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CONTROLLABILITY OF DAMPED SECOND-ORDER NEUTRAL FUNCTIONAL DIFFERENTIAL SYSTEMS WITH IMPULSES

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Abstract. In this paper, the Sadovskii fixed point theorem and the theory of strongly continuous cosine families of operators are used to investigate the controllability of damped second order neutral system with impulses. Examples are provided to show the application of the result.

1. INTRODUCTION

The study of impulsive functional differential equations is linked to their utility in simulating processes and phenomena subject to short-time perturbations during their evolution. The perturbations are performed discretely and their duration is negligible in comparison with the total duration of the processes. That is why the perturbations are considered to take place instantaneously in the form of impulses. The theory of impulsive systems provides a general framework for mathematical modeling of many real world phenomena. Moreover, these impulsive phenomena can also be found in fields such as information science, electronics, fed-batch culture infermentative production, robotics and telecommunications (see [10, 21] and references therein).

In recent years, the study of impulsive control systems has received increasing interest. Due to its importance several authors have investigated the controllability of impulsive systems (see [18, 20, 28]).

This paper is mainly concerned with the study of controllability of damped second order impulsive neutral system of the form

$$(1)\qquad \begin{aligned} \frac{d}{dt}[x'(t)-p(t,x(t))] &= Ax(t)+Gx'(t)+Bu(t)+f(t,x(t),x(h(t))),\\ &\quad t\in J=[0,b],\ t\neq t_i,\ i=1,2,\dots,n, \end{aligned}$$

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(2) $$x(0) = \zeta, \quad x'(0) = \eta$$

(3) $$\triangle x(t_i) = I_i(x(t_i)),\ i = 1, 2, \ldots, n,$$

(4) $$\triangle x'(t_i) = \tilde{I}_i(x(t_i)),\ i = 1, 2, \ldots, n.$$

where A is the infinitesimal generator of a strongly continuous cosine family of bounded linear operators $(C(t))_{t\in\mathbb{R}}$ defined on a Banach space X. The control function $u(\cdot)$ is given in $L^2(J, U)$, a Banach space of admissible control functions with U as a Banach space and $B : U \to X$ as a bounded linear operator ; G is a bounded linear operator on a Banach space X with $D(G) \subset D(A)$; $0 < t_1 < \ldots < t_n < b$ are prefixed numbers ; $f(\cdot),\ p(\cdot),\ h(\cdot),\ I_i(\cdot)$ and $\tilde{I}_i(\cdot)$ are appropriate continuous functions and the jump $\triangle\xi(t)$ of the function $\xi(\cdot)$ at t defined by $\triangle\xi(t) = \xi(t^+) - \xi(t^-)$.

Abstract neutral differential equations arise in many areas of applied mathematics. There are many contributions relative to this topic and we refer the readers to [1, 4] and the monograph [27]. The study of abstract deterministic second-order evolution equations governed by the generator of a strongly continuous cosine family was initiated by Fattorini [6], and subsequently studied by Travis and Webb [24, 25]. Models of abstract second order systems can be found in $[13-15]$. Concerning first and second order equations with damping were discussed in [3, 5, 7, 8, 11, 13, 22, 23, 26]. With the help of fixed point argument several authors have investigated the problem of controllability of second order nonlinear systems with and without impulses $[2, 16-18, 20]$. In this paper, we study the controllability of second order nonlinear systems without imposing compactness condition on the semigroup of cosine family and this fact is the main motivation for this paper.

2. Preliminaries

In this section, we will briefly recall some basic definitions, notations, lemmas and properties that will be used in the paper.

Throughout this paper, $(X, \|\cdot\|)$ is a Banach space and A is the infinitesimal generator of a strongly continuous cosine family of bounded linear operators $(C(t))_{t\in\mathbb{R}}$ on Banach space X. We denote by $(S(t))_{t\in\mathbb{R}}$ the sine function associated with $(C(t))_{t\in\mathbb{R}}$ which is defined by

$$S(t)x = \int_0^t C(s)x ds, \quad \text{for } x \in X, \quad t \in \mathbb{R}.$$

Moreover, M and N are positive constants such that $\|C(t)\| \leq M$ and $\|S(t)\| \leq N$, for every $t \in J$.

In this work, $[D(A)]$ is the space $D(A)$ endowed with the graph norm $\|x\|_A = \|x\| + \|Ax\|$, $x \in D(A)$. E stands for the space of all vectors $x \in X$ for which the function $C(\cdot)x$ is of class C^1 on $\mathbb{R}$. It was proved by Kisynski [9] that E endowed with the norm $\|x\|_E = \|x\| + \sup_{0 \le t \le 1} \|AS(t)x\|$, $x \in E$, is a Banach space. The operator valued function

$$\mathcal{F}(t) = \begin{bmatrix} C(t) & S(t) \\ AS(t) & C(t) \end{bmatrix}$$

is a strongly continuous group of linear operators on the space $E \times X$ generated by the operator $\mathcal{A} = \begin{bmatrix} 0 & I \\ A & 0 \end{bmatrix}$ defined on $D(A) \times E$. It follows that $AS(t) : E \to X$ is a bounded linear operator and $AS(t)x \to 0$, $t \to 0$, for each $x \in E$. Furthermore, if $x : [0, \infty) \to X$ is a locally integrable function, then the function $y(t) = \int_0^t S(t-s)x(s)ds$ defines an E-valued continuous function. This is an immediate consequence of the fact that

$$\int_0^t \mathcal{F}(t-s) \begin{bmatrix} 0 \\ x(s) \end{bmatrix} ds = \Big[\int_0^t S(t-s)x(s)ds, \int_0^t C(t-s)x(s)ds \Big]^T$$

defines an $E \times X$-valued continuous function.

