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SUBADDITIVITY OF SOME FUNCTIONALS ASSOCIATED TO JENSEN'S INEQUALITY WITH APPLICATIONS

S. S. Dragomir, Y. J. Cho* and J. K. Kim

Abstract. Some new results related to Jensen's celebrated inequality for convex functions defined on convex sets in linear spaces are given. Applications for the arithmetic mean-geometric mean inequality are provided as well.

1. Introduction

Let C be a convex subset of the linear space X and f a convex function on C. If I denotes a finite subset of the set $\mathbb{N}$ of natural numbers, $x_i \in C, p_i \geq 0$ for $i \in I$ and $P_I := \sum_{i \in I} p_i > 0$, then we have

$$f\left(\frac{1}{P_I}\sum_{i\in I} p_i x_i\right) \leq \frac{1}{P_I}\sum_{i\in I} p_i f(x_i), \tag{1.1}$$

which is well known in the literature as *Jensen's inequality*.

The Jensen inequality for convex functions plays a crucial role in the Theory of Inequalities due to the fact that other inequalities such as the generalised triangle inequality, the arithmetic mean-geometric mean inequality, Hölder and Minkowski inequalities, Ky Fan's inequality etc. can be obtained as particular cases of it. For more details on Jensen's inequality, see[1, 4-16].

In order to simplify the presentation, we introduce the following notations (see also [14]):

$$\begin{aligned}
F(C,\mathbb{R}) &:= \text{the linear space of all real functions on } C,\\
F^+(C,\mathbb{R}) &:= \{f \in F(C,\mathbb{R}) : f(x) > 0 \text{ for all } x \in C\},\\
P_f(\mathbb{N}) &:= \{I \subset \mathbb{N}\text{: } I \text{ is finite}\},\\
J(\mathbb{R}) &:= \left\{p = \{p_i\}_{i\in\mathbb{N}},\ p_i \in \mathbb{R} \text{ are such that } P_I \neq 0 \text{ for all } I \in P_f(\mathbb{N})\right\},
\end{aligned}$$

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*Corresponding author.

and

$$J^{+}(\mathbb{R}) := \{p \in J(\mathbb{R}) : p_i \geq 0 \text{ for all } i \in \mathbb{N}\},$$
$$J_{*}(C) := \left\{x = \{x_i\}_{i\in\mathbb{N}} : x_i \in C \text{ for all } i \in \mathbb{N}\right\}$$

and

$$Conv(C, \mathbb{R}) := \text{ the cone of all convex functions defined on } C,$$

respectively.

In [14] the authors considered the following functional associated with the Jensen inequality:

$$J(f, I, p, x) := \sum_{i\in I} p_i f(x_i) - P_I f\left(\frac{1}{P_I}\sum_{i\in I} p_i x_i\right), \tag{1.2}$$

where $f \in F(C, \mathbb{R})$, $I \in P_f(\mathbb{N})$, $p \in J^{+}(\mathbb{R})$, $x \in J_{*}(C)$. They established some quasi-linearity and monotonicity properties and applied the obtained results for norm and means inequalities.

The following result concerning the properties of the functional $J(f, I, \cdot, x)$ as a *function of weights* holds (see [14, Theorem 2.4]):

Theorem 1. *Let* $f \in Conv(C, \mathbb{R}), I \in P_f(\mathbb{N})$ *and* $x \in J_{*}(C)$.

(i) *If* $p, q \in J^{+}(\mathbb{R})$ *then*

$$J(f, I, p+q, x) \geq J(f, I, p, x) + J(f, I, q, x)\,(\geq 0) \tag{1.3}$$

i.e., $J(f, I, \cdot, x)$ *is superadditive on* $J^{+}(\mathbb{R})$*;*

(ii) *If* $p, q \in J^{+}(\mathbb{R})$ *with* $p \geq q$, *meaning that* $p_i \geq q_i$ *for each* $i \in \mathbb{N}$*, then*

$$J(f, I, p, x) \geq J(f, I, q, x)\,(\geq 0) \tag{1.4}$$

i.e., $J(f, I, \cdot, x)$ *is monotonic nondecreasing on* $J^{+}(\mathbb{R})$.

