ and some space-like orthonormal vector fields $\mathbf{e}_2(s), \cdots, \mathbf{e}_r(s)$ along α satisfying

$$\langle A(s), \mathbf{e}_i(s)\rangle = \langle B(s), \mathbf{e}_i(s)\rangle = \langle B'(s), \mathbf{e}_i(s)\rangle = 0 \tag{5.2}$$

for all s and $i = 2, 3, \cdots, r$.

Such a ruled submanifold is called a *ruled submanifold of the null scroll kind* or simply an *NS-kind ruled submanifold.*

Then, we have similar equations as (4.8)-(4.10) with a function $a = a(s)$. By a straightforward computation the Laplacian Δ of M is given by

$$\Delta = 2\frac{\partial^2}{\partial s \partial t_1} + \frac{\partial P}{\partial t_1}\frac{\partial}{\partial t_1} + P\frac{\partial^2}{\partial t_1^2} - \sum_{i=2}^{r} \frac{\partial^2}{\partial t_i^2},$$

where we have put

$$P = \langle x_s, x_s\rangle = 2\sum_{i=2}^{r} t_i U_i(s) + (2at_1 - \sum_{i=2}^{r} t_i V_i(s))^2 + \sum_{i,j=2}^{r}\sum_{b=r+2}^{m-1} t_i t_j z_{b,i} z_{b,j}.$$

Thus, the Laplacian of the immersion x is obtained as

$$\Delta x = 2B'(s) + \frac{\partial P}{\partial t_1} B(s),$$

or equivalently, the mean curvature vector field H is given by

$$H(s,t_1,t_2,\cdots,t_r) = -\frac{2a(s)}{r+1}\{(a(s)t_1 - \sum_{i=2}^{r} t_i V_i(s))B(s) + \mathbf{e}_{r+1}(s)\},$$

where $\mathbf{e}_{r+1}(s) = \frac{B'(s)}{||B'(s)||}$ and $V_i(s) = \langle \mathbf{e}_i'(s), \mathbf{e}_{r+1}(s)\rangle$ for $i = 2, 3, \cdots, r$. Then, the shape operator A_H has the form

$$\begin{pmatrix} \frac{2a^2}{r+1} & 0 & 0 & \cdots & 0 \\ \varphi & \frac{2a^2}{r+1} & -\frac{2a}{r+1}V_2 & \cdots & -\frac{2a}{r+1}V_r \\ \frac{2a}{r+1}V_2 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \\ \frac{2a}{r+1}V_r & 0 & 0 & \cdots & 0 \end{pmatrix}, \tag{5.3}$$

where

$$\varphi = \varphi(s,t_1,t_2,\cdots,t_r) = \frac{2a(s)}{r+1}\{a(s)\sum_{i=2}^{r} t_i U_i(s) + \sum_{i=2}^{r} t_i V_i'(s) - U_{r+1}(s)\}.$$

Let us identify the matrix

$$\begin{pmatrix} \frac{2a^2}{r+1} & 0 & 0 & \cdots & 0 \\ \varphi & \frac{2a^2}{r+1} & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

with

$$\begin{pmatrix} \frac{2a^2}{r+1} & 0 \\ \varphi & \frac{2a^2}{r+1} \end{pmatrix},$$

which is denoted by $\tilde{A}_H$. We call the operator $\tilde{A}_H$ the *restricted shape operator* associated with the mean curvature vector field H. Then, $\tilde{A}_H$ has the minimal polynomial $(x - \frac{2a^2}{r+1})^2$ everywhere if $a(s)U_{r+1}(s) \neq 0$, which is derived from the constant term of the linear function φ with respect to $t_1, t_2, \cdots, t_r$. Such a minimal polynomial is called the *restricted minimal polynomial* of $\tilde{A}_H$.

Suppose the restricted shape operator $\tilde{A}_H$ associated with the mean curvature vector field H of an NS-kind ruled submanifold has the restricted minimal polynomial $(x-\lambda)^2$ for some non-zero real number λ. Then, by (5.3), a is a non-zero constant. Therefore, it is a non-minimal BS-kind ruled submanifold. Thus, we have

Theorem 5.1. *Let M be an NS-kind ruled submanifold in Lorentzian m-space $\mathbb{L}^m$. Then, M is a BS-kind ruled submanifold if the restricted shape operator $\tilde{A}_H$ associated with the mean curvature vector field H has the restricted minimal polynomial of the form $(x-\lambda)^2$ for some non-zero constant λ.*

Corollary 5.2. ([9]). *Let M be a null scroll in Lorentzian m-space $\mathbb{L}^m$. Then, M is a generalized B-scroll if the shape operator A_H associated with the mean curvature vector field H has the minimal polynomial of the form $(x-\lambda)^2$ for some non-zero constant λ.*

ACKNOWLEDGMENT

The authors would like to express their sincere thanks to the referee for his valuable suggestions that would improve this paper.

REFERENCES

1. C. Baikoussis and D. E. Blair, On the Gauss map of ruled surfaces, *Glasgow Math. J.*, **34** (1992), 355-359.
2. C. Baikoussis, B.-Y. Chen and L. Verstraelen, Ruled surfaces and tubes with finite type Gauss map, *Tokyo J. Math.*, **16** (1993), 341-348.
3. J. M. Barbosa, M. Dajczer and I. P. Jorge, Minimal ruled submanifolds in spaces of constant curvature, *Indiana Univ. Math. J.*, **33** (1984), 531-547.
4. B.-Y. Chen, *Total mean curvature and submanifolds of finite type*, World Scientific Publ., New Jersey, 1984.
5. B.-Y. Chen, A report on submanifolds of finite type, *Soochow J. Math.*, **22** (1996), 117-337.
6. B.-Y. Chen and P. Piccinni, Submanifolds with finite type Gauss map, *Bull. Austral. Math. Soc.*, **3** (1987), 161-186.
7. F. Dillen, Rueld submanifolds of finite type, *Proc. Amer. Math. Soc.*, **114** (1992), 795-798.
8. L. K. Graves, Codimension one isometric immersions between Lorentz spaces, *Trans. Amer. Math. Soc.*, **252** (1979), 367-392.
9. D.-S. Kim, Y. H. Kim and D. W. Yoon, Characterization of generalized B-scrolls and cylinders over finite type curves, *Ind. J. pure appl. Math.*, **34** (2003), 1521-1532.
10. D.-S. Kim, Y. H. Kim and D. W. Yoon, Finite type ruled surfaces in Lorentz-Minkowski space, *Taiwan. J. Math.*, **11** (2007), 1-13.
11. Y. H. Kim and D. W. Yoon, On the Gauss map of ruled surfaces in Minkowski-space, *Rocky Mount. J. Math.*, **35** (2005), 1555-1581.

Dong-Soo Kim
Department of Mathematics
Chonnam National University
Kwangju 500-757
Korea
E-mail: dosokim@chonnam.chonnam.ac.kr

Young Ho Kim
Department of Mathematics
Kyungpook National University
Taegu 702-701
Korea
E-mail: yhkim@knu.ac.kr