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ULAM-HYERS STABILITY FOR OPERATORIAL EQUATIONS AND INCLUSIONS VIA NONSELF OPERATORS

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Abstract. Using the weakly Picard operator technique, we present some abstract Ulam-Hyers stability results for operatorial equations and inclusions involving nonself single-valued and multivalued operators.

1. Introduction

Let (X, d) be a metric space, $\mathcal{P}(X)$ be the family of all subsets of X and consider the following families of subsets of X:

$$P(X) := \{Y \in \mathcal{P}(X) | \ Y \neq \emptyset\}, \ P_b(X) := \{Y \in P(X) | \ Y \text{ is bounded}\},$$

$$P_{cl}(X) := \{Y \in P(X) | \ Y \text{ is closed}\}, \ P_{cp}(X) := \{Y \in P(X) | \ Y \text{ is compact}\}.$$

We will denote by $\bar{B}(x_0, r)$ the closure of $B(x_0, r)$ in (X, d), where $B(x_0, r) := \{x \in X | d(x_0, x) < r\}$ is the open ball centered in $x_0 \in X$ with radius $r > 0$ and by $\tilde{B}(x_0, r)$ the closed ball centered in $x_0 \in X$ with radius $r > 0$, i.e., $\tilde{B}(x_0, r) := \{x \in X | d(x_0, x) \leq r\}$.

If (X, d) is a metric space, then the gap functional in $P(X)$ is defined as

$$D_d : P(X) \times P(X) \to \mathbb{R}_+, \ D_d(A, B) = \inf\{d(a, b) \mid a \in A, \ b \in B\}.$$

In particular, if $x_0 \in X$ then $D_d(x_0, B) := D_d(\{x_0\}, B)$.

We will denote by H the generalized Pompeiu-Hausdorff functional on $P(X)$, defined as

$$H_d : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\}, \ H_d(A, B) = \max\{\sup_{a \in A} D_d(a, B), \sup_{b \in B} D_d(b, A)\}.$$

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Let (X, d) be a metric space. If $F : X \to P(X)$ is a multivalued operator, then $x \in X$ is called fixed point for F if and only if $x \in F(x)$. The set $Fix(F) := \{x \in X| \ x \in F(x)\}$ is called the fixed point set of T, while $SFix(F) = \{x \in X| \ \{x\} = F(x)\}$ is called the strict fixed point set of F.

Let Y be a nonempty set and $T, S : X \to P(Y)$ be two multivalued operators. An element $x^* \in X$ is a coincidence point for T and S if $T(x^*) \cap S(x^*) \neq \emptyset$. We denote by $C(T, S)$ the set of all coincidence points for T and S.

Let $T, S : X \to P(X)$ be two multivalued operators. An element $x^* \in X$ is called a common fixed point for T and S if $x^* \in T(x^*) \cap S(x^*)$. We denote by $CM(T, S) := Fix(T) \cap Fix(S)$ the set of all common fixed points for the multivalued operators T and S.

For a multivalued opertor $T : X \to P(Y)$ we will denote by

$$\mathrm{Graph}(T) := \{(x, y) \in X \times Y : y \in T(x)\}$$

The graphic of T. Notice that $t : X \to Y$ is a selection for $T : X \to P(Y)$ if $t(x) \in T(x)$, for each $x \in X$. Also, $T : X \to P(Y)$ is said to be onto if and only if for each $y \in Y$ there exists $x \in X$ such that $y \in T(x)$.

In particular, when F (or T and S) is a singlevalued operator, we obtain the similar well-known concepts in fixed point theory.

For the following notions see I. A. Rus [16] and [14], I. A. Rus, A. Petruşel, A. Sîntămărian [23] and A. Petruşel [13].

Definition 1.1. Let (X, d) be a metric space and $f : X \to X$ be an operator. By definition, f is a weakly Picard operator (briefly WPO) if the sequence $(f^n(x))_{n \in \mathbb{N}}$ of successive approximations for f starting from $x \in X$ converges, for all $x \in X$ and its limit is a fixed point of f.

If f is WPO, then we consider the operator

$$f^\infty : X \to X \text{ defined by } f^\infty(x) := \lim_{n \to \infty} f^n(x).$$

Notice that $f^\infty(X) = Fix(f)$.

Definition 1.2. Let (X, d) be a metric space, $f : X \to X$ be a WPO and $c > 0$ be a real number. By definition, the operator f is said to be a c-weakly Picard operator (briefly c-WPO) if and only if

$$d(x, f^\infty(x)) \leq c \ d(x, f(x)), \text{ for all } x \in X.$$

Definition 1.3. Let (X, d) be a metric space, and $F : X \to P_{cl}(X)$ be a multivalued operator. By definition, F is a multivalued weakly Picard (briefly MWP) operator if for each $x \in X$ and each $y \in F(x)$ there exists a sequence $(x_n)_{n\in\mathbb{N}}$ such that:

(i) $x_0 = x$, $x_1 = y$;

(ii) $x_{n+1} \in F(x_n)$, for each $n \in \mathbb{N}$;

(iii) the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent and its limit is a fixed point of F.

Remark 1.1. A sequence $(x_n)_{n\in\mathbb{N}}$ satisfying the condition *(*i*)* and *(*ii*)*, in the Definition 1.3 is called a sequence of successive approximations of F starting from $(x, y) \in Graph(F)$.

If $F : X \to P(X)$ is a MWP operator, then we define $F^\infty : Graph(F) \to P(FixF)$ by the formula $F^\infty(x, y) := \{\ z \in Fix(F) \mid$ there exists a sequence of successive approximations of F starting from (x, y) that converges to $z\ \}$.