The behavior of this functional as an *index set function* is incorporated in the following (see [14, Theorem 2.1]):

Theorem 2. *Let* $f \in Conv(C, \mathbb{R})$, $p \in J^{+}(\mathbb{R})$ *and* $x \in J_{*}(C)$.

(i) *If* $I, H \in P_f(\mathbb{N})$ *with* $I \cap H = \varnothing$, *then*

$$J(f, I \cup H, p, x) \geq J(f, I, p, x) + J(f, H, p, x)\,(\geq 0), \tag{1.5}$$

i.e., $J(f, \cdot, p, x)$ *is superadditive as an index set function on* $P_f(\mathbb{N})$*;*

(ii) *If* $I, H \in P_f(\mathbb{N})$ *with* $H \subset I$, *then*

$$J(f, I, p, x) \geq J(f, H, p, x) (\geq 0), \tag{1.6}$$

i.e., $J(f, \cdot, p, x)$ *is monotonic nondecreasing as an index set function on* $P_f(\mathbb{N})$.

As pointed out in [14], the above Theorem 2 is a generalisation of the Vasić-Mijalković result for convex functions of a real variable obtained in [16] and therefore creates the possibility to obtain vectorial inequalities as well.

For applications of the above results to logarithmic convex functions, to norm inequalities, in relation with the arithmetic mean-geometric mean inequality and with other classical results, see [14].

Motivated by the above results, we introduce in the present paper another functional associated to Jensen's discrete inequality, establish its subadditivity properties as both a function of weights and an index set function and use it for some particular cases that provide inequalities of interest. Applications related to the arithmetic mean - geometric mean celebrated inequality are provided as well.

2. Some Subadditivity Properties for the Weights

We consider the more general functional

$$D(f, I, p, x; \Psi) := P_I \Psi \left[\frac{1}{P_I} \sum_{i \in I} p_i f(x_i) - f\left(\frac{1}{P_I} \sum_{i \in I} p_i x_i \right) \right], \tag{2.1}$$

where $f \in Conv(C, \mathbb{R})$, $I \in P_f(\mathbb{N})$, $p \in J^+(\mathbb{R})$, $x \in J_*(C)$ and $\Psi : [0, \infty) \to \mathbb{R}$ is a function whose properties will determine the behavior of the functional D as follows. Obviously, for $\Psi(t) = t$ we recapture from D the functional J considered in [14].

First of all we observe that, by Jensen's inequality, the functional D is well defined and *positive homogeneous* in the third variable, i.e.,

$$D(f, I, \alpha p, x; \Psi) = \alpha D(f, I, p, x; \Psi),$$

for any $\alpha > 0$ and $p \in J^+(\mathbb{R})$.

The following result concerning the subadditivity of the functional D as a function of weights holds:

Theorem 3. *Let* $f \in Conv(C, \mathbb{R}), I \in P_f(\mathbb{N})$ *and* $x \in J_*(C)$. *Assume that* $\Psi : [0, \infty) \to \mathbb{R}$ *is monotonic nonincreasing and convex where it is defined. If* $p, q \in J^+(\mathbb{R})$ *then*

$$D(f, I, p+q, x; \Psi) \leq D(f, I, p, x; \Psi) + D(f, I, q, x; \Psi), \tag{2.2}$$

i.e., D *is subadditive as a function of weights.*

Proof. Let $p, q \in J^{+}(\mathbb{R})$. It is easy to see that, by the convexity of the function f on C, we have

