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ASYMPTOTICS OF MINIMIZERS OF VARIATIONAL PROBLEMS IN A MULTI-CONNECTED DOMAIN ASSOCIATED WITH THE THEORY OF LIQUID CRYSTALS

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Abstract. We consider the asymptotic behavior of minimizers of the Ericksen functional with the Dirichlet boundary condition in a general domain. We examine the asymptotic behavior of minimizers of the functional as some of the elastic coefficients tend to the infinity, according to the de Gennes predictions with respect to nematic smectic-A phase transition.

1. Introduction

According to the Oseen-Frank theory, the nematic phase of liquid crystals can be described by a director field $\boldsymbol{n} : \Omega \to \mathbb{S}^2$ which is a minimizer of the Oseen-Frank energy functional

$$\mathcal{W}_{\mathrm{OF}}(\boldsymbol{n}) = \int_\Omega W_{\mathrm{OF}}(\boldsymbol{n}, \nabla \boldsymbol{n}) dx$$

where $\Omega \subset \mathbb{R}^3$ is a smooth bounded domain occupied by the liquid crystal sample, and

$$W_{\mathrm{OF}}(\boldsymbol{n}, \nabla \boldsymbol{n}) = \frac{k_1}{2}|\mathrm{div}\, \boldsymbol{n}|^2 + \frac{k_2}{2}|\boldsymbol{n} \cdot \mathrm{curl}\, \boldsymbol{n}|^2 + \frac{k_3}{2}|\boldsymbol{n} \times \mathrm{curl}\, \boldsymbol{n}|^2 + \frac{k_2 + k_4}{2}[\mathrm{Tr}(\nabla \boldsymbol{n})^2 - (\mathrm{div}\, \boldsymbol{n})^2].$$

Here k_1, k_2 and k_3 are positive material constants, which are called spray constant, twist constant and bend constant, respectively. Since we consider the Dirichlet boundary condition, we can drop the term $\mathrm{Tr}(\nabla \boldsymbol{n})^2 - (\mathrm{div}\, \boldsymbol{n})^2$ (cf. Hardt et al. [8]). By this Oseen-Frank model, one can clarify the point defect of the nematic

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liquid crystals. However, for the line defect, it is necessary to consider the Ericksen model

$$\mathcal{W}_{\mathrm{E}}(s, \boldsymbol{n}) = \int_\Omega W_{\mathrm{E}}(s, \boldsymbol{n})dx$$

where

$$W_{\mathrm{E}}(s, \boldsymbol{n}) = \frac{s^2}{2} W_{\mathrm{OF}}(\boldsymbol{n}, \nabla \boldsymbol{n}) + \frac{k_5}{2}|\nabla s|^2 + \frac{k_6}{2}|\nabla s \cdot \boldsymbol{n}|^2 + \psi(s).$$

Here s is a scale function called the degree of orientation. Moreover, the Ericksen model is prefer to represent the phase transition from nematic phase to smectic-A phase.

Pan and Qi [13] considered a simplified energy functional

$$\mathcal{W}(\boldsymbol{u}) = \int_\Omega \left\{ \frac{k_1}{2}|\nabla \boldsymbol{u}|^2 + \frac{k_2}{2}|\operatorname{curl} \boldsymbol{u}|^2 + \psi(|\boldsymbol{u}|) \right\} dx.$$

They assumed that ψ is a non-negative C^1 function on $\mathbb{R}$ such that $\lim_{s\to\infty} \psi(s) = +\infty$. Under this setting, they considered the following problem due to the de Gennes prediction. For a given vector field $\mathcal{H} \in H^{1/2}(\Gamma, \mathbb{R}^3)$ and for fixed $k_1 > 0$, let $\boldsymbol{u}(k_2)$ be a minimizer of $\mathcal{W}$ on

$$H(\Omega, \mathcal{H}) = \{\boldsymbol{u} \in H^1(\Omega, \mathbb{R}^3); \boldsymbol{u} = \mathcal{H} \text{ on } \Gamma\}.$$

(P) Examine the analysis of the minimal energy $\mathcal{W}(\boldsymbol{u}(k_2))$ and the asymptotics of the minimizer $\boldsymbol{u}(k_2)$ as $k_2 \to \infty$. More precisely,

(P1) As $k_2 \to \infty$, is the total energy $\mathcal{W}(\boldsymbol{u}(k_2))$ bounded ?

(P2) If (P1) is yes, does $\boldsymbol{u}(k_2)$ converge (in some sense) to a limit which is a minimizer or stationary point of the functional

$$\mathcal{I}(\boldsymbol{u}) = \int_\Omega \left\{ \frac{k_1}{2}|\nabla \boldsymbol{u}|^2 + \psi(|\boldsymbol{u}|) \right\} dx$$

with $\operatorname{curl} \boldsymbol{u} = \boldsymbol{0}$? They clarified the problem (P) in the case where Ω is simply connected domain.

However, in the case where Ω is a multi-connected domain, it remains unknown.

In this paper we shall consider the case where Ω is a multi-connected domain. Related to this direction, we refer to Aramaki [1], Pan [9], [10], [11], [12]. To handle multi-connected domains, we must assume that the funciton ψ satisfies a stronger condition. Thus we assume that ψ is a non-negative function on $[0, \infty)$ and $\psi(s)$ is divergent faster than s^2 at the infinity. That is to say, there exist constants $c, M > 0$ such that

$$\psi(s) \ge cs^2 \quad \text{for } s \ge M. \tag{1.1}$$

We note that the important function $\psi(s) = \lambda(1 - s^2)^2 \ (\lambda > 0)$ satisfies (1.1) (cf. Aviles and Giga [3]).

It is advisable to treat the full trace as the boundary condition, but we are obliged to treat the restriction of the trace. For a given vector field $\mathcal{H}$ on $\Gamma = \partial\Omega$, we denote the tangential component of $\mathcal{H}$ by $\mathcal{H}_T$: $\mathcal{H}_T = \mathcal{H} - (\mathcal{H} \cdot \boldsymbol{\nu})\boldsymbol{\nu}$ where $\boldsymbol{\nu}$ is the outer unit normal vector field to Γ. We assume that

$$(1.2) \qquad \mathcal{H}_T \in H^{1/2}(\Gamma, \mathbb{R}^3) \cap L^\infty(\Gamma, \mathbb{R}^3).$$

We define the space

$$\mathcal{H}^1_t(\Omega, \mathbb{R}^3, \mathcal{H}_T) = \{\boldsymbol{u} \in H^1(\Omega, \mathbb{R}^3); \boldsymbol{u}_T = \mathcal{H}_T \text{ on } \Gamma\}$$

where $\boldsymbol{u}_T$ denotes the tangential component of the trace of $\boldsymbol{u}$ to Γ. Instead of $\mathcal{W}(\boldsymbol{u})$ and $\mathcal{I}(\boldsymbol{u})$, we consider a simplified functional

$$(1.3) \qquad \mathcal{W}_0(\boldsymbol{u}) = \int_\Omega \left\{ \frac{k_1}{2}|\operatorname{div}\boldsymbol{u}|^2 + \frac{k_2}{2}|\operatorname{curl}\boldsymbol{u}|^2 + \psi(|\boldsymbol{u}|) \right\} dx$$

and

$$(1.4) \qquad \mathcal{I}_0(\boldsymbol{u}) = \int_\Omega \left\{ \frac{k_1}{2}|\operatorname{div}\boldsymbol{u}|^2 + \psi(|\boldsymbol{u}|) \right\} dx$$

on $\mathcal{H}^1_t(\Omega, \mathbb{R}^3, \mathcal{H}_T)$.

We set our problems as follows.

(Q) Examine the analysis of the minimal energy $\mathcal{W}_0(\boldsymbol{u}(k_2))$ and the asymptotics of the minimizer $\boldsymbol{u}(k_2)$ as $k_2 \to \infty$ for fixed $k_1 > 0$. More precisely,

(Q.1) As $k_2 \to \infty$, is the total energy $\mathcal{W}_0(\boldsymbol{u}(k_2))$ bounded ?

(Q.2) If (Q1) is yes, does $\boldsymbol{u}(k_2)$ converge (in some sense) to a limit which is a minimizer or stationary point of the functional $\mathcal{I}_0(\boldsymbol{u})$ with $\operatorname{curl}\boldsymbol{u} = \mathbf{0}$?

Fortunately, Pan [12] showed that the variational problem

$$(1.5) \qquad \mathcal{R}_t(\mathcal{H}_T) := \inf_{\boldsymbol{u} \in \mathcal{H}^1_t(\Omega, \mathbb{R}^3, \mathcal{H}_T)} \|\operatorname{curl}\boldsymbol{u}\|^2_{L^2(\Omega)}$$

is achieved, and the set $\Sigma_t(\mathcal{H}_T)$ of all minimizers of (1.5) is given by

$$(1.6) \qquad \Sigma_t(\mathcal{H}_T) = \{\overline{\boldsymbol{u}} + \boldsymbol{w}; \boldsymbol{w} \in \mathcal{H}^1_t(\Omega, \mathbb{R}^3, \mathbf{0}), \operatorname{curl}\boldsymbol{w} = \mathbf{0} \text{ in } \Omega\}$$

where $\overline{\boldsymbol{u}} \in \mathcal{H}^1_t(\Omega, \mathbb{R}^3, \mathcal{H}_T)$ is a fixed minimizer of (1.5). Thus we define

$$(1.7) \qquad a(\mathcal{H}_T) = \inf_{\boldsymbol{u} \in \mathcal{H}^1_t(\Omega, \mathbb{R}^3, \mathcal{H}_T)} \mathcal{I}_0(\boldsymbol{u}).$$

and

$$b(\mathcal{H}_T) = \inf_{\boldsymbol{u}\in\Sigma_t(\mathcal{H}_T)} \mathcal{I}_0(\boldsymbol{u}). \tag{1.8}$$

In the present paper, we examine the problem (Q) where Ω is a bounded, multi-connected domain in $\mathbb{R}^3$ with a smooth boundary.