The existence of solutions for the second order abstract Cauchy problem

$$x''(t) = Ax(t) + h(t), \quad \sigma \le t \le \mu, \tag{5}$$

$$x(0) = z, \quad x'(0) = w, \tag{6}$$

where $h : [\sigma, \mu] \to X$ is an integrable function, has been discussed in [24]. Similarly the existence of solutions of semilinear second order abstract Cauchy problems has been treated in [25]. We only mention here that the function $x : [\sigma, \mu] \to X$ given by

$$x(t) = C(t-\sigma)z + S(t-\sigma)w + \int_\sigma^t S(t-s)h(s)ds, \quad \sigma \le t \le \mu, \tag{7}$$

is called a mild solution of (5)-(6) and that when $z \in E$, $x(\cdot)$ is continuously differentiable and

$$x'(t) = AS(t-\sigma)z + C(t-\sigma)w + \int_\sigma^t C(t-s)h(s)ds, \quad \sigma \le t \le \mu. \tag{8}$$

For additional details on the cosine function theory, we refer the reader to [6, 24, 25].

The terminology and notations are those generally used in functional analysis. In particular, for Banach spaces $(Z, \|\cdot\|_Z)$, $(W, \|\cdot\|_W)$, the notation $\mathcal{L}(Z, W)$ stands for the Banach space of bounded linear operators from Z into W and we abbreviate to $\mathcal{L}(Z)$ whenever $Z = W$. Moreover $B_r(x : Z)$ denotes the closed ball with center at x and radius $r > 0$ in Z and we write simply B_r when no confusion arises.

The following lemma is crucial in the proof of our main theorem.

Lemma 2.1. ([16, Lemma 3.1]). *Assume that $(Hf1), (Hf2), (W)$ hold. Then the operator*

$$Ny(t) = \int_0^t S(t-s)[f(s, y(s)) + (\mathcal{B}u_y)(s)]ds, \quad t \in [0, b],$$

is completely continuous.

The key tool in our approach is following fixed-point theorem.

Lemma 2.2. ([19, Sadovskii's Fixed Point Theorem]). *Let F be a condensing operator on a Banach space X. If $F(D) \subset D$ for a convex, closed and bounded set D of X, then F has a fixed point in D.*

Definition 2.3. The system $(1)-(4)$ is said to be controllable on the interval J, if for every $\zeta \in D(A), \eta \in E$ and $x_1 \in X$, there exists a control $u \in L^2(J, U)$ such that the mild solution $x(t)$ of $(1)-(4)$ satisfies $x(b) = x_1$.

3. Controllability Results

In this section, we state and prove our main results. We begin by studying the following abstract damped second order neutral system.

3.1. Neutral Systems

This section is concerned with the result on controllability of damped second order abstract Cauchy problem of the form

$$\frac{d}{dt}[x'(t) - p(t, x(t))] = Ax(t) + Gx'(t) + Bu(t) + f(t, x(t), x(h(t))), \quad t \in J = [0, b], \tag{9}$$

$$x(0) = \zeta, \quad x'(0) = \eta \tag{10}$$

where A, B, G, f, h and p are defined as in equations $(1)-(4)$. Let $C = C([0, b] : X)$ and $C^1 = C^1([0, b] : X)$ be the Banach space of continuous X valued functions on J and is endowed with the supremum norm.

If $x(\cdot)$ is a solution of (9)-(10), then from (7), we adopt the following concept of mild solution,

$$\begin{aligned} x(t) = {} & C(t)\zeta + S(t)[\eta - p(0, \zeta)] + \int_0^t C(t-s)p(s, x(s))ds \\ & + \int_0^t S(t-s)\big[Gx'(s) + Bu(s) + f(s, x(s), x(h(s)))\big]ds, \ t \in J. \end{aligned} \tag{11}$$

The above expression is equivalent to the following definition.

Definition 3.1. A function $x \in C$ is said to be a mild solution of (9)-(10) if

$$\begin{aligned} x(t) &= (C(t) - S(t)G)\zeta + S(t)[\eta - p(0,\zeta)] \\ &\quad + \int_0^t C(t-s)\big[p(s,x(s)) + Gx(s)\big]ds \\ &\quad + \int_0^t S(t-s)\big[Bu(s) + f(s,x(s),x(h(s)))\big]ds, \ t \in J. \end{aligned}$$

We shall need the following assumptions :

(H1) The function $f : J \times X^2 \to X$ satisfies the following conditions :

(i) $f(t,\cdot,\cdot) : X \times X \to X$ is continuous a.e. $t \in J$. For every $x, y \in X$, the function $f(\cdot,x,y) : J \to X$ is strongly measurable.
(ii) For each $t \in J$, the function $f(t,\cdot,\cdot) : X \times X \to X$ is completely continuous.
(iii) There is a function $m \in L^1(J,[0,\infty))$ and non-decreasing function $W \in C([0,\infty);(0,\infty))$ such that, for all $t \in J$ and every $\phi, \psi \in X$,

$$\|f(t,\phi,\psi)\| \le m(t)W(\|\phi\| + \|\psi\|)$$

(iv) For every positive constant l, there exists an $w_l \in L^1(J)$ such that

$$\sup_{\|\phi\|,\|\psi\| \le l} \|f(t,\phi,\psi)\| \le w_l(t), \text{ for a.a. } t \in J.$$

(H2) B is a continuous operator from U to X and the linear operator $W : L^2(J,U) \to X$, defined by

$$Wu = \int_0^b S(b-s)Bu(s)ds,$$

has a bounded invertible operator W^{-1} which takes values in $L^2(J,U)/kerW$ such that $\|B\| \le M_1$ and $\|W^{-1}\| \le M_2$, for some positive constants M_1, M_2.