$$
\begin{aligned}
&\frac{1}{P_I+Q_I}\sum_{i\in I}(p_i+q_i)\,f(x_i)-f\left(\frac{1}{P_I+Q_I}\sum_{i\in I}(p_i+q_i)\,x_i\right)\\
&=\frac{P_I\left(\frac{1}{P_I}\sum_{i\in I}p_if(x_i)\right)+Q_I\left(\frac{1}{Q_I}\sum_{i\in I}q_if(x_i)\right)}{P_I+Q_I}\\
&\quad-f\left(\frac{P_I\left(\frac{1}{P_I}\sum_{i\in I}p_ix_i\right)+Q_I\left(\frac{1}{Q_I}\sum_{i\in I}q_ix_i\right)}{P_I+Q_I}\right)\\
&\geq\frac{P_I\left(\frac{1}{P_I}\sum_{i\in I}p_if(x_i)\right)+Q_I\left(\frac{1}{Q_I}\sum_{i\in I}q_if(x_i)\right)}{P_I+Q_I}\\
&\quad-\frac{P_If\left(\frac{1}{P_I}\sum_{i\in I}p_ix_i\right)+Q_If\left(\frac{1}{Q_I}\sum_{i\in I}q_ix_i\right)}{P_I+Q_I}\\
&=\frac{P_I\left[\frac{1}{P_I}\sum_{i\in I}p_if(x_i)-f\left(\frac{1}{P_I}\sum_{i\in I}p_ix_i\right)\right]}{P_I+Q_I}\\
&\quad+\frac{Q_I\left[\frac{1}{Q_I}\sum_{i\in I}q_if(x_i)-f\left(\frac{1}{Q_I}\sum_{i\in I}q_ix_i\right)\right]}{P_I+Q_I}.
\end{aligned}\tag{2.3}
$$

Since Ψ is monotonic nonincreasing, then by (2.3) we have

$$
\begin{aligned}
&\Psi\left[\frac{1}{P_I+Q_I}\sum_{i\in I}(p_i+q_i)\,f(x_i)-f\left(\frac{1}{P_I+Q_I}\sum_{i\in I}(p_i+q_i)\,x_i\right)\right]\\
&\leq\Psi\left\{\frac{P_I\left[\frac{1}{P_I}\sum_{i\in I}p_if(x_i)-f\left(\frac{1}{P_I}\sum_{i\in I}p_ix_i\right)\right]}{P_I+Q_I}\right.\\
&\quad\left.+\frac{Q_I\left[\frac{1}{Q_I}\sum_{i\in I}q_if(x_i)-f\left(\frac{1}{Q_I}\sum_{i\in I}q_ix_i\right)\right]}{P_I+Q_I}\right\}.
\end{aligned}\tag{2.4}
$$

Now, on utilising the convexity property of Ψ we also have

$$
\begin{aligned}
&\Psi\left\{\frac{P_I\left[\frac{1}{P_I}\sum_{i\in I}p_if(x_i)-f\left(\frac{1}{P_I}\sum_{i\in I}p_ix_i\right)\right]}{P_I+Q_I}\right.\\
&\quad\left.+\frac{Q_I\left[\frac{1}{Q_I}\sum_{i\in I}q_if(x_i)-f\left(\frac{1}{Q_I}\sum_{i\in I}q_ix_i\right)\right]}{P_I+Q_I}\right\}
\end{aligned}\tag{2.5}
$$

$$\leq \frac{P_I \Psi \left[\frac{1}{P_I} \sum_{i \in I} p_i f(x_i) - f\left(\frac{1}{P_I} \sum_{i \in I} p_i x_i \right) \right]}{P_I + Q_I} + \frac{Q_I \Psi \left[\frac{1}{Q_I} \sum_{i \in I} q_i f(x_i) - f\left(\frac{1}{Q_I} \sum_{i \in I} q_i x_i \right) \right]}{P_I + Q_I}.$$

Finally, on making use of (2.4) and (2.5), we deduce the desired inequality (2.2). ■

Obviously, there are many examples of functions $\Psi : [0, \infty) \to \mathbb{R}$ that are monotonically decreasing and convex on the interval $[0, \infty)$. In what follows we give some examples that are of interest.