In order to treat the case where Ω is multi-connected, throughout this paper we assume that Ω satisfies the following conditions.

(O.1) Ω is a bounded domain in $\mathbb{R}^3$ with boundary $\partial\Omega = \Gamma$ which is a manifold of class C^r $(r \geq 2)$ of dimension 2, and Ω is locally situated on one side of Γ. Moreover, Γ has a finite number of connected components $\Gamma_1, \Gamma_2, \ldots, \Gamma_{m+1}$ where $m \geq 0$ and Γ_{m+1} denoting the boundary of the infinite connected component of $\mathbb{R}^3 \setminus \overline{\Omega}$.

(O.2) There exist n manifolds $\Sigma_1, \Sigma_2, \ldots, \Sigma_n$, $(n \geq 0)$ of dimension 2 of class C^r such that $\Sigma_i \cap \Sigma_j = \emptyset$ $(i \neq j)$ and they are not tangential to Γ such that $\dot{\Omega} = \Omega \setminus \Sigma$ where $\Sigma = \cup_{j=1}^n \Sigma_i$ is simply connected and Lipschitzian.

The number n is called the first Betti number which is equal to the number of handles of Ω and m is called the second Betti number which is equal to the number of holes. We say that Ω is simply connected if $n = 0$, and that Ω has no holes if $m = 0$. If we define the spaces

$$\mathbb{H}_1(\Omega) = \{\boldsymbol{u} \in L^2(\Omega, \mathbb{R}^3)\,;\operatorname{curl}\boldsymbol{u} = \boldsymbol{0}, \operatorname{div}\boldsymbol{u} = 0 \text{ in } \Omega, \boldsymbol{u}\cdot\boldsymbol{\nu}|_\Gamma = 0\},$$

$$\mathbb{H}_2(\Omega) = \{\boldsymbol{u} \in L^2(\Omega, \mathbb{R}^3)\,;\operatorname{curl}\boldsymbol{u} = \boldsymbol{0}, \operatorname{div}\boldsymbol{u} = 0 \text{ in } \Omega, \boldsymbol{u}\times\boldsymbol{\nu}|_\Gamma = 0\},$$

it is well known that

$$\dim \mathbb{H}_1(\Omega) = n \quad \text{and} \quad \dim \mathbb{H}_2(\Omega) = m.$$

Moreover, $\mathbb{H}_i(\Omega)$ $(i = 1, 2)$ is a closed subspace of $L^2(\Omega, \mathbb{R}^3)$, and $\mathbb{H}_i(\Omega) \subset H^1(\Omega, \mathbb{R}^3)$ for $i = 1, 2$. Furthermore, if Γ is of class $C^{r,\theta}$ with $r \geq 2$ and $0 < \theta < 1$, then

$$\mathbb{H}_i(\Omega) \subset C^{r-1,\theta}(\overline{\Omega}, \mathbb{R}^3) \quad (i = 1, 2).$$

For these facts, see Dautray and Lions [6], Girault and Raviart [7] and Temam [14].

We are in a position to state the theorems.

Theorem 1.1. *Let Ω be a bounded domain in $\mathbb{R}^3$ satisfying (O.1) and (O.2) and assume that $\mathcal{H}_T \in H^{1/2}(\Gamma, \mathbb{R}^3) \cap L^\infty(\Gamma, \mathbb{R}^3)$. Then*

$$\mathcal{A}_0(\mathcal{H}_T) := \inf_{\boldsymbol{u}\in\mathcal{H}^1_t(\Omega,\mathbb{R}^3,\mathcal{H}_T)} \mathcal{W}_0(\boldsymbol{u}) \tag{1.9}$$

is achieved.

The answer for the problem (Q) is the following.

Theorem 1.2. *Let Ω be a bounded domain in $\mathbb{R}^3$ satisfying (O.1) and (O.2) and assume that $\mathcal{H}_T \in H^{1/2}(\Gamma, \mathbb{R}^3) \cap L^\infty(\Gamma, \mathbb{R}^3)$. Let $\boldsymbol{u}(k_2)$ be a minimizer of $\mathcal{A}_0(\mathcal{H}_T)$ in $\mathcal{H}^1_t(\Omega, \mathbb{R}^3, \mathcal{H}_T)$ for fixed $k_1 > 0$. Then for any sequence $\{k_2^{(j)}\}$ so that $k_2^{(j)} \to \infty$ as $j \to \infty$, there exists a subsequence (still denoted by $\{k_2^{(j)}\}$) such that $\boldsymbol{u}(k_2^{(j)}) \to \overline{\boldsymbol{u}}$ strongly in $H^1(\Omega, \mathbb{R}^3)$ as $j \to \infty$ where $\overline{\boldsymbol{u}}$ satisfies that*

$$\mathcal{R}_t(\mathcal{H}_T) = \|\mathrm{curl}\,\overline{\boldsymbol{u}}\|^2_{L^2(\Omega,\mathbb{R}^3)}, \quad \textit{and} \quad \mathcal{I}_0(\overline{\boldsymbol{u}}) = b(\mathcal{H}_T).$$

Compared with the problem (Q), we can also set an another problem as follows.

(R) Examine the analysis of the minimal energy $\mathcal{W}_0(\boldsymbol{u}(k_1))$ and the asymptotics of the minimizer $\boldsymbol{u}(k_1)$ as $k_1 \to \infty$ for fixed $k_2 > 0$. More precisely,

(R.1) As $k_1 \to \infty$, is the total energy $\mathcal{W}_0(\boldsymbol{u}(k_1))$ bounded ?

(R.2) If (R1) is yes, does $\boldsymbol{u}(k_1)$ converge (in some sense) to a limit which is a minimizer or stationary point of the functional

$$\mathcal{J}_0(\boldsymbol{u}) = \int_\Omega \left\{ \frac{k_2}{2} |\mathrm{curl}\,\boldsymbol{u}|^2 + \psi(|\boldsymbol{u}|) \right\} dx \tag{1.10}$$

with $\mathrm{div}\,\boldsymbol{u} = 0$?

For this problem, we assume that

$$\mathcal{H}_n := \mathcal{H} \cdot \boldsymbol{\nu} \in H^{1/2}(\Gamma) \cap L^\infty(\Gamma)$$

and define

$$\mathcal{H}^1_n(\Omega, \mathbb{R}^3, \mathcal{H}_n) = \{\boldsymbol{u} \in H^1(\Omega, \mathbb{R}^3); \boldsymbol{u} \cdot \boldsymbol{v} = \mathcal{H}_n \,\mathrm{on}\, \Gamma\}.$$

In our previous paper Aramaki [2], we showed that the variational problem

$$\mathcal{D}_n(\mathcal{H}_n) := \inf_{\boldsymbol{u} \in \mathcal{H}^1_n(\Omega, \mathbb{R}^3, \mathcal{H}_n)} \|\mathrm{div}\,\boldsymbol{u}\|^2_{L^2(\Omega)} \tag{1.11}$$

is achieved, and the set $\Sigma_n(\mathcal{H}_n)$ of all minimizers of (1.11) is given by

$$\Sigma_n(\mathcal{H}_n) = \{\overline{\boldsymbol{u}} + \boldsymbol{w}; \boldsymbol{w} \in \mathcal{H}^1_n(\Omega, \mathbb{R}^3, 0), \mathrm{div}\,\boldsymbol{w} = 0 \text{ in } \Omega\} \tag{1.12}$$

where $\overline{\boldsymbol{u}} \in \mathcal{H}^1_n(\Omega, \mathbb{R}^3, \mathcal{H}_n)$ is a fixed minimizer of (1.11). Thus we define

$$c(\mathcal{H}_n) = \inf_{\boldsymbol{u} \in \mathcal{H}^1_n(\Omega, \mathbb{R}^3, \mathcal{H}_n)} \mathcal{J}_0(\boldsymbol{u}). \tag{1.13}$$

and

$$d(\mathcal{H}_n) = \inf_{\boldsymbol{u} \in \Sigma_n(\mathcal{H}_n)} \mathcal{J}_0(\boldsymbol{u}). \tag{1.14}$$

We can also get the following.

Theorem 1.3. *Let Ω be a bounded domain in $\mathbb{R}^3$ satisfying (O.1) and (O.2) and assume that $\mathcal{H}_n \in H^{1/2}(\Gamma) \cap L^\infty(\Gamma)$. Then*

$$\mathcal{B}_0(\mathcal{H}_n) := \inf_{\boldsymbol{u} \in \mathcal{H}_n^1(\Omega, \mathbb{R}^3, \mathcal{H}_n)} \mathcal{W}_0(\boldsymbol{u})$$

is achieved.

For the problem (R), we can get

Theorem 1.4. *Let Ω be a bounded domain in $\mathbb{R}^3$ satisfying (O.1) and (O.2) and assume that $\mathcal{H}_n \in H^{1/2}(\Gamma) \cap L^\infty(\Gamma)$. Let $\boldsymbol{u}(k_1)$ be a minimizer of $\mathcal{B}_0(\mathcal{H}_n)$ in $\mathcal{H}_n^1(\Omega, \mathbb{R}^3, \mathcal{H}_n)$ for fixed $k_2 > 0$. Then for any sequence $\{k_1^{(j)}\}$ so that $k_1^{(j)} \to \infty$ as $j \to \infty$, there exists a subsequence (still denoted by $\{k_1^{(j)}\}$) such that $\boldsymbol{u}(k_1^{(j)}) \to \overline{\boldsymbol{u}}$ strongly in $H^1(\Omega, \mathbb{R}^3)$ as $j \to \infty$ where $\overline{\boldsymbol{u}}$ satisfies that*

$$\mathcal{D}_n(\mathcal{H}_n) = \|\mathrm{div}\, \overline{\boldsymbol{u}}\|_{L^2(\Omega)}^2, \quad \textit{and} \quad \mathcal{J}_0(\overline{\boldsymbol{u}}) = d(\mathcal{H}_n).$$

The plan of this paper is as follows. In section 2, we give some basic estimates and regularities according to [6]. Section 3 is devoted to the proofs of Theorem 1.1, 1.2 and state a further remark. In section 4, we give the proofs of Theorem 1.3, 1.4 and state a further remark.

