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$L(3, 2, 1)$-LABELING OF GRAPHS

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Abstract. Given a graph G, an $L(3, 2, 1)$-*labeling* of G is a function f from the vertex set $V(G)$ to the set of all nonnegative integers such that $|f(u) - f(v)| \geqslant 1$ if $d(u, v) = 3$, $|f(u) - f(v)| \geqslant 2$ if $d(u, v) = 2$ and $|f(u) - f(v)| \geqslant 3$ if $d(u, v) = 1$. For a nonnegative integer k, a k-$L(3, 2, 1)$-*labeling* is an $L(3, 2, 1)$-labeling such that no label is greater than k. The $L(3, 2, 1)$-*labeling number* of G, denoted by $\lambda_{3,2,1}(G)$, is the smallest number k such that G has a k-$L(3, 2, 1)$-labeling. We study the $L(3, 2, 1)$-labelings of graphs in this paper. We give upper bounds for the $L(3, 2, 1)$-labeling numbers of general graphs and trees, and consider the $L(3, 2, 1)$-labeling numbers of several classes of graphs, such as the Cartesian product of paths and cycles, and the powers of paths.

1. INTRODUCTION

The $L(2, 1)$-labeling problem proposed by Griggs and Roberts [9] is a variation of the frequency assignment problem introduced by Hale [5]. Suppose we are given a number of transmitters or stations. The $L(2, 1)$-labeling problem is to assign frequencies (nonnegative integers) to the transmitters so that "close" transmitters must receive different frequencies and "very close" transmitters must receive frequencies that are at least two frequencies apart.

To formulate the problem in graphs, the transmitters are represented by the vertices of a graph; two vertices are "very close" if they are adjacent in the graph and "close" if they are of distance two in the graph. More precisely, an $L(2, 1)$-*labeling* of a graph G is a function f from the vertex set $V(G)$ to the set of all nonnegative

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integers such that $|f(u) - f(v)| \geqslant 1$ if $d(u, v) = 2$ and $|f(u) - f(v)| \geqslant 2$ if $d(u, v) = 1$. For a nonnegative integer k, a k-$L(2, 1)$-*labeling* is an $L(2, 1)$-labeling such that no label is greater than k. The $L(2, 1)$-*labeling number* of G, denoted by $\lambda(G)$, is the smallest number k such that G has a k-$L(2, 1)$-labeling.

The $L(2, 1)$-labeling problem has been studied extensively over the past decade. Griggs and Yeh [4] showed that the $L(2, 1)$-labeling problem is NP-complete for general graphs. They proved that $\lambda(G) \leqslant \Delta^2(G) + 2\Delta(G)$ and conjectured that $\lambda(G) \leqslant \Delta^2(G)$ for general graphs. Chang and Kuo [1] proved that $\lambda(G) \leqslant \Delta^2(G) + \Delta(G)$ and gave a polynomial-time algorithm for the $L(2, 1)$-labeling problem on trees. The upper bound for general graphs was improved to $\lambda(G) \leqslant \Delta^2(G) + \Delta(G) - 1$ by Král and Skrekovski [7], and was further improved to $\lambda(G) \leqslant \Delta^2(G) + \Delta(G) - 2$ by Gonçalves [3]. Hasunuma et al. [6] gave an $O(n^{1.75})$ algorithm for the $L(2, 1)$-labeling problem on trees. There are also many results concerning this problem, for a good survey, see [11].

Liu and Shao [8] considered the following generalization of $L(2, 1)$-labeling problem, which they called the $L(3, 2, 1)$-*labeling problem*: Given a graph G, an $L(3, 2, 1)$-*labeling* of G is a function f from the vertex set $V(G)$ to the set of all nonnegative integers such that $|f(u) - f(v)| \geqslant 1$ if $d(u, v) = 3$, $|f(u) - f(v)| \geqslant 2$ if $d(u, v) = 2$ and $|f(u) - f(v)| \geqslant 3$ if $d(u, v) = 1$. For a nonnegative integer k, a k-$L(3, 2, 1)$-*labeling* is an $L(3, 2, 1)$-labeling such that no label is greater than k. The $L(3, 2, 1)$-*labeling number* of G, denoted by $\lambda_{3,2,1}(G)$, is the smallest number k such that G has a k-$L(3, 2, 1)$-labeling.

Shao [10] studied the $L(3, 2, 1)$-labeling of Kneser graphs, extremely irregular graphs, Halin graphs, and gave bounds for the $L(3, 2, 1)$-labeling numbers of these classes of graphs. Liu and Shao [8] studied the $L(3, 2, 1)$-labeling of planar graphs, and showed that $\lambda_{3,2,1}(G) \leq 15(\Delta^2 - \Delta + 1)$ if G is a planar graph of maximum degree Δ. Clipperton et al. [2] determined the $L(3, 2, 1)$-labeling numbers for paths, cycles, caterpillars, n-ary trees, complete graphs and complete bipartite graphs, and showed that $\lambda_{3,2,1}(G) \leq \Delta^3 + \Delta^2 + 3\Delta$ for any graph G with maximum degree Δ.

In this paper, we study the $L(3, 2, 1)$-labeling numbers of several classes of graphs. We give some basic properties in Section two, and give upper bounds for the $L(3, 2, 1)$-labeling numbers of general graphs and trees in Section three. In Section four, we study the the $L(3, 2, 1)$-labeling numbers of the Cartesian product of paths and cycles. And, in the last section, we study the $L(3, 2, 1)$-labeling numbers of the powers of paths.

2. Preliminaries

Lemma 1. *If H is a subgraph of G, then $\lambda_{3,2,1}(H) \leq \lambda_{3,2,1}(G)$.*

Lemma 2. *If f is a k-$L(3, 2, 1)$-labeling of G, then the function $f' : V(G) \to \{0, 1, ..., k\}$, defined by $f'(v) = k - f(v)$, is also a k-$L(3, 2, 1)$-labeling of G.*

Lemma 3. *For a star* $S_n = \{v\} + \overline{K_n}$, $\lambda_{3,2,1}(S_n) = 2n + 1$. *Moreover, if* f *is a* $(2n + 1)$-$L(3, 2, 1)$*-labeling of* S_n, *then* $f(v) = 0$ *or* $2n + 1$.

Corollary 4. *For any graph* G *with* $\Delta(G) = \Delta > 0$, $\lambda_{3,2,1}(G) \geq 2\Delta + 1$. *Moreover, if* $\lambda_{3,2,1}(G) = 2\Delta + 1$ *and* f *is a* $(2\Delta+1)$-$L(3, 2, 1)$*-labeling of* G, *then for all* $v \in V(G)$ *with* $\deg(v) = \Delta$, $f(v) \in \{0, 2\Delta + 1\}$.

Corollary 5. *Given a graph* G *with* $\Delta(G) = \Delta$. *If there exist* v_1, v_2, v_3 *in* $V(G)$, *such that* $\deg(v_i) = \Delta$, *and* $d(v_i, v_j) \leq 3$ *for all* $1 \leq i, j \leq 3$, *then* $\lambda_{3,2,1}(G) \geq 2\Delta + 2$.

Lemma 6. *Given a graph* G *with* $\Delta(G) = \Delta$. *If* $\lambda_{3,2,1}(G) = 2\Delta + 2$, *and* f *is a* $(2\Delta + 2)$-$L(3, 2, 1)$*-labeling of* G, *then for all* $v \in V(G)$ *with* $\deg(v) = \Delta$, $f(v) \in \{0, 1, 3, ..., 2\Delta - 1, 2\Delta + 1, 2\Delta + 2\}$.

Corollary 7. *Given a graph* G *with* $\Delta(G) = \Delta$. *If* $\lambda_{3,2,1}(G) = 2\Delta + 2$, *and there exist* $v, v_1, v_2, v_3 \in V(G)$, *such that* $\deg(v) = \deg(v_1) = \deg(v_2) = \deg(v_3) = \Delta$, *and* $vv_1, vv_2, vv_3 \in E(G)$, *then for all* $(2\Delta + 2)$-$L(3, 2, 1)$*-labeling* f *of* G, $f(v) \in \{0, 2\Delta + 2\}$.

3. Upper Bounds for the $L(3, 2, 1)$-Labeling Numbers of General Graphs and Trees

Given a graph G and a vertex v in $V(G)$, the *kth-neighborhood* of v in G, denoted $N_G^k(v)$, is defined by $N_G^k(v) = \{u \mid d_G(u, v) = k\}$. If G is the only graph we considered, we use $N^k(v)$ to replace $N_G^k(v)$ for simplicity. And, when $k = 1$, we simply write $N_G(v)$ in stead of $N_G^k(v)$.

For a fixed integer k, a k*-stable set* of a graph G is a subset S of $V(G)$ such that every two distinct vertices in S are of distance greater than k.

Theorem 8. *If* G *is a graph with maximum degree* Δ, $\lambda_{3,2,1}(G) \leq \Delta^3 + 2\Delta$.