Definition 1.4. Let (X, d) be a metric space and $F : X \to P(X)$ be a MWP operator. Then, F is called a c-multivalued weakly Picard operator (briefly c-MWP operator) if and only if there exists a selection f^∞ of F^∞ such that

$$d(x, f^\infty(x, y)) \leq c\ d(x, y),\ \text{for all } (x, y) \in Graph(F).$$

For the theory of weakly Picard operators, see [16] for the singlevalued case and [23] and [13] for the multivalued one.

The purpose of this paper is to extend and generalize some results given in [14], concerning the Ulam-Hyers stability of some operatorial equations and inclusions by using the weakly Picard operator technique.

2. Ulam-hyers Stability for Fixed Point Equations and Inclusions with Non-self Operators

Let (X, d) be a metric space, Y be a nonempty subset of X and $f : Y \to X$ be an operator. In this section we shall use the following notations and notions (see [14, 3]):

$I(f) := \{Z \subset Y \mid f(Z) \subset Z,\ Z \neq \emptyset\}$ - the set of all invariant subsets of f

$(MI)_f$ - the maximal invariant subset of f, i.e., $(MI)_f := \bigcup_{Z\in I(f)} Z$;

$(AB)_f(x^*) := \{x \in Y \mid f^n(x)$ is defined for all $n \in \mathbb{N}$ and $f^n(x) \xrightarrow{d} x^* \in Fix(f)\}$ - the attraction basin of $x^* \in Fix(f)$ with respect to f

$(AB)_f := \bigcup_{x^*\in Fix(f)} (AB)_f(x^*)$ - the attraction basin of f.

Definition 2.1. (A. Chiş-Novac, R. Precup, I. A. Rus [3]). By definition, $f : Y \to X$ is called a nonself weakly Picard operator if $Fix(f) \neq \emptyset$ and $(MI)_f = (AB)_f$. If $Fix(f) = \{x^*\}$, then a nonself weakly Picard operator is said to be nonself Picard operator.

Definition 2.2. (A. Chiş-Novac, R. Precup, I. A. Rus [3]). For each nonself weakly Picard operator $f : Y \to X$ we define the operator $f^\infty : (AB)_f \to Fix(f) \subset (AB)_f$, by $f^\infty(x) = \lim\limits_{n\to\infty} f^n(x)$.

Definition 2.3. (A. Chiş-Novac, R. Precup, I. A. Rus [3]). Let $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be an increasing function which is continuous in 0 and $\psi(0) = 0$. An operator $f : Y \to X$ is said to be a nonself ψ-weakly Picard operator if it is nonself weakly Picard operator and

$$d(x, f^\infty(x)) \leq \psi(d(x, f(x))), \text{ for all } x \in (MI)_f.$$

In the case that $\psi(t) := ct$ (for some $c > 0$), for each $t \in \mathbb{R}_+$, we say that f is c-weakly Picard operator.

For some examples of nonself weakly Picard operators and ψ-weakly Picard operators, see [3].

If $f : Y \to X$ is an operator, let us consider the fixed point equation

$$x = f(x), \ x \in Y \tag{2.1}$$

and the inequation

$$d(y, f(y)) \leq \varepsilon. \tag{2.2}$$

Definition 2.4. (I. A. Rus [14]). The equation (2.1) is called generalized Ulam-Hyers stable if there exists $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ increasing, continuous in 0 and $\psi(0) = 0$ such that for each $\varepsilon > 0$ and for each solution $y^* \in (AB)_f$ of (2.2) there exists a solution x^* of the fixed point equation (2.1) such that

$$d(y^*, x^*) \leq \psi(\varepsilon).$$

If there exists $c > 0$ such that $\psi(t) := ct$, for each $\in \mathbb{R}_+$, the equation (2.1) is said to be Ulam-Hyers stable.

The following abstract result is presented in [14].

Theorem 2.1. (I.A. Rus [14]). *Let (X, d) be a metric space, Y be a nonempty subset of X and $f : Y \to X$ be a ψ-weakly Picard operator. Then, the fixed point equation (2.1) is generalized Ulam-Hyers stable. In particular, if f is c-weakly Picard operator, then the equation (2.1) is Ulam-Hyers stable.*

Proof. Let $\varepsilon > 0$ and $y^* \in (AB)_f$ be a solution of (2.2), i.e., $d(y^*, f(y^*)) \leq \varepsilon$. Since f is a ψ-weakly Picard operator, for each $x \in (MI)_f$ we have

$$d(x, f^\infty(x)) \leq \psi(d(x, \psi(x))).$$

Hence, taking into account that $(MI)_f = (AB)_f$, we can choose $x^* := f^\infty(y^*)$ and thus we get that x^* is a solution of the fixed point equation (2.1) and

$$d(y^*, x^*) \leq \psi(\varepsilon). \qquad \blacksquare$$

We will present now some consequences of the above result. We need first some definitions, see [15] for details.

A mapping $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is called a comparison function if it is increasing and $\varphi^k(t) \to 0$ as $k \to +\infty$. As a consequence, we also have $\varphi(t) < t$, for each $t > 0$, $\varphi(0) = 0$ and φ is continuous in 0. The mapping $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be a strict comparison function if it is strictly increasing and $\sum\limits_{n=1}^{\infty} \varphi^n(t) < +\infty$, for each $t > 0$.