2. Preliminaries

In this section, we give some properties which are needed for the proofs from Theorem 1.1 to 1.4. We state the regularities and some estimates which are well known (cf. [6] and also [14]). Let Ω be a bounded domain in $\mathbb{R}^3$ with C^{k+2} boundary $\Gamma = \partial\Omega$ $(k \geq 0)$. Then we see that

$$\begin{aligned} H^{k+1}(\Omega, \mathbb{R}^3) &= \{\boldsymbol{u} \in L^2(\Omega, \mathbb{R}^3)\,;\mathrm{curl}\, \boldsymbol{u} \in H^k(\Omega, \mathbb{R}^3), \\ &\qquad \mathrm{div}\, \boldsymbol{u} \in H^k(\Omega), \boldsymbol{u} \cdot \boldsymbol{\nu}|_\Gamma \in H^{k+1/2}(\Gamma)\} \end{aligned} \tag{2.1}$$

or

$$\begin{aligned} H^{k+1}(\Omega, \mathbb{R}^3) &= \{\boldsymbol{u} \in L^2(\Omega, \mathbb{R}^3)\,;\mathrm{curl}\, \boldsymbol{u} \in H^k(\Omega, \mathbb{R}^3), \\ &\qquad \mathrm{div}\, \boldsymbol{u} \in H^k(\Omega), \boldsymbol{u} \times \boldsymbol{\nu}|_\Gamma \in H^{k+1/2}(\Gamma, \mathbb{R}^3)\}. \end{aligned} \tag{2.2}$$

Here $\cdot|_\Gamma$ means the trace operator to Γ and $\boldsymbol{\nu}$ denotes the unit outer normal vector field on Γ. Moreover, the following estimate holds : There exists a constant $C > 0$ such that

$$\begin{aligned} \|\boldsymbol{u}\|_{H^{k+1}(\Omega,\mathbb{R}^3)} &\leq C\{\|\boldsymbol{u}\|_{L^2(\Omega,\mathbb{R}^3)} + \|\mathrm{curl}\, \boldsymbol{u}\|_{H^k(\Omega,\mathbb{R}^3)} \\ &\qquad + \|\mathrm{div}\, \boldsymbol{u}\|_{H^k(\Omega)} + \|\boldsymbol{u} \cdot \boldsymbol{\nu}\|_{H^{k+1/2}(\Gamma)}\} \end{aligned} \tag{2.3}$$

or

$$\|\boldsymbol{u}\|_{H^{k+1}(\Omega,\mathbb{R}^3)} \le C\{\|\boldsymbol{u}\|_{L^2(\Omega,\mathbb{R}^3)} + \|\operatorname{curl}\boldsymbol{u}\|_{H^k(\Omega,\mathbb{R}^3)} + \|\operatorname{div}\boldsymbol{u}\|_{H^k(\Omega)} + \|\boldsymbol{u}\times\boldsymbol{\nu}\|_{H^{k+1/2}(\Gamma,\mathbb{R}^3)}\} \tag{2.4}$$

for any $\boldsymbol{u} \in H^{k+1}(\Omega,\mathbb{R}^3)$, respectively. Of course, the right hand sides of (2.3) and (2.4) are estimated by $H^{k+1}(\Omega,\mathbb{R}^3)$ norm of $\boldsymbol{u}$.

Here we note that if Ω is simply connected, we can omit the term $\|\boldsymbol{u}\|_{L^2(\Omega,\mathbb{R}^3)}$ in the right hand side of (2.3), and if Ω has no holes, we can omit the term $\|\boldsymbol{u}\|_{L^2(\Omega,\mathbb{R}^3)}$ in the right hand side of (2.4) (cf. [2] and Bates and Pan [4]).

3. Proof of Theorem 1.1 and Theorem 1.2

In this section, we shall prove Theorem 1.1 and Theorem 1.2. In order to do so, we first show the existence of the minimizer of $b(\mathcal{H}_T)$ defined in (1.8).

Proposition 3.1. $b(\mathcal{H}_T) = \inf_{\boldsymbol{u}\in\Sigma_t(\mathcal{H}_T)} \mathcal{I}_0(\boldsymbol{u})$ *is achieved.*

Proof. Let $\{\boldsymbol{u}_n\} \subset \Sigma_t(\mathcal{H}_T)$ be a minimizing sequence of $b(\mathcal{H}_T)$. Then we can write $\boldsymbol{u}_n = \overline{\boldsymbol{u}} + \boldsymbol{w}_n$ where $\overline{\boldsymbol{u}}$ is a fixed minimizer of (1.5) and $\boldsymbol{w}_n \in \mathcal{H}^1_t(\Omega,\mathbb{R}^3,\mathbf{0})$ satisfies that $\operatorname{curl}\boldsymbol{w}_n = \mathbf{0}$ in Ω. Then we may assume that

$$\int_\Omega \left\{\frac{k_1}{2}|\operatorname{div}\overline{\boldsymbol{u}} + \operatorname{div}\boldsymbol{w}_n|^2 + \psi(|\overline{\boldsymbol{u}} + \boldsymbol{w}_n|)\right\} dx = b(\mathcal{H}_t) + o(1)$$

as $n \to \infty$. Since $\psi \ge 0$, we see that $\{\operatorname{div}\boldsymbol{w}_n\}$ is bounded in $L^2(\Omega)$. Since it follows from the hypothesis (1.1) that $cs^2 \le cM^2 + \psi(s)$. Therefore we have

$$\begin{aligned} c\textstyle\int_\Omega |\overline{\boldsymbol{u}} + \boldsymbol{w}_n|^2 dx &\le cM^2|\Omega| + \int_\Omega \psi(|\overline{\boldsymbol{u}} + \boldsymbol{w}_n|)dx \\ &\le cM^2|\Omega| + b(\mathcal{H}_T) + o(1). \end{aligned} \tag{3.1}$$

It follows that $\{\boldsymbol{w}_n\}$ is bounded in $L^2(\Omega,\mathbb{R}^3)$. Since $\operatorname{curl}\boldsymbol{w}_n = \mathbf{0}$ in Ω and $\boldsymbol{w}_{nT} = \mathbf{0}$ on Γ, it follows from (2.4) that $\{\boldsymbol{w}_n\}$ is boudend in $H^1(\Omega,\mathbb{R}^3)$. Passing to a subsequence, we may assume that $\boldsymbol{w}_n \to \boldsymbol{w}_0$ weakly in $H^1(\Omega,\mathbb{R}^3)$, strongly in $L^2(\Omega,\mathbb{R}^3)$ and a.e. in Ω. Since $\operatorname{curl}\boldsymbol{w}_n = \mathbf{0}$ in Ω and $\boldsymbol{w}_{nT} = \mathbf{0}$ on Γ, we see that $\operatorname{curl}\boldsymbol{w}_0 = \mathbf{0}$ in Ω and $\boldsymbol{w}_{0T} = \mathbf{0}$ on Γ. Thus we have $\boldsymbol{u}_0 := \overline{\boldsymbol{u}} + \boldsymbol{w}_0 \in \Sigma_t(\mathcal{H}_T)$ and $\boldsymbol{u}_n \to \boldsymbol{u}_0$ weakly in $H^1(\Omega,\mathbb{R}^3)$. Therefore, since $\operatorname{div}\boldsymbol{u}_n \to \operatorname{div}\boldsymbol{u}_0$ weakly in $L^2(\Omega)$,

$$\int_\Omega |\operatorname{div}\boldsymbol{u}_0|^2 dx \le \liminf_{n\to\infty} \int_\Omega |\operatorname{div}\boldsymbol{u}_n|^2 dx.$$

Moreover, since $\boldsymbol{u}_n \to \boldsymbol{u}_0$ a.e. in Ω, it follows from the Fatou lemma that

$$\int_\Omega \psi(|\boldsymbol{u}_0|) \le \liminf_{n\to\infty} \int_\Omega \psi(|\boldsymbol{u}_n|)dx.$$

Thus we have

$$\mathcal{I}_0(\boldsymbol{u}_0) \le \liminf_{n\to\infty} \mathcal{I}_0(\boldsymbol{u}_n) = b(\mathcal{H}_T).$$

This means that $\boldsymbol{u}_0$ is a minimizer of $b(\mathcal{H}_T)$. ∎

Remark 3.2. Since the minimizer $\boldsymbol{u}_0$ of $b(\mathcal{H}_T)$ satisfies that $\operatorname{curl}\boldsymbol{u}_0 = \operatorname{curl}\overline{\boldsymbol{u}}$ in Ω and $\boldsymbol{u}_{0T} = \overline{\boldsymbol{u}}_T = \mathcal{H}_T$ on Γ, we can replace $\overline{\boldsymbol{u}}$ in (1.6) with $\boldsymbol{u}_0$. Thus we can assume that $\overline{\boldsymbol{u}}$ in (1.6) satisfies that

$$\mathcal{R}_t(\mathcal{H}_T) = \int_\Omega |\operatorname{curl}\overline{\boldsymbol{u}}|^2 dx, \quad \mathcal{I}_0(\overline{\boldsymbol{u}}) = b(\mathcal{H}_T). \tag{3.2}$$

We are in a position to prove Theorem 1.1. For the brevity of notation, we put $k_2 = 1/\varepsilon^2$ and write

$$\mathcal{W}_\varepsilon(\boldsymbol{u}) = \int_\Omega \left\{ \frac{k_1}{2}|\operatorname{div}\boldsymbol{u}|^2 + \frac{1}{2\varepsilon^2}|\operatorname{curl}\boldsymbol{u}|^2 + \psi(|\boldsymbol{u}|) \right\} dx.$$

Then we have

Proposition 3.3. *Assume that $\mathcal{H}_T \in H^{1/2}(\Gamma, \mathbb{R}^3) \cap L^\infty(\Gamma, \mathbb{R}^3)$. Then*

$$\mathcal{A}_\varepsilon(\mathcal{H}_T) = \inf_{\boldsymbol{u}\in\mathcal{H}_t^1(\Omega,\mathbb{R}^3,\mathcal{H}_T)} \mathcal{W}_\varepsilon(\boldsymbol{u})$$

is achieved.