Proof. Consider the following labeling scheme on $V(G)$. Initially, all vertices are unlabeled. Let $S_{-2} = S_{-1} = \emptyset$. When S_{i-2} and S_{i-1} is determined and not all vertices in G are labeled, let

$$\begin{aligned} F_i = \{&x \in V(G) \mid x \text{ is unlabeled and } d(x, y) \geq 2 \text{ for all } y \in S_{i-2} \\ &\text{and } d(x, z) \geq 3 \text{ for all } z \in S_{i-1}\}. \end{aligned}$$

Choose a maximal 3-stable subset S_i of F_i. Note that if $F_i = \emptyset$, (i.e., for any unlabeled vertex x, there exists some $y \in S_{i-2}$ with $d(x, y) < 2$ or some $z \in S_{i-1}$ with $d(x, y) < 3$) $S_i = \emptyset$. In any case, label all vertices in S_i by i. Then increase i by 1, and continue the process until all vertices are labeled. Assume that k is the maximum label used and choose a vertex x whose label is k. Let

$$\begin{aligned}
I_1 &= \{i \mid 0 \le i \le k-1 \text{ and } d(x,y) = 1 \text{ for some } y \in S_i\},\\
I_2 &= \{i \mid 0 \le i \le k-1 \text{ and } d(x,y) = 2 \text{ for some } y \in S_i\},\\
I_3 &= \{i \mid 0 \le i \le k-1 \text{ and } d(x,y) \le 3 \text{ for some } y \in S_i\},\\
I_4 &= \{i \mid 0 \le i \le k-1 \text{ and } d(x,y) \ge 4 \text{ for all } y \in S_i\}.
\end{aligned}$$

It is clear that $|I_3| + |I_4| = k$. Since the total number of vertices y with $1 \le d(x,y) \le 3$ is at most $\deg(x) + \sum_{y \in N(x)} (\deg(y) - 1) + \sum_{z \in N^2(x)} (\deg(z) - 1) \le \Delta + \Delta(\Delta-1) + \Delta(\Delta-1)^2 = \Delta^3 - \Delta^2 + \Delta$, we have $|I_3| \le \Delta^3 - \Delta^2 + \Delta$. Similarly, we have $|I_1| \le \Delta$ and $|I_2| \le \Delta^2 - \Delta$. For any $i \in I_4$, $x \notin F_i$, for otherwise, $S_i \cup \{x\}$ is a 3-stable set, which will contradict to the choice of S_i. That is, $d(x,y) = 2$ for some $y \in S_{i-1}$ or $d(x,z) = 1$ for some $z \in S_{i-2} \cup S_{i-1}$. Thus $|I_4| \le |I_2| + 2|I_1|$, and so

$$\begin{aligned}
k &= |I_3| + |I_4|\\
&\le |I_3| + |I_2| + 2|I_1|\\
&\le \Delta^3 - \Delta^2 + \Delta + \Delta^2 - \Delta + 2\Delta\\
&= \Delta^3 + 2\Delta. \qquad \blacksquare
\end{aligned}$$

We now consider the upper bound of the $L(3,2,1)$-labeling numbers of trees. Given a rooted tree T with root v, the *height* of T, denoted $h(T)$, is defined by $h(T) = \max\{d(u,v) \mid u \in V(T)\}$. A rooted tree T with $V(T) = \{v_{ij} \mid 1 \le i \le h+1, 1 \le j \le n^{i-1}\}$ and $E(T) = \{v_{ij}v_{(i+1)k} \mid 1 \le i \le h, (j-1)n+1 \le k \le jn\}$ is called a *complete n-ary tree of height h*. Griggs and Yeh [4] studied the $L(2,1)$-labeling numbers of trees and showed that if T is a tree with $\Delta(T) = \Delta$, then $\Delta + 1 \le \lambda(T) \le \Delta + 2$. Chang and Kuo [1] gave a polynomial-time algorithm to determine whether $\lambda(T) = \Delta + 1$ or $\Delta + 2$ if T is a tree with $\Delta(T) = \Delta$. Clipperton et al. [2] studied the complete n-ary trees and gave the following result.

Theorem 9. ([2]). *If T is a complete n-ary tree of height $h \ge 3$, then* $\lambda_{3,2,1}(T) = 2n + 5$.

In fact, for any tree T with $\Delta(T) = \Delta$, we have

Lemma 10. *If T is a rooted tree with root v, and $\Delta(T) = \Delta$, then $\lambda_{3,2,1}(T) \le 2\Delta + 3$. Moreover, there exists a $(2\Delta+3)$-$L(3,2,1)$-labeling f of T such that $f(u) \equiv d(u,v) \pmod 2$ for all $u \in V(T)$.*

Proof. We prove this by induction on the height h of T. The conclusion clearly holds for $h \le 2$. Suppose the conclusion holds for all rooted trees of height h with $2 \le h \le l$, and let T be a rooted tree with root v of height $h = l + 1$.

Consider the subtree T' of T which is obtained from T by deleting all the leaves of T other than v. Then, since $h(T') = l$, by the induction hypothesis, there exists a $(2\Delta(T')+3)$-$L(3,2,1)$-labeling f' of T' such that $f'(u) \equiv d(u,v) \pmod 2$ for all $u \in V(T')$. Note that since $\Delta(T') \le \Delta$, f' is a $(2\Delta+3)$-$L(3,2,1)$-labeling of T'.

Now, let $\{v_i \mid 1 \le i \le m\}$ be the set of leaves of T'. For all i, $1 \le i \le m$, let $\{u_i\} = N_{T'}(v_i)$, $a_i = f'(u_i)$, $b_i = f'(v_i)$, and let $A_i = N_T(v_i) - \{u_i\}$, $B_i = \{j \mid 0 \le j \le 2\Delta+3,\ j \not\equiv f'(v_i) \pmod 2\} - \{a_i, b_i - 1, b_i + 1\}$. Since for each i, $1 \le i \le m$, $|A_i| \le \Delta - 1$ and $|B_i| \ge \Delta - 1$, there exists a one-to-one function h_i from A_i to B_i. Define a function $f : V(T) \to \{0, 1, 2, \cdots, 2\Delta+3\}$ by

$$f(v) = \begin{cases} f'(v), & \text{if } v \in V(T'), \\ h_i(v), & \text{if } v \in A_i. \end{cases}$$

Then, clearly, f is a $(2\Delta+3)$-$L(3,2,1)$-labeling of T which satisfies $f(u) \equiv d(u,v)$ $\pmod 2$ for all $u \in V(T)$. Thus the conclusion also holds for $h = l+1$. By the principle of mathematical induction, the conclusion holds for any tree T. ∎

By Corollary 4 and Lemma 10, we have

Theorem 11. *For any tree T with $\Delta(T) = \Delta$, $2\Delta + 1 \le \lambda_{3,2,1}(T) \le 2\Delta + 3$.*

4. $L(3,2,1)$-Labelings of Cartesian Product of Paths and Cycles

Given k graphs $G_1, G_2, \cdots, G_k$, the *Cartesian product* of these k graphs, denoted by $G_1 \times G_2 \times \cdots \times G_k$, is a graph with

$$\begin{aligned} & V(G_1 \times G_2 \times \cdots \times G_k) \\ = \ & V(G_1) \times V(G_2) \times \cdots \times V(G_k), \\ & E(G_1 \times G_2 \times \cdots \times G_k) \\ = \ & \{(u_1, u_2, \cdots, u_k)(v_1, v_2, \cdots, v_k) \mid u_l, v_l \in V(G_l) \\ & \text{for all } l,\ 1 \le l \le k,\ u_i v_i \in E(G_i) \text{ for some } i \text{ and} \\ & u_j = v_j \text{ for all } j \ne i\}. \end{aligned}$$

We consider the $L(3,2,1)$-labeling numbers of Cartesian product of paths and cycles in this section. From now on, in convenience, when consider the graph $P_{m_1} \times P_{m_2} \times \cdots \times P_{m_k}$, we always assume that

$$\begin{aligned} & V(P_{m_1} \times P_{m_2} \times \cdots \times P_{m_k}) \\ = \ & \{(i_1, i_2, \cdots, i_k) \mid 1 \le i_l \le m_l \text{ for all } l,\ 1 \le l \le k\}, \end{aligned}$$

and

$$\begin{aligned}&E(P_{m_1} \times P_{m_2} \times \cdots \times P_{m_k})\\ &= \{(i_1, i_2, \cdots, i_k)(j_1, j_2, \cdots, j_k) \mid \sum_{l=1}^{k} |i_l - j_l| = 1\}.\end{aligned}$$

And, in order to simplify the notation, when consider a label of a vertex $(i_1, i_2, \cdots, i_k)$, we use $f(i_1, i_2, \cdots, i_k)$ to replace $f((i_1, i_2, \cdots, i_k))$.

Clipperton et al. [2] studied the $L(3, 2, 1)$-labeling numbers of cycles and gave the following result.