Example 1. Consider the function $\Psi : [0, \infty) \to (0, \infty)$ defined by $\Psi(t) = \exp(-t)$. Obviously this function is strictly decreasing and strictly convex on the interval $[0, \infty)$ and we can consider the functional

$$E(f, I, p, x) := D(f, I, p, x; \exp(-\cdot)) = \frac{P_I \exp\left[f\left(\frac{1}{P_I} \sum_{i \in I} p_i x_i \right) \right]}{\left\{ \prod_{i \in I} \exp[p_i f(x_i)] \right\}^{\frac{1}{P_I}}}. \tag{2.6}$$

Since the functional $E(f, I, \cdot, x)$ is subadditive, then we can state the following interesting inequality for convex functions

$$\begin{aligned} &\frac{(P_I + Q_I) \exp\left[f\left(\frac{1}{P_I + Q_I} \sum_{i \in I} (p_i + q_i) x_i \right) \right]}{\left\{ \prod_{i \in I} \exp[(p_i + q_i) f(x_i)] \right\}^{\frac{1}{P_I + Q_I}}} \\ &\leq \frac{P_I \exp\left[f\left(\frac{1}{P_I} \sum_{i \in I} p_i x_i \right) \right]}{\left\{ \prod_{i \in I} \exp[p_i f(x_i)] \right\}^{\frac{1}{P_I}}} + \frac{Q_I \exp\left[f\left(\frac{1}{Q_I} \sum_{i \in I} q_i x_i \right) \right]}{\left\{ \prod_{i \in I} \exp[q_i f(x_i)] \right\}^{\frac{1}{Q_I}}} \end{aligned} \tag{2.7}$$

for any $p, q \in J^+(\mathbb{R})$.

Example 2. Now assume that $f \in Conv(C, \mathbb{R})$ and $x \in J_*(C)$ are selected such that

$$\frac{1}{P_I} \sum_{i \in I} p_i f(x_i) > f\left(\frac{1}{P_I} \sum_{i \in I} p_i x_i \right)$$

for any $I \in P_f(\mathbb{N})$ with $card(I) \geq 2$ and $p \in J^+(\mathbb{R})$ (notice that is enough to assume that f is strictly convex and x is not constant). If we consider the function $\Psi : (0, \infty) \to (0, \infty)$ defined by $\Psi(t) = t^{-\alpha}$ with $\alpha > 0$, then obviously this function is strictly decreasing and strictly convex on the interval $(0, \infty)$ and we can consider the functional

$$\begin{aligned} W(f, I, p, x) &:= D\left(f, I, p, x; (\cdot)^{-\alpha}\right) \\ &= \frac{P_I}{\left[\frac{1}{P_I} \sum_{i \in I} p_i f(x_i) - f\left(\frac{1}{P_I} \sum_{i \in I} p_i x_i \right) \right]^{\alpha}}. \end{aligned} \tag{2.8}$$

Since the functional $E(f,I,\cdot,x)$ is subadditive, we can state the following interesting inequality for convex functions

$$\begin{aligned}&\frac{P_I+Q_I}{\left[\frac{1}{P_I+Q_I}\sum_{i\in I}(p_i+q_i)f(x_i)-f\left(\frac{1}{P_I+Q_I}\sum_{i\in I}(p_i+q_i)x_i\right)\right]^{\alpha}}\\ &\leq\frac{P_I}{\left[\frac{1}{P_I}\sum_{i\in I}p_if(x_i)-f\left(\frac{1}{P_I}\sum_{i\in I}p_ix_i\right)\right]^{\alpha}}\\ &\quad+\frac{Q_I}{\left[\frac{1}{Q_I}\sum_{i\in I}q_if(x_i)-f\left(\frac{1}{Q_I}\sum_{i\in I}q_ix_i\right)\right]^{\alpha}}\end{aligned}\tag{2.9}$$

for any $p,q\in J^{+}(\mathbb{R})$ such that the involved denominators are not zero.

Corollary 1. *Let $f\in Conv(C,\mathbb{R}),I\in P_f(\mathbb{N})$ and $x\in J_*(C)$. Assume that $\Xi:[0,\infty)\to(0,\infty)$. We define the new functional*

$$M(f,I,p,x;\Xi):=\left\{\Xi\left[\frac{1}{P_I}\sum_{i\in I}p_if(x_i)-f\left(\frac{1}{P_I}\sum_{i\in I}p_ix_i\right)\right]\right\}^{P_I}.\tag{2.10}$$

If $\Xi:[0,\infty)\to(0,\infty)$ is monotonic nonincreasing and logarithmic convex, i.e. $\ln(\Xi)$ is a convex function, then for any $p,q\in J^{+}(\mathbb{R})$ we have

$$M(f,I,p+q,x;\Xi)\leq M(f,I,p,x;\Xi)\cdot M(f,I,q,x;\Xi),\tag{2.11}$$

i.e., the functional is submultiplicative as a function of weights.

