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ON POSITIVE DEFINITE SOLUTIONS OF THE MATRIX EQUATION $X + A^*X^{-q}A = Q(0 < q \leq 1)$

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Abstract. Consider the nonlinear matrix equation $X + A^*X^{-q}A = Q$ where $0 < q \leq 1$. A new sufficient condition for this equation to have positive definite solution is provided and two iterative methods for the maximal positive definite solution are proposed. Applying the theory of condition number developed by Rice, an explicit expression of the condition number of the maximal positive definite solution is obtained. The theoretical results are illustrated by numerical examples.

1. INTRODUCTION

In this paper, we investigate the nonlinear matrix equation

$$X+A^*X^{-q}A = Q, \tag{1.1}$$

where $0 < q \leq 1$, A is nonsingular and Q is positive definite. Eq.(1.1) can be viewed as a natural extension of the scalar equation $x + a^2/x^q = 1$. Following [1], this matrix equation can be appeared during solving systems of linear equations of the form:

$$Mx = f, \tag{1.2}$$

where the positive definite matrix M arises from a finite difference approximation to an elliptic differential equation. As an example, let $M = \begin{pmatrix} Q & A \\ A^* & Q \end{pmatrix}$ in which we can rewrite M as $M = \tilde{M} + \mathrm{diag}[Q - X^q, 0]$ where $\tilde{M} = \begin{pmatrix} X^q & A \\ A^* & Q \end{pmatrix}$. We can decompose $\tilde{M}$, for solving Eq.(1.2), to the LU decomposition

$$\tilde{M} = \begin{pmatrix} X^q & A \\ A^* & Q \end{pmatrix} = \begin{pmatrix} I & 0 \\ A^*X^{-q} & I \end{pmatrix} \begin{pmatrix} X^q & A \\ 0 & X \end{pmatrix}.$$

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For such a decomposition to exist, the matrix X must be a solution of the matrix equation $X + A^*X^{-q}A = Q$. The solution of the system $\tilde{M}y = f$ is transformed to the the solution of two linear systems that have lower triangular block coefficient matrix and upper triangular block coefficient matrix, respectively. The Woodbury formula can be applied to compute the solution of Eq.(1.2). Moreover, this type of nonlinear matrix equations have many applications in control theory, dynamic programming, statistics, stochastic filtering, nano research, see for example [1-5] and the references therein.

The case that $q = 1$ has been studied extensively by many authors [1-7]. Also there are many contributions in the literature to the theory, applications and numerical solutions of Eq.(1.1) in the case that q is a positive integer, see for instance [8,9] and the references therein. The first attempt to solve Eq.(1.1) for $q \neq 1$ was made for $q = \frac{1}{2}$ and $Q = I$ by El-Sayed [10]. In [11], M.A.Ramadan investigated the equation $X + A^*X^{-\frac{1}{2^m}}A = I$. Then V.I.Hasanov investigated Eq.(1.1) for $q = 1/n, n \in N$ [12]. Eq.(1.1) with arbitrary $0 < q \leq 1$ and $q \geq 1$ were considered in [13,14] and [15], respectively, where several necessary conditions and sufficient conditions for the existence of the positive definite solutions were derived. When $Q = I$, [16] treat iterative methods for $X + A^*X^{-q}A = I$. Zhenyun Peng and his coauthors [17,18] discussed Eq.(1.1) with two cases: $q \geq 1$ and $0 < q < 1$. They established iterative methods to obtain the positive definite solutions and considered the convergence rates of the considered methods.

Based on these, in this paper, we continue to study the matrix equation $X + A^*X^{-q}A = Q$ with $0 < q \leq 1$ and Q positive definite. We derive a new sufficient condition for Eq.(1.1) to have positive definite solutions and also we propose two iterative methods for obtaining the maximal positive definite solution. Moreover, by the theory of condition number developed by Rice, we give an explicit expression of the condition number of the maximal positive definite solution to Eq.(1.1).

The following notations are used throughout the rest of the paper. Denote the set of $n \times n$ Hermitian matrices by $H^{n\times n}$. The notation $A \geq 0 (A > 0)$ means that A is Hermitian positive semidefinite (positive definite). For $A, B \in H^{n\times n}$, we write $A \geq B (A > B)$ if $A - B \geq 0 (> 0)$. We denote by $\sigma_1(A)$ and $\sigma_{\mathrm{m}}(A)$ the maximal and minimal singular values of A, respectively. Similarly, $\lambda_1(A)$ and $\lambda_{\mathrm{m}}(A)$ means the maximal and the minimal eigenvalues of A, respectively. For $n \times n$ square matrix $A = (a_1, ..., a_n) = (a_{ij})$ and a matrix B, $A \otimes B = (a_{ij}B)$ is the Kronecker product, and $\mathrm{vec}(A) = (a_1^T, ..., a_n^T)^T$. Unless otherwise noted, the symbol $\|\cdot\|$ denotes the matrix spectral norm (i.e., $\|A\| = \sqrt{\rho(AA^*)} = \sigma_1(A)$), and $\|\cdot\|_F$ the Frobenius norm.

The paper is organized as follows. In Section 2, we give a new sufficient condition for Eq.(1.1) to have positive definite solutions and we offer two iterative methods for the maximal positive definite solution. An explicit expression of the condition number for the maximal solution is obtained in Section 3. Finally, several numerical examples are given in Section 4.

2. Positive Definite Solutions

In this section, we provide a sufficient condition for Eq.(1.1) to have positive definite solutions and also we propose two iterative methods for obtaining the maximal positive definite solution.