Theorem 12. *[2] For any cycle C_n, $n \geq 3$,*

$$\lambda_{3,2,1}(C_n) = \begin{cases} 6, & \text{if } n = 3, \\ 7, & \text{if } n \text{ is even}, \\ 8, & \text{if } n \text{ is odd and } n \neq 3, 7, \\ 9, & \text{if } n = 7. \end{cases}$$

Given an integer $k \geq 2$, we use the symbol $\overline{i_k}$ to denote the number $i \bmod k$.

Theorem 13. *For all $n \geq 2$,*

$$\lambda_{3,2,1}(P_2 \times P_n) = \begin{cases} 7, & \text{if } n = 2, \\ 8, & \text{if } n = 3, 4, \\ 9, & \text{if } n \geq 5. \end{cases}$$

Proof. Since $P_2 \times P_2 = C_4$, $\lambda_{3,2,1}(P_2 \times P_2) = 7$ follows from Theorem 12. For $n \geq 3$, consider the labeling f of $P_2 \times P_n$ defined by $f(i, j) = \overline{(5i + 3j - 6)_{10}}$ for all i, j, $1 \leq i \leq 2$, $1 \leq j \leq n$. Then, it is easy to verify that f is an $L(3, 2, 1)$-labeling of $P_2 \times P_n$. Since $\max\{f(i, j) \mid (i, j) \in V(P_2 \times P_n)\} = 8$ when $n = 3, 4$, and $\max\{f(i, j) \mid (i, j) \in V(P_2 \times P_n)\} = 9$ when $n \geq 5$, we have $\lambda_{3,2,1}(P_2 \times P_n) \leq 8$ when $n = 3, 4$, and $\lambda_{3,2,1}(P_2 \times P_n) \leq 9$ when $n \geq 5$.

Now, to prove this theorem, by Lemma 1, we only need to show that $\lambda_{3,2,1}(P_2 \times P_3) \geq 8$ and $\lambda_{3,2,1}(P_2 \times P_5) \geq 9$. Suppose $\lambda_{3,2,1}(P_2 \times P_3) \leq 7$, and f is a 7-$L(3, 2, 1)$-labeling of $P_2 \times P_3$. Then by Corollary 4, we have $\{f(1, 2), f(2, 2)\} = \{0, 7\}$. However, this implies $\{f(1, 1), f(2, 1)\} = \{f(1, 3), f(2, 3)\} = \{2, 5\}$, a contradiction. Thus $\lambda_{3,2,1}(P_2 \times P_3) \geq 8$. If $\lambda_{3,2,1}(P_2 \times P_5) \leq 8$, let f be an 8-$L(3, 2, 1)$-labeling of $P_2 \times P_5$, and let $S = \{(i, j) \mid i = 1, 2,\ j = 2, 3, 4\}$, then $f(i, j) \in \{0, 1, 3, 5, 7, 8\}$ for all $(i, j) \in S$ by Lemma 6. Therefore, $f(S) = \{0, 1, 3, 5, 7, 8\}$ since f is an $L(3, 2, 1)$-labeling of $P_2 \times P_5$. However, since $\{f(1, 3), f(2, 3)\} = \{0, 8\}$ by Corollary 7, $f(i, j) \neq 1$ for all $(i, j) \in S$, a contradiction. Hence $\lambda_{3,2,1}(P_2 \times P_5) \geq 9$. ■

Lemma 14. $\lambda_{3,2,1}(P_m \times P_n) \leq 11$ *if $n \geq m \geq 3$. Furthermore,* $\lambda_{3,2,1}(P_3 \times P_n) \leq 10$ *when $n = 4, 5$.*

Proof. Consider the labeling f of $P_m \times P_n$ defined by $f(i,j) = \overline{(3i+5j-4)_{12}}$. Then, it is easy to verify that f is an $L(3,2,1)$-labeling of $P_m \times P_n$, hence $\lambda_{3,2,1}(P_m \times P_n) \leq 11$ for all m,n with $n \geq m \geq 3$. Note that when $m = 3$ and $n = 4,5$, $\max\{f(i,j) \mid (i,j) \in V(P_m \times P_n)\} = 10$. Hence $\lambda_{3,2,1}(P_3 \times P_n) \leq 10$ for $n = 4,5$. ∎

Lemma 15. *Let* $V(C_4) = \{v_1, v_2, v_3, v_4\}$ *and* $E(C_4) = \{v_1v_2, v_2v_3, v_3v_4, v_4v_1\}$. *If* f *is a* 9-$L(3,2,1)$*-labeling of* C_4*, then* $\{f(v_1), f(v_3)\} \neq \{3,7\}$, $\{2,6\}$. *And, if* f *is a* 10-$L(3,2,1)$*-labeling of* C_4*, then* $\{f(v_1), f(v_3)\} \neq \{3,8\}$, $\{1,8\}$, $\{4,8\}$.

Lemma 16. $\lambda_{3,2,1}(P_3 \times P_4) = 10$. *Moreover, if* f *is a* 10-$L(3,2,1)$*-labeling of* $P_3 \times P_4$*, then* $\{f(2,2), f(2,3)\} = \{0,5\}$ *or* $\{f(2,2), f(2,3)\} = \{5,10\}$.

Proof. $\lambda_{3,2,1}(P_3 \times P_4) \leq 10$ follows from Lemma 14. If $\lambda_{3,2,1}(P_3 \times P_4) = 9$, let f be a 9-$L(3,2,1)$-labeling of $P_3 \times P_4$, then, by Corollary 4, we have $\{f(2,2), f(2,3)\} = \{0,9\}$. Without loss of generality, we may assume that $f(2,2) = 0$ and $f(2,3) = 9$. Since $\{f(1,2), f(2,1), f(3,2)\} = \{3,5,7\}$, by Lemma 15, $\{f(1,2), f(2,1)\} \neq \{3,7\}$ and $\{f(2,1), f(3,2)\} \neq \{3,7\}$, thus $f(2,1) = 5$, and, without loss of generality, we may assume that $f(3,2) = 3$ and $f(1,2) = 7$. By a similar argument, we must have $f(3,3) = 6$, $f(2,4) = 4$, and $f(1,3) = 2$. However, in this case, no numbers can be be assigned to the vertex $(1,4)$, a contradiction. Hence $\lambda_{3,2,1}(P_3 \times P_4) = 10$.

Now, if f is a 10-$L(3,2,1)$-labeling of $P_3 \times P_4$, by Lemma 6, $\{f(2,2), f(2,3)\} \subseteq \{0,1,3,5,7,9,10\}$.

Claim 1. $3 \notin \{f(2,2), f(2,3)\}$ and $7 \notin \{f(2,2), f(2,3)\}$.

Proof of Claim 1. Suppose, to the contrary, $3 \in \{f(2,2), f(2,3)\}$. Without loss of generality, we may assume that $f(2,2) = 3$. Since $\{f(i,j) \mid (i,j) \in N((2,2))\} = \{0,6,8,10\}$, by Lemma 6, we have $f(2,3) \in \{0,10\}$. Consider the following two cases.

Case 1. $f(2,3) = 0$.

In this case, if $f(2,1) = 6$, then $\{f(1,2), f(3,2)\} = \{8,10\}$. But this implies $f(1,1) = f(3,1) = 1$, a contradiction. Hence $f(2,1) \neq 6$. Without loss of generality, we assume that $f(3,2) = 6$. Now, if $f(1,2) = 8$, then $f(2,1) = 10$ and $f(3,1) = 1$, which implies $f(1,1) = 5$. But then, no number can be assigned to the vertex $(1,3)$, a contradiction. Hence if $f(3,2) = 6$, we must have $f(1,2) = 10$ and $f(2,1) = 8$. In this case, we have $f(3,1) = 1$ and $f(3,3) = 9$. But this implies $f(1,1) = 5$, and so $f(1,3) = 7$. Thus $f(2,4) = 5$, and so $f(1,4) = f(3,4) = 2$, also a contradiction. Hence this case is impossible.

Case 2. $f(2,3) = 10$.

In this case, since the diameter of the subgraph induced by the vertices in $V(P_3 \times P_4) - \{(1,1),(3,1),(1,4),(3,4)\}$ is 3 and $f(i,j) = 0$ for some $(i,j) \in$

$N((2,2))$, $\{f(i,j) \mid (i,j) \in N((2,3))\} = \{1,3,5,7\}$. If $f(2,4) = 1$, then, without loss of generality, we may assume that $f(3,3) = 5$ and $f(1,3) = 7$. However, this implies $f(1,2) = 0$ and $f(3,4) = 8$. But then, no number can be assigned to the vertex $(3,2)$, a contradiction. If $f(2,4) = 5$, then, without loss of generality, we may assume that $f(3,3) = 1$ and $f(1,3) = 7$. However, this implies $f(1,2) = 0$ and $f(3,4) = 8$. Hence $f(3,2) = 6$ and $f(2,1) = 8$. But then, no number can be assigned to the vertex $(3,1)$, a contradiction. If $f(2,4) = 7$, then, without loss of generality, we may assume that $f(3,3) = 1$ and $f(1,3) = 5$. Under this condition, if $\{f(1,2), f(2,1)\} = \{0,8\}$, then no number can be assigned to the vertex $(1,1)$. Therefore, since $f(1,2) \neq 6$ and $f(3,2) \neq 0$, we must have $f(2,1) = 6$, $f(1,2) = 0$, and $f(3,2) = 8$. But then, no number can be assigned to the vertex $(3,1)$, a contradiction. Hence this case is also impossible.