(H3) The function $h : J \to J$ is continuous and $h(t) \le t$ for every $t \in J$.

(H4) The function $p : J \times X \to X$ is completely continuous and there exists a constant $L_p > 0$ such that

$$\|p(t,\phi_1) - p(t,\phi_2)\| \le L_p\|\phi_1 - \phi_2\|, \ \ (t,\phi_i) \in J \times X, \ \ i = 1,2.$$

(H5) There exist positive constants k_1, k_2 such that $\|p(t,\phi)\| \le k_1\|\phi\| + k_2$, for every $(t,\phi) \in J \times X$.

Theorem 3.2. Assume that conditions $(H1)-(H5)$ are satisfied. Then the system (9)-(10) is controllable on J provided that

$$(1+bNM_1M_2)\Big[bM(L_p+\|G\|)+N\lim_{l\to\infty}\inf\frac{W(2l)}{l}\int_0^b m(s)ds\Big]<1.$$

Proof. Consider the space $Z=\{x\in C=C([0,b]:X)\}$ endowed with the uniform convergence topology. Using the hypothesis $(H2)$, for an arbitrary function $x(\cdot)$, define the control

$$\begin{aligned}u(t) \;=\; & W^{-1}\Bigg[x_1-(C(b)-S(b)G)\zeta-S(b)[\eta-p(0,\zeta)]\\ &-\int_0^b C(b-s)\big[p(s,x(s))+Gx(s)\big]ds\\ &-\int_0^b S(b-s)f(s,x(s),x(h(s)))ds\Bigg](t).\end{aligned}$$

Using this control, we shall show that the operator $\Phi:Z\to Z$ defined by

$$\begin{aligned}(\Phi x)(t) \;=\; & (C(t)-S(t)G)\zeta+S(t)[\eta-p(0,\zeta)]+\int_0^t C(t-s)\\ &\big[p(s,x(s))+Gx(s)\big]ds+\int_0^t S(t-s)f(s,x(s),x(h(s)))ds\\ &+\int_0^t S(t-\xi)BW^{-1}\Bigg[x_1-(C(b)-S(b)G)\zeta-S(b)[\eta-p(0,\zeta)]\\ &-\int_0^b C(b-s)\big[p(s,x(s))+Gx(s)\big]ds\\ &-\int_0^b S(b-s)f(s,x(s),x(h(s)))ds\Bigg](\xi)d\xi,\quad t\in J,\end{aligned}$$

has a fixed point $x(\cdot)$. This fixed point is then a mild solution of the system (9)-(10). Clearly $(\Phi x)(b)=x_1$ which means that the control u steers the system from the initial state ζ to x_1 in time b, provided we can obtain a fixed point of the operator Φ which implies that the system is controllable. From the assumptions, it is easy to see that Φ is well defined and continuous.

Next we affirm that there exists $l>0$ such that $\Phi(B_l(0,Z))\subseteq B_l(0,Z)$. If we assume that this assertion is false, then for each $l>0$, we can choose $x_l\in B_l(0,Z)$ and $t_l\in J$ such that $\|\Phi x_l(t_l)\|>l$. Consequently,

$$\begin{aligned}l \;<\; & \|\Phi x_l(t_l)\|\\ \le\; & (M+N\|G\|)\|\zeta\|+N\big[\|\eta\|+\|p(0,\zeta)\|\big]+M\|G\|\int_0^{t_l}\|x_l(s)\|ds\end{aligned}$$

$$
\begin{aligned}
&+M\int_0^{t_l}\Big[\|p(s,x_l(s))-p(s,0)\|+\|p(s,0)\|\Big]ds\\
&+N\int_0^{t_l}m(s)W(\|x_l(s)\|+\|x_l(h(s))\|)ds+NM_1M_2\int_0^{t_l}\Bigg[\|x_1\|\\
&+(M+N\|G\|)\|\zeta\|+N\big[\|\eta\|+\|p(0,\zeta)\|\big]+M\|G\|\int_0^b\|x_l(s)\|ds\\
&+M\int_0^b\Big[\|p(s,x_l(s))-p(s,0)\|+\|p(s,0)\|\Big]ds\\
&+N\int_0^b m(s)W(\|x_l(s)\|+\|x_l(h(s))\|)ds\Bigg]d\xi,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
l \le\ &(M+N\|G\|)\|\zeta\|+N\big[\|\eta\|+\|p(0,\zeta)\|\big]+bM\|G\|l+bM\big[L_pl+\|p(s,0)\|\big]\\
&+NW(2l)\int_0^b m(s)ds+bNM_1M_2\Bigg[\|x_1\|+(M+N\|G\|)\|\zeta\|+N\big[\|\eta\|\\
&+\|p(0,\zeta)\|\big]+bM\|G\|l+bM\big[L_pl+\|p(s,0)\|\big]+NW(2l)\int_0^b m(s)ds\Bigg]
\end{aligned}
$$

and hence

$$
1 \le (1+bNM_1M_2)\Big[bM(L_p+\|G\|)+N\lim_{l\to\infty}\inf\frac{W(2l)}{l}\int_0^b m(s)ds\Big]
$$

which is absurd.