To prove our main results, we need the following lemmas:

Lemma 2.1. ([19]). *Let A, B be positive definite. Then for any unitary invariant norm $|||\cdot|||$, we have*

$$|||B^tA^tB^t||| \leq |||(BAB)^t|||, \quad \text{if } 0\leq t\leq 1;$$

$$|||(BAB)^t||| \leq |||B^tA^tB^t|||, \quad \text{if } t\geq 1.$$

Lemma 2.2. ([20]). *If $A>B>0(\text{or } A\geq B>0)$, then $A^q>B^q>0(\text{or } A^q\geq B^q>0)$ for all $q\in(0,1]$, and $0<A^q<B^q(\text{or } 0<A^q\leq B^q)$ for all $q\in[-1,0)$.*

Lemma 2.3. ([20]). *For any Hermite matrices $X, Y\geq bI>0$, $q>0$, we have*

$$\|X^{-q}-Y^{-q}\|\leq qb^{-(q+1)}\|X-Y\|.$$

Lemma 2.4. ([3]). *If Eq. (1.1) has a positive definite solution X, then*

$$\sigma_{\mathrm{m}}^2(Q^{-q/2}AQ^{-1/2})\leq \frac{q^q}{(q+1)^{q+1}} \quad \text{and} \quad X\leq \alpha Q,$$

where α is a solution of the scalar equation $x^q(1-x)=\sigma_{\mathrm{m}}^2(Q^{-q/2}AQ^{-1/2})$ in $[\frac{q}{q+1},1]$.

Lemma 2.5. ([3]). *If C and P are Hermitian matrices of the same order with $P>0$, then $CPC+P^{-1}\geq 2C$.*

Now consider the following scalar equations:

$$x^q(1-x)=\sigma_{\mathrm{m}}^2(Q^{-q/2}AQ^{-1/2}), \tag{2.1}$$

$$x^q(1-x)=\sigma_1^2(Q^{-q/2}AQ^{-1/2}). \tag{2.2}$$

Let

$$f(x)=x^q(1-x),\ x\in[0,1].$$

Obviously, $f(x)$ is monotonically increasing on $[0,\frac{q}{q+1}]$, monotonically decreasing on $[\frac{q}{q+1},1]$, and

$$\max_{x\in[0,1]} f(x)=f(\frac{q}{q+1})=\frac{q^q}{(q+1)^{q+1}}.$$

Thus, if

$$\sigma_1^2(Q^{-q/2}AQ^{-1/2}) < \frac{q^q}{(q+1)^{q+1}}, \tag{2.3}$$

then the scalar equation (2.1) has two positive solutions α_1, α_2 $(\alpha_1 < \frac{q}{q+1} < \alpha_2)$ and equation (2.2) has two positive solutions β_1, β_2 $(\beta_1 < \frac{q}{q+1} < \beta_2)$. It is not difficult to verify that

$$0 < \alpha_1 \leq \beta_1 < \frac{q}{q+1} < \beta_2 \leq \alpha_2 < 1. \tag{2.4}$$

Denote the following matrix sets:

$$\begin{aligned}
\varphi_1 &= \{X > 0 | X < \alpha_1 Q\}, \\
\varphi_2 &= \{X > 0 | \alpha_1 Q \leq X \leq \beta_1 Q\}, \\
\varphi_3 &= \{X > 0 | \beta_1 Q < X < \beta_2 Q\}, \\
\varphi_4 &= \{X > 0 | \beta_2 Q \leq X \leq \alpha_2 Q\}, \\
\varphi_5 &= \{X > 0 | \alpha_2 Q < X < Q\}.
\end{aligned}$$

Then we have the following result.

Theorem 2.1. *Suppose that* $\sigma_1^2(Q^{-q/2}AQ^{-1/2}) < \frac{q^q}{(q+1)^{q+1}}$.

(1) *Matrix Eq.(1.1) has no positive definite solution in* $\varphi_1, \varphi_3, \varphi_5$*;*

(2) *Matrix Eq.(1.1) has a positive definite solution in* φ_4*;*

(3) *If* $\|A\|^2\|Q^{-1}\|^{q+1} < \frac{q^q}{(q+1)^{q+1}}$*, then the positive definite solution* X_L *in* φ_4 *is unique, and* X_L *can be obtained by the following iteration:*

$$\begin{cases} X_{k+1} = Q - A^* X_k^{-q} A \\ X_0 = \gamma Q, \quad \gamma \in [\dfrac{q}{q+1}, 1]. \end{cases} \tag{2.5}$$

Moreover, X_L *is the maximal positive definite solution of Eq.* (1.1).

Proof. (1) Let X be any positive definite solution of Eq.(1.1). We obtain from Lemma 2.1 that

$$\lambda_{\mathrm{m}}(Q^{\frac{-q}{2}}X^qQ^{\frac{-q}{2}}) = \frac{1}{\|Q^{\frac{q}{2}}X^{-q}Q^{\frac{q}{2}}\|} \geqslant \frac{1}{\|Q^{\frac{1}{2}}X^{-1}Q^{\frac{1}{2}}\|^q} = \lambda_{\mathrm{m}}^q(Q^{\frac{-1}{2}}XQ^{\frac{-1}{2}}).$$

According to Weyl's inequality[19], we have

$$\begin{aligned}
\lambda_{\rm m}(Q^{-1/2}XQ^{-1/2}) &= \lambda_{\rm m}(I-Q^{-1/2}A^*X^{-q}AQ^{-1/2})\\
&= 1-\lambda_1(Q^{-1/2}A^*X^{-q}AQ^{-1/2})\\
&= 1-\lambda_1(Q^{-1/2}A^*Q^{-q/2}Q^{q/2}X^{-q}Q^{q/2}Q^{-q/2}AQ^{-1/2})\\
&\geq 1-\lambda_1(Q^{-1/2}A^*Q^{-q}AQ^{-1/2})\lambda_1(Q^{q/2}X^{-q}Q^{q/2})\\
&= 1-\frac{\lambda_1(Q^{-1/2}A^*Q^{-q}AQ^{-1/2})}{\lambda_{\rm m}(Q^{-q/2}X^qQ^{-q/2})}\\
&\geq 1-\frac{\lambda_1(Q^{-1/2}A^*Q^{-q}AQ^{-1/2})}{\lambda_{\rm m}^q(Q^{-1/2}XQ^{-1/2})}.
\end{aligned}$$