Proof. Let $\{\boldsymbol{u}_n\} \subset \mathcal{H}_t^1(\Omega, \mathbb{R}^3, \mathcal{H}_T)$ be a minimizing sequence of $\mathcal{A}_\varepsilon(\mathcal{H}_T)$. That is to say $\mathcal{W}_\varepsilon(\boldsymbol{u}_n) \to \mathcal{A}_\varepsilon(\mathcal{H}_T)$ as $n \to \infty$. Then $\{\operatorname{div}\boldsymbol{u}_n\}$ is bounded in $L^2(\Omega)$ and $\{\operatorname{curl}\boldsymbol{u}_n\}$ is bounded in $L^2(\Omega, \mathbb{R}^3)$. Moreover, since $\int_\Omega \psi(|\boldsymbol{u}_n|)dx$ is bounded, $\{\boldsymbol{u}_n\}$ is bounded in $L^2(\Omega, \mathbb{R}^3)$ from (1.1). Since $\boldsymbol{u}_{nT} = \mathcal{H}_T$ on Γ, it follows from (2.4) that $\{\boldsymbol{u}_n\}$ is bounded in $H^1(\Omega, \mathbb{R}^3)$. Passing to a subsequence, we may assume that $\boldsymbol{u}_n \to \boldsymbol{u}_\varepsilon$ weakly in $H^1(\Omega, \mathbb{R}^3)$, strongly in $L^2(\Omega, \mathbb{R}^3)$ and a.e. in Ω. Since $\boldsymbol{u}_{nT} = \mathcal{H}_T$ on Γ, it follows that $\boldsymbol{u}_{\varepsilon T} = \mathcal{H}_T$ on Γ. Thus $\boldsymbol{u}_\varepsilon \in \mathcal{H}_t^1(\Omega, \mathbb{R}^3, \mathcal{H}_T)$. Therefore, we have

$$\mathcal{W}_\varepsilon(\boldsymbol{u}_\varepsilon) \le \liminf_{n\to\infty} \mathcal{W}_\varepsilon(\boldsymbol{u}_n) = \mathcal{A}_\varepsilon(\mathcal{H}_T).$$

Hence $\boldsymbol{u}_\varepsilon$ is a minimizer of $\mathcal{A}_\varepsilon(\mathcal{H}_T)$. ∎

From this proposition, we can easily get Theorem 1.1.

Here we examine the Euler-Lagrange equation for minimizers $\boldsymbol{u}_\varepsilon$ of $\mathcal{A}_\varepsilon(\mathcal{H}_T)$. For any $\boldsymbol{v} \in \mathcal{H}_t^1(\Omega, \mathbb{R}^3, \mathbf{0})$, since $\boldsymbol{u}_\varepsilon + t\boldsymbol{v} \in \mathcal{H}_t^1(\Omega, \mathbb{R}^3, \mathcal{H}_T)$ for any $t \in \mathbb{R}$, we have

$$\begin{aligned} 0 &= \left.\frac{d}{dt}\right|_{t=0} \mathcal{W}_\varepsilon(\boldsymbol{u}_\varepsilon + t\boldsymbol{v}) \\ &= \int_\Omega \Big\{ k_1(\operatorname{div}\boldsymbol{u}_\varepsilon)(\operatorname{div}\boldsymbol{v}) + \frac{1}{\varepsilon^2}\operatorname{curl}\boldsymbol{u}_\varepsilon \cdot \operatorname{curl}\boldsymbol{v} + \frac{\psi'(|\boldsymbol{u}_\varepsilon|)}{|\boldsymbol{u}_\varepsilon|}\boldsymbol{u}_\varepsilon \cdot \boldsymbol{v} \Big\} dx. \end{aligned}$$

In particular, if we take $\boldsymbol{v} \in C_0^\infty(\Omega, \mathbb{R}^3)$, we have, weakly

$$\begin{cases} -k_1 \nabla(\operatorname{div} \boldsymbol{u}_\varepsilon) + \frac{1}{\varepsilon^2}\operatorname{curl}^2 \boldsymbol{u}_\varepsilon + \xi(|\boldsymbol{u}_\varepsilon|)\boldsymbol{u}_\varepsilon = \mathbf{0} & \text{in} \quad \Omega, \\ \boldsymbol{u}_{\varepsilon T} = \mathcal{H}_T & \text{on} \quad \Gamma \end{cases}$$

where $\xi(s) = \psi'(s)/s$.

Proof of Theorem 1.2. Let $\boldsymbol{u}_\varepsilon$ be a minimizer of $\mathcal{W}_\varepsilon$ in $\mathcal{H}^1_t(\Omega, \mathbb{R}^3, \mathcal{H}_T)$. For any $\boldsymbol{u} \in \mathcal{H}^1_t(\Omega, \mathbb{R}^3, \mathcal{H}_T)$, we have

$$\mathcal{I}_0(\boldsymbol{u}) \geq a(\mathcal{H}_T) \quad \text{and} \quad \int_\Omega |\operatorname{curl} \boldsymbol{u}|^2 dx \geq \mathcal{R}_t(\mathcal{H}_T).$$

Thus we see that

$$\mathcal{W}_\varepsilon(\boldsymbol{u}) = \mathcal{I}_0(\boldsymbol{u}) + \frac{1}{2\varepsilon^2}\int_\Omega |\operatorname{curl} \boldsymbol{u}|^2 dx \geq a(\mathcal{H}_T) + \frac{1}{2\varepsilon^2}\mathcal{R}_t(\mathcal{H}_T).$$

Let $\overline{\boldsymbol{u}}$ be a minimizer of (1.5) satisfying (3.2). Then

$$\mathcal{W}_\varepsilon(\overline{\boldsymbol{u}}) = \mathcal{I}_0(\overline{\boldsymbol{u}}) + \frac{1}{2\varepsilon^2}\int_\Omega |\operatorname{curl} \overline{\boldsymbol{u}}|^2 dx = b(\mathcal{H}_T) + \frac{1}{2\varepsilon^2}\mathcal{R}_t(\mathcal{H}_T).$$

Therefore, we have

$$a(\mathcal{H}_T) + \frac{1}{2\varepsilon^2}\mathcal{R}_t(\mathcal{H}_T) \leq \mathcal{A}_\varepsilon(\mathcal{H}_T) = \mathcal{W}_\varepsilon(\boldsymbol{u}_\varepsilon) \leq b(\mathcal{H}_T) + \frac{1}{2\varepsilon^2}\mathcal{R}_t(\mathcal{H}_T). \tag{3.3}$$

Thus we have

$$\begin{aligned} a(\mathcal{H}_T) &\leq \mathcal{I}_0(\boldsymbol{u}_\varepsilon) \\ &= \mathcal{W}_\varepsilon(\boldsymbol{u}_\varepsilon) - \frac{1}{2\varepsilon^2}\int_\Omega |\operatorname{curl} \boldsymbol{u}_\varepsilon|^2 dx \\ &\leq b(\mathcal{H}_T) - \frac{1}{2\varepsilon^2}\left\{\int_\Omega |\operatorname{curl} \boldsymbol{u}_\varepsilon|^2 dx - \mathcal{R}_t(\mathcal{H}_T)\right\} \\ &\leq b(\mathcal{H}_T) = \mathcal{I}_0(\overline{\boldsymbol{u}}). \end{aligned}$$

It follows that

$$a(\mathcal{H}_T) \leq \mathcal{I}_0(\boldsymbol{u}_\varepsilon) \leq b(\mathcal{H}_T). \tag{3.4}$$

We also have

$$\begin{aligned} \int_\Omega |\operatorname{curl} \boldsymbol{u}_\varepsilon|^2 dx &= 2\varepsilon^2[\mathcal{W}_\varepsilon(\boldsymbol{u}_\varepsilon) - \mathcal{I}_0(\boldsymbol{u}_\varepsilon)] \\ &\leq 2\varepsilon^2\left[b(\mathcal{H}_T) + \frac{1}{2\varepsilon^2}\mathcal{R}_t(\mathcal{H}_T) - \mathcal{I}_0(\boldsymbol{u}_\varepsilon)\right] \\ &= \mathcal{R}_t(\mathcal{H}_T) + 2\varepsilon^2[b(\mathcal{H}_T) - \mathcal{I}_0(\boldsymbol{u}_\varepsilon)] \\ &\leq \mathcal{R}_t(\mathcal{H}_T) + 2\varepsilon^2[b(\mathcal{H}_T) - a(\mathcal{H}_T)]. \end{aligned}$$