From the two cases above, we have $3 \notin \{f(2,2), f(2,3)\}$. By Lemma 2, we also have $7 \notin \{f(2,2), f(2,3)\}$.

Claim 2. $1 \notin \{f(2,2), f(2,3)\}$ and $9 \notin \{f(2,2), f(2,3)\}$.

Proof of Claim 2. Suppose, to the contrary, $1 \in \{f(2,2), f(2,3)\}$. Without loss of generality, we may assume that $f(2,2) = 1$. Since $\{f(i,j)\ (i,j) \in N((2,2))\} = \{4,6,8,10\}$, by Lemma 6, we have $f(2,3) = 10$. By Lemma 15, we have $\{f(1,2), f(3,2)\} = \{4,8\}$. Hence, without loss of generality, we may assume that $f(1,2) = 4$, $f(2,1) = 6$, and $f(3,2) = 8$. Therefore, $f(3,1) = 3$ and $f(1,3) = 7$, which implies $f(3,3) = 5$, and so $f(2,4) = 3$. However, in this case, we have $f(1,4) = f(3,4) = 0$, a contradiction. Thus $1 \notin \{f(2,2), f(2,3)\}$. By Lemma 2, we also have $9 \notin \{f(2,2), f(2,3)\}$.

Claim 3. If $f(2,2) = 0$ and $f(2,3) = 10$, then there exists $(i,j) \in N((2,2))$, $f(i,j) = 3$.

Proof of Claim 3. If $f(i,j) \geq 4$ for all $(i,j) \in N((2,2))$, then $\{f(i,j) \mid (i,j) \in N((2,2))\} = \{4,6,8,10\}$. By Lemma 15, we have $\{f(1,2), f(3,2)\} = \{4,8\}$. Without loss of generality, we assume that $f(1,2) = 4$, $f(2,1) = 6$, and $f(3,2) = 8$. Hence $f(1,3) = 7$, and so $\{f(2,4), f(3,3)\} = \{2,5\}$ or $\{f(2,4), f(3,3)\} = \{3,5\}$. In either case, no number can be assigned to the vertex $(3,4)$, a contradiction.

By Claim 1 and Claim 2, we know that $\{f(2,2), f(2,3)\} \subseteq \{0,5,10\}$. If $\{f(2,2), f(2,3)\} = \{0,10\}$, then, without loss of generality, we may assume that $f(2,2) = 0$ and $f(2,3) = 10$. By Claim 3, there exists $(i,j) \in N((2,2))$, $f(i,j) = 3$. By Lemma 15, we have $\{f(1,2), f(2,1)\} \neq \{3,8\}$ and $\{f(2,1), f(3,2)\} \neq \{3,8\}$. If $\{f(1,2), f(2,1)\} = \{3,7\}$, then no number can be assigned to the vertex $(1,1)$, a contradiction. Hence $\{f(1,2), f(2,1)\} \neq \{3,7\}$. Similarly, $\{f(2,1), f(3,2)\} \neq \{3,7\}$. Hence $f(2,1) \neq 3$. Without loss of generality, we may assume that $f(1,2) = 3$. In this case, if $f(1,3) = 6$, then $\{f(2,4), f(3,3)\} = \{4,8\}$, which will contradict to Lemma 15. Hence $f(1,3) = 7$, and so $f(1,4) = 1$. Therefore,

$f(2,4) \in \{4,5\}$, and so $f(3,3) = 2$. Thus $f(3,4) = 8$ and $f(3,2) \in \{5,6\}$, which implies $f(2,1) = 8$. But this will contradict to Lemma 15, thus $\{f(2,2), f(2,3)\} \neq \{0,10\}$, and so either $\{f(2,2), f(2,3)\} = \{0,5\}$ or $\{f(2,2), f(2,3)\} = \{5,10\}$. ■

Combining Lemma 14 and Lemma 16, we have

Lemma 17. $\lambda_{3,2,1}(P_3 \times P_5) = 10$. *Moreover, if f is a 10-$L(3,2,1)$-labeling of $P_3 \times P_5$, then* $(f(2,2), f(2,3), f(2,4)) = (0,5,10)$ *or* $(10,5,0)$.

Theorem 18. *For all $n \geq 3$,*

$$\lambda_{3,2,1}(P_3 \times P_n) = \begin{cases} 9, & \text{if } n = 3, \\ 10, & \text{if } n = 4,5, \\ 11, & \text{if } n \geq 6. \end{cases}$$

Proof. $\lambda_{3,2,1}(P_3 \times P_n) = 10$ for $n = 4,5$ follows from Lemma 1, Lemma 14 and Lemma 16. For $n \geq 6$, by Lemma 14, we have $\lambda_{3,2,1}(P_3 \times P_n) \leq 11$. If $\lambda_{3,2,1}(P_3 \times P_6) = 10$, then, by Lemma 17, for any 10-$L(3,2,1)$-labeling f of $P_3 \times P_6$, we have $f(2,3) = 5$ and $f(2,4) = 5$(the subgraph induced by $\{(i,j) \mid 1 \leq i \leq 3,\ 2 \leq j \leq 6\}$ is the graph $P_3 \times P_5$), a contradiction. Hence $\lambda_{3,2,1}(P_3 \times P_6) \geq 11$, and so $\lambda_{3,2,1}(P_3 \times P_n) = 11$ for all $n \geq 6$.

For $n = 3$, since $\Delta(P_3 \times P_3) = 4$, we have $\lambda_{3,2,1}(P_3 \times P_3) \geq 9$ by Corollary 4. It is easy to verify that the labeling $f : V(P_3 \times P_3) \to \{0,1,2,\cdots,9\}$, defined by $f(i,j) = \overline{(5i+3j-6)_{10}}$ for all i, j, $1 \leq i \leq 2$, $1 \leq j \leq 3$, and $f(3,1) = 4$, $f(3,2) = 9$, $f(3,3) = 6$, is a 9-$L(3,2,1)$-labeling of $P_3 \times P_3$. Thus $\lambda_{3,2,1}(P_3 \times P_3) = 9$. ■

Theorem 19. *For all m, n with $n \geq m \geq 4$, $\lambda_{3,2,1}(P_m \times P_n) = 11$.*

Proof. By Lemma 14, $\lambda_{3,2,1}(P_m \times P_n) \leq 11$ for all m, n. If $\lambda_{3,2,1}(P_4 \times P_4) = 10$, then, by Lemma 16, for any 10-$L(3,2,1)$-labeling f of $P_4 \times P_4$, we have $5 \in \{f(2,2), f(2,3)\}$ and $5 \in \{f(3,2), f(3,3)\}$(the subgraph induced by $\{(i,j) \mid 2 \leq i \leq 4,\ 1 \leq j \leq 4\}$ is the graph $P_3 \times P_4$), a contradiction. Hence $\lambda_{3,2,1}(P_4 \times P_4) \geq 11$, and so $\lambda_{3,2,1}(P_m \times P_n) = 11$ for all m, n with $n \geq m \geq 4$. ■

Combining the theorems above, we have

Theorem 20. *For all m, n with $n \geq m \geq 2$,*

$$\lambda_{3,2,1}(P_m \times P_n) = \begin{cases} 7, & \text{if } (m,n) = (2,2), \\ 8, & \text{if } (m,n) = (2,3), (2,4), \\ 9, & \text{if } (m,n) = (3,3), \text{ or } m = 2 \text{ and } n \geq 5, \\ 10, & \text{if } (m,n) = (3,4), (3,5), \\ 11, & \text{if } m = 3 \text{ and } n \geq 6, \text{ or } n \geq m \geq 4. \end{cases}$$

The *lattice* $\Gamma_\square$ is a graph with $V(\Gamma_\square) = \{(a,b) \mid a,b \in \mathbb{Z}\}$ and $E(\Gamma_\square) = \{(a,b)(c,d) \mid |a-c|+|b-d| = 1,\ a,b,c,d \in \mathbb{Z}\}$. It is easy to see that the labeling of $P_m \times P_n$, given in Lemma 14, can be extended as an $L(3,2,1)$-labeling of $\Gamma_\square$. Therefore, since $P_m \times P_n$ can be viewed as a subgraph of $\Gamma_\square$, by Lemma 1 and Theorem 20, we have

Theorem 21. $\lambda_{3,2,1}(\Gamma_\square) = 11$.