Thus $\sigma_1^2(Q^{-q/2}AQ^{-1/2}) = \lambda_1(Q^{-1/2}A^*Q^{-q}AQ^{-1/2}) \geq [1-\lambda_{\rm m}(Q^{-1/2}XQ^{-1/2})]$ $\lambda_{\rm m}^q(Q^{-1/2}XQ^{-1/2})$, which gives $\lambda_{\rm m}(Q^{-1/2}XQ^{-1/2}) \leq \beta_1$ or $\lambda_{\rm m}(Q^{-1/2}XQ^{-1/2}) \geq \beta_2$. Therefore, Eq.(1.1) has no positive definite solution in φ_3.

Similarly, one can show that $\alpha_1 \leq \lambda_1(Q^{-1/2}XQ^{-1/2}) \leq \alpha_2$ which means that Eq.(1.1) has no positive definite solution in $\varphi_1,\ \varphi_5$.

(2) Consider the following mapping G :

$$G(X) = Q - A^*X^{-q}A.$$

For any $X\in\varphi_4$, according to Lemma 2.2, we have $X^{-q} \leq \frac{1}{\beta_2^q}Q^{-q}$ and then

$$\begin{aligned}
\lambda_{\rm m}(Q^{-1/2}G(X)Q^{-1/2}) &= \lambda_{\rm m}(I-Q^{-1/2}A^*X^{-q}AQ^{-1/2})\\
&\geq \lambda_{\rm m}(I-\frac{1}{\beta_2^q}Q^{-1/2}A^*Q^{-q}AQ^{-1/2})\\
&\geq \lambda_{\rm m}[I-\frac{1}{\beta_2^q}\sigma_1^2(Q^{-q/2}AQ^{-1/2})I]\\
&= \lambda_{\rm m}[I-\frac{1}{\beta_2^q}\beta_2^q(1-\beta_2)I] = \beta_2.
\end{aligned}$$

Similarly, we get $\lambda_1(Q^{-1/2}G(X)Q^{-1/2}) \leq \alpha_2$. Thus $\beta_2 I \leq Q^{-1/2}G(X)Q^{-1/2} \leq \alpha_2 I$ which gives $\beta_2 Q \leq G(X) \leq \alpha_2 Q$, namely, $G(X)\in\varphi_4$. By Schauder fixed point theorem, we know that $G(X)$ has a fixed point in φ_4. That is, Eq.(1.1) has a positive definite solution in φ_4.

(3) Denote $\Omega = \{X>0 : \frac{q}{q+1}Q \leq X \leq Q]$. Obviously, $G(X) = Q - A^*X^{-q}A \leq Q$ for any $X\in\Omega$. Combining Lemma 2.2 with the assumption $\|A\|^2\|Q^{-1}\|^{q+1} < \frac{q^q}{(q+1)^{q+1}}$, we have

$$\begin{aligned}
G(X) &= Q - A^*X^{-q}A\\
&\geq Q^{1/2}[I-\frac{(1+q)^q}{q^q}Q^{-1/2}A^*Q^{-q}AQ^{-1/2}I]Q^{1/2}\\
&\geq Q^{1/2}[I-\frac{(1+q)^q}{q^q}\sigma_1^2(Q^{-q/2}AQ^{-1/2})I]Q^{1/2}
\end{aligned}$$

$$\begin{aligned}&\geq Q^{1/2}[1-\frac{(1+q)^q}{q^q}\|A\|^2\|Q^{-1}\|^{q+1}]Q^{1/2}\\&>\frac{q}{q+1}Q,\end{aligned}$$

which gives $G(\Omega)\subseteq\Omega$. Moreover, for any $X, Y\in\Omega$,

$$X, Y\geq\frac{q}{q+1}\lambda_{\mathrm{m}}(Q)I=\frac{q}{(q+1)\|Q^{-1}\|}I.$$

According to Lemma 2.3, we have

$$\begin{aligned}\|G(X)-G(Y)\| &= \|A^*(X^{-q}-Y^{-q})A\|\leq\|A\|^2\|X^{-q}-Y^{-q}\|\\&\leq\frac{(q+1)^{q+1}}{q^q}\|Q^{-1}\|^{q+1}\|A\|^2\|X-Y\|<\|X-Y\|,\end{aligned}$$

which means that $G(X)$ is a contraction on Ω. By Banach's fixed-point theorem, $G(X)$ has a unique fixed point on Ω, i.e., matrix equation (1.1) has a unique positive definite solution $X_L\in\Omega$, and X_L can be obtained by iteration (2.5). Combining the fact that $\varphi_4\subset\Omega$, we obtain that the unique positive definite solution X_L is in φ_4.

Next, we prove that X_L is the maximal positive definite solution.

Let X be an arbitrary positive definite solution to Eq.(1.1). Obviously, $X\leq Q$. Since $G(X)$ is monotonically increasing, then

$$X=G(X)\leq G(Q),\quad X=G^k(X)\leq G^k(Q)\to X_L,\ k\to\infty.$$

Consequently, $X\leq X_L$ which means that X_L is the maximal positive definite solution of Eq.(1.1). ∎

Corollary 2.1. *If* $\|A\|^2\|Q^{-1}\|^{q+1}<\frac{q^q}{(q+1)^{q+1}}$, *then the maximal positive definite solution* X_L *exists and satisfies*

$$\|X_L^{-1}\|<(1+\frac{1}{q})\|Q^{-1}\|.$$

Moreover, for any other positive definite solution X *of matrix Eq.(1.1), we have*

$$\|X^{-1}\|>(1+\frac{1}{q})\|Q^{-1}\|.$$

Proof. Since $\|A\|^2\|Q^{-1}\|^{q+1}<\frac{q^q}{(q+1)^{q+1}}$, then the unique maximal positive definite solution X_L exists and satisfies $X_L\geq\beta_2Q>\frac{q}{q+1}Q$ according to Theorem 2.1. Thus $X_L^{-1}<(1+\frac{1}{q})Q^{-1}$ which gives $\|X_L^{-1}\|<(1+\frac{1}{q})\|Q^{-1}\|$.