From this, we have

$$\limsup_{\varepsilon\to 0}\int_\Omega |\mathrm{curl}\,\boldsymbol{u}_\varepsilon|^2 dx \le \mathcal{R}_t(\mathcal{H}_T) \tag{3.5}$$

and so $\{\mathrm{curl}\,\boldsymbol{u}_\varepsilon\}$ is boudend in $L^2(\Omega,\mathbb{R}^3)$. Since

$$\int_\Omega \left\{\frac{k_1}{2}|\mathrm{div}\,\boldsymbol{u}_\varepsilon|^2 + \psi(|\boldsymbol{u}_\varepsilon|)\right\}dx = \mathcal{I}_0(\boldsymbol{u}_\varepsilon) \le b(\mathcal{H}_T),$$

we see that $\{\mathrm{div}\,\boldsymbol{u}_\varepsilon\}$ is bounded in $L^2(\Omega)$ and $\{\boldsymbol{u}_\varepsilon\}$ is bounded in $L^2(\Omega,\mathbb{R}^3)$. Since $\boldsymbol{u}_{\varepsilon T} = \mathcal{H}_T$ on Γ, it follows from (2.4) that $\{\boldsymbol{u}_\varepsilon\}$ is bounded in $H^1(\Omega,\mathbb{R}^3)$. Passing to a subsequence, we may assume that $\boldsymbol{u}_\varepsilon \to \boldsymbol{u}^*$ weakly in $H^1(\Omega,\mathbb{R}^3)$, strongly in $L^2(\Omega,\mathbb{R}^3)$ and a.e. in Ω. Since $\mathrm{curl}\,\boldsymbol{u}_\varepsilon \to \mathrm{curl}\,\boldsymbol{u}^*$ weakly in $L^2(\Omega,\mathbb{R}^3)$, it follows from (3.5) that

$$\begin{aligned}\int_\Omega |\mathrm{curl}\,\boldsymbol{u}^*|^2 dx &\le \liminf_{\varepsilon\to 0}\int_\Omega |\mathrm{curl}\,\boldsymbol{u}_\varepsilon|^2 dx\\ &\le \limsup_{\varepsilon\to 0}\int_\Omega |\mathrm{curl}\,\boldsymbol{u}_\varepsilon|^2 dx\\ &\le \mathcal{R}_t(\mathcal{H}_T).\end{aligned}$$

Since $\boldsymbol{u}^*_T = \mathcal{H}_T$, we see that $\boldsymbol{u}^* \in \mathcal{H}^1_t(\Omega,\mathbb{R}^3,\mathcal{H}_T)$ and we have

$$\mathcal{R}_t(\mathcal{H}_T) \le \int_\Omega |\mathrm{curl}\,\boldsymbol{u}^*|^2 dx,$$

we get

$$\mathcal{R}_t(\mathcal{H}_T) = \int_\Omega |\mathrm{curl}\,\boldsymbol{u}^*|^2 dx.$$

Therefore, we have $\boldsymbol{u}^* \in \Sigma_t(\mathcal{H}_T)$. Since $\boldsymbol{u}_\varepsilon \to \boldsymbol{u}^*$ a.e. in Ω, we have

$$\int_\Omega \psi(|\boldsymbol{u}^*|)dx \le \liminf_{\varepsilon\to 0}\int_\Omega \psi(|\boldsymbol{u}_\varepsilon|)dx.$$

Hence we see that

$$b(\mathcal{H}_T) \le \mathcal{I}_0(\boldsymbol{u}^*) \le \liminf_{\varepsilon\to 0}\mathcal{I}_0(\boldsymbol{u}_\varepsilon) \le \limsup_{\varepsilon\to 0}\mathcal{I}_0(\boldsymbol{u}_\varepsilon) \le b(\mathcal{H}_T).$$

Thus since we have $\mathcal{I}_0(\boldsymbol{u}^*) = b(\mathcal{H}_T)$, we see that $\boldsymbol{u}^*$ satisfies (3.2). So, we can take $\boldsymbol{u}^* = \overline{\boldsymbol{u}}$ in (1.6) and $\boldsymbol{u}_\varepsilon \to \overline{\boldsymbol{u}}$ weakly in $H^1(\Omega,\mathbb{R}^3)$, strongly in $L^2(\Omega,\mathbb{R}^3)$ and a.e. in Ω, and $\mathcal{I}_0(\boldsymbol{u}_\varepsilon) \to b(\mathcal{H}_T) = \mathcal{I}_0(\overline{\boldsymbol{u}})$. We have

$$\begin{aligned}\frac{k_1}{2}\int_\Omega |\operatorname{div}\overline{\boldsymbol{u}}|^2 dx &= \mathcal{I}_0(\overline{\boldsymbol{u}}) - \int_\Omega \psi(|\overline{\boldsymbol{u}}|)dx \\ &\geq \mathcal{I}_0(\overline{\boldsymbol{u}}) - \liminf_{\varepsilon\to 0}\int_\Omega \psi(|\boldsymbol{u}_\varepsilon|)dx \\ &= \mathcal{I}_0(\overline{\boldsymbol{u}}) - \liminf_{\varepsilon\to 0}\left\{\mathcal{I}_0(\boldsymbol{u}_\varepsilon) - \frac{k_1}{2}\int_\Omega |\operatorname{div}\boldsymbol{u}_\varepsilon|^2 dx\right\} \\ &= \mathcal{I}_0(\overline{\boldsymbol{u}}) - \lim_{\varepsilon\to 0}\mathcal{I}_0(\boldsymbol{u}_\varepsilon) + \limsup_{\varepsilon\to 0}\frac{k_1}{2}\int_\Omega |\operatorname{div}\boldsymbol{u}_\varepsilon|^2 dx \\ &= \limsup_{\varepsilon\to 0}\frac{k_1}{2}\int_\Omega |\operatorname{div}\boldsymbol{u}_\varepsilon|^2 dx.\end{aligned}$$

Therefore, we have

$$\lim_{\varepsilon\to 0}\int_\Omega |\operatorname{div}\boldsymbol{u}_\varepsilon|^2 dx = \int_\Omega |\operatorname{div}\overline{\boldsymbol{u}}|^2 dx$$

and

$$\lim_{\varepsilon\to 0}\int_\Omega |\operatorname{curl}\boldsymbol{u}_\varepsilon|^2 dx = \int_\Omega |\operatorname{curl}\overline{\boldsymbol{u}}|^2 dx.$$

Since $\operatorname{div}\boldsymbol{u}_\varepsilon \to \operatorname{div}\overline{\boldsymbol{u}}$ weakly in $L^2(\Omega)$ and $\operatorname{curl}\boldsymbol{u}_\varepsilon \to \operatorname{curl}\overline{\boldsymbol{u}}$ weakly in $L^2(\Omega,\mathbb{R}^3)$, we see that $\operatorname{div}\boldsymbol{u}_\varepsilon \to \operatorname{div}\overline{\boldsymbol{u}}$ strongly in $L^2(\Omega)$ and $\operatorname{curl}\boldsymbol{u}_\varepsilon \to \operatorname{curl}\overline{\boldsymbol{u}}$ strongly in $L^2(\Omega,\mathbb{R}^3)$. Since $\boldsymbol{u}_\varepsilon \to \overline{\boldsymbol{u}}$ strongly in $L^2(\Omega,\mathbb{R}^3)$ and $\boldsymbol{u}_{\varepsilon T} = \overline{\boldsymbol{u}}_T$ on Γ, it follows from (2.4) that $\boldsymbol{u}_\varepsilon \to \overline{\boldsymbol{u}}$ strongly in $H^1(\Omega,\mathbb{R}^3)$. This completes the proof of Theorem 1.2.

We get an immediate consequence of Theorem 1.2.

Corollary 3.4. *Assume that the hypotheses of Theorem 1.2 hold and let $\boldsymbol{u}_\varepsilon$ be a minimizer of $\mathcal{W}_\varepsilon$. Then we have the following.*

(i) The total energy $\mathcal{W}_\varepsilon(\boldsymbol{u}_\varepsilon)$ is bounded as $\varepsilon \to 0$, if and only if $\mathcal{R}_t(\mathcal{H}_T) = 0$.

(ii) If $\mathcal{R}_t(\mathcal{H}_T) = 0$, for any convergent sequence of $\{\boldsymbol{u}_\varepsilon\}$ with $\varepsilon \to 0$, there exist $\overline{\phi} \in H^2(\Omega)$ and $\boldsymbol{h} \in \mathbb{H}_1(\Omega)$ such that $\boldsymbol{u}_\varepsilon \to \overline{\boldsymbol{u}} := \nabla\overline{\phi} + \boldsymbol{h}$ strongly in $H^1(\Omega,\mathbb{R}^3)$ and $(\nabla\overline{\phi} + \boldsymbol{h})_T = \mathcal{H}_T$ on Γ.

In particular case where Ω is simply connected, we can write $\overline{\boldsymbol{u}} = \nabla\overline{\phi}$ where $\overline{\phi} \in H^2(\Omega)$ and satisfies that $(\nabla\overline{\phi})_T = \mathcal{H}_T$ on Γ.

Proof. (i) Since

$$\int_\Omega |\operatorname{curl}\boldsymbol{u}_\varepsilon|^2 dx = 2\varepsilon^2[\mathcal{W}_\varepsilon(\boldsymbol{u}_\varepsilon) - \mathcal{I}_0(\boldsymbol{u}_\varepsilon)] \leq 2\varepsilon^2\mathcal{W}_\varepsilon(\boldsymbol{u}_\varepsilon),$$

if $\mathcal{W}_\varepsilon(\boldsymbol{u}_\varepsilon)$ is bounded,

$$\mathcal{R}_t(\mathcal{H}_T) = \int_\Omega |\operatorname{curl}\overline{\boldsymbol{u}}|^2 dx \leq \liminf_{\varepsilon\to 0}\int_\Omega |\operatorname{curl}\boldsymbol{u}_\varepsilon|^2 dx = 0.$$

Conversely, if $\mathcal{R}_t(\mathcal{H}_T) = 0$, then from (3.3) $\mathcal{W}_\varepsilon(\boldsymbol{u}_\varepsilon) \le b(\mathcal{H}_T)$. Thus $\mathcal{W}_\varepsilon(\boldsymbol{u}_\varepsilon)$ is bounded.