By Lemma 16 and the labeling given in Lemma 14, we also have

Theorem 22. $\lambda_{3,2,1}(C_m \times P_n) = 11$ *if* $m \equiv 0 \pmod 4$ *and* $n \geq 3$.

Theorem 23. $\lambda_{3,2,1}(C_m \times C_n) = 11$ *if* $m \equiv 0 \pmod 4$ *and* $n \equiv 0 \pmod{12}$.

We now consider the $L(3,2,1)$-labeling numbers of $P_2 \times C_n$. The following lemma is easy to verify.

Lemma 24. *If f is a 9-$L(3,2,1)$-labeling of $P_2 \times C_n$, then for all $u \in V(P_2 \times C_n)$ with $f(u) = 2$, $f(v) \in \{5,7,9\}$ for all $v \in N(u)$. And for all $w \in V(P_2 \times C_n)$ with $f(w) = 7$, $f(x) \in \{0,2,4\}$ for all $x \in N(w)$.*

Lemma 25. *If f is a 9-$L(3,2,1)$-labeling of $P_2 \times C_n$ and $uv \in E(P_2 \times C_n)$, then $\{f(u), f(v)\} \neq \{0,8\}$.*

Proof. Suppose, to the contrary, there exists $uv \in E(P_2 \times C_n)$, such that $\{f(u), f(v)\} = \{0,8\}$. Let w, x be the vertices in $P_2 \times C_n$ such that the subgraph induced by $\{u,v,w,x\}$ is isomorphic to C_4. Without loss of generality, we may assume that $vw, wx, xu \in E(P_2 \times C_n)$ and $f(u) = 0$, $f(v) = 8$. Since f is a 9-$L(3,2,1)$-labeling, by Lemma 24, we have $(f(w), f(x)) = (3,6)$. However, in this case, no number can be assigned to the vertex y with $y \in N(w) - \{v,x\}$, a contradiction. ■

Lemma 26. *If f is a 9-$L(3,2,1)$-labeling of $P_2 \times C_n$ and there exist $u = (i,j), v = (k,l) \in V(P_2 \times C_n)$, such that $d(u,v) = 2$ and $\{f(u), f(v)\} = \{0,8\}$, then $|i-k| = 1$.*

Proof. Suppose, to the contrary, the conclusion false. Without loss of generality, we may assume that $u = (1,1)$, $v = (1,3)$, and $(f(u), f(v)) = (0,8)$. Then $f(1,2) \in \{3,4,5\}$. If $f(1,2) = 4$, then no number can be assigned to the vertex $(2,2)$. If $f(1,2) = 5$, then $f(2,2) = 2$, and thus no number can be assigned to the vertex $(2,3)$. Hence $f(1,2) = 3$. But this implies $f(2,2) = 6$, $f(2,3) = 1$, and $f(2,4) = 4$. However, in this case, no number can be assigned to the vertex $(1,4)$, a contradiction. Hence if $u = (i,j)$, $v = (k,l)$, $d(u,v) = 2$ and $\{f(u), f(v)\} = \{0,8\}$, we must have $|i-k| = 1$. ■

Lemma 27. *If f is a 9-$L(3,2,1)$-labeling of $P_2 \times C_n$ and $f(1,1) = 0$, then $f(2,1) \notin \{4,6,7\}$.*

Proof. If $f(2,1) = 4$, then $\{f(2,2), f(2,n)\} = \{7,9\}$. Without loss of generality, we may assume that $f(2,2) = 7$. But then, no number can be assigned to the vertex $(1,2)$, a contradiction. Hence $f(2,1) \neq 4$. If $f(2,1) = 6$, then $f(i,j) \in \{2,3\}$ for some $(i,j) \in \{(2,2),(2,n)\}$. Without loss of generality, we may assume that $f(2,2) \in \{2,3\}$. In this case, $f(2,n) = 9$ and $f(1,2) = 8$, therefore, no number can be assigned to the vertex $(2,3)$, a contradiction. Now, if $f(2,1) = 7$, then, by Lemma 24, $f(i,j) = 2$ for some $(i,j) \in \{(2,2),(2,n)\}$. Without loss of generality, we may assume that $f(2,2) = 2$. Again, by Lemma 24, $\{f(1,2), f(2,3)\} = \{5,9\}$. But then, no number can be assigned to the vertex $(1,3)$, also a contradiction. Thus if f is a 9-$L(3,2,1)$-labeling of $P_2 \times C_n$ and $f(1,1) = 0$, $f(2,1) \notin \{4,6,7\}$. ■

Lemma 28. *If f is a 9-$L(3,2,1)$-labeling of $P_2 \times C_n$ and $f(1,1) = 0$, then there exists $(i,j) \in N^2((1,1))$, $f(i,j) = 8$.*

Proof. Suppose, to the contrary, for all $(i,j) \in N^2((1,1))$, $f(i,j) \neq 8$. By Lemma 25 and Lemma 27, $f(2,1) \in \{3,5,9\}$. If $f(2,1) = 3$, then there exists $(i,j) \in \{(2,2),(2,n)\}$, such that $f(i,j) \in \{6,7\}$. Without loss of generality, we may assume that $f(2,2) \in \{6,7\}$. In this case, $f(2,n) = 9$. But then, no numbers can be assigned to the vertex $(1,2)$, a contradiction. If $f(2,1) = 5$, then $\{f(2,2), f(2,n)\} = \{2,9\}$. Without loss of generality, we may assume that $f(2,2) = 2$ and $f(2,n) = 9$. Since $d((2,n),(2,3)) = d((2,n),(1,2)) = 3$, $9 \notin \{f(2,3), f(1,2)\}$. But this will contradict to Lemma 24. Thus $f(2,1) \neq 5$.

Now, assume that $f(2,1) = 9$. Since f is a 9-$L(3,2,1)$-labeling, $\{f(2,2), f(2,n)\} = \{2,4\}$, $\{2,5\}$, $\{2,6\}$, $\{3,5\}$, $\{3,6\}$ or $\{4,6\}$. We consider the following cases.

Case 1. $\{f(2,2), f(2,n)\} = \{2,5\}$ or $\{3,5\}$.

Without loss of generality, we may assume that $f(2,2) = 5$. In this case, no number can be assigned to the vertex $(1,2)$, a contradiction. Hence this case is impossible.

Case 2. $\{f(2,2), f(2,n)\} = \{3,6\}$.

Without loss of generality, we may assume that $f(2,n) = 3$ and $f(2,2) = 6$. In this case, no number can be assigned to the vertex $(1,2)$, a contradiction. Hence this case is impossible.

Case 3. $\{f(2,2), f(2,n)\} = \{2,4\}$ or $\{4,6\}$.

Without loss of generality, we may assume that $f(2,2) = 4$. Thus $f(1,2) = 7$ and $f(2,3) = 1$. But then, no number can be assigned to the vertex $(1,3)$, a contradiction. Thus this case is also impossible.

Case 4. $\{f(2,2), f(2,n)\} = \{2,6\}$.

Without loss of generality, we may assume that $f(2,2) = 6$. Thus $f(1,2) = 3$ and $f(2,3) = 1$. Since $d((1,1),(1,3)) = 2$, no number can be assigned to the vertex $(1,3)$, a contradiction. Therefore, this case is also impossible.

From the argument above, there exists $(i,j) \in N^2((1,1))$, $f(i,j) = 8$. ■

Lemma 29. *If f is a 9-$L(3,2,1)$-labeling of $P_2 \times C_n$ and $f(1,1) = 0$, $f(2,2) = 8$, then $f(2,1) = 5$ and $f(1,2) = 3$.*

Proof. Since f is a 9-$L(3,2,1)$-labeling, either $(f(2,1), f(1,2)) = (3,5)$ or $(f(2,1), f(1,2)) = (5,3)$. If $(f(2,1), f(1,2)) = (3,5)$, then $f(2,3) = 1$, and thus no number can be assigned to the vertex $(1,3)$, a contradiction. Hence $f(2,1) = 5$ and $f(1,2) = 3$. ■

Lemma 30. *If f is a 9-$L(3,2,1)$-labeling of $P_2 \times C_n$ and $f(1,1) = 0$, then $f(2,1) = 5$, and either $f(2,2) = 8$, $f(1,2) = 3$, or $f(2,n) = 8$ and $f(1,n) = 3$.*

Proof. By Lemma 28, there exists $(i,j) \in N^2((1,1))$, $f(i,j) = 8$. Thus by Lemma 25 and Lemma 26, either $f(2,2) = 8$ or $f(2,n) = 8$. If $f(2,2) = 8$, then by Lemma 29, $f(2,1) = 5$ and $f(1,2) = 3$. Similarly, if $f(2,n) = 8$, then $f(2,n) = 5$ and $f(1,n) = 3$. ■