Let $l>0$ be such that $\Phi(B_l(0,Z))\subset B_l(0,Z)$. In order to prove that Φ is a condensing map on $B_l(0,Z)$ into $B_l(0,Z)$. We introduce the decomposition $\Phi=\Phi_1+\Phi_2$ where

$$
\begin{aligned}
\Phi_1x(t) &= (C(t)-S(t)G)\zeta+S(t)[\eta-p(0,\zeta)]\\
&\quad+\int_0^t C(t-s)\big[p(s,x(s))+Gx(s)\big]ds,\\
\Phi_2x(t) &= \int_0^t S(t-s)[f(s,x(s),x(h(s)))+Bu(s)]ds.
\end{aligned}
$$

Now

$$
\begin{aligned}
\|Bu(s)\| \le\ &M_1M_2\Bigg[\|x_1\|+(M+N\|G\|)\|\zeta\|+N\big[\|\eta\|+k_1\|\zeta\|+k_2\big]\\
&+M\int_0^b\Big[\|G\|\|x(s)\|+k_1\|x(s)\|+k_2\Big]ds+N\int_0^b w_l(s)ds\Bigg]
\end{aligned}
$$

$$\leq M_1M_2\Big[\|x_1\|+(M+N\|G\|)\|\zeta\|+N\big[\|\eta\|+k_1\|\zeta\|+k_2\big] \\ +bM\Big[l(\|G\|+k_1)+k_2\Big]+N\int_0^b w_l(s)ds\Big]=C_o$$

Here we can apply the same technique that is used in Lemma 2.1. From the hypothesis $(H1),(H2)$ and $(H3)$, we infer that $\Phi_2(\cdot)$ is completely continuous on $B_l(0,Z)$. Moreover, for $x_1,x_2\in B_l(0,Z)$, we see that

$$\|\Phi_1x_1-\Phi_1x_2\|\leq\Big[bM(L_p+\|G\|)\Big]\|x_1-x_2\|,$$

which shows that Φ_1 is a contraction and $\Phi(\cdot)$ is a condensing map on $B_l(0,Z)$.

Now, from Lemma 2.2, the operator Φ has a fixed point in Z. This means that any fixed point of Φ is a mild solution of the problem (9)-(10). Thus the system (9)-(10) is controllable on J.

3.2. Impulsive Neutral Systems

In this section, we study impulsive control problems for damped second-order systems. Such problems arise naturally from a wide variety of applications, such as spacecraft maneuver[12], mathematical in epidemiology, automatic control systems and inspection process in operation research. It is the purpose of this section to establish controllability conditions for nonlinear damped system with impulses.

Now we consider some additional concepts and notations concerning impulsive differential equations. In what follows we put $t_0=0, t_{n+1}=b$ and we denote $\mathcal{PC}([\mu,\tau];X)=\{\phi:[\mu,\tau]\to X:\phi(\cdot)$ is continuous at $t\neq t_i,\ \phi(t_i^-)=\phi(t_i)$ and $\phi(t_i^+)$ exists for all $i=1,2,\ldots,n\}$. In this paper, the notation $\mathcal{PC}$ stands for the space formed by all functions $\phi\in\mathcal{PC}([0,b];X)$. The norm $\|\cdot\|_{\mathcal{PC}}$ of the space $\mathcal{PC}$ is defined by $\|\phi\|_{\mathcal{PC}}=\sup_{s\in I}\|\phi(s)\|$. It is clear that $(\mathcal{PC},\|\cdot\|_{\mathcal{PC}})$ is a Banach space. Similarly, $\mathcal{PC}^1$ will be the space of the functions $\phi(\cdot)\in\mathcal{PC}$ such that $\phi(\cdot)$ is continuously differentiable on $J\backslash\{t_i:i=1,\ldots,n\}$ and the lateral derivatives $\phi'_R(t)=\lim_{s\to0^+}\frac{\phi(t+s)-\phi(t^+)}{s}, \phi'_L(t)=\lim_{s\to0^-}\frac{\phi(t+s)-\phi(t^-)}{s}$ are continuous functions on $[t_i,t_{i+1})$ and $(t_i,t_{i+1}]$ respectively. Next, for $\phi\in\mathcal{PC}^1$ we represent by $\phi'(t)$ the left derivative at $t\in(0,b]$ and by $\phi'(0)$ the right derivative at zero.

For $\phi\in\mathcal{PC}$ we denote by $\tilde{\phi}_i, i=0,1,\ldots,n$, the unique continuous function $\tilde{\phi}_i\in C([t_i,t_{i+1}];X)$ such that

$$\tilde{\phi}_i(t)=\begin{cases}\phi(t), & \text{for } t\in(t_i,t_{i+1}],\\ \phi(t_i^+), & \text{for } t=t_i.\end{cases}$$

We consider the damped second-order neutral functional differential equation with impulses of the form $(1)-(4)$. Motivated from the expression (11), let us begin by introducing the following definition.

Definition 3.3. A function $x \in \mathcal{PC}$ is said to be a mild solution of $(1)-(4)$, if

$$\begin{aligned} x(t) \;=\;& C(t)\zeta + S(t)\big[\eta - p(0,\zeta)\big] + \int_0^t C(t-s)p(s,x(s))ds \\ &+ \sum_{i=0}^{j-1}\big[S(t-t_{i+1})Gx(t_{i+1}^-) - S(t-t_i)Gx(t_i^+)\big] - S(t-t_j)Gx(t_j^+) \\ &+ \int_0^t C(t-s)Gx(s)ds + \int_0^t S(t-s)[Bu(s) + f(s,x(s),x(h(s)))]ds \\ &+ \sum_{t_i<t} C(t-t_i)I_i(x(t_i)) + \sum_{t_i<t} S(t-t_i)\tilde{I}_i(x(t_i)), \end{aligned}$$

for all $t \in [t_j, t_{j+1}]$ and every $j = 0, \ldots, n$.