Let $X\neq X_L$ be any positive definite solution to Eq.(1.1). Then $X<\frac{q}{q+1}Q$ from the proof of theorem 2.1 (3) and consequently $\|X^{-1}\|>(1+\frac{1}{q})\|Q^{-1}\|$. ∎

Definition 2.1. ([21]). Suppose that the sequence $\{X_k\}$ of matrices converges to the matrix B. We say that this sequence converges linearly to B, if there exists a number $\sigma \in (0,1)$ and $p=1$ such that

$$\overline{\lim}\frac{\|X_{k+1}-B\|}{\|X_k-B\|^p}=\sigma.$$

The number σ is called the rate of convergence.

If the above holds with $\sigma=0$ and $p=1$ or $p>1$, then the sequence is said to converge superlinearly.

In particular, if $p=2$ and $\sigma<\infty$, the sequence is said to be quadratic rate of convergence.

Corollary 2.2. *If $\|A\|^2\|Q^{-1}\|^{q+1}<\frac{q^q}{(q+1)^{q+1}}$, then the sequence $\{X_k\}$ of positive definite matrices generated by (2.5) converges to the maximal solution X_L with at least linear convergence rate.*

Proof. By (2.5), we have $X_{k+1}=Q-A^*X_k^{-q}A$ and $X_{k+1}\geq\frac{q}{q+1}Q\geq\frac{q}{q+1}\|Q^{-1}\|^{-1}I$ for each $k=1,2,\ldots$. Using Lemma 2.3, we then obtain that

$$\begin{aligned}\|X_{k+1}-X_L\| &= \|A^*(X_L^{-q}-X_k^{-q})A\|\leq q(\frac{q}{q+1}\|Q^{-1}\|^{-1})^{-(q+1)}\|A\|^2\|X_k-X_L\|\\ &= \frac{(q+1)^{q+1}}{q^q}\|A\|^2\|Q^{-1}\|^{q+1}\|X_k-X_L\|.\end{aligned}$$

Denote $0<\theta=\frac{(q+1)^{q+1}}{q^q}\|A\|^2\|Q^{-1}\|^{q+1}<1$. Then we have

$$\overline{\lim}\frac{\|X_{k+1}-X_L\|}{\|X_k-X_L\|}\leq\theta,$$

which completes the proof. ■

Next, we give an inversion-free iteration for X_L.

$$\begin{cases} Y_0=(\gamma Q)^{-1}, \quad \gamma\in[\alpha_2,1]\\ X_k=Q-A^*Y_k^qA,\\ Y_{k+1}=2Y_k-Y_kX_kY_k.\end{cases} \tag{2.6}$$

where α_2 is defined in (2.4).

Theorem 2.2. *If Eq. (1.1) has a positive definite solution, then the sequence $\{X_k\}$ generated by (2.6) is monotone decreasing and converges to the maximal positive definite solution X_L.*

Proof. Suppose X is a positive definite solution of Eq.(1.1), then $X \leq \alpha_2 Q$ according to Lemma 2.4. Combining this with Lemma 2.2, we have

$$Y_0 = (\gamma Q)^{-1} \leq (\alpha_2 Q)^{-1} \leq X^{-1}, \quad X_0 = Q - A^* Y_0^q A \geq Q - A^* X^{-q} A = X.$$

Appling Lemma 2.5, we obtain that

$$Y_1 = 2Y_0 - Y_0 X_0 Y_0 \leq X_0^{-1} \leq X^{-1}.$$

Since

$$X_0 = Q - \frac{A^* Q^{-q} A}{\gamma^q} \leq Q^{1/2}[1 - \frac{\sigma_{\mathrm{m}}^2(Q^{-q/2} A Q^{-1/2})}{\gamma^q}]Q^{1/2} \leq \gamma Q = Y_0^{-1}$$

from the definition of γ, then

$$Y_1 - Y_0 = Y_0 - Y_0 X_0 Y_0 = Y_0[Y_0^{-1} - X_0]Y_0 \geq 0.$$

It follows that

$$X_1 = Q - A^* Y_1^q A \geq Q - A^* X^{-q} A = X$$

and

$$X_1 - X_0 = -A^* Y_1^q A + A^* Y_0^q A = A^*(Y_0^q - Y_1^q)A \leq 0.$$

Thus $X_0 \geq X_1 \geq X$ and $Y_0 \leq Y_1 \leq X^{-1}$. Suppose that $X_{k-1} \geq X_k \geq X, \quad Y_{k-1} \leq Y_k \leq X^{-1}$, then

$$\begin{aligned} Y_{k+1} &= 2Y_k - Y_k X_k Y_k \leq X_k^{-1} \leq X^{-1}, \\ X_{k+1} &= Q - A^* Y_k^q A \geq Q - A^* X^{-q} A = X, \end{aligned}$$

and

$$\begin{aligned} Y_{k+1} - Y_k &= Y_k(Y_k^{-1} - X_k)Y_k \geq Y_k(Y_k^{-1} - X_{k-1})Y_k \geq 0, \\ Y_{k+1} &= 2Y_k - Y_k X_k Y_k \leq X_k^{-1} \leq X^{-1}. \end{aligned}$$

According to Lemma 2.2, we get

$$X_{k+1} - X_k = A^*(Y_k^q - Y_{k+1}^q)A \leq 0.$$

Therefore, $X_k \geq X_{k+1} \geq X$ and $Y_k \leq Y_{k+1} \leq X^{-1}$ for each $k = 0, 1, 2, \ldots$. Thus the sequences $\{X_k\}$ and $\{Y_k\}$ are both convergent. Denote $X_l = \lim_{k\to\infty} X_k$ and $Y = \lim_{k\to\infty} Y_k$. Taking limit in iteration (2.6) leads to $Y = X_l^{-1}$ and $X_l + A^* X_l^{-q} A = Q$. Moreover, since $X_k \geq X$ for all $k = 0, 1, 2, \ldots$, then $X_l = X_L$ where X_L is the maximal positive definite solution of Eq.(1.1). ∎

Remark 2.1. Let $\gamma = 1$ in (2.6), then we obtain Algorithm 2.1 in [18].