Recall that if (X, d) is a metric space, Y is a nonempty subset of X and $f : Y \to X$ is an operator, then f is called:

(i) α-contraction if $\alpha \in [0, 1[$ and

$$d(f(x_1), f(x_2)) \leq \alpha d(x_1, x_2) \text{ for all } x_1, x_2 \in Y.$$

(ii) φ-contraction if $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a comparison function and

$$d(f(x_1), f(x_2)) \leq \varphi(d(x_1, x_2)) \text{ for all } x_1, x_2 \in Y.$$

Theorem 2.2. *Let (X, d) be a complete metric space, $x_0 \in X$, $r > 0$ and $f : \tilde{B}(x_0, r) \to X$ be an $\alpha-$contraction, such that $d(x_0, f(x_0)) \leq (1 - \alpha)r$.*
Then the fixed point equation (2.1) is Ulam-Hyers stable.

Proof. It is easy to see that $(MI)_f = (AB)_f = \tilde{B}(x_0, r)$ and hence, by Banach-Caccioppoli fixed point principle, we have that $Fix(f) = \{x^*\}$ and for each $x \in \tilde{B}(x_0, r)$

$$d(x, x^*) \leq \frac{1}{1 - \alpha} d(x, f(x)).$$

Thus, f is a c-WPO with $c := \frac{1}{1-\alpha} > 0$. Hence, by Theorem 2.1 the fixed point equation (2.1) is Ulam-Hyers stable. $\blacksquare$

Another result of this type is the following.

Theorem 2.3. *Let* (X, d) *be a complete metric space,* $x_0 \in X$*,* $r > 0$ *and* $f : \tilde{B}(x_0, r) \to X$ *be a* φ*-contraction, such that* $d(x_0, f(x_0)) \leq r - \varphi(r)$*. Suppose also that the function* $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ $\psi(t) := t - \varphi(t)$ *is strictly continuous and onto. Then, the fixed point equation (2.1) is generalized Ulam-Hyers stable.*

Proof. Notice that, by our hypotheses, we have $(MI)_f = (AB)_f = \tilde{B}(x_0, r)$ and hence, by Matkowski-Rus fixed point principle (see [9] and [15]), we have that $Fix(f) = \{x^*\}$. Then, for each $x \in \tilde{B}(x_0, r)$ we have

$$d(x, x^*) \leq d(x, f(x)) + d(f(x), x^*) \leq d(x, f(x)) + \varphi(d(x, x^*)).$$

Notice that $\psi^{-1} : \mathbb{R}_+ \to \mathbb{R}_+$ exists, is increasing, continuous at 0 and $\psi^{-1}(0) = 0$. Thus,

$$d(x, x^*) \leq \psi^{-1}(d(x, f(x))), \text{ for each } x \in \tilde{B}(x_0, r)$$

proving that f is a nonself ψ^{-1}-weakly Picard operator. Hence, by Theorem 2.1 the fixed point equation (2.1) is generalized Ulam-Hyers stable. ■

Remark 2.2. If $f : \tilde{B}(x_0, r) \to X$, then similar results concerning the Ulam-Hyers stablility of the fixed point equation (2.1) can be given for:

(a) generalized contractions of Ćirić-Reich-Rus type, i.e., there exists $\alpha, \beta, \gamma \in \mathbb{R}_+$ with $\alpha + \beta + \gamma < 1$ such that

$$d(f(x), f(y)) \leq \alpha d(x, y) + \beta d(x, f(x)) + \gamma d(y, f(y)), \text{ for all } x, y \in \tilde{B}(x_0, r),$$

where $c := \frac{1-\beta}{1-\alpha-\beta-\gamma} > 0$;

(b) generalized contractions of Ćirić type, i.e., there exists $q \in [0, \frac{1}{2}[$, such that for all $x, y \in \tilde{B}(x_0, r)$ one have

$$d(f(x), f(y)) \leq q \max\{d(x, y), d(x, f(x)), d(y, f(y)), d(x, f(y)), d(y, f(x))\},$$

where $c := \dfrac{1-q}{1-2q}$.

For details, rigorous statements and other results see [3].

We will consider now the multivalued case.

Let (X, d) be a metric space, Y be a nonempty subset of X and $F : Y \to P(X)$ be a multivalued operator.

In the sequel, we shall use the following notations and notions: $I(F) := \{Z \subset Y : F(Z) \subset Z,\ Z \neq \emptyset\}$ - the set of all invariant subsets of F;

$(MI)_F$ - the maximal invariant subset of F, i.e., $(MI)_F := \bigcup\limits_{Z \in I(F)} Z$;

$(AB)_F(x^*) := \{x \in Y :$ for each $y \in F(x),$ there exists in Y a sequence, $(x_n)_{n \in \mathbb{N}},$

of successive approximations for F starting from (x, y), which converges to $x^*\}$
- the attraction basin of $x^* \in Fix(F)$ with respect to F;

$(AB)_F := \bigcup_{x^* \in Fix(F)} (AB)_F(x^*)$ - the attraction basin of F.

Definition 2.5. Let (X, d) be a metric space, $Y \in P(X)$ and $F : Y \to P(X)$ be a multivalued operator. By definition, F is a nonself multivalued weakly Picard operator if $Fix(F) \neq \emptyset$ and $(MI)_F = (AB)_F$.
If $Y = X$, then F having the above properties is said to be a multivalued weakly Picard operator.

Let $F : Y \to P(X)$ be a nonself multivalued weakly Picard operator. Denote

$$D_F^\infty := \{(x, y) \in X \times X : x \in (AB)_F \text{ and } y \in F(x)\}.$$

Then, we consider the multivalued operator $F^\infty : D_F^\infty \to P(Fix(F))$ defined by the following formula:
$F^\infty(x, y)$:= the set of all fixed points of F that are limits of a successive approximations sequence starting from (x, y).