Furthermore we assume the following conditions :

(H6) There are positive constants K_i, L_i such that

$$\|I_i(\phi_1) - I_i(\phi_2)\| \le K_i\|\phi_1 - \phi_2\|, \;\; \phi_j \in X, \;\; j = 1,2, \;\; i = 1,2,\cdots,n,$$
$$\|\tilde{I}_i(\phi_1) - \tilde{I}_i(\phi_2)\| \le L_i\|\phi_1 - \phi_2\|, \;\; \phi_j \in X, \;\; j = 1,2, \;\; i = 1,2,\cdots,n.$$

(H7) The maps $I_i, \tilde{I}_i : X \to X, i = 1,2,\cdots,n$ are completely continuous and there exist continuous non-decreasing functions $\mu_i, \sigma_i : [0,\infty) \to (0,\infty)$, $i = 1,2,\cdots,n$, such that

$$\begin{aligned} \|I_i(\phi)\| &\le \mu_i(\|\phi\|) \\ \|\tilde{I}_i(\phi)\| &\le \sigma_i(\|\phi\|), \quad \phi \in X. \end{aligned}$$

Theorem 3.4. *If the assumptions* $(H1)-(H7)$ *are satisfied, then the system* $(1)-(4)$ *is controllable on* J *provided that*

$$\begin{aligned} &(1 + bNM_1M_2)\Big[bML_p + (3N + bM)\|G\| \\ &+ N\lim_{l\to\infty}\inf\frac{W(2l)}{l}\int_0^b m(s)ds + \sum_{i=1}^n(MK_i + NL_i)\Big] < 1. \end{aligned}$$

Proof. Consider the space $Y = \{x \in \mathcal{PC} = \mathcal{PC}([0,b] : X)\}$ endowed with the uniform convergence topology. Using the assumption $(H2)$, for an arbitrary function $x(\cdot)$, define the control

$$
\begin{aligned}
u(t) \;=\; W^{-1}\Bigg[& x_1 - C(b)\zeta - S(b)\big[\eta - p(0,\zeta)\big] - \int_0^b C(b-s)p(s,x(s))ds \\
& + S(b-t_j)Gx(t_j^+) - \sum_{i=0}^{j-1}\big[S(b-t_{i+1})Gx(t_{i+1}^-) - S(b-t_i)Gx(t_i^+)\big] \\
& - \int_0^b C(b-s)Gx(s)ds - \int_0^b S(b-s)f(s,x(s),x(h(s)))ds \\
& - \sum_{i=1}^{n} C(b-t_i)I_i(x(t_i)) - \sum_{i=1}^{n} S(b-t_i)\tilde{I}_i(x(t_i))\Bigg](t).
\end{aligned}
$$

Using this control we shall show that the operator $\Phi : \mathcal{PC} \to \mathcal{PC}$ defined by

$$
\begin{aligned}
& (\Phi x)(t) \\
= \; & C(t)\zeta + S(t)\big[\eta - p(0,\zeta)\big] + \int_0^t C(t-s)p(s,x(s))ds \\
& + \sum_{i=0}^{j-1}\big[S(t-t_{i+1})Gx(t_{i+1}^-) - S(t-t_i)Gx(t_i^+)\big] - S(t-t_j)Gx(t_j^+) \\
& + \int_0^t C(t-s)Gx(s)ds + \int_0^t S(t-s)f(s,x(s),x(h(s)))ds \\
& + \int_0^t S(t-\xi)BW^{-1}\Bigg[x_1 - C(b)\zeta - S(b)\big[\eta - p(0,\zeta)\big] \\
& - \int_0^b C(b-s)p(s,x(s))ds + S(b-t_j)Gx(t_j^+) \\
& - \sum_{i=0}^{j-1}\big[S(b-t_{i+1})Gx(t_{i+1}^-) - S(b-t_i)Gx(t_i^+)\big] \\
& - \int_0^b C(b-s)Gx(s)ds - \int_0^b S(b-s)f(s,x(s),x(h(s)))ds \\
& - \sum_{i=1}^{n} C(b-t_i)I_i(x(t_i)) - \sum_{i=1}^{n} S(b-t_i)\tilde{I}_i(x(t_i))\Bigg](\xi)d\xi \\
& + \sum_{t_i<t} C(t-t_i)I_i(x(t_i)) + \sum_{t_i<t} S(t-t_i)\tilde{I}_i(x(t_i)),\; t \in [t_j,t_{j+1}],\; j = 0,\ldots,n,
\end{aligned}
$$

has a fixed point $x(\cdot)$. This fixed point is then a mild solution of the system $(1)-(4)$. Clearly $(\Phi x)(b) = x_1$ which means that the control u steers the system from the initial state ζ to x_1 in time b, provided we can obtain a fixed point of the operator Φ which implies that the system is controllable. From the assumptions, it is easy to see that Φ is well defined and continuous.