3. Condition Number of the Maximal Solution X_L

By the theory of condition number developed by Rice[22], we give in this section an explicit expression of the condition number of the maximal positive definite solution X_L.

3.1. The complex case

Lemma 3.1. ([23]). *Let X be any $n\times n$ positive definite matrix,* $0<q\leq 1$. *Then*

(i) $X^{-q}=\frac{\sin q\pi}{\pi}\int_0^\infty(\lambda I+X)^{-1}\lambda^{-q}d\lambda,$

(ii) $X^{-q}=\frac{\sin q\pi}{q\pi}\int_0^\infty(\lambda I+X)^{-1}X(\lambda I+X)^{-1}\lambda^{-q}d\lambda.$

Denote the perturbed matrix equation of (1.1) by

$$\tilde{X}+\tilde{A}^*\tilde{X}^{-q}\tilde{A}=\tilde{Q}, \tag{3.1}$$

where $\tilde{A}$ and $\tilde{Q}$ are the slightly perturbed matrices of the matrices A and Q, respectively. From Theorem 2.1, we see that if $\|\Delta A\|$ and $\|\Delta Q\|$ are sufficiently small, then the maximal positive solution to the perturbed matrix equation (3.1) exists. Let $\tilde{X}_L$ be the maximal positive definite solution to Eq.(3.1). Denote $\Delta A=\tilde{A}-A$, $\Delta Q=\tilde{Q}-Q$ and $\Delta X=\tilde{X}_L-X_L$. Subtracting (1.1) from (3.1) gives rise to

$$\begin{aligned}&\Delta X+\tilde{A}^*\tilde{X}_L^{-q}\tilde{A}-A^*X_L^{-q}A=\Delta Q,\ \text{ i.e.,}\\&\Delta X+A^*(\tilde{X}_L^{-q}-X_L^{-q})A+\tilde{A}^*(\tilde{X}_L^{-q}-X_L^{-q})\Delta A+\Delta A^*(\tilde{X}_L^{-q}-X_L^{-q})A\\=\;&\Delta Q-(\Delta A^*X_L^{-q}A+\Delta A^*X_L^{-q}\Delta A+A^*X_L^{-q}\Delta A).\end{aligned} \tag{3.2}$$

Using Lemma 3.1, we have

$$\begin{aligned}&\tilde{X}_L^{-q}-X_L^{-q}\\=\;&\frac{\sin q\pi}{\pi}\int_0^\infty[(\lambda I+X_L+\Delta X)^{-1}-(\lambda I+X_L)^{-1}]\lambda^{-q}d\lambda\\=\;&\frac{\sin q\pi}{\pi}\int_0^\infty-(\lambda I+X_L)^{-1}\Delta X(\lambda I+X_L+\Delta X)^{-1}\lambda^{-q}d\lambda\\=\;&\frac{\sin q\pi}{\pi}\int_0^\infty-(\lambda I+X_L)^{-1}\Delta X(\lambda I+X_L)^{-1}\lambda^{-q}d\lambda\\&+\frac{\sin q\pi}{\pi}\int_0^\infty(\lambda I+X_L)^{-1}\Delta X(\lambda I+X_L+\Delta X)^{-1}\Delta X(\lambda I+X_L)^{-1}\lambda^{-q}d\lambda\end{aligned}. \tag{3.3}$$

Combining (3.3) with (3.2), we obtain that

$$\Delta X-\frac{\sin q\pi}{\pi}\int_0^\infty A^*(\lambda I+X_L)^{-1}\Delta X(\lambda I+X_L)^{-1}\lambda^{-q}Ad\lambda=E+h(\Delta X), \tag{3.4}$$

where

$$\begin{aligned} E &= \Delta Q - (\Delta A^* X_L^{-q} A + \Delta A^* X_L^{-q} \Delta A + A^* X_L^{-q} \Delta A), \\ h(\Delta X) &= -\frac{\sin q\pi}{\pi} A^* \int_0^\infty (\lambda I + X_L)^{-1} \Delta X (\lambda I + X_L + \Delta X)^{-1} \\ &\quad \Delta X (\lambda I + X_L)^{-1} \lambda^{-q} d\lambda A \\ &\quad + \frac{\sin q\pi}{\pi} \tilde{A}^* \int_0^\infty (\lambda I + X_L)^{-1} \Delta X (\lambda I + X_L + \Delta X)^{-1} \lambda^{-q} d\lambda \Delta A \\ &\quad + \frac{\sin q\pi}{\pi} \Delta A^* \int_0^\infty (\lambda I + X_L)^{-1} \Delta X (\lambda I + X_L + \Delta X)^{-1} \lambda^{-q} d\lambda A. \end{aligned}$$

Lemma 3.2. *Let* $\|A\|^2 \|Q^{-1}\|^{q+1} < \frac{q^q}{(q+1)^{q+1}}$. *Then the linear operator* $\boldsymbol{L} : H^{n\times n} \to H^{n\times n}$ *defined by*

$$\boldsymbol{L} W = W - \frac{\sin q\pi}{\pi} \int_0^\infty A^* (\lambda I + X_L)^{-1} W (\lambda I + X_L)^{-1} A \lambda^{-q} d\lambda \tag{3.5}$$

is invertible.