Definition 2.6. Let (X, d) be a metric space and $Y \in P(X)$. Let $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be an increasing function which is continuous in 0 and $\psi(0) = 0$. Then $F : Y \to P(X)$ is said to be a nonself multivalued ψ-weakly Picard operator if it is a nonself multivalued weakly Picard operator and there exists a selection $f^\infty : D_F^\infty \to Fix(F)$ of F^∞ such that

$$d(x, f^\infty(x, y)) \leq \psi(d(x, y)), \quad \text{for all } (x, y) \in D_F^\infty.$$

If $Y = X$, then F having the above property is said to be a multivalued ψ-weakly Picard operator. If there exists $c > 0$ such that $\psi(t) = ct$, for each $t \in \mathbb{R}_+$, then we say that F is a nonself multivalued c-weakly Picard operator.

Definition 2.7. Let (X, d) be a metric space, Y be a nonempty subset of X and $F : Y \to P(X)$ be a multivalued operator. The fixed point inclusion

$$x \in F(x), \; x \in Y \tag{2.3}$$

is called generalized Ulam-Hyers stable if and only if there exists $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ increasing, continuous in 0 and $\psi(0) = 0$ such that for each $\varepsilon > 0$ and for each solution $y^* \in (AB)_F$ of the inequation

$$D(y, F(y)) \leq \varepsilon \tag{2.4}$$

there exists a solution x^* of the fixed point inclusion (2.3) such that

$$d(y^*, x^*) \leq \psi(\varepsilon).$$

If there exists $c > 0$ such that $\psi(t) := ct$, for each $t \in \mathbb{R}_+$, then the fixed point inclusion (2.3) is said to be Ulam-Hyers stable.

The following theorem is an abstract result concerning the Ulam-Hyers stability of the fixed point inclusion (2.3) with nonself multivalued operators with compact values.

Theorem 2.4. *Let (X, d) be a metric space, Y be a nonempty subset of X and $F : Y \to P_{cp}(X)$ be a nonself multivalued ψ-weakly Picard operator. Then, the fixed point inclusion (2.3) is generalized Ulam-Hyers stable.*

Proof. Let $\varepsilon > 0$ and $y^* \in (AB)_F$ be a solution of (2.4), i.e., $D(y^*, F(y^*)) \leq \varepsilon$. Let $u^* \in F(y^*)$ such that $d(y^*, u^*) = D(y^*, F(y^*))$. Since F is a nonself multivalued ψ-weakly Picard operator, for each $(x, y) \in D_F^\infty$ we have

$$d(x, f^\infty(x, y)) \leq \psi(d(x, y)).$$

Hence, taking into account that $(y^*, u^*) \in D_F^\infty$, we can choose $x^* := f^\infty(y^*, u^*)$ and thus we get that x^* is a solution of the fixed point inclusion (2.3) and

$$d(y^*, x^*) = d(y^*, f^\infty(y^*, u^*)) \leq \psi(d(y^*, u^*)) \leq \psi(\varepsilon). \qquad \blacksquare$$

In particular, if the multivalued operator is self, then Theorem 2.4 gives a theorem concerning Ulam-Hyers stability of the fixed point inclusion with multivalued self operators, which was presented in [14]. We list here this result.

Corollary 2.1. *Let (X, d) be a metric space and $F : X \to P_{cp}(X)$ be a multivalued ψ-weakly Picard operator. Then, the fixed point inclusion (2.3) is generalized Ulam-Hyers stable.*

We will present now some consequences of the above result. We need first some definitions.

Definition 2.8. Let (X, d), (Y, d') be metric spaces and $F : X \to P_{cl}(Y)$ be a multivalued operator. Then, F is called:

(i) a-contraction, if $a \in [0, 1[$ and $H_{d'}(F(x_1), F(x_2)) \leq ad(x_1, x_2)$, for all $x_1, x_2 \in X$;

(ii) φ-contraction, if $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a strict comparison function and for all $x_1, x_2 \in X$ we have that $H_{d'}(F(x_1), F(x_2)) \leq \varphi(d(x_1, x_2))$;

Theorem 2.5. *Let (X, d) be a complete metric space, $x_0 \in X$ and $r > 0$. Let $F : \tilde{B}(x_0; r) \to P_{cp}(X)$ be a multivalued a-contraction such that $H(x_0, F(x_0)) < (1 - a)r$. Then, the fixed point inclusion (2.3) is Ulam-Hyers stable.*

Proof. By Theorem 4.5 in [8], the set $\tilde{B}(x_0; r)$ is invariant with respect to F, i.e., $(MI)_F = \tilde{B}(x_0; r)$. Thus, by Nadler's contraction principle (see [10]), we get that F is a nonself multivalued weakly Picard operator. Moreover, F is a nonself multivalued c-weakly Picard operator with $c := \frac{1}{1-a}$ (see [23]). Hence, Theorem 2.4 applies and the conclusion follows. ■

The following result is known in the literature as Węgrzyk's theorem (see [25]).

Theorem 2.6. *Let (X, d) be a complete metric space and $F : X \to P_{cl}(X)$ be a multivalued φ-contraction. Then F is a multivalued weakly Picard operator.*

A Ulam-Hyers stability result for nonself multivalued φ-contractions is the following.