Lemma 31. *If f is a 9-$L(3,2,1)$-labeling of $P_2 \times C_n$ and $f(1,1) = \overline{i_{10}}$, $f(2,1) = \overline{(i+5)_{10}}$, $f(2,2) = \overline{(i+8)_{10}}$, $f(1,2) = \overline{(i+3)_{10}}$, then $f(2,3) = \overline{(i+1)_{10}}$ and $f(1,3) = \overline{(i+6)_{10}}$.*

Proof. Clearly, $f(2,3) \notin \{\overline{i_{10}}, \overline{(i+3)_{10}}, \overline{(i+4)_{10}}, \overline{(i+5)_{10}}, \overline{(i+6)_{10}}, \overline{(i+8)_{10}}\}$. If $f(2,3) = \overline{(i+2)_{10}}$, then $i = 7$, and so $f(1,1) = 7$, $f(2,1) = 2$, $f(2,2) = 5$ and $f(2,3) = 9$. By Lemma 24, we have $f(2,n) = 9$, a contradiction. If $f(2,3) = \overline{(i+7)_{10}}$, then $i = 2$, and so $f(1,1) = 2$, $f(2,2) = 0$, $f(1,2) = 5$ and $f(2,3) = 9$. But then, no number can be assigned to the vertex $(1,3)$, a contradiction. If $f(2,3) = \overline{(i+9)_{10}}$, then $i = 1$, and so $f(1,1) = 1$, $f(2,1) = 6$, $f(2,2) = 9$, $f(1,2) = 4$ and $f(2,3) = 0$. But then, by Lemma 27, no number can be assigned to the vertex $(1,3)$, also a contradiction. Hence $f(2,3) = \overline{(i+1)_{10}}$. By a similar argument, we also have $f(1,3) = \overline{(i+6)_{10}}$. ■

Theorem 32. $\lambda_{3,2,1}(P_2 \times C_n) = 9$ *if and only if* $n \equiv 0 \pmod{10}$.

Proof. If $\lambda_{3,2,1}(P_2 \times C_n) = 8$, then, by Corollary 7, for all 8-$L(3,2,1)$-labeling f of $P_2 \times C_n$, $f(i,j) \in \{0,8\}$ for all i, j, $i = 1, 2$, $1 \le j \le n$, which is impossible. Hence $\lambda_{3,2,1}(P_2 \times C_n) \ge 9$. When $n \equiv 0 \pmod{10}$, it is easy to see that the function f, given in the proof of Theorem 13, is a 9-$L(3,2,1)$-labeling of $P_2 \times C_n$. Hence $\lambda_{3,2,1}(P_2 \times C_n) = 9$ if $n \equiv 0 \pmod{10}$.

Conversely, if $\lambda_{3,2,1}(P_2 \times C_n) = 9$ and f is a 9-$L(3,2,1)$-labeling of $P_2 \times C_n$, then, since there exists (i,j), $f(i,j) = 0$, by Lemma 30, without loss of generality, we may assume that $f(1,1) = 0$, $f(2,1) = 5$, $f(2,2) = 8$ and $f(1,2) = 3$. Thus by Lemma 31, $f(i,j) = \overline{(5i+3j-8)_{10}}$ for all $(i,j) \in V(P_2 \times C_n)$. Since $f(1,1) = \overline{[f(1,n)+3]_{10}}$, we have $n \equiv 0 \pmod{10}$. ∎

Theorem 33. *If n is even and $n \not\equiv 0 \pmod{10}$, $n \neq 6$, $\lambda_{3,2,1}(P_2 \times C_n) = 10$.*

Proof. By Theorem 32, if $n \not\equiv 0 \pmod{10}$, then $\lambda_{3,2,1}(P_2 \times C_n) \geq 10$. If $n \equiv 0 \pmod 4$, then, it is easy to verify that the labeling $f : V(P_2 \times C_n) \rightarrow \{0,1,2,\cdots,10\}$, defined by $f(i,j) = \overline{(5i+3j-7)_{12}}$, is a 10-$L(3,2,1)$-labeling of $P_2 \times C_n$. If $n \equiv 2 \pmod 4$, $n \not\equiv 0 \pmod{10}$, and $n \neq 6$, then, it is easy to verify that the labeling $f_1 : V(P_2 \times C_n) \rightarrow \{0,1,2,\cdots,10\}$, defined by

$$f_1(i,j) = \begin{cases} \overline{(5i+3j+3)_{10}}, & \text{if } 1 \leq j \leq 10, \\ \overline{(5i+3j-1)_{12}}, & \text{if } 11 \leq j \leq n, \end{cases}$$

is a 10-$L(3,2,1)$-labeling of $P_2 \times C_n$. Hence $\lambda_{3,2,1}(P_2 \times C_n) \leq 10$ if n is even and $n \not\equiv 0 \pmod{10}$, $n \neq 6$. ∎

Theorem 34. *$\lambda_{3,2,1}(P_2 \times C_n) \leq 11$ if $n \equiv 1 \pmod 4$, $n \geq 21$, or $n \equiv 3 \pmod 4$, $n \geq 11$.*

Proof. For $n \equiv 1 \pmod 4$, $n \geq 21$, define $f_1 : V(P_2 \times C_n) \rightarrow \mathbb{N} \cup \{0\}$ as

$$f_1(i,j) = \begin{cases} \overline{(3j+5i+5)_{10}}, & \text{if } 1 \leq i \leq 2 \text{ and } 1 \leq j \leq 14, \\ \overline{(3j-1)_{13}}, & \text{if } i = 1 \text{ and } 15 \leq j \leq 21, \\ 10, & \text{if } i = 2 \text{ and } j = 15, \\ \overline{(3j+2)_{10}}, & \text{if } i = 2 \text{ and } 16 \leq j \leq 21, \\ \overline{(7i+3j)_{12}}, & \text{otherwise.} \end{cases}$$

Then, it is easy to verify that f_1 is an $L(3,2,1)$-labeling of $P_2 \times C_n$, and $\max\{f_1(v) \mid v \in V(P_2 \times C_n)\} = 11$. Hence $\lambda_{3,2,1}(P_2 \times C_n) \leq 11$ if $n \equiv 1 \pmod 4$, $n \geq 21$.

Similarly, if $n \equiv 3 \pmod 4$ and $n \geq 11$, then, it is easy to verify that the labeling $f_2 : V(P_2 \times C_n) \rightarrow \mathbb{N} \cup \{0\}$, defined by

$$f_2(i,j) = \begin{cases} \overline{(3j-1)_{13}}, & \text{if } i = 1 \text{ and } 1 \leq j \leq 7, \\ \overline{(3j+2)_{13}}, & \text{if } i = 1 \text{ and } 8 \leq j \leq 10, \\ \overline{(3j+4)_{13}}, & \text{if } i = 2 \text{ and } 1 \leq j \leq 6, \\ \overline{(3j+7)_{13}}, & \text{if } i = 2 \text{ and } 7 \leq j \leq 10, \\ \overline{(7i+3j+5)_{12}}, & \text{otherwise,} \end{cases}$$

is an $L(3,2,1)$-labeling of $P_2 \times C_n$. Since we also have $\max\{f_2(v) \mid v \in V(P_2 \times C_n)\} = 11$ in this case, $\lambda_{3,2,1}(P_2 \times C_n) \leq 11$ if $n \equiv 3 \pmod 4$, $n \geq 11$. ∎

By using a similar labeling scheme as in the proof of Lemma 14, for those graphs which are the Cartesian product of paths, we have

Lemma 35. *If* $G = P_{m_1} \times P_{m_2} \times \cdots \times P_{m_k}$, *then* $\lambda_{3,2,1}(G) \leq 4k+3$.

Proof. Define a labeling f of G as

$$f(i_1, i_2, \cdots, i_k) = \overline{\left(\sum_{l=1}^{k}(2l+1)i_l\right)}_{4k+4}.$$

Then, for $u = (i_1, i_2, \cdots, i_k), v = (j_1, j_2, \cdots, j_k) \in V(G)$, if $d(u,v) = 1$, then, since $\overline{(2i+1)}_{4k+4} \neq 0, 1, 2$ for all i, $1 \leq i \leq k$, $|f(u) - f(v)| \geq 3$. If $d(u,v) = 2$, then, since $\overline{[(2i+1)+(2j+1)]}_{4k+4} \neq 0, 1$, and $\overline{[(2i+1)-(2l+1)]}_{4k+4} \neq 0, 1$ for all i, j, l, $1 \leq i, j, l \leq k$, $i \neq l$(note that $i = j$ is possible), $|f(u) - f(v)| \geq 2$. If $d(u,v) = 3$, then, since $\overline{[(2i+1) \pm (2j+1) \pm (2l+1)]}_{4k+4} \neq 0$ for all i, j, l, $1 \leq i, j, l \leq k$(note that $i = j$, or $i = j = l$, are possible), $|f(u) - f(v)| \geq 1$. Hence f is an $L(3,2,1)$-labeling of G, and so $\lambda_{3,2,1}(G) \leq 4k+3$. ∎

Theorem 36. *For the graph* $G = P_{m_1} \times P_{m_2} \times \cdots \times P_{m_k}$ *with* $k \geq 3$ *and* $m_k \geq m_{k-1} \geq \cdots \geq m_1 \geq 3$, *if* $m_{k-2} \geq 4$, *or* $(m_{k-2}, m_{k-1}) = (3, 4)$ *and* $m_k \geq 6$, *then* $\lambda_{3,2,1}(G) = 4k+3$.