Proof. Consider the function $\Psi=\ln(\Xi)$ which is convex and, obviously

$$D(f,I,p,x;\Psi)=\ln M(f,I,p,x;\Xi).$$

The inequality (2.11) follows now by (2.2) and the details are omitted. ∎

Example 3. We consider the Dirichlet series generated by a nonnegative sequence $a_n,n\geq 1$ namely $\delta:(0,\infty)\to(0,\infty)$ given by

$$\delta(s):=\sum_{n=1}^{\infty}\frac{a_n}{n^{s+1}}.\tag{2.12}$$

An important example of such series is the Zeta function defined by

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} \quad \text{for all } s > 1.$$

It is known that the function δ is monotonic nondecreasing and logarithmic convex on $(0, \infty)$ (see for instance [3]). Therefore, for any Dirichlet series of the form (2.12) we have the inequalities

$$\begin{aligned}
&\left\{\delta\left[\frac{1}{P_I+Q_I}\sum_{i\in I}(p_i+q_i)\,f(x_i)-f\left(\frac{1}{P_I+Q_I}\sum_{i\in I}(p_i+q_i)\,x_i\right)\right]\right\}^{P_I+Q_I} \\
&\le \left\{\delta\left[\frac{1}{P_I}\sum_{i\in I}p_i f(x_i)-f\left(\frac{1}{P_I}\sum_{i\in I}p_i x_i\right)\right]\right\}^{P_I} \\
&\quad\times\left\{\delta\left[\frac{1}{Q_I}\sum_{i\in I}q_i f(x_i)-f\left(\frac{1}{Q_I}\sum_{i\in I}q_i x_i\right)\right]\right\}^{Q_I}
\end{aligned} \tag{2.13}$$

for any $p, q \in J^{+}(\mathbb{R})$.

3. Some Subadditivity Properties for the Index

The following result concerning the superadditivity and monotonicity of the functional D as an index set function holds:

Theorem 4. *Let $f \in Conv(C, \mathbb{R})$, $p \in J^{+}(\mathbb{R})$ and $x \in J_{*}(C)$. Assume that $\Psi : [0, \infty) \to \mathbb{R}$ is monotonic nonincreasing and convex where it is defined. If $I, H \in P_f(\mathbb{N})$ with $I \cap H = \varnothing$, then*

$$D(f, I \cup H, p, x; \Psi) \le D(f, I, p, x; \Psi) + D(f, H, p, x; \Psi), \tag{3.1}$$

i.e., $D(f, \cdot, p, x; \Psi)$ is subadditive as an index set function on $P_f(\mathbb{N})$.

Proof. Let $I, H \in P_f(\mathbb{N})$ with $I \cap H = \varnothing$. By the convexity of the function f on C, we have successively

$$\begin{aligned}
&\frac{1}{P_{I\cup H}}\sum_{k\in I\cup H}p_k f(x_k)-f\left(\frac{1}{P_{I\cup H}}\sum_{k\in I\cup H}p_k x_k\right) \\
&= \frac{P_I\left(\frac{1}{P_I}\sum_{i\in I}p_i f(x_i)\right)+P_H\left(\frac{1}{P_H}\sum_{j\in H}p_j f(x_j)\right)}{P_I+P_H} \\
&\quad -f\left(\frac{P_I\left(\frac{1}{P_I}\sum_{i\in I}p_i x_i\right)+P_H\left(\frac{1}{P_H}\sum_{j\in H}p_j x_j\right)}{P_I+P_H}\right) \\
&\ge \frac{P_I\left(\frac{1}{P_I}\sum_{i\in I}p_i f(x_i)\right)+P_H\left(\frac{1}{P_H}\sum_{j\in H}p_j f(x_j)\right)}{P_I+P_H}
\end{aligned} \tag{3.2}$$