Next we claim that there exists $l > 0$ such that $\Phi(B_l(0, \mathcal{PC})) \subseteq B_l(0, \mathcal{PC})$. If this property is false, then for every $l > 0$, there exist $x_l \in B_l(0, \mathcal{PC}), j = \{0, \ldots, n\}$ and $t_l \in [t_j, t_{j+1}]$ such that $\|\Phi x_l(t_l)\| > l$. Consequently,

$$
\begin{aligned}
l &< \|\Phi x_l(t_l)\| \\
&\le M\|\zeta\| + N\big[\|\eta\| + \|p(0,\zeta)\|\big] + M\int_0^{t_l}\Big[\|p(s,x_l(s)) - p(s,0)\| \\
&\quad + \|p(s,0)\|\Big]ds + N\|G\|\sum_{i=0}^{j-1}\Big[\|x_l(t_{i+1}^-)\| + \|x_l(t_i^+)\|\Big] + N\|G\|\|x_l(t_j^+)\| \\
&\quad + M\|G\|\int_0^{t_l}\|x_l(s)\|ds + N\int_0^{t_l} m(s)W(\|x_l(s)\| + \|x_l(h(s))\|)ds + NM_1M_2 \\
&\qquad \int_0^{t_l}\Bigg[\|x_1\| + M\|\zeta\| + N\big[\|\eta\| + \|p(0,\zeta)\|\big] + M\int_0^b\Big[\|p(s,x_l(s)) - p(s,0)\| \\
&\quad + \|p(s,0)\|\Big]ds + N\|G\|\sum_{i=0}^{j-1}\Big[\|x_l(t_{i+1}^-)\| + \|x_l(t_i^+)\|\Big] + N\|G\|\|x_l(t_j^+)\| \\
&\quad + M\|G\|\int_0^b\|x_l(s)\|ds + N\int_0^b m(s)W(\|x_l(s)\| + \|x_l(h(s))\|)ds \\
&\quad + M\sum_{i=1}^n\Big[\|I_i(x_l(t_i)) - I_i(0)\| + \|I_i(0)\|\Big] + N\sum_{i=1}^n\Big[\|\tilde{I}_i(x_l(t_i)) - \tilde{I}_i(0)\| \\
&\quad + \|\tilde{I}_i(0)\|\Big]\Bigg]d\xi + M\sum_{i=1}^n\Big[\|I_i(x_l(t_i)) - I_i(0)\| + \|I_i(0)\|\Big] \\
&\quad + N\sum_{i=1}^n\Big[\|\tilde{I}_i(x_l(t_i)) - \tilde{I}_i(0)\| + \|\tilde{I}_i(0)\|\Big],
\end{aligned}
$$

which implies that

$$
\begin{aligned}
l \le\ & M\|\zeta\| + N\big[\|\eta\| + \|p(0,\zeta)\|\big] + bM\big[L_pl + \|p(s,0)\|\big] \\
& + 3N\|G\|l + bM\|G\|l + NW(2l)\int_0^b m(s)ds + bNM_1M_2\Bigg[\|x_1\| \\
& + M\|\zeta\| + N\big[\|\eta\| + \|p(0,\zeta)\|\big] + bM\big[L_pl + \|p(s,0)\|\big] + 3N\|G\|l \\
& + bM\|G\|l + NW(2l)\int_0^b m(s)ds + \sum_{i=1}^n\Big[(MK_i + NL_i)l + M\|I_i(0)\| \\
& + N\|\tilde{I}_i(0)\|\Big]\Bigg] + \sum_{i=1}^n\Big[(MK_i + NL_i)l + M\|I_i(0)\| + N\|\tilde{I}_i(0)\|\Big]
\end{aligned}
$$

and hence

$$\begin{aligned} 1 \;\le\; & (1 + bNM_1M_2)\Big[bML_p + (3N + bM)\|G\| \\ & + N \lim_{l\to\infty} \inf \frac{W(2l)}{l}\int_0^b m(s)ds + \sum_{i=1}^{n}(MK_i + NL_i)\Big] \end{aligned}$$

which contradicts our assumption.

Let $l > 0$ be such that $\varPhi(B_l(0,\mathcal{PC})) \subset B_l(0,\mathcal{PC})$. In order to prove that $\varPhi$ is a condensing map on $B_l(0,\mathcal{PC})$ into $B_l(0,\mathcal{PC})$. We introduce the decomposition $\varPhi = \varPhi_1 + \varPhi_2$ where

$$\begin{aligned} \varPhi_1 x(t) \;=\; & C(t)\zeta + S(t)\big[\eta - p(0,\zeta)\big] + \int_0^t C(t-s)p(s,x(s))ds \\ & + \sum_{i=0}^{j-1}\big[S(t-t_{i+1})Gx(t_{i+1}^-) - S(t-t_i)Gx(t_i^+)\big] \\ & - S(t-t_j)Gx(t_j^+) + \int_0^t C(t-s)Gx(s)ds \\ & + \sum_{t_i<t} C(t-t_i)I_i(x(t_i)) + \sum_{t_i<t} S(t-t_i)\tilde{I}_i(x(t_i)), \\ \varPhi_2 x(t) \;=\; & \int_0^t S(t-s)[f(s,x(s),x(h(s))) + Bu(s)]ds. \end{aligned}$$

Now

$$\begin{aligned} \|Bu(s)\| \;\le\; & M_1M_2\Bigg[\|x_1\| + M\|\zeta\| + N\big[\|\eta\| + k_1\|\zeta\| + k_2\big] \\ & + M\int_0^b (k_1\|x(s)\| + k_2)\,ds + N\|G\|\|x(t_j^+)\| \\ & + N\|G\|\sum_{i=0}^{j-1}\Big[\|x(t_{i+1}^-)\| + \|x(t_i^+)\|\Big] + M\|G\|\int_0^b \|x(s)\|ds \\ & + N\int_0^b w_l(s)ds + M\sum_{i=1}^{n}\mu_i\|x(t_i)\| + N\sum_{i=1}^{n}\sigma_i\|x(t_i)\|\Bigg] \\ \le\; & M_1M_2\Bigg[\|x_1\| + M\|\zeta\| + N\big[\|\eta\| + k_1\|\zeta\| + k_2\big] \\ & + bM\,(k_1 l + k_2) + (3N + bM)\|G\| l \\ & + N\int_0^b w_l(s)ds + \sum_{i=1}^{n} l(M\mu_i + N\sigma_i)\Bigg] = \tilde{C}_o. \end{aligned}$$