Theorem 2.7. *Let (X, d) be a complete metric space, $x_0 \in X$ and $r > 0$. Let $F : \tilde{B}(x_0; r) \to P_{cp}(X)$ be a multivalued φ-contraction such that the function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ given by $\psi(t) = t - \varphi(t)$ is strictly increasing and onto. Suppose $H(x_0, F(x_0)) < r - \varphi(r)$ and $SFix(F) \neq \emptyset$. Then, the fixed point inclusion (2.3) is generalized Ulam-Hyers stable.*

Proof. Since F is a φ-contraction, using the assumption $H(x_0, F(x_0)) < r - \varphi(r)$, we obtain (see [8]) that the set $\tilde{B}(x_0; r)$ is invariant with respect to F, i.e., $(MI)_F = \tilde{B}(x_0; r)$. Thus, by Węgrzyk's Theorem 2.6, we get that $F : \tilde{B}(x_0; r) \to P_{cp}(X)$ is a nonself multivalued weakly Picard operator.

Moreover, F is a nonself multivalued ψ^{-1}-weakly Picard operator. Indeed, let $x^* \in SFix(F)$ and $x \in Fix(F)$ be arbitrary. Then $d(x, x^*) = D(x, F(x^*)) \leq H(F(x), F(x^*)) \leq \varphi(d(x, x^*))$. By the properties of φ we get that $d(x, x^*) = 0$ and hence $Fix(F) \subset \{x^*\}$. Since $SFix(F) \subset Fix(F)$, we get that $Fix(F) = SFix(F) = \{x^*\}$. Hence for each $x \in \tilde{B}(x_0; r)$ and $y \in F(x)$ we have

$$d(x, x^*) \leq d(x, y) + H(F(x), F(x^*)) \leq d(x, y) + \varphi(d(x, x^*)).$$

Thus, since ψ is a strictly increasing bijection we obtain that

$$d(x, x^*) \leq \psi^{-1}(d(x, y)), \text{ for each } (x, y) \in \tilde{B}(x_0; r).$$

Thus, Theorem 2.4 applies and the conclusion follows. ■

A similar concept will be given in the last part of the section.

We denote by

$(SAB)_F(x^*) := \{x \in Y : F^n(x) \text{ is defined and } F^n(x) \xrightarrow{H} \{x^*\}\}$ - the strict attraction basin of $x^* \in SFix(F)$ with respect to F;

$(SAB)_F := \bigcup\limits_{x^* \in SFix(F)} (SAB)_F(x^*)$ - the strict attraction basin of F.

Definition 2.9. Let (X, d) be a metric space, $Y \in P(X)$ and $F : Y \to P(X)$ be a multivalued operator. By definition, F is a nonself multivalued Picard operator if $SFix(F) = Fix(F) = \{x^*\}$ and $(MI)_F = (SAB)_F$.

Definition 2.10. Let $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be an increasing function which is continuous in 0 and $\psi(0) = 0$. Then $F : Y \to P(X)$ is said to be a nonself multivalued ψ-Picard operator if it is a nonself multivalued Picard operator and

$$d(x, x^*) \leq \psi(H(x, F(x))), \quad \text{for all } x \in (SAB)_F.$$

If there exists $c > 0$ such that $\psi(t) = ct$, for each $t \in \mathbb{R}_+$, then we say that F is a nonself multivalued c-Picard operator.

Moreover, if $Y = X$, then F is said multivalued ψ-Picard operator, respectively multivalued c-Picard operator.

Definition 2.11. Let (X, d) be a metric space, Y be a nonempty subset of X and $F : Y \to P(X)$ be a multivalued operator. The strict fixed point inclusion

$$\{x\} = F(x), \ x \in Y \tag{2.5}$$

is called generalized Ulam-Hyers stable if and only if there exists $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ increasing, continuous in 0 and $\psi(0) = 0$ such that for each $\varepsilon > 0$ and for each solution $y^* \in (SAB)_F$ of the inequation

$$H(y, F(y)) \leq \varepsilon \tag{2.6}$$

there exists a solution x^* of the strict fixed point inclusion (2.5) such that

$$d(y^*, x^*) \leq \psi(\varepsilon).$$

If there exists $c > 0$ such that $\psi(t) := ct$, for each $t \in \mathbb{R}_+$, then the strict fixed point inclusion (2.5) is said to be Ulam-Hyers stable.

Remark 2.3. It is worth to note that the above definition can briefly re-written as follows: the strict fixed point inclusion is generalized Ulam-Hyers stable if and only if the fixed point (set) equation

$$\{x\} = F(x), \ x \in Y$$

is generalized Ulam-Hyers stable in $(P_{cl}(X), H)$.

The following theorem is an abstract result concerning the Ulam-Hyers stability of the strict fixed point inclusion (2.5) with nonself multivalued operators with closed values.

Theorem 2.8. *Let (X, d) be a metric space, Y be a nonempty subset of X and $F : Y \to P_{cl}(X)$ be a nonself multivalued ψ-Picard operator. Then, the strict fixed point inclusion (2.5) is generalized Ulam-Hyers stable.*

Proof. Let $\varepsilon > 0$ and $y^* \in (SAB)_F$ be a solution of (2.6), i.e., $H(y^*, F(y^*)) \leq \varepsilon$. Since F is a nonself multivalued ψ-Picard operator, we have

$$d(x, x^*) \leq \psi(H(x, F(x))), \text{ for all } x \in (SAB)_F.$$

Hence $d(y^*, x^*) \leq \psi(H(y^*, F(y^*))) \leq \psi(\varepsilon)$. ∎

As a consequence of Theorem 2.8, we immediately obtain:

Theorem 2.9. *Let (X, d) be a complete metric space, $x_0 \in X$ and $r > 0$. Let $F : \tilde{B}(x_0; r) \to P_{cl}(X)$ be a multivalued a-contraction such that $H(x_0, F(x_0)) < (1-a)r$ and $SFix(F) \neq \emptyset$. Then, the strict fixed point inclusion (2.5) is Ulam-Hyers stable.*

Proof. By the contraction condition and using the fact that $H(x_0, F(x_0)) < (1-a)r$ we obtain that $(MI)_F = \tilde{B}(x_0; r)$. Since $SFix(F) \neq \emptyset$, we obtain (see I.A. Rus [17]) that $Fix(F) = SFix(F) = \{x^*\}$. Hence, F is a nonself multivalued Picard operator.