$$-\frac{P_I f\left(\frac{1}{P_I}\sum_{i\in I} p_i x_i\right) + P_H f\left(\frac{1}{P_H}\sum_{j\in H} p_j x_j\right)}{P_I + P_H}$$

$$= \frac{P_I\left[\frac{1}{P_I}\sum_{i\in I} p_i f(x_i) - f\left(\frac{1}{P_I}\sum_{i\in I} p_i x_i\right)\right]}{P_I + P_H}$$

$$+\frac{P_H\left[\frac{1}{P_H}\sum_{j\in H} p_j f(x_j) - f\left(\frac{1}{P_H}\sum_{j\in H} p_j x_j\right)\right]}{P_I + P_H}.$$

Since Ψ is monotonic nonincreasing, then by (3.2) we have

$$\Psi\left[\frac{1}{P_{I\cup H}}\sum_{k\in I\cup H} p_k f(x_k) - f\left(\frac{1}{P_{I\cup H}}\sum_{k\in I\cup H} p_k x_k\right)\right]$$

$$\leq \Psi\left\{\frac{P_I\left[\frac{1}{P_I}\sum_{i\in I} p_i f(x_i) - f\left(\frac{1}{P_I}\sum_{i\in I} p_i x_i\right)\right]}{P_I + P_H}\right. \tag{3.3}$$

$$\left.+\frac{P_H\left[\frac{1}{P_H}\sum_{j\in H} p_j f(x_j) - f\left(\frac{1}{P_H}\sum_{j\in H} p_j x_j\right)\right]}{P_I + P_H}\right\}.$$

Utilising the convexity of the function Ψ we also have that

$$\Psi\left\{\frac{P_I\left[\frac{1}{P_I}\sum_{i\in I} p_i f(x_i) - f\left(\frac{1}{P_I}\sum_{i\in I} p_i x_i\right)\right]}{P_I + P_H}\right.$$

$$\left.+\frac{P_H\left[\frac{1}{P_H}\sum_{j\in H} p_j f(x_j) - f\left(\frac{1}{P_H}\sum_{j\in H} p_j x_j\right)\right]}{P_I + P_H}\right\} \tag{3.4}$$

$$\leq \frac{P_I \Psi\left[\frac{1}{P_I}\sum_{i\in I} p_i f(x_i) - f\left(\frac{1}{P_I}\sum_{i\in I} p_i x_i\right)\right]}{P_I + P_H}$$

$$+\frac{P_H \Psi\left[\frac{1}{P_H}\sum_{j\in H} p_j f(x_j) - f\left(\frac{1}{P_H}\sum_{j\in H} p_j x_j\right)\right]}{P_I + P_H},$$

which together with (3.3) produces the desired result (3.1) ■

Example 4. With the assumptions in Example 1 and utilising (3.1), we have the inequality

$$\begin{aligned}(3.5)\qquad &\frac{P_{I\cup H}\exp\left[f\left(\frac{1}{P_{I\cup H}}\sum_{i\in I\cup H}p_ix_i\right)\right]}{\left\{\prod_{i\in I\cup H}\exp\left[p_if\left(x_i\right)\right]\right\}^{\frac{1}{P_{I\cup H}}}}\\ &\leq\frac{P_I\exp\left[f\left(\frac{1}{P_I}\sum_{i\in I}p_ix_i\right)\right]}{\left\{\prod_{i\in I}\exp\left[p_if\left(x_i\right)\right]\right\}^{\frac{1}{P_I}}}+\frac{P_H\exp\left[f\left(\frac{1}{P_H}\sum_{i\in H}p_ix_i\right)\right]}{\left\{\prod_{i\in H}\exp\left[p_if\left(x_i\right)\right]\right\}^{\frac{1}{P_H}}},\end{aligned}$$

for any $I,H\in P_f\left(\mathbb{N}\right)$ with $I\cap H=\varnothing$.