(ii) If $\mathcal{R}_t(\mathcal{H}_T) = 0$, then $\operatorname{curl} \overline{\boldsymbol{u}} = \mathbf{0}$ in Ω. Therefore, it is well known (cf. [6]) that we can write $\overline{\boldsymbol{u}} = \nabla\overline{\phi} + \boldsymbol{h}$ where $\overline{\phi} \in H^2(\Omega)$ and $\boldsymbol{h} \in \mathbb{H}_1(\Omega)$. If Ω is simply connected, it suffices to note that $\mathbb{H}_1(\Omega) = \{\mathbf{0}\}$. ■

From Corollary 3.4, we see that the problems (Q.1) and (Q.2) are yes, if and only if $\mathcal{R}_t(\mathcal{H}_T) = 0$.

4. Proof of Theorem 1.3 and Theorem 1.4

In this section, we consider the case where a given vector field $\mathcal{H}$ on Γ satisfies $\mathcal{H}_n = \mathcal{H}\cdot\boldsymbol{\nu} \in H^{1/2}(\Gamma) \cap L^\infty(\Gamma)$. We define

$$\mathcal{H}^1_n(\Omega, \mathbb{R}^3, \mathcal{H}_n) = \{\boldsymbol{u} \in H^1(\Omega, \mathbb{R}^3); \boldsymbol{u}\cdot\boldsymbol{\nu}\big|_\Gamma = \mathcal{H}_n\}$$

and

$$\mathcal{D}_n(\mathcal{H}_n) = \inf_{\boldsymbol{u}\in\mathcal{H}^1_n(\Omega,\mathbb{R}^3,\mathcal{H}_n)} \|\operatorname{div} \boldsymbol{u}\|^2_{L^2(\Omega)}. \tag{4.1}$$

According to [2], we see that $\mathcal{D}_n(\mathcal{H}_n)$ is achieved and the set $\Sigma_n(\mathcal{H}_n)$ of all minimizers is given by

$$\Sigma_n(\mathcal{H}_n) = \{\overline{\boldsymbol{u}} + \boldsymbol{w}; \boldsymbol{w} \in \mathcal{H}^1_n(\Omega, \mathbb{R}^3, \mathbf{0}), \operatorname{div} \boldsymbol{w} = 0\} \tag{4.2}$$

where $\overline{\boldsymbol{u}}$ is a fixed minimizer of $\mathcal{D}_n(\mathcal{H}_n)$ in $\mathcal{H}^1_n(\Omega, \mathbb{R}^3, \mathcal{H}_n)$. As in section 3, we define

$$\mathcal{J}_0(\boldsymbol{u}) = \int_\Omega \left\{\frac{k_2}{2}|\operatorname{curl} \boldsymbol{u}|^2 dx + \psi(|\boldsymbol{u}|)\right\} dx,$$

and

$$c(\mathcal{H}_n) = \inf_{\boldsymbol{u}\in\mathcal{H}^1_n(\Omega,\mathbb{R}^3,\mathcal{H}_n)} \mathcal{J}_0(\boldsymbol{u}),$$

and

$$d(\mathcal{H}_n) = \inf_{\boldsymbol{u}\in\Sigma_n(\mathcal{H}_n)} \mathcal{J}_0(\boldsymbol{u}).$$

Proposition 4.1. $d(\mathcal{H}_T) = \inf_{\boldsymbol{u}\in\Sigma_n(\mathcal{H}_n)} \mathcal{J}_0(\boldsymbol{u})$ *is achieved.*

Proof. Let $\{\boldsymbol{u}_k\} \subset \Sigma_n(\mathcal{H}_n)$ be a minimizing sequence of $d(\mathcal{H}_n)$. Then we can write $\boldsymbol{u}_k = \overline{\boldsymbol{u}} + \boldsymbol{w}_k$ where $\overline{\boldsymbol{u}}$ is a fixed minimizer of (4.1) and $\boldsymbol{w}_k \in \mathcal{H}^1_n(\Omega, \mathbb{R}^3, \mathbf{0})$ satisfies $\operatorname{div} \boldsymbol{w}_k = 0$ in Ω. Then we may assume that

$$\int_\Omega \left\{\frac{k_2}{2}|\operatorname{curl} \overline{\boldsymbol{u}} + \operatorname{curl} \boldsymbol{w}_k|^2 + \psi(|\overline{\boldsymbol{u}} + \boldsymbol{w}_k|)\right\} dx = d(\mathcal{H}_n) + o(1)$$

as $k \to \infty$. Since $\psi \geq 0$, we see that $\{\operatorname{curl} \boldsymbol{w}_k\}$ is bounded in $L^2(\Omega, \mathbb{R}^3)$. From the hypothesis (1.1) and (3.1),

$$c \int_\Omega |\overline{\boldsymbol{u}} + \boldsymbol{w}_k|^2 dx \leq cM^2|\Omega| + \int_\Omega \psi(|\overline{\boldsymbol{u}} + \boldsymbol{w}_k|)dx \leq cM^2|\Omega| + d(\mathcal{H}_n) + o(1).$$

It follows that $\{\boldsymbol{w}_k\}$ is bounded in $L^2(\Omega, \mathbb{R}^3)$. Since $\operatorname{div} \boldsymbol{w}_k = 0$ in Ω and $\boldsymbol{w}_k \cdot \boldsymbol{\nu} = 0$ on Γ, it follows from (2.3) that $\{\boldsymbol{w}_k\}$ is bounded in $H^1(\Omega, \mathbb{R}^3)$. Passing to a subsequence, we may assume that $\boldsymbol{w}_k \to \boldsymbol{w}_0$ weakly in $H^1(\Omega, \mathbb{R}^3)$, strongly in $L^2(\Omega, \mathbb{R}^3)$ and a.e in Ω. Since $\operatorname{div} \boldsymbol{w}_k = 0$ in Ω and $\boldsymbol{w}_k \cdot \boldsymbol{\nu} = 0$ on Γ, we see that $\operatorname{div} \boldsymbol{w}_0 = 0$ in Ω and $\boldsymbol{w}_0 \cdot \boldsymbol{\nu} = 0$ on Γ. Thus we have $\boldsymbol{u}_0 := \overline{\boldsymbol{u}} + \boldsymbol{w}_0 \in \Sigma_n(\mathcal{H}_n)$ and $\boldsymbol{u}_k \to \boldsymbol{u}_0$ weakly in $H^1(\Omega, \mathbb{R}^3)$. Therefore, since $\operatorname{curl} \boldsymbol{u}_k \to \operatorname{curl} \boldsymbol{u}_0$ weakly in $L^2(\Omega, \mathbb{R}^3)$,

$$\int_\Omega |\operatorname{curl} \boldsymbol{u}_0|^2 dx \leq \liminf_{k \to \infty} \int_\Omega |\operatorname{curl} \boldsymbol{u}_k|^2 dx.$$

Moreover, since $\boldsymbol{u}_k \to \boldsymbol{u}_0$ a.e. in Ω, it follows from the Fatou lemma that

$$\int_\Omega \psi(|\boldsymbol{u}_0|) \leq \liminf_{k \to \infty} \int_\Omega \psi(|\boldsymbol{u}_k|)dx.$$

Thus we have

$$\mathcal{J}_0(\boldsymbol{u}_0) \leq \liminf_{k \to \infty} \mathcal{J}_0(\boldsymbol{u}_k) = d(\mathcal{H}_n).$$

This means that $\boldsymbol{u}_0$ is a minimizer of $d(\mathcal{H}_n)$. ■

Remark 4.2. Since the minimizer $\boldsymbol{u}_0$ of $d(\mathcal{H}_n)$ satisfies that $\operatorname{div} \boldsymbol{u}_0 = \operatorname{div} \overline{\boldsymbol{u}}$ in Ω and $\boldsymbol{u}_0 \cdot \boldsymbol{\nu} = \overline{\boldsymbol{u}} \cdot \boldsymbol{\nu} = \mathcal{H}_n$ on Γ, we can replace $\overline{\boldsymbol{u}}$ in (4.2) with $\boldsymbol{u}_0$. Thus we can assume that $\overline{\boldsymbol{u}}$ in (4.2) satisfies that

$$\mathcal{D}_n(\mathcal{H}_n) = \int_\Omega |\operatorname{div} \overline{\boldsymbol{u}}|^2 dx, \quad \mathcal{J}_0(\overline{\boldsymbol{u}}) = d(\mathcal{H}_n). \tag{4.3}$$

We are in a position to prove Theorem 1.3. For the brevity of notation, we put $k_1 = 1/\varepsilon^2$ and write

$$\mathcal{V}_\varepsilon(\boldsymbol{u}) := \int_\Omega \left\{ \frac{1}{2\varepsilon^2}|\operatorname{div} \boldsymbol{u}|^2 + \frac{k_2}{2}|\operatorname{curl} \boldsymbol{u}|^2 + \psi(|\boldsymbol{u}|) \right\} dx.$$

Then we have

Proposition 4.3. *Assume that* $\mathcal{H}_n \in H^{1/2}(\Gamma) \cap L^\infty(\Gamma)$. *Then*

$$\mathcal{B}_\varepsilon(\mathcal{H}_n) := \inf_{\boldsymbol{u} \in \mathcal{H}_n^1(\Omega, \mathbb{R}^3, \mathcal{H}_n)} \mathcal{V}_\varepsilon(\boldsymbol{u})$$

is achieved.