Proof. $\lambda_{3,2,1}(G) \leq 4k+3$ follows from Lemma 35. If $\lambda_{3,2,1}(G) = 4k+2$, then, by Corollary 7, for any $(4k+2)$-$L(3,2,1)$-labeling f of G, $f(i_1, i_2, \cdots, i_k) \in \{0, 4k+2\}$ for all $(i_1, i_2, \cdots, i_k) \in V(G)$ with $2 \leq i_j \leq n-1$, $1 \leq j \leq k$, which is impossible since either $m_k \geq m_{k-1} \geq m_{k-2} \geq 4$, or $(m_{k-2}, m_{k-1}) = (3, 4)$ and $m_k \geq 6$. Thus $\lambda_{3,2,1}(G) \geq 4k+3$, and so $\lambda_{3,2,1}(G) = 4k+3$. ∎

Given a positive integer n, the n*-cube* Q_n is defined by $Q_n = G_1 \times G_2 \times \cdots \times G_n$, where $G_i = P_2$ for all i, $1 \leq i \leq n$. By Corollary 7 and the labeling given in Lemma 35, we also have

Theorem 37. *For all* $n \geq 3$, $2n+3 \leq \lambda_{3,2,1}(Q_n) \leq 4n+3$.

5. $L(3,2,1)$-Labelings of Powers of Paths

Given a graph G, the r*-th power* of G, denoted by G^r, is a graph with $V(G^r) = V(G)$ and $E(G^r) = \{uv \mid u, v \in V(G) \text{ and } d_G(u,v) \leq r\}$.

We study the $L(3,2,1)$-labeling numbers of P_n^r in this section. For convenience, when consider the graph P_n^r, we always assume that $V(P_n^r) = \{v_i \mid 1 \leq i \leq n\}$ and $E(P_n^r) = \{v_i v_j \mid 1 \leq i, j \leq n, |i-j| \leq r\}$. For an $L(3,2,1)$-labeling f of a graph G, we let $H_{G,f}$ be the set defined by $H_{G,f} = \{0,1,2,\cdots,a\}\backslash f(V(G))$, where $a = \max\{f(v) \mid v \in V(G)\}$. Clearly, if f is an $L(3,2,1)$-labeling of G and $\max\{f(v) \mid v \in V(G)\} = a$, then $a = |f(V(G))| + |H_{G,f}| - 1$.

For an integer k, we use $n(k)$ to denote the set $\{k-1, k, k+1\}$.

Lemma 38. *If* $r+2 \leq n \leq 3r$, *then* $\lambda_{3,2,1}(P_n^r) = n + 2r$.

Proof. If $r+2 \leq n \leq 2r+2$, define a labeling f of P_n^r as

$$f(v_i) = \begin{cases} 4i-2, & \text{if } 1 \leq i \leq n-r-1, \\ 3i+n-r-3, & \text{if } n-r \leq i \leq r+1, \\ 4(i-r-2), & \text{if } r+2 \leq i \leq n, \end{cases}$$

and if $2r+3 \leq n \leq 3r$, define a labeling f of P_n^r as

$$f(v_i) = \begin{cases} 5i-2, & \text{if } 1 \leq i \leq n-2r-2, \\ 4i+n-2r-4, & \text{if } n-2r-1 \leq i \leq r+1, \\ 5(i-r-2), & \text{if } r+2 \leq i \leq n-r, \\ 4i+n-6r-10, & \text{if } n-r+1 \leq i \leq 2r+2, \\ 5i-10r-13, & \text{if } 2r+3 \leq i \leq n. \end{cases}$$

In either case, it is easy to verify that f is an $L(3,2,1)$-labeling of P_n^r. Hence $\lambda_{3,2,1}(P_n^r) \leq n+2r$ if $r+2 \leq n \leq 3r$.

To prove the lower bound, let f be an $L(3,2,1)$-labeling of P_n^r, and $\max\{f(v) \mid v \in V(P_n^r)\} = a$. Since $n \leq 3r$, for all i,j with $\left\lfloor \frac{n-r}{2} \right\rfloor \leq i,j \leq \left\lfloor \frac{n-r}{2} \right\rfloor + r + 1$, $i \neq j$, we have $d(v_i, v) \leq 2$ for all $v \in V(P_n^r)$, and $d(v_i, v_j) = 1$ if $\{i,j\} \neq \{\left\lfloor \frac{n-r}{2} \right\rfloor, \left\lfloor \frac{n-r}{2} \right\rfloor + r + 1\}$. Hence $n(f(v_i)) \cap f(V(P_n^r)) = \{f(v_i)\}$, $n(f(v_i)) \cap n(f(v_j)) = \emptyset$ if $\{i,j\} \neq \{\left\lfloor \frac{n-r}{2} \right\rfloor, \left\lfloor \frac{n-r}{2} \right\rfloor + r+1\}$, and $|n(f(v_{\lfloor \frac{n-r}{2} \rfloor})) \cap n(f(v_{\lfloor \frac{n-r}{2} \rfloor + r+1}))| \leq 1$. Therefore,

$$\begin{aligned} |H_{P_n^r,f}| &\geq \left| \left(\left(\bigcup_{i=\lfloor \frac{n-r}{2} \rfloor}^{\lfloor \frac{n-r}{2} \rfloor + r+1} n(f(v_i)) \right) \bigcap \{0,1,2,\cdots,a\} \right) \backslash f(V(P_n^r)) \right| \\ &\geq 2r+1. \end{aligned}$$

Since $\operatorname{diam}(P_n^r) \leq 3$, $|f(V(P_n^r))| = n$. Thus $a = |f(V(P_n^r))| + |H_{P_n^r,f}| - 1 \geq n + 2r$, and so $\lambda_{3,2,1}(P_n^r) \geq n + 2r$. ■

From now on, in convenience, when consider the graph P_n^r, we let $S = \{v_i \mid r+1 \leq i \leq 2r+1\}$.

Lemma 39. $\lambda_{3,2,1}(P_{3r+1}^r) = 5r$. *Moreover, if* f *is a* $(5r)$-$L(3,2,1)$*-labeling of* P_{3r+1}^r, *then* $f(S) = \{0, 5, \cdots, 5r\}$.

Proof. By Lemma 1 and Lemma 38, $\lambda_{3,2,1}(P^r_{3r+1}) \geq \lambda_{3,2,1}(P^r_{3r}) = 5r$. Let f be a labeling of P^r_{3r+1}, defined by

$$f(v_i) = \begin{cases} 5i-2, & \text{if } 1 \leq i \leq r, \\ 5(i-r-1), & \text{if } r+1 \leq i \leq 2r+1, \\ 5(i-2r-1)-3, & \text{if } 2r+2 \leq i \leq 3r+1. \end{cases}$$

Then, clearly, f is a $(5r)$-$L(3,2,1)$-labeling of P^r_{3r+1}. Hence $\lambda_{3,2,1}(P^r_{3r+1}) \leq 5r$, and so $\lambda_{3,2,1}(P^r_{3r+1}) = 5r$.

If f is a $(5r)$-$L(3,2,1)$-labeling of P^r_{3r+1}, let $f(S) = \{a_0, a_1, \cdots, a_r\}$, where $a_0 < a_1 < \cdots < a_r$.

Claim. $a_0 = 0$, $a_r = 5r$, and for each i, $0 \leq i \leq r-1$, there exists exactly one vertex v_j in $\{v_1, v_2, \cdots, v_r\}$, and exactly one vertex $v_{j'}$ in $\{v_{2r+2}, v_{2r+3}, \cdots, v_{3r+1}\}$, such that $a_i < f(v_j), f(v_{j'}) < a_{i+1}$.