Proof. It suffices to show that for any matrix $V \in H^{n\times n}$, the following equation

$$\mathbf{L} W = V \tag{3.6}$$

has a unique solution. Define the operator **M** : $H^{n\times n} \to H^{n\times n}$ by

$$\begin{aligned} \mathbf{M} Z &= \frac{\sin q\pi}{\pi} \int_0^\infty X_L^{-1/2} A^* (\lambda I + X_L)^{-1} \\ &\quad X_L^{1/2} Z X_L^{1/2} (\lambda I + X_L)^{-1} A X_L^{-1/2} \lambda^{-q} d\lambda, \quad Z \in H^{n\times n}. \end{aligned}$$

Let $Y = X_L^{-1/2} W X_L^{-1/2}$. Thus (3.6) is equivalent to

$$Y - \mathbf{M} Y = X_L^{-1/2} V X_L^{-1/2}. \tag{3.7}$$

Notice that $\|X_L^{-1}\| < \frac{q+1}{q} \|Q^{-1}\|$. According to Lemma 3.1 (ii), we have

$$\begin{aligned} \|\mathbf{M} Y\| &= \|\frac{\sin q\pi}{\pi} \int_0^\infty X_L^{-1/2} A^* (\lambda I + X_L)^{-1} X_L^{1/2} Y X_L^{1/2} (\lambda I + X_L)^{-1} A X_L^{-1/2} \lambda^{-q} d\lambda\| \\ &\le \|\frac{\sin q\pi}{\pi} \int_0^\infty X_L^{-1/2} A^* (\lambda I + X_L)^{-1} X_L (\lambda I + X_L)^{-1} A X_L^{-1/2} \lambda^{-q} d\lambda\| \cdot \|Y\| \\ &= q\|Y\| \cdot \|X_L^{-1/2} A^* X_L^{-q} A X_L^{-1/2}\| \le q\|Y\| \cdot \|X_L^{-1/2} A^* X_L^{-q/2}\|^2 \\ &\le q\|Y\| \cdot \|A\|^2 \|X_L^{-1}\|^{q+1} \\ &\le q\|Y\| \cdot \|A\|^2 (\frac{q+1}{q})^{q+1} \|Q^{-1}\|^{q+1} < \|Y\|. \end{aligned}$$

Then $\|\mathbf{M}\| < 1$ which implies that $\mathbf{I}-\mathbf{M}$ is invertible. Therefore, for any matrix $V \in H^{n\times n}$, equation (3.7) has a unique solution Y. Thus equation (3.6) has a unique solution W for any $V \in H^{n\times n}$ which implies that the operator $\mathbf{L}$ is invertible. The proof is then completed. ■

Let $B = X_L^{-q}A$, we can rewrite (3.4) as

$$\Delta X = \tilde{X}_L - X_L = \mathbf{L}^{-1}(\Delta Q - B^*\Delta A - \Delta A B^*) - \mathbf{L}^{-1}(\Delta A^* X_L^{-q}\Delta A) + \mathbf{L}^{-1}(h(\Delta X)).$$

Then we have

$$\begin{aligned}\Delta X &= \tilde{X}_L - X_L = \mathbf{L}^{-1}(\Delta Q - B^*\Delta A - \Delta A B^*)\\ &\quad + O(\|(\Delta A, \Delta Q)\|_F^2), \quad (\Delta A, \Delta Q) \to 0.\end{aligned} \tag{3.8}$$

By Rice's condition number theory, we define the condition number of the maximal positive definite solution X_L of Eq.(1.1) as follows:

$$C(X_L) = \lim_{\delta\to 0} \sup_{\|(\frac{\Delta A}{\mu}, \frac{\Delta Q}{\rho})\|_F \leq \delta} \frac{\|\Delta X\|_F}{\xi\delta}, \tag{3.9}$$

where ξ, μ, ρ are positive parameters. Taking $\xi = \mu = \rho = 1$ in (3.9) gives the absolute condition number $C_{\text{abs}}(X_L)$ and taking $\xi = \|X_L\|_F, \mu = \|A\|_F, \rho = \|Q\|_F$ gives the relative condition number $C_{\text{rel}}(X_L)$.

Substituting (3.8) into (3.9), we get

$$\begin{aligned} C(X_L) &= \frac{1}{\xi} \max_{\substack{(\frac{\Delta A}{\mu}, \frac{\Delta Q}{\rho}) \neq 0 \\ \Delta A \in C^{n\times n}, \Delta Q \in H^{n\times n}}} \frac{\|\mathbf{L}^{-1}(\Delta Q - B^*\Delta A - (\Delta A)^* B)\|_F}{\|(\frac{\Delta A}{\mu}, \frac{\Delta Q}{\rho})\|_F} \\ &= \frac{1}{\xi} \max_{\substack{(E,H)\neq 0 \\ E\in C^{n\times n}, H \in H^{n\times n}}} \frac{\|\mathbf{L}^{-1}(\rho H - \mu(B^*E + E^*B))\|_F}{\|(E,H)\|_F}. \\ &= \frac{1}{\xi} \max_{\substack{(E,H)\neq 0 \\ E\in C^{n\times n}, H \in H^{n\times n}}} \frac{\|\mathbf{L}^{-1}(\rho H - \mu(B^*E + E^*B))\|_F}{\|(-E,H)\|_F}. \\ &= \frac{1}{\xi} \max_{\substack{(K,H)\neq 0 \\ K\in C^{n\times n}, H \in H^{n\times n}}} \frac{\|\mathbf{L}^{-1}(\rho H + \mu(B^*K + K^*B))\|_F}{\|(K,H)\|_F}. \end{aligned}$$