Here we can apply the same technique that is used in Lemma 2.1. From the hypothesis $(H1), (H2)$ and $(H3)$, we infer that $\Phi_2(\cdot)$ is completely continuous on $B_l(0, \mathcal{PC})$. Moreover, for $x, z \in B_l(0, \mathcal{PC})$, we see that

$$\|\Phi_1 x - \Phi_1 z\|_{\mathcal{PC}} \leq \Big[bML_p + (3N + bM)\|G\| + \sum_{i=1}^{n}(MK_i + NL_i)\Big]\|x - z\|_{\mathcal{PC}},$$

which shows that Φ_1 is a contraction and $\Phi(\cdot)$ is a condensing map on $B_l(0, \mathcal{PC})$.

Now, from Lemma 2.2, Φ has a fixed point in $\mathcal{PC}$. This means that any fixed point of Φ is a mild solution of the problem $(1) - (4)$. This completes the proof.

Corollary 3.5. If all conditions of Theorem 3.4 hold except that $(H6)$ and $(H7)$ replaced by the following one,
(C1) : there exist positive constants a_i, b_i, c_i, d_i and constants $\theta_i, \delta_i \in (0, 1), i = 1, 2, \ldots, n$ such that for each $\phi \in X$,

$$\|I_i(\phi)\| \leq a_i + b_i(\|\phi\|)^{\theta_i}, \quad i = 1, 2, \ldots, n,$$

and

$$\|\tilde{I}_i(\phi)\| \leq c_i + d_i(\|\phi\|)^{\delta_i}, \quad i = 1, 2, \ldots, n,$$

then the system $(1) - (4)$ is controllable provided that

$$(1 + bNM_1M_2)\Big[bML_p + (3N + bM)\|G\| + N \lim_{l\to\infty} \inf \frac{W(2l)}{l}\int_0^b m(s)ds\Big] < 1.$$

4. Examples

In this section, we apply some of the results established in this paper. Let $X = L^2([0, \pi])$ and let A be an operator defined by $Aw = w''$ with domain $D(A) = \{w \in H^2(0, \pi) : w(0) = w(\pi) = 0\}$. It is well known that A is the infinitesimal generator of a strongly continuous cosine function $(C(t))_{t\in\mathbb{R}}$ on X. Moreover, A has a discrete spectrum with eigenvalues of the form $-n^2$, $n \in \mathbb{N}$, and the corresponding normalized eigenfunctions given by $e_n(\xi) := (\frac{2}{\pi})^{(\frac{1}{2})} sin(n\xi)$. Also the following properties hold :

(a) The set of functions $\{e_n : n \in \mathbb{N}\}$ forms an orthonormal basis of X.

(b) If $w \in D(A)$, then $Aw = \sum_{n=1}^{\infty} -n^2 < w, e_n > e_n$.

(c) For $w \in X, C(t)w = \sum_{n=1}^{\infty} cos(nt) < w, e_n > e_n$. The associated sine family is given by $S(t)w = \sum_{n=1}^{\infty} \frac{sin(nt)}{n} < w, e_n > e_n, w \in X$.

(d) If Ψ is the group of translations on X defined by $\Psi(t)x(\xi) = \tilde{x}(\xi + t)$, where $\tilde{x}(\cdot)$ is the extension of $x(\cdot)$ with period 2π, then $C(t) = \frac{1}{2}[\Psi(t) + \Psi(-t)]$; $A = B^2$ where B is the infinitesimal generator of Ψ and $E = \{x \in H^1(0, \pi) : x(0) = x(\pi) = 0\}$, see [6] for more details.

4.1. Second-order nonlinear system

Consider the following damped second order neutral differential equation with control $\hat{\mu}(t,\cdot)$

$$\frac{\partial}{\partial t}\left[\frac{\partial}{\partial t}u(t,\xi) - P(t,\xi,u(t,\xi))\right] = \frac{\partial^2}{\partial \xi^2}u(t,\xi) + \alpha\frac{\partial}{\partial t}u(t,\xi) \tag{12}$$
$$+ \int_0^{\xi} \beta(s)\frac{\partial}{\partial t}u(t,s)ds + \hat{\mu}(t,\xi) + F(t,\xi,u(t,\xi),u(h(t),\xi)),$$

for $t \in J = [0,b], \xi \in [0,\pi]$, subject to the initial conditions

$$\begin{aligned} u(t,0) &= u(t,\pi) = 0, \quad t \in J, \\ u(0,\xi) &= u_0(\xi), \quad u_t(0,\xi) = u_1(\xi), \quad 0 \le \xi \le \pi, \quad t \in J. \end{aligned}$$

We have to show that there exists a control $\hat{\mu}$ which steers (12) from any specified initial state to the final state in a Banach space X.