Proof of the Claim. Since $\mathrm{diam}(P^r_{3r+1}) = 3$, we have $|f(V(P^r_{3r+1}))| = 3r+1$. Hence $|H_{P^r_{3r+1},f}| = 5r - |f(V(P^r_{3r+1}))| + 1 = 2r$, since f is a $(5r)$-$L(3,2,1)$-labeling of P^r_{3r+1}. For $v_i, v_j \in S$, since $d(v_i, v) \leq 2$ for all $v \in V(P^r_{3r+1})$, and $d(v_i, v_j) = 1$, we have $n(f(v_i)) \cap f(V(P^r_{3r+1})) = \{f(v_i)\}$ and $n(f(v_i)) \cap n(f(v_j)) = \emptyset$. Therefore, since $|H_{P^r_{3r+1},f}| = 2r$, we have $\{0, 5r\} \subseteq f(S)$ and $H_{P^r_{3r+1},f} = \{a_i + 1 \mid 0 \leq i \leq r-1\} \cup \{a_i - 1 \mid 1 \leq i \leq r\}$. Thus if $v \notin S$, there exists i, $0 \leq i \leq r-1$, such that $a_i < f(v) < a_{i+1}$. If v_l, v_m are two vertices in $\{v_1, v_2, \cdots, v_r\}$, such that $a_i < f(v_l) < f(v_m) < a_{i+1}$ for some i, $0 \leq i \leq r-1$, and for all vertex v_q in $\{v_1, v_2, \cdots, v_r\}$, $f(v_q) < f(v_l)$ or $f(v_q) > f(v_m)$, then, since $d(v_l, v_m) = 1$, and $f(v_l)+1, f(v_m)-1 \notin H_{P^r_{3r+1},f}$, there exist $v_{l'}, v_{m'}$ in $\{v_{2r+2}, v_{2r+3}, \cdots, v_{3r+1}\}$, such that $f(v_{l'}) = f(v_l)+1$ and $f(v_{m'}) = f(v_m)-1$. But this implies that $f(v_{l'})+1 \in H_{P^r_{3r+1},f}$, a contradiction. Hence for all i, $0 \leq i \leq r-1$, there exists exactly one vertex v_j in $\{v_1, v_2, \cdots, v_r\}$, such that $a_i < f(v_j) < a_{i+1}$. Similarly, for all i, $0 \leq i \leq r-1$, there exists exactly one vertex $v_{j'}$ in $\{v_{2r+2}, v_{2r+3}, \cdots, v_{3r+1}\}$, such that $a_i < f(v_{j'}) < a_{i+1}$.

By the Claim, for all i, $0 \leq i \leq r-1$, $a_{i+1} - a_i \geq 5$. Since f is a $(5r)$-$L(3,2,1)$-labeling of P^r_{3r+1}, $a_{i+1} - a_i = 5$ for all i, $0 \leq i \leq r-1$. Since $a_0 = 0$, we have $f(S) = \{0, 5, \cdots, 5r\}$. ∎

Lemma 40. *If f is an $L(3,2,1)$-labeling of P^r_{3r+1}, and $\max\{f(v) \mid v \in V(P^r_{3r+1})\} = a$, then $a \geq 5r+2$ if $\{0, a\} \cap f(S) = \emptyset$.*

Proof. For $v_i, v_j \in S$, $v_i \neq v_j$, $v \in V(P^r_{3r+1})$, we have $d(v_i, v) \leq 2$ and $d(v_i, v_j) = 1$. Hence $n(f(v_i)) \cap f(V(P^r_{3r+1})) = \{f(v_i)\}$ and $n(f(v_i)) \cap n(f(v_j)) = \emptyset$. Therefore, if $\{0, a\} \cap f(S) = \emptyset$, then

$$
\begin{aligned}
|H_{P^r_{3r+1},f}| &\geq \left| \left(\left(\bigcup_{v \in S} n(f(v)) \right) \bigcap \{0,1,2,\cdots,a\} \right) \setminus f(V(P^r_{3r+1})) \right| \\
&\geq 2r+2.
\end{aligned}
$$

Since $\mathrm{diam}(P^r_{3r+1}) = 3$, $|f(V(P^r_{3r+1}))| = 3r+1$. Thus $a = |f(V(P^r_{3r+1}))| + |H_{P^r_{3r+1},f}| - 1 \geq (3r+1)+(2r+2)-1 = 5r+2$. ∎

Lemma 41. *If* $3r+2 \leq n \leq 5r+2$, *then* $\lambda_{3,2,1}(P^r_n) = 5r+1$.

Proof. Let f be a labeling of P^r_n, defined by $f(v_i) = \overline{(5i-1)_{5r+3}}$ for all i, $1 \leq i \leq n$. Clearly, f is a $(5r+1)$-$L(3,2,1)$-labeling of P^r_n. Hence $\lambda_{3,2,1}(P^r_n) \leq 5r+1$.

To prove the lower bound, by Lemma 1, we only need to show that $\lambda_{3,2,1}(P^r_{3r+2}) \geq 5r+1$. Suppose, to the contrary, $\lambda_{3,2,1}(P^r_{3r+2}) \leq 5r$. Let f be a $(5r)$-$L(3,2,1)$-labeling of P^r_{3r+2}, and let G be the subgraph of P^r_{3r+2} induced by $\{v_1, v_2, \cdots, v_{3r+1}\}$, H be the subgraph of P^r_{3r+2} induced by $\{v_2, v_3, \cdots, v_{3r+2}\}$. By Lemma 39, since $f|_{V(G)}$ is a $(5r)$-$L(3,2,1)$-labeling of G and $f|_{V(H)}$ is a $(5r)$-$L(3,2,1)$-labeling of H, we have $f(S) = \{0, 5, \cdots, 5r\} = \{f(v_{r+2}), f(v_{r+3}), \cdots, f(v_{2r+2})\}$, which implies $f(v_{r+1}) = f(v_{2r+2})$. However, $d(v_{r+1}, v_{2r+2}) = 2$, a contradiction. Thus $\lambda_{3,2,1}(P^r_{3r+2}) \geq 5r+1$, and so $\lambda_{3,2,1}(P^r_n) = 5r+1$ if $3r+2 \leq n \leq 5r+2$. ∎

Lemma 42. $\lambda_{3,2,1}(P^r_n) = 5r+2$ *if* $n \geq 5r+3$.

Proof. Clearly, the labeling f, given in the proof of Lemma 41, is a $(5r+2)$-$L(3,2,1)$-labeling of P^r_n. Hence $\lambda_{3,2,1}(P^r_n) \leq 5r+2$.

To prove the lower bound, by Lemma 1, we only need to show that $\lambda_{3,2,1}(P^r_{5r+3}) \geq 5r+2$. Suppose, to the contrary, $\lambda_{3,2,1}(P^r_{5r+3}) \leq 5r+1$. Let f be a $(5r+1)$-$L(3,2,1)$-labeling of P^r_{5r+3}, and let G_i be the subgraph of P^r_{5r+3} induced by $\{v_i, v_{i+1}, \cdots, v_{3r+i}\}$ for all i, $1 \leq i \leq 2r+3$. Since $f|_{V(G_1)}$ is a $(5r+1)$-$L(3,2,1)$-labeling of G_1, by Lemma 40, there exists $v_\alpha \in S$, such that $f(v_\alpha) \in \{0, 5r+1\}$. Without loss of generality, we assume that $f(v_\alpha) = 0$. Since $f|_{V(G_{\alpha-r+1})}$ is a $(5r+1)$-$L(3,2,1)$-labeling of $G_{\alpha-r+1}$, by Lemma 40, there exists $v_\beta \in \{v_{\alpha+1}, v_{\alpha+2}, \cdots, v_{\alpha+r+1}\}$, such that $f(v_\beta) \in \{0, 5r+1\}$. Since $d(v_\alpha, v_\beta) \leq 2$ and $f(v_\alpha) = 0$, we have $f(v_\beta) = 5r+1$. By a similar argument, there exists $v_\gamma \in \{v_{\beta+1}, v_{\beta+2}, \cdots, v_{\beta+r+1}\}$, such that $f(v_\gamma) = 0$. Now, since $\gamma - \beta \leq r+1$, $\beta - \alpha \leq r+1$, we have $\gamma - \alpha \leq 2r+2$. But this implies $d(v_\alpha, v_\gamma) \leq 3$, a contradiction. Hence $\lambda_{3,2,1}(P^r_{5r+3}) \geq 5r+2$, and so $\lambda_{3,2,1}(P^r_n) = 5r+2$ if $n \geq 5r+3$. ∎

Since $P^r_n = K_n$ for $r \geq n-1$, by Lemma 38, Lemma 39, Lemma 41 and Lemma 42, we have

Theorem 43. *If* $n, r \geq 1$, *then*

$$\lambda_{3,2,1}(P_n^r) = \begin{cases} 3n-3, & \textit{if } n \leq r+1, \\ n+2r, & \textit{if } r+2 \leq n \leq 3r, \\ 5r, & \textit{if } n = 3r+1, \\ 5r+1, & \textit{if } 3r+2 \leq n \leq 5r+2, \\ 5r+2, & \textit{if } n \geq 5r+3. \end{cases}$$

Clipperton et al. [2] determined the $L(3,2,1)$-labeling numbers of paths, by setting $r=1$ in Theorem 43, we also have

Theorem 44. [2]. *For any* $n \geq 2$,

$$\lambda_{3,2,1}(P_n) = \begin{cases} 3, & \textit{if } n = 2, \\ 5, & \textit{if } n = 3,4, \\ 6, & \textit{if } n = 5,6,7, \\ 7, & \textit{if } n \geq 8. \end{cases}$$

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