Let L be the matrix of the operator $\mathbf{L}$. Then it is not difficult to see that

$$L = I \otimes I - \frac{\sin q\pi}{\pi} \int_0^\infty [(\lambda I + X_L)^{-1}A]^T \otimes [(\lambda I + X_L)^{-1}A]^* \lambda^{-q} d\lambda.$$

Denote by $w = \mathrm{vec}(K) = u + iv, \eta = \mathrm{vec}(H) = a + ib, g_1 = (a^T, b^T)^T, g_2 = (u^T, v^T)^T, a, b, u, v \in R^{n^2}$.

$$L^{-1}(I \otimes B^*) = L^{-1}(I \otimes (X_L^{-q}A)^*) = U_1 + i\Omega_1,$$
$$L^{-1}(B^T \otimes I)\Pi = L^{-1}((X_L^{-q}A)^T \otimes I)\Pi = U_2 + i\Omega_2,$$

where $U_1, U_2, \Omega_1, \Omega_2 \in R^{n^2\times n^2}$, and Π is the vec-permutation matrix, such that $\mathrm{vec}(K^T) = \Pi\mathrm{vec}K$. Denote

$$L^{-1} = S+i\Sigma, S, \Sigma \in R^{n^2\times n^2}, \quad U_c = \begin{bmatrix} U_1+U_2 & \Omega_2-\Omega_1 \\ \Omega_1+\Omega_2 & U_1-U_2 \end{bmatrix}, \quad S_c = \begin{bmatrix} S & -\Sigma \\ \Sigma & S \end{bmatrix}.$$

Then we obtain that

$$\begin{aligned}
&C(X_L)\\
&= \frac{1}{\xi} \max_{\substack{(K,H)\neq 0\\ E\in C^{n\times n}, H\in H^{n\times n}}} \frac{\|\mathbf{L}^{-1}(\rho H + \mu(B^*K + K^*B))\|_F}{\|(K,H)\|_F}\\
&= \frac{1}{\xi} \max_{\substack{(K,H)\neq 0\\ K\in C^{n\times n}, H\in H^{n\times n}}} \frac{\|\rho L^{-1}\mathrm{vec}(H) + \mu L^{-1}\mathrm{vec}(B^*K + K^*B)\|}{\|\mathrm{vec}(K,H)\|}\\
&= \frac{1}{\xi} \max_{\substack{(K,H)\neq 0\\ K\in C^{n\times n}, H\in H^{n\times n}}} \frac{\|\rho L^{-1}\mathrm{vec}(H)+\mu L^{-1}(I\otimes B^*)\mathrm{vec}(K)+\mu L^{-1}(B^T\otimes I)\mathrm{vec}(K^*)\|}{\|\mathrm{vec}(K,H)\|}\\
&= \frac{1}{\xi} \max_{\substack{(K,H)\neq 0\\ K\in C^{n\times n}, H\in H^{n\times n}}} \frac{\|\rho(S+i\Sigma)(a+ib)+\mu(U_1+i\Omega_1)(u+iv)+\mu(U_2+i\Omega_2)(u-iv)\|}{\|\mathrm{vec}(K,H)\|}\\
&= \frac{1}{\xi} \max_{\substack{(K,H)\neq 0\\ K\in C^{n\times n}, H\in H^{n\times n}}} \frac{\|\rho S_c \begin{pmatrix} a \\ b \end{pmatrix} + \mu U_c \begin{pmatrix} u \\ v \end{pmatrix}\|}{\|\mathrm{vec}(K,H)\|}\\
&= \frac{1}{\xi} \max_{(g_1,g_2)\neq 0} \frac{\|\rho S_c g_2 + \mu U_c g_1\|}{\|\begin{pmatrix} g_1 \\ g_2 \end{pmatrix}\|} = \frac{1}{\xi}\max_{g\neq 0} \frac{\|(\rho S_c, \mu U_c)g\|}{\|g\|} = \frac{1}{\xi}\|(\rho S_c, \mu U_c)\|.
\end{aligned}$$

Theorem 3.1. *Let* $\|A\|^2\|Q^{-1}\|^{q+1} < \frac{q^q}{(q+1)^{q+1}}$. *Then the condition number* $C(X_L)$ *defined by (3.9) has the following explicit expression*

$$C(X_L) = \frac{1}{\xi}\|(\rho S_c, \mu U_c)\|, \tag{3.10}$$

where S_c, U_c *are defined as above.*

Remark 3.1. From (3.10), we have the relative condition number

$$C_{\mathrm{rel}}(X_L) = \frac{\|(\|Q\|_F S_c, \|A\|_F U_c)\|}{\|X_L\|_F}. \tag{3.11}$$

3.2. The real case

In this section we consider the real case, i.e., all the coefficient matrices A, Q of Eq.(1.1) are real. In such a case the corresponding maximal solution X_L is also real. Similar to Theorem 3.1, we obtain the following theorem.

Theorem 3.2. *Let A, Q be real. Suppose that $\|A\|^2\|Q^{-1}\|^{q+1} < \frac{q^q}{(q+1)^{q+1}}$. Then the condition number $C(X_L)$ defined by (3.9) has the explicit expression*

$$C(X_L) = \frac{1}{\xi}\|(\rho S_r, \mu U_r)\|, \tag{3.12}$$

where

$$S_r = (I \otimes I - \frac{\sin q\pi}{\pi}\int_0^\infty [(\lambda I + X_L)^{-1}A]^T \otimes [(\lambda I + X_L)^{-1}A]^T \lambda^{-q} d\lambda.)^{-1},$$

$$U_r = S_r[I \otimes (A^T X_L^{-q}) + ((A^T X_L^{-q}) \otimes I)\Pi].$$

Remark 3.2. In the real case the relative condition number is given by

$$C_{\rm rel}(X_L) = \frac{\|(\|Q\|_F S_r, \|A\|_F U_r)\|}{\|X_L\|_F}. \tag{3.13}$$

4. Numerical Experiments

In this section, some simple examples are given to illustrate the results of the previous sections. All the tests are carried out using MATLAB 7.1 with machine precision around 10^{-16}. The practical stopping criterion used is the residual $\|X + A^*X^{-q}A - Q\| < 10^{-10}$.