Then, for each $x \in \tilde{B}(x_0; r)$ we have $d(x, x^*) \leq D(x, F(x)) + H(F(x), F(x^*)) \leq D(x, F(x)) + ad(x, x^*)$. Hence

$$d(x, x^*) \leq \frac{1}{1-a} D(x, F(x)) \leq \frac{1}{1-a} H(x, F(x)), \text{ for each } x \in \tilde{B}(x_0; r).$$

Thus, F is a nonself multivalued c-Picard operator with $c := \frac{1}{1-a}$. The conclusion follows from Theorem 2.8. ∎

3. Some Applications to Operatorial Inclusions

As a first application, let us consider the following integral inclusion of Fredholm type.

$$x(t) \in \int_a^b K(t, s, x(s))ds + g(t), \ t \in [a, b]. \tag{3.1}$$

Throughout this section we will denote by $\|\cdot\|$ the supremum norm in $C([a, b], \mathbb{R}^n)$.

The main result concerning the stability of the Fredholm integral incusion (3.1) is the following.

Theorem 3.1. *Let* $K : [a,b] \times [a,b] \times \mathbb{R}^n \to P_{cl,cv}(\mathbb{R}^n)$ *and* $g : [a,b] \to \mathbb{R}^n$ *such that:*

(a) *there exists an integrable function* $M : [a,b] \to \mathbb{R}_+$ *such that for each* $t \in [a,b]$ *and* $u \in \mathbb{R}^n$ *we have* $K(t,s,u) \subset M(s)B(0;1)$*, a.e.* $s \in [a,b]$*;*

(b) *for each* $u \in \mathbb{R}^n$ $K(\cdot,\cdot,u) : [a,b] \times [a,b] \to P_{cl,cv}(\mathbb{R}^n)$ *is jointly measurable;*

(c) *for each* $(s,u) \in [a,b] \times \mathbb{R}^n$ $K(\cdot,s,u) : [a,b] \to P_{cl,cv}(\mathbb{R}^n)$ *is lower semi-continuous;*

(d) *there exists a continuous function* $p : [a,b] \times [a,b] \to \mathbb{R}_+$ *with* $\sup_{t\in[a,b]} \int_a^b p(t,s)ds \leq \alpha < 1$ *such that for each* $(t,s) \in [a,b] \times [a,b]$ *and each* $u, v \in \mathbb{R}^n$ *we have that*

$$H(K(t,s,u), K(t,s,v)) \leq p(t,s) \cdot |u - v|; \tag{3.2}$$

(e) g *is continuous.*

Then the follwing conclusions hold:

(a) *the integral inclusion* (3.1) *has least one solution, i.e., there exists* $x^* \in C([a,b],\mathbb{R}^n)$ *which satisfies* (3.1)*, for each* $t \in [a,b]$.

(b) *The integral inclusion* (3.1) *is Ulam-Hyers stable, i.e., there exists* $c > 0$*, such that for each* $\varepsilon > 0$ *and for any* ε*-solution* y *of* (3.1)*, i.e., any* $y \in C([a,b],\mathbb{R}^n)$ *for which there exists* $u \in C([a,b],\mathbb{R}^n)$ *such that*

$$u(t) \in \int_a^b K(t,s,y(s))ds + g(t), \ t \in [a,b]$$

and

$$|u(t) - y(t)| \leq \varepsilon, \text{ for each } t \in [a,b]),$$

there exists a solution x^* *of the integral inclusion* (3.1) *such that*

$$|y(t) - x^*(t)| \leq c \cdot \varepsilon, \text{ for each } t \in [a,b].$$

Proof. (a) Define the multivalued operator $T : C([a,b],\mathbb{R}^n) \to \mathcal{P}(C([a,b],\mathbb{R}^n))$ by

$$T(x) := \left\{ v \in C([a,b],\mathbb{R}^n) |\ v(t) \in \int_a^b K(t,s,x(s))ds + g(t), \ t \in [a,b] \right\}.$$

Then, (3.1) is equivalent to the fixed point inclusion

$$x \in T(x), \ x \in C([a,b],\mathbb{R}^n). \tag{3.3}$$

The proof is organized in several steps.

1. $T(x) \in P_{cp}(C([a,b],\mathbb{R}^n))$.

From (e) and Theorem 2 in Rybiński [24] we have that for each $x \in C([a,b],\mathbb{R}^n)$ there exists $k(t,s) \in K(t,s,x(s))$, for all $(t,s) \in [a,b]$, such that $k(t,s)$ is integrable with respect to s and continuous with respect to t. Then $v(t) := \int_a^b k(t,s)ds + g(t)$, has the property $v \in T(x)$. Moreover, from (a) and (b), via Theorem 8.6.3. in Aubin and Frankowska [1], we get that $T(x)$ is a compact set, for each $x \in C([a,b],\mathbb{R}^n)$.

2. $H(T(x_1),T(x_2)) \leq \alpha \cdot \|x_1 - x_2\|$, for each $x_1, x_2 \in C([a,b],\mathbb{R}^n)$.