Example 5. With the assumptions in Example 1 and making use of (3.1), we also have the inequality

$$\begin{aligned}(3.6)\qquad &\frac{P_{I\cup H}}{\left[\frac{1}{P_{I\cup H}}\sum_{i\in I\cup H}p_if\left(x_i\right)-f\left(\frac{1}{P_{I\cup H}}\sum_{i\in I\cup H}p_ix_i\right)\right]^{\alpha}}\\ &\leq\frac{P_I}{\left[\frac{1}{P_I}\sum_{i\in I}p_if\left(x_i\right)-f\left(\frac{1}{P_I}\sum_{i\in I}p_ix_i\right)\right]^{\alpha}}\\ &\quad+\frac{P_H}{\left[\frac{1}{P_H}\sum_{i\in H}p_if\left(x_i\right)-f\left(\frac{1}{P_H}\sum_{i\in H}p_ix_i\right)\right]^{\alpha}}\end{aligned}$$

for any $I,H\in P_f\left(\mathbb{N}\right)$ with $I\cap H=\varnothing$ and such that the involved denominators are not zero.

If we use the superadditivity property, then we can state the following result as well:

Corollary 2. *Let $f\in Conv\left(C,\mathbb{R}\right),p\in J^{+}\left(\mathbb{R}\right)$ and $x\in J_{*}\left(C\right)$. Assume that $\Psi:[0,\infty)\to\mathbb{R}$ is monotonic nonincreasing and convex where it is defined. Then*

$$\begin{aligned}(3.7)\qquad &P_{2n}\Psi\left[\frac{1}{P_{2n}}\sum_{i=1}^{2n}p_if\left(x_i\right)-f\left(\frac{1}{P_{2n}}\sum_{i=1}^{2n}p_ix_i\right)\right]\\ &\geq\sum_{i=1}^{n}p_{2i}\Psi\left[\frac{1}{\sum_{i=1}^{n}p_{2i}}\sum_{i=1}^{n}p_{2i}f\left(x_{2i}\right)-f\left(\frac{1}{\sum_{i=1}^{n}p_{2i}}\sum_{i=1}^{n}p_{2i}x_{2i}\right)\right]\\ &\quad+\sum_{i=1}^{n}p_{2i-1}\Psi\left[\frac{1}{\sum_{i=1}^{n}p_{2i-1}}\sum_{i=1}^{n}p_{2i-1}f\left(x_{2i-1}\right)\right.\\ &\quad\left.-f\left(\frac{1}{\sum_{i=1}^{n}p_{2i-1}}\sum_{i=1}^{n}p_{2i-1}x_{2i-1}\right)\right]\end{aligned}$$

and

$$\begin{aligned}(3.8)\qquad & P_{2n+1}\Psi\left[\frac{1}{P_{2n+1}}\sum_{i=1}^{2n+1}p_i f(x_i)-f\left(\frac{1}{P_{2n+1}}\sum_{i=1}^{2n+1}p_i x_i\right)\right]\\ &\geq\sum_{i=1}^{n}p_{2i}\Psi\left[\frac{1}{\sum_{i=1}^{n}p_{2i}}\sum_{i=1}^{n}p_{2i}f(x_{2i})-f\left(\frac{1}{\sum_{i=1}^{n}p_{2i}}\sum_{i=1}^{n}p_{2i}x_{2i}\right)\right]\\ &\quad+\sum_{i=1}^{n}p_{2i+1}\Psi\left[\frac{1}{\sum_{i=1}^{n}p_{2i+1}}\sum_{i=1}^{n}p_{2i+1}f(x_{2i+1})\right.\\ &\quad\left.-f\left(\frac{1}{\sum_{i=1}^{n}p_{2i+1}}\sum_{i=1}^{n}p_{2i+1}x_{2i+1}\right)\right],\end{aligned}$$

where $P_{2n} := \sum_{i=1}^{2n} p_i$ *and* $P_{2n+1} := \sum_{i=1}^{2n+1} p_i$.