Example 4.1. Consider Eq.(1.1) with the case $q = 0.3$, and the matrices A and Q as follows:

$$A=\begin{pmatrix} -0.9 & 0.9 & 1.7 & -2.4 & 1.5 \\ 1.1 & 2.1 & 0.8 & 1.5 & 1.8 \\ -1.8 & 1.9 & -1.4 & 1.7 & -1.8 \\ 1.4 & -1.7 & 0.8 & 1.4 & 1.5 \\ 0.5 & 1.3 & 1.5 & -1.2 & 1.3 \end{pmatrix},\ Q=\begin{pmatrix} 68.6 & 28.8 & 21.2 & 25.2 & 21.6 \\ 28.8 & 52.4 & 9.6 & 10.8 & 20.4 \\ 21.2 & 9.6 & 38.0 & 12.0 & 13.2 \\ 25.2 & 10.8 & 12.0 & 48.9 & 9.6 \\ 21.6 & 20.4 & 13.2 & 9.6 & 40.4 \end{pmatrix}.$$

By computation, $\|A\|^2\|Q^{-1}\|^{q+1} = 0.4792 < \frac{q^q}{(q+1)^{q+1}} = 0.4955$, $\frac{q}{q+1} = 0.2308$, $\beta_2 = 0.6824$ and $\alpha_2 = 0.9989$. According to Theorem 2.1, take $\gamma = 0.3$, using iteration (2.5) and iterating 11 steps, we get the maximal positive definite solution to matrix Eq.(1.1):

$$X_L \approx X_{11} = \begin{pmatrix} 65.7420 & 30.2878 & 19.8317 & 24.2997 & 19.1178 \\ 30.2878 & 48.0157 & 9.8016 & 10.5172 & 20.8473 \\ 19.8317 & 9.8016 & 35.2826 & 14.3132 & 10.1500 \\ 24.2997 & 10.5172 & 14.3132 & 43.0937 & 10.8167 \\ 19.1178 & 20.8473 & 10.1500 & 10.8167 & 36.3518 \end{pmatrix}$$

with the residual $\|X_{11} + A^* X_{11}^{-q} A - Q\| = 6.4720e-012$. Moreover, from $\lambda_{\rm m}(X_{11} - \beta_2 Q) = 0.7440$ and $\lambda_{\rm m}(\alpha_2 Q - X_{11}) = 7.0749e-004$, we know that $X_L \in [\beta_2 Q, \alpha_2 Q]$.

Example 4.2. Consider Eq.(1.1) with the case $q = 0.7$, and the matrix Q is the same as in Example 4.1 and A is as follows:

$$A = \begin{pmatrix} -1.8 & 1.9 & 3.4 & -4.5 & 3.1 \\ 2.3 & 4.2 & 1.7 & 2.1 & 3.7 \\ -3.6 & 3.8 & -2.9 & 3.4 & -3.7 \\ 2.9 & -3.4 & 1.7 & 2.8 & 3.1 \\ 1.1 & 2.7 & 3.1 & -2.4 & 2.6 \end{pmatrix}.$$

By computation, $\sigma_1^2(Q^{-1/2} A^* Q^{-q} A Q^{-1/2}) - \frac{q^q}{(q+1)^{q+1}} = -0.01031$, $\alpha_2 = 0.998428$. Take $\gamma = 0.9985$, according to Theorem 2.2, using iteration (2.6) and iterating 14 steps, we get

$$X_L \approx X_{14} = \begin{pmatrix} 65.2247 & 30.6246 & 19.6229 & 24.2256 & 18.7216 \\ 30.6246 & 48.2006 & 10.4208 & 10.6308 & 21.6694 \\ 19.6229 & 10.4208 & 35.2090 & 14.6426 & 10.1819 \\ 24.2256 & 10.6308 & 14.6426 & 42.8684 & 11.0073 \\ 18.7216 & 21.6694 & 10.1819 & 11.0073 & 36.3604 \end{pmatrix}$$

with the residual $\|X_{14} + A^* X_{14}^{-q} A - Q\| = 2.039676e-011$. Moreover, by computing the minimal eigenvalue $\lambda_{\rm m}(X_k - X_{k-1})$, we get that the sequence $\{X_k\}$ is monotonically decreasing.

Example 4.3. Consider Eq.(1.1) with $q = 0.5$ and

$$A = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 1.1 & 0 \\ 0 & 1.2 \end{pmatrix},$$

where $a = 0.45 + 10^{-k}$. Some numerical results on $C_{\rm rel}(X_L)$ are listed in the following table where $C_{\rm rel}(X_L)$ is the relative condition number of the maximal positive definite solution.

Results for Example 4.3 with different vales of k.

k	1	2	3	4	5	6	7
$\|A\|^2\|Q^{-1}\|^{q+1}-\frac{q^q}{(q+1)^{q+1}}$	-0.1227	-0.2015	-0.2086	-0.2093	-0.2094	-0.2094	-0.2094
$C_{\rm rel}(X_L)$	1.2589	1.1665	1.1589	1.1581	1.1580	1.1580	1.1580

From the numerical results in the second line, we see that the condition of Remark 3.2 is always satisfied for each $k = 1, 2, ...7$. Then according to Remark 3.2, we can compute the relative condition number $C_{\rm rel}(X_L)$ of the maximal positive definite solution by (3.13). The numerical results listed in the third line show that the maximal positive definite solution X_L is well-conditioned in such cases.

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