Here α is prefixed real number. In the sequel $h \in C([0,b],[0,b]), h(t) \le t$ for every $t \in J, \beta \in L^2([0,\pi])$ and the following conditions hold :

(a) The function F satisfies the following conditions :
   (i) $F(t,\xi,\cdot) : \mathbb{R}^2 \to \mathbb{R}$ is continuous a.e. $(t,\xi) \in \mathbb{R} \times [0,\pi]$.
   (ii) For every $w_1, w_2 \in \mathbb{R}$, the function $F(\cdot, w_1, w_2) : \mathbb{R} \times [0,\pi] \to \mathbb{R}$ is strongly measurable.
   (iii) There exists a continuous function $\Upsilon_F(\cdot) : \mathbb{R} \times [0,\pi] \to [0,\infty^+)$ such that

$$|F(t,\xi,w_1,w_2)| \le \Upsilon_F(t,\xi)(|w_1| + |w_2|), \quad w_i \in \mathbb{R}, \quad t \in \mathbb{R}, \quad \xi \in [0,\pi].$$

(b) The function P satisfies the following conditions :
   (i) $P(t,\xi,\cdot) : \mathbb{R} \to \mathbb{R}$ is continuous a.e. $(t,\xi) \in \mathbb{R} \times [0,\pi]$.
   (ii) For every $w \in \mathbb{R}$, the function $P(\cdot, w) : \mathbb{R} \times [0,\pi] \to \mathbb{R}$ is strongly measurable.
   (iii) There exists a continuous function $\Upsilon_P(\cdot) : \mathbb{R} \times [0,\pi] \to [0,\infty^+)$ such that, for every $w_i \in \mathbb{R}, i = 1,2,\ (t,\xi) \in \mathbb{R} \times [0,\pi]$,

$$|P(t,\xi,w_1) - P(t,\xi,w_2)| \le \Upsilon_P(t,\xi)|w_1 - w_2|.$$

Assume that the bounded linear operator $B : U \subset J \to X$ is defined by

$$(Bu)(t)(\xi) = \hat{\mu}(t,\xi), \quad \xi \in [0,\pi].$$

Define the operators $G : X \to X,\ f : J \times X^2 \to X$ and $p : J \times X \to X$ by

$$\begin{aligned} Gx(\xi) &= \alpha x(\xi) + \int_0^\xi \beta(s)x(s)ds, \\ f(t,x,y)(\xi) &= F(t,\xi,x(\xi),y(\xi)), \\ p(t,x)(\xi) &= P(t,\xi,x(\xi)). \end{aligned}$$

Further, the linear operator W is given by

$$(Wu)(\xi) = \sum_{n=1}^{\infty} \int_0^\pi \frac{1}{n} \sin ns(\hat{\mu}(s,\xi), e_n)e_n ds, \quad \xi \in [0,\pi].$$

Assume that this operator has a bounded inverse operator W^{-1} in $L^2(J,U)/kerW$. Moreover the functions $f,\ p$ and G are bounded linear operators with $\|G\|_{\mathcal{L}(X)} \leq |\alpha| + \|\beta\|_{L^2(0,b)},\ \|f(t,x,y)(\theta)\| \leq \sup_{\theta \in [0,\pi]} \varUpsilon_F(t,\theta)(\|x\| + \|y\|)$, for all $t \in J,\ x, y \in X$. It is easy to see that $p(\cdot)$ satisfies the assumption $\|p(t,\psi_1) - p(t,\psi_2)\| \leq L_p\|\psi_1 - \psi_2\|$, $t \in J,\ \psi_1, \psi_2 \in X$ and that $L_p = \sup_{\xi \in [0,\pi]} \varUpsilon_P(s,\xi);\ s \in J$. Now the equation (12) can be written in the abstract form (9)-(10). so that an application of Theorem 3.2. yields the controllability on J.

4.2. Second-order impulsive system

Consider the following second order impulsive Cauchy problem with control $\hat{\mu}(t,\cdot)$

$$\begin{aligned} &\frac{\partial}{\partial t}\left[\frac{\partial}{\partial t}u(t,\xi) - P(t,\xi,u(t,\xi))\right] = \frac{\partial^2}{\partial \xi^2}u(t,\xi) + \alpha\frac{\partial}{\partial t}u(t,\xi) \\ &+ \int_0^\xi \beta(s)\frac{\partial}{\partial t}u(t,s)ds + \hat{\mu}(t,\xi) + F(t,\xi,u(t,\xi),u(h(t),\xi)), \end{aligned} \tag{13}$$

for $t \in J = [0,b], \xi \in [0,\pi]$, subject to the initial conditions

$$\begin{aligned} u(t,0) &= u(t,\pi) = 0, \quad t \in J, \\ u(0,\xi) &= u_0(\xi), \quad u_t(0,\xi) = u_1(\xi), \quad 0 \leq \xi \leq \pi, \quad t \in J, \\ \triangle u(t_i,\cdot)(\xi) &= \alpha_i u(t_i,\xi), \quad \xi \in [0,\pi], \\ \triangle\frac{\partial u(t_i,\cdot)}{\partial t}(\xi) &= \beta_i u(t_i,\xi), \quad \xi \in [0,\pi], \end{aligned}$$

where $0 < t_1 < t_2 < \ldots < b,\ \alpha_i,\ \beta_i,\ i = 1,\ldots,n$, are prefixed real numbers and the functions $h,\ \alpha,\ \beta$ are as defined in Example 4.1. Assume that the condition

(a) and (b) of the previous example hold. Define the functions $G,\ f,\ p,\ B$ and W as in Example 4.1 and the maps $I_i,\ \tilde{I}_i : X \to X,\ i = 1, \ldots, n$, by $I_i(x)(\xi) = \alpha_i x(\xi),\ \tilde{I}_i(x)(\xi) = \beta_i x(\xi)$, the system (13) can be modelled as $(1)-(4)$. Moreover the functions I_i and $\tilde{I}_i$ are bounded linear operators on X, $\|I_i\|_{\mathcal{L}(X)} \leq \alpha_i$ and $\|\tilde{I}_i\|_{\mathcal{L}(X)} \leq \beta_i$ for every $i = 1, \ldots, n$. Thus by Theorem 3.4, the second order impulsive system $(1) - (4)$ is controllable.

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