Proof. Let $\{\boldsymbol{u}_k\} \subset \mathcal{H}^1_n(\Omega, \mathbb{R}^3, \mathcal{H}_n)$ be a minimizing sequence of $\mathcal{B}_\varepsilon(\mathcal{H}_n)$. That is to say $\mathcal{V}_\varepsilon(\boldsymbol{u}_k) \to \mathcal{B}_\varepsilon(\mathcal{H}_n)$ as $k \to \infty$. Then $\{\operatorname{curl} \boldsymbol{u}_k\}$ is bounded in $L^2(\Omega, \mathbb{R}^3)$ and $\{\operatorname{div} \boldsymbol{u}_k\}$ is bounded in $L^2(\Omega)$. Moreover, since $\int_\Omega \psi(|\boldsymbol{u}_k|)dx$ is bounded, $\{\boldsymbol{u}_k\}$ is bounded in $L^2(\Omega, \mathbb{R}^3)$ from (1.1). Since $\boldsymbol{u}_k \cdot \boldsymbol{\nu} = \mathcal{H}_n$ on Γ, it follows from (2.3) that $\{\boldsymbol{u}_k\}$ is bounded in $H^1(\Omega, \mathbb{R}^3)$. Passing to a subsequence, we may assume that $\boldsymbol{u}_k \to \boldsymbol{u}_\varepsilon$ weakly in $H^1(\Omega, \mathbb{R}^3)$, strongly in $L^2(\Omega, \mathbb{R}^3)$ and a.e. in Ω. Since $\boldsymbol{u}_k \cdot \boldsymbol{\nu} = \mathcal{H}_n$ on Γ, it follows that $\boldsymbol{u}_\varepsilon \cdot \boldsymbol{\nu} = \mathcal{H}_n$ on Γ. Thus $\boldsymbol{u}_\varepsilon \in \mathcal{H}^1_n(\Omega, \mathbb{R}^3, \mathcal{H}_n)$. Therefore, we have

$$\mathcal{V}_\varepsilon(\boldsymbol{u}_\varepsilon) \leq \liminf_{k\to\infty} \mathcal{V}_\varepsilon(\boldsymbol{u}_k) = \mathcal{B}_\varepsilon(\mathcal{H}_n).$$

Hence $\boldsymbol{u}_\varepsilon$ is a minimizer of $\mathcal{B}_\varepsilon(\mathcal{H}_n)$. ■

From this proposition, we can easily get Theorem 1.3.

Here we examine the Euler-Lagrange equation for minimizers $\boldsymbol{u}_\varepsilon$ of $\mathcal{B}_\varepsilon(\mathcal{H}_n)$. For any $\boldsymbol{v} \in \mathcal{H}^1_n(\Omega, \mathbb{R}^3, \mathbf{0})$, since $\boldsymbol{u}_\varepsilon + t\boldsymbol{v} \in \mathcal{H}^1_n(\Omega, \mathbb{R}^3, \mathcal{H}_n)$ for any $t \in \mathbb{R}$, we have

$$\begin{aligned} 0 = \frac{d}{dt}\Big|_{t=0} & \mathcal{V}_\varepsilon(\boldsymbol{u}_\varepsilon + t\boldsymbol{v}) \\ &= \int_\Omega \Big\{\frac{1}{\varepsilon^2}(\operatorname{div} \boldsymbol{u}_\varepsilon)(\operatorname{div} \boldsymbol{v}) + k_2 \operatorname{curl} \boldsymbol{u}_\varepsilon \cdot \operatorname{curl} \boldsymbol{v} + \frac{\psi'(|\boldsymbol{u}_\varepsilon|)}{|\boldsymbol{u}_\varepsilon|}\boldsymbol{u}_\varepsilon \cdot \boldsymbol{v}\Big\} dx. \end{aligned}$$

In particular, if we take $\boldsymbol{v} \in C^\infty_0(\Omega, \mathbb{R}^3)$, we have, weakly

$$\begin{cases} -\frac{1}{\varepsilon^2}\nabla(\operatorname{div} \boldsymbol{u}_\varepsilon) + k_2 \operatorname{curl}^2 \boldsymbol{u}_\varepsilon + \xi(|\boldsymbol{u}_\varepsilon|)\boldsymbol{u}_\varepsilon = \mathbf{0} & \text{in} \quad \Omega, \\ \boldsymbol{u}_\varepsilon \cdot \boldsymbol{\nu} = \mathcal{H}_n & \text{on} \quad \Gamma \end{cases}$$

where $\xi(s) = \psi'(s)/s$.

Proof of Theorem 1.4. Let $\boldsymbol{u}_\varepsilon$ be a minimizer of $\mathcal{V}_\varepsilon$ in $\mathcal{H}^1_n(\Omega, \mathbb{R}^3, \mathcal{H}_n)$. For any $\boldsymbol{u} \in \mathcal{H}^1_n(\Omega, \mathbb{R}^3, \mathcal{H}_n)$, we have

$$\mathcal{J}_0(\boldsymbol{u}) \geq c(\mathcal{H}_n) \quad \text{and} \quad \int_\Omega |\operatorname{div} \boldsymbol{u}|^2 dx \geq \mathcal{D}_n(\mathcal{H}_n).$$

Thus we see that

$$\mathcal{V}_\varepsilon(\boldsymbol{u}) = \mathcal{J}_0(\boldsymbol{u}) + \frac{1}{2\varepsilon^2}\int_\Omega |\operatorname{div} \boldsymbol{u}|^2 dx \geq c(\mathcal{H}_n) + \frac{1}{2\varepsilon^2}\mathcal{D}_n(\mathcal{H}_n).$$

Let $\overline{\boldsymbol{u}}$ be a minimizer of (4.1) satisfying (4.3). Then

$$\mathcal{V}_\varepsilon(\overline{\boldsymbol{u}}) = \mathcal{J}_0(\overline{\boldsymbol{u}}) + \frac{1}{2\varepsilon^2}\int_\Omega |\operatorname{div} \overline{\boldsymbol{u}}|^2 dx = d(\mathcal{H}_n) + \frac{1}{2\varepsilon^2}\mathcal{D}_n(\mathcal{H}_n).$$

Therefore, we have

$$c(\mathcal{H}_n) + \frac{1}{2\varepsilon^2}\mathcal{D}_n(\mathcal{H}_n) \le \mathcal{B}_\varepsilon(\mathcal{H}_n) = \mathcal{V}_\varepsilon(\boldsymbol{u}_\varepsilon) \le d(\mathcal{H}_n) + \frac{1}{2\varepsilon^2}\mathcal{D}_n(\mathcal{H}_n). \tag{4.4}$$

Thus we have

$$\begin{aligned} c(\mathcal{H}_n) &\le \mathcal{J}_0(\boldsymbol{u}_\varepsilon) \\ &= \mathcal{V}_\varepsilon(\boldsymbol{u}_\varepsilon) - \frac{1}{2\varepsilon^2}\int_\Omega |\operatorname{div} \boldsymbol{u}_\varepsilon|^2 dx \\ &\le d(\mathcal{H}_n) - \frac{1}{2\varepsilon^2}\left\{\int_\Omega |\operatorname{div} \boldsymbol{u}_\varepsilon|^2 dx - \mathcal{D}_n(\mathcal{H}_n)\right\} \\ &\le d(\mathcal{H}_n) = \mathcal{J}_0(\overline{\boldsymbol{u}}). \end{aligned}$$

From this inequality, we see that

$$c(\mathcal{H}_n) \le \mathcal{J}_0(\boldsymbol{u}_\varepsilon) \le d(\mathcal{H}_n). \tag{4.5}$$

We also have

$$\begin{aligned} \int_\Omega |\operatorname{div} \boldsymbol{u}_\varepsilon|^2 dx &= 2\varepsilon^2[\mathcal{V}_\varepsilon(\boldsymbol{u}_\varepsilon) - \mathcal{J}_0(\boldsymbol{u}_\varepsilon)] \\ &\le 2\varepsilon^2\left[d(\mathcal{H}_n) + \frac{1}{2\varepsilon^2}\mathcal{D}_n(\mathcal{H}_n) - \mathcal{J}_0(\boldsymbol{u}_\varepsilon)\right] \\ &= \mathcal{D}_n(\mathcal{H}_n) + 2\varepsilon^2[d(\mathcal{H}_n) - \mathcal{J}_0(\boldsymbol{u}_\varepsilon)] \\ &\le \mathcal{D}_n(\mathcal{H}_n) + 2\varepsilon^2[d(\mathcal{H}_n) - c(\mathcal{H}_n)]. \end{aligned}$$

From this, we have

$$\limsup_{\varepsilon\to 0}\int_\Omega |\operatorname{div} \boldsymbol{u}_\varepsilon|^2 dx \le \mathcal{D}_n(\mathcal{H}_n) \tag{4.6}$$

and so $\{\operatorname{div} \boldsymbol{u}_\varepsilon\}$ is bounded in $L^2(\Omega)$. Since

$$\int_\Omega \left\{\frac{k_2}{2}|\operatorname{curl} \boldsymbol{u}_\varepsilon|^2 + \psi(|\boldsymbol{u}_\varepsilon|)\right\}dx = \mathcal{J}_0(\boldsymbol{u}_\varepsilon) \le d(\mathcal{H}_n),$$

we see that $\{\operatorname{curl} \boldsymbol{u}_\varepsilon\}$ is bounded in $L^2(\Omega, \mathbb{R}^3)$ and $\{\boldsymbol{u}_\varepsilon\}$ is bounded in $L^2(\Omega, \mathbb{R}^3)$. Since $\boldsymbol{u}_\varepsilon \cdot \boldsymbol{\nu} = \mathcal{H}_n$ on Γ, it follows from (2.3) that $\{\boldsymbol{u}_\varepsilon\}$ is bounded in $H^1(\Omega, \mathbb{R}^3)$. Passing to a subsequence, we may assume that $\boldsymbol{u}_\varepsilon \to \boldsymbol{u}^*$ weakly in $H^1(\Omega, \mathbb{R}^3)$, strongly in $L^2(\Omega, \mathbb{R}^3)$ and a.e. in Ω. Since $\operatorname{div} \boldsymbol{u}_\varepsilon \to \operatorname{div} \boldsymbol{u}^*$ weakly in $L^2(\Omega)$, it follows from (4.6) that

$$\begin{aligned} \int_\Omega |\operatorname{div} \boldsymbol{u}^*|^2 dx &\le \liminf_{\varepsilon\to 0}\int_\Omega |\operatorname{div} \boldsymbol{u}_\varepsilon|^2 dx \\ &\le \limsup_{\varepsilon\to 0}\int_\Omega |\operatorname{div} \boldsymbol{u}_\varepsilon|^2 dx \\ &\le \mathcal{D}_n(\mathcal{H}_n). \end{aligned}$$