Notice first that one may suppose (without affecting the generality of the Lipschitz condition) that the inequality (3.2) is strict. Let $x_1, x_2 \in C([a,b],\mathbb{R}^n)$ and $v_1 \in T(x_1)$. Then $v_1(t) \in \displaystyle\int_a^b K(t,s,x_1(s))ds + g(t)$, $t \in [a,b]$. It follows that $v_1(t) = \displaystyle\int_a^b k_1(t,s)ds + g(t)$, $t \in [a,b]$, for some $k_1(t,s) \in K(t,s,x_1(s))$, $(t,s) \in [a,b] \times [a,b]$.

From (d) we have $H(K(t,s,x_1(s)),K(t,s,x_2(s)) < p(t,s)|x_1(s) - x_2(s)| \leq p(t,s)\|x_1 - x_2\|$. Thus, there exists $w \in K(t,s,x_2(s))$ such that $|k_1(t,s) - w| \leq p(t,s)\|x_1 - x_2\|$, for $t,s \in [a,b]$.

Let us define $U : [a,b] \times [a,b] \to \mathcal{P}(\mathbb{R}^n)$, by $U(t,s) = \{w|\ |k_1(t,s) - w| \leq p(t,s)\|x_1 - x_2\|\}$. Since the multi-valued operator $V(t,s) := U(t,s) \cap K(t,s,x_2(s))$ is jointly measurable and lower semi-continuous in t there exists $k_2(t,s)$ a selection for V, jointly measurable (and, hence, integrable in s) and continuous in t. Hence, $k_2(t,s) \in K(t,s,x_2(s))$ and $|k_1(t,s) - k_2(t,s)| \leq p(t,s)\|x_1 - x_2\|$, for each $t,s \in [a,b]$.

Consider $v_2(t) = \displaystyle\int_a^b k_2(t,s)ds + g(t)$, $t \in [a,b]$. Then, we have:

$|v_1(t) - v_2(t)| \leq \int_a^b |k_1(t,s) - k_2(t,s)|ds \leq \int_a^b p(t,s)\|x_1 - x_2\|ds \leq \alpha\|x_1 - x_2\|$.

A similar relation can be obtained by interchanging the roles of x_1 and x_2. Thus the second step follows.

The first conclusion follows by Covitz-Nadler's fixed point theorem, see [4].

(b) We will prove that the fixed point inclusion problem (3.3) is Ulam-Hyers stable. Indeed, let $\varepsilon > 0$ and $y \in C([a,b],\mathbb{R}^n)$ for which there exists $u \in C([a,b],\mathbb{R}^n)$ such that

$$u(t) \in \int_a^b K(t,s,y(s))ds + g(t),\ t \in [a,b]$$

and

$$\|u - y\| \leq \varepsilon.$$

Then $D_{\|\cdot\|}(y,T(y)) \leq \varepsilon$. Moreover, since T is a multivalued α-contraction, we obtain that T is a multivalued c-weakly Picard operator with $c := \frac{1}{1-\alpha}$. The conclusion follows by Corollary 2.1. ■

A second application concerns an integral inclusion of Volterra type.

$$x(t) \in \int_a^t K(t,s,x(s))ds + g(t),\ t \in [a,b]. \tag{3.4}$$

By a similar method, we can prove the following.

Theorem 3.2. *Let* $K : [a,b] \times [a,b] \times \mathbb{R}^n \to P_{cl,cv}(\mathbb{R}^n)$ *and* $g : [a,b] \to \mathbb{R}^n$ *such that:*

(a) there exists an integrable function $M : [a,b] \to \mathbb{R}_+$ *such that for each* $t \in [a,b]$ *and* $u \in \mathbb{R}^n$ *we have* $K(t,s,u) \subset M(s)B(0;1)$*, a.e.* $s \in [a,b]$*;*

(b) for each $u \in \mathbb{R}^n$ $K(\cdot,\cdot,u) : [a,b]\times[a,b] \to P_{cl,cv}(\mathbb{R}^n)$ *is jointly measurable;*

(c) for each $(s,u) \in [a,b] \times \mathbb{R}^n$ $K(\cdot,s,u) : [a,b] \to P_{cl,cv}(\mathbb{R}^n)$ *is lower semi-continuous;*

(d) there exists a continuous function $p : [a,b] \to \mathbb{R}_+$ *such that for each* $(t,s) \in [a,b] \times [a,b]$ *and each* $u, v \in \mathbb{R}^n$ *we have that*

$$H(K(t,s,u), K(t,s,v)) \le p(s) \cdot |u - v|; \tag{3.5}$$

(e) g *is continuous.*

Then the follwing conclusions hold:

(a) the integral inclusion (3.4) *has at least one solution, i.e., there exists* $x^{8} \in C([a,b],\mathbb{R}^n)$ *which satisfies* (3.4) *for each* $t \in [a,b]$*;*

(b) The integral inclusion (3.4) *is Ulam-Hyers stable, i.e., there exists* $c > 0$ *such that for each* $\varepsilon > 0$ *and for any* ε*-solution* y *of* (3.4)*, i.e., any* $y \in C([a,b],\mathbb{R}^n)$ *for which there exists* $u \in C([a,b],\mathbb{R}^n)$ *such that*

$$u(t) \in \int_a^t K(t,s,y(s))ds + g(t),\ t \in [a,b]$$

and

$$|u(t) - y(t)| \le \varepsilon,\ \text{for each } t \in [a,b]),$$

there exists a solution x^* *of the integral inclusion* (3.4) *such that*

$$|y(t) - x^*(t)| \le c \cdot \varepsilon,\ \text{for each } t \in [a,b].$$

Proof. We consider the multi-valued operator $T : C([a,b],\mathbb{R}^n) \to \mathcal{P}(C([a,b],\mathbb{R}^n))$

$$T(x) := \left\{ v \in C([a,b],\mathbb{R}^n) |\ v(t) \in \int_a^t K(t,s,x(s))ds + g(t),\ t \in [a,b] \right\}.$$