Since $\boldsymbol{u}^* \cdot \boldsymbol{\nu} = \mathcal{H}_n$ on Γ, we see that $\boldsymbol{u}^* \in \mathcal{H}^1_n(\Omega, \mathbb{R}^3, \mathcal{H}_n)$. Thus since we have

$$\mathcal{D}_n(\mathcal{H}_n) \le \int_\Omega |\mathrm{div}\, \boldsymbol{u}^*|^2 dx,$$

we get

$$\mathcal{D}_n(\mathcal{H}_n) = \int_\Omega |\mathrm{div}\, \boldsymbol{u}^*|^2 dx.$$

Therefore, we have $\boldsymbol{u}^* \in \Sigma_n(\mathcal{H}_n)$. Since $\boldsymbol{u}_\varepsilon \to \boldsymbol{u}^*$ a.e. in Ω, we have

$$\int_\Omega \psi(|\boldsymbol{u}^*|)dx \le \liminf_{\varepsilon\to 0} \int_\Omega \psi(|\boldsymbol{u}_\varepsilon|)dx.$$

Hence we see that

$$d(\mathcal{H}_n) \le \mathcal{J}_0(\boldsymbol{u}^*) \le \liminf_{\varepsilon\to 0} \mathcal{J}_0(\boldsymbol{u}_\varepsilon) \le \limsup_{\varepsilon\to 0} \mathcal{J}_0(\boldsymbol{u}_\varepsilon) \le d(\mathcal{H}_n).$$

Thus since we have $\mathcal{J}_0(\boldsymbol{u}^*) = d(\mathcal{H}_n)$, we see that $\boldsymbol{u}^*$ satisfies (4.3). So, we can take $\boldsymbol{u}^* = \overline{\boldsymbol{u}}$ in (4.2) and $\boldsymbol{u}_\varepsilon \to \overline{\boldsymbol{u}}$ weakly in $H^1(\Omega, \mathbb{R}^3)$, strongly in $L^2(\Omega, \mathbb{R}^3)$ and a.e. in Ω, and $\mathcal{J}_0(\boldsymbol{u}_\varepsilon) \to d(\mathcal{H}_n) = \mathcal{J}_0(\overline{\boldsymbol{u}})$. We have

$$\begin{aligned}
\frac{k_2}{2}\int_\Omega |\mathrm{curl}\, \overline{\boldsymbol{u}}|^2 dx &= \mathcal{J}_0(\overline{\boldsymbol{u}}) - \int_\Omega \psi(|\overline{\boldsymbol{u}}|)dx \\
&\ge \mathcal{J}_0(\overline{\boldsymbol{u}}) - \liminf_{\varepsilon\to 0} \int_\Omega \psi(|\boldsymbol{u}_\varepsilon|)dx \\
&= \mathcal{J}_0(\overline{\boldsymbol{u}}) - \liminf_{\varepsilon\to 0} \left\{ \mathcal{J}_0(\boldsymbol{u}_\varepsilon) - \frac{k_2}{2}\int_\Omega |\mathrm{curl}\, \boldsymbol{u}_\varepsilon|^2 dx \right\} \\
&= \mathcal{J}_0(\overline{\boldsymbol{u}}) - \lim_{\varepsilon\to 0} \mathcal{J}_0(\boldsymbol{u}_\varepsilon) + \limsup_{\varepsilon\to 0} \frac{k_2}{2}\int_\Omega |\mathrm{curl}\, \boldsymbol{u}_\varepsilon|^2 dx \\
&= \limsup_{\varepsilon\to 0} \frac{k_2}{2}\int_\Omega |\mathrm{curl}\, \boldsymbol{u}_\varepsilon|^2 dx.
\end{aligned}$$

Since $\mathrm{curl}\, \boldsymbol{u}_\varepsilon \to \mathrm{curl}\, \overline{\boldsymbol{u}}$ weakly in $L^2(\Omega, \mathbb{R}^3)$, we have

$$\int_\Omega |\mathrm{curl}\, \overline{\boldsymbol{u}}|^2 dx \le \liminf_{\varepsilon\to 0} \int_\Omega |\mathrm{curl}\, \boldsymbol{u}_\varepsilon|^2 dx.$$

Therefore, we have

$$\lim_{\varepsilon\to 0} \int_\Omega |\mathrm{curl}\, \boldsymbol{u}_\varepsilon|^2 dx = \int_\Omega |\mathrm{curl}\, \overline{\boldsymbol{u}}|^2 dx$$

and we also have

$$\lim_{\varepsilon\to 0} \int_\Omega |\mathrm{div}\, \boldsymbol{u}_\varepsilon|^2 dx = \int_\Omega |\mathrm{div}\, \overline{\boldsymbol{u}}|^2 dx.$$

Since $\operatorname{curl}\boldsymbol{u}_\varepsilon \to \operatorname{curl}\overline{\boldsymbol{u}}$ weakly in $L^2(\Omega, \mathbb{R}^3)$ and $\operatorname{div}\boldsymbol{u}_\varepsilon \to \operatorname{div}\overline{\boldsymbol{u}}$ weakly in $L^2(\Omega)$, we see that $\operatorname{curl}\boldsymbol{u}_\varepsilon \to \operatorname{curl}\overline{\boldsymbol{u}}$ strongly in $L^2(\Omega, \mathbb{R}^3)$ and $\operatorname{div}\boldsymbol{u}_\varepsilon \to \operatorname{div}\overline{\boldsymbol{u}}$ strongly in $L^2(\Omega)$. Since $\boldsymbol{u}_\varepsilon \to \overline{\boldsymbol{u}}$ strongly in $L^2(\Omega, \mathbb{R}^3)$ and $\boldsymbol{u}_\varepsilon \cdot \boldsymbol{\nu} = \overline{\boldsymbol{u}} \cdot \boldsymbol{\nu}$ on Γ, it follows from (2.3) that $\boldsymbol{u}_\varepsilon \to \overline{\boldsymbol{u}}$ strongly in $H^1(\Omega, \mathbb{R}^3)$. This completes the proof of Theorem 1.4.

We get an immediate consequence of Theorem 1.4.

Corollary 4.4. *Assume that the hypotheses of Theorem 1.4 hold and let* $\boldsymbol{u}_\varepsilon$ *be a minimizer of* $\mathcal{V}_\varepsilon$. *Then we have the following.*

(i) *The total energy* $\mathcal{V}_\varepsilon(\boldsymbol{u}_\varepsilon)$ *is bounded as* $\varepsilon \to 0$, *if and only if* $\mathcal{D}_n(\mathcal{H}_n) = 0$.

(ii) *If* $\mathcal{D}_n(\mathcal{H}_n) = 0$, *for any convergent sequence of* $\{\boldsymbol{u}_\varepsilon\}$ *with* $\varepsilon \to 0$, *there exist* $\overline{\boldsymbol{w}} \in H^2(\Omega, \mathbb{R}^3)$ *and* $\boldsymbol{h} \in \mathbb{H}_2(\Omega)$ *such that* $\boldsymbol{u}_\varepsilon \to \overline{\boldsymbol{u}} := \operatorname{curl}\overline{\boldsymbol{w}} + \boldsymbol{h}$ *strongly in* $H^1(\Omega, \mathbb{R}^3)$ *and* $(\operatorname{curl}\overline{\boldsymbol{w}} + \boldsymbol{h}) \cdot \boldsymbol{\nu} = \mathcal{H}_n$ *on* Γ.

In particular case where Ω *has no holes, the above* $\overline{\boldsymbol{u}}$ *can be written by* $\overline{\boldsymbol{u}} = \operatorname{curl}\overline{\boldsymbol{w}}$ *where* $\boldsymbol{\nu} \cdot \operatorname{curl}\overline{\boldsymbol{w}} = \mathcal{H}_n$ *on* Γ.

Proof.

(i) Since
$$\int_\Omega |\operatorname{div}\boldsymbol{u}_\varepsilon|^2 dx = 2\varepsilon^2[\mathcal{V}_\varepsilon(\boldsymbol{u}_\varepsilon) - \mathcal{J}_0(\boldsymbol{u}_\varepsilon)] \le 2\varepsilon^2 \mathcal{V}_\varepsilon(\boldsymbol{u}_\varepsilon),$$
if $\mathcal{V}_\varepsilon(\boldsymbol{u}_\varepsilon)$ is bounded,
$$\mathcal{D}_n(\mathcal{H}_n) = \int_\Omega |\operatorname{div}\overline{\boldsymbol{u}}|^2 dx \le \liminf_{\varepsilon \to 0} \int_\Omega |\operatorname{div}\boldsymbol{u}_\varepsilon|^2 dx = 0.$$
Conversely, if $\mathcal{D}_n(\mathcal{H}_n) = 0$, then from (4.4) $\mathcal{V}_\varepsilon(\boldsymbol{u}_\varepsilon) \le d(\mathcal{H}_n)$. Thus $\mathcal{V}_\varepsilon(\boldsymbol{u}_\varepsilon)$ is bounded.

(ii) If $\mathcal{D}_n(\mathcal{H}_n) = 0$, then $\operatorname{div}\overline{\boldsymbol{u}} = 0$ in Ω. Therefore, it is well known (cf. [6]) that we can write $\overline{\boldsymbol{u}} = \operatorname{curl}\overline{\boldsymbol{w}} + \boldsymbol{h}$ where $\overline{\boldsymbol{w}} \in H^2(\Omega, \mathbb{R}^3)$ and $\boldsymbol{h} \in \mathbb{H}_2(\Omega)$. In the case where Ω has no holes, it suffices to note that $\mathbb{H}_2(\Omega) = \{\mathbf{0}\}$. ■

From Corollary 4.4, we see that the problems (R.1) and (R.2) are yes if and only if $\mathcal{D}_n(\mathcal{H}_n) = 0$.

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