Then, (3.4) is equivalent to the fixed point inclusion

$$x \in T(x),\ x \in C([a,b],\mathbb{R}^n). \tag{3.6}$$

As in the proof of Theorem 3.1 we obtain $T(x) \in P_{cp}(C([a,b],\mathbb{R}^n))$. Next, we will prove that T is a multivalued contraction on $C([a,b],\mathbb{R}^n)$.

Notice first that one may suppose (without affecting the generality of the Lipschitz condition) that the inequality (3.5) is strict. Let $x_1, x_2 \in C([a,b],\mathbb{R}^n)$ and $v_1 \in T(x_1)$. Then $v_1(t) \in \displaystyle\int_a^t K(t,s,x_1(s))ds + g(t)$, $t \in [a,b]$. It follows that $v_1(t) = \displaystyle\int_a^b k_1(t,s)ds + g(t)$, $t \in [a,b]$, for some $k_1(t,s) \in K(t,s,x_1(s))$, $(t,s) \in [a,b] \times [a,b]$.

From (d) we have $H(K(t,s,x_1(s)), K(t,s,x_2(s))) < p(s)|x_1(s) - x_2(s)|$. Thus, there exists $w \in K(t,s,x_2(s))$ such that $|k_1(t,s) - w| \leq p(s)|x_1(s) - x_2(s)|$, for $t,s \in [a,b]$.

Let us define $U : [a,b] \times [a,b] \to \mathcal{P}(\mathbb{R}^n)$, by $U(t,s) = \{w |\ |k_1(t,s) - w| \leq p(t,s)|x_1(s) - x_2(s)|\}$. Since the multivalued operator $V(t,s) := U(t,s) \cap K(t,s,x_2(s))$ is jointly measurable and lower semi-continuous in t there exists $k_2(t,s)$ a selection for V, jointly measurable (and, hence, integrable in s) and continuous in t. Hence, $k_2(t,s) \in K(t,s,x_2(s))$ and $|k_1(t,s) - k_2(t,s)| \leq p(s)|x_1(s) - x_2(s)|$, for each $t,s \in [a,b]$.

Consider $v_2(t) = \displaystyle\int_a^t k_2(t,s)ds + g(t)$, $t \in [a,b]$. We denote by $\|\cdot\|_B$ a Bielecki-type norm in $C([a,b],\mathbb{R}^n)$, given by $\|x\|_B := \sup\limits_{t\in[a,b]} (|x(t)|e^{-\tau q(t)})$, where $q(t) := \int_a^t p(s)ds$.

Then, for each $t \in [a,b]$, we have:

$|v_1(t) - v_2(t)| \leq \int_a^t |k_1(t,s) - k_2(t,s)|ds \leq \int_a^t p(s)|x_1(s) - x_2(s)|ds = \int_a^t p(s)e^{\tau q(s)}|x_1(s) - x_2(s)|e^{-\tau q(s)}ds \leq \int_a^t p(s)e^{\tau q(s)}\|x_1 - x_2\|_B ds = \frac{1}{\tau}\|x_1 - x_2\|_B(e^{\tau q(t)} - e^{\tau q(a)}) \leq \frac{1}{\tau}\|x_1 - x_2\|_B e^{\tau q(t)}$. Thus, we immediately get

$$\|v_1 - v_2\|_B \leq \frac{1}{\tau}\|x_1 - x_2\|_B.$$

A similar relation can be obtained by interchanging the roles of x_1 and x_2. By choosing now $\tau > 1$ we get that $H_{\|\cdot\|_B}(T(x_1), T(x_2)) \leq \frac{1}{\tau}\|x_1 - x_2\|_B$, which proves that T is a multivalued contraction with constant $\alpha := \frac{1}{\tau}$. Hence, conclusion (a) follows by Covitz-Nadler's fixed point theorem [4].

For the second conclusion, let $\varepsilon > 0$ and $y \in C([a,b],\mathbb{R}^n)$ for which there exists $u \in C([a,b],\mathbb{R}^n)$ such that

$$u(t) \in \int_a^t K(t,s,y(s))ds + g(t),\ t \in [a,b]$$

and

$$|u(t) - y(t)| \leq \varepsilon,\ \text{for each } t \in [a,b].$$

Notice that

$$\|\cdot\|_B \leq \|\cdot\| \leq \|\cdot\|_B \, e^{\tau q(b)}.$$

Then, we obtain that $\|u-y\|_B \leq \|u-y\| \leq \varepsilon$. Thus, $D_{\|\cdot\|_B}(y, T(y)) \leq \varepsilon$. Moreover, since T is a multivalued α-contraction with respect to $\|\cdot\|_B$, we obtain that T is a multivalued c-weakly Picard operator with $c := \frac{1}{1-\alpha}$. The conclusion (b) is a consequence of Corollary 2.1. Hence, there exists a solution x^* of the integral inclusion (3.4) such that

$$\|y - x^*\|_B \leq c\varepsilon.$$

Hence,

$$|y(t) - x^*(t)| \leq ce^{\tau q(b)}\varepsilon, \text{ for each } t \in [a,b]. \qquad \blacksquare$$

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