The following submultiplicity result also holds:

Corollary 3. *Let* $f \in Conv(C,\mathbb{R}), I \in P_f(\mathbb{N})$ *and* $x \in J_*(C)$. *Assume that* $\Xi : [0,\infty) \to (0,\infty)$ *is monotonic nonincreasing and logarithmic convex. If* $I, H \in P_f(\mathbb{N})$ *with* $I \cap H = \varnothing$, *then*

$$M(f, I\cup H, p, x;\Xi) \leq M(f, I, p, x;\Xi)\cdot M(f, H, p, x;\Xi), \tag{3.9}$$

i.e., $M(f,\cdot,p,x;\Xi)$ *is submultiplicative as an index set function on* $P_f(\mathbb{N})$*;*

4. Applications for the Arithmetic Mean-geometric Mean Inequality

For two sequences of positive numbers p and x, we use the notations

$$A(p,x,I) := \frac{1}{P_I}\sum_{i\in I}p_i x_i \quad \text{and} \quad G(p,x,I) := \left(\prod_{i\in I}x_i^{p_i}\right)^{\frac{1}{P_I}},$$

where I is a finite set of indices and $A(p,x,I)$ is the *arithmetic mean* while $G(p,x,I)$ is the *geometric mean* of the numbers x_i with the weights p_i, $i \in I$.

It is well known that

$$A(p,x,I) \geq G(p,x,I), \tag{4.1}$$

which is known in the literature as the *arithmetic mean-geometric mean inequality*. For various results related to this inequality we recommend the monograph [2] and the references therein.

For the convex function $f : (0,\infty) \to \mathbb{R}$, $f(t) := -\ln(t)$, consider the functional

$$L(I,p,x;\Psi) := D(-\ln(\cdot), I,p,x;\Psi) := P_I \Psi\left[\ln\left(\frac{A(p,x,I)}{G(p,x,I)}\right)\right]. \tag{4.2}$$

We can state the following.

Proposition 1. *Let* $f \in Conv(C,\mathbb{R}), I \in P_f(\mathbb{N})$ *and* $x \in J_*(C)$. *Assume that* $\Psi : [0,\infty) \to \mathbb{R}$ *is monotonic nonincreasing and convex where it is defined.*

(i) *If* $p,q \in J^+(\mathbb{R})$, *then*

$$L(I,p+q,x;\Psi) \le L(I,p,x;\Psi) + L(I,q,x;\Psi) \tag{4.3}$$

i.e., L *is subadditive as a function of weights.*

(ii) *If* $I,H \in P_f(\mathbb{N})$ *with* $I \cap H = \varnothing$, *then*

$$L(I\cup H,p,x;\Psi) \le L(I,p,x;\Psi) + L(H,p,x;\Psi), \tag{4.4}$$

i.e., L *is subadditive as an index set function on* $P_f(\mathbb{N})$.

Utilising these inequalities, we can state the following results concerning the arithmetic and geometric means:

Example 6. Consider the function $\Psi : [0,\infty) \to (0,\infty)$ defined by $\Psi(t) = \exp(-t)$. Obviously this function is strictly decreasing and strictly convex on the interval $[0,\infty)$ and we can consider the functional

$$L_e(I,p,x) := L(I,p,x;\exp(-\cdot)) = \frac{P_I G(p,x,I)}{A(p,x,I)}. \tag{4.5}$$

By Proposition 1 above, we have that L_e is both additive as a weights and index set functional].

We can give the following example as well:

Example 7. If we consider the function $\Psi : (0,\infty) \to (0,\infty)$ defined by $\Psi(t) = t^{-\alpha}$ with $\alpha > 0$, then obviously this function is strictly decreasing and strictly convex on the interval $(0,\infty)$ and we can consider the functional

$$W_{\ln,\alpha}(I,p,x) := L\left(I,p,x;(\cdot)^{-\alpha}\right) = P_I\left[\ln\left(\frac{A(p,x,I)}{G(p,x,I)}\right)\right]^{-\alpha}. \tag{4.6}$$

By the above Proposition 1 we have that $W_{\ln,\alpha}$ is both additive as a weights and index set functional.

Now, for positive sequences x we introduce the notation

$$G\left(p, x^{x}, I\right):=\left(\prod_{i \in I} x_{i}^{p_{i} x_{i}}\right)^{\frac{1}{P_{I}}}, \tag{4.7}$$

which is the geometric mean of the sequence having the terms $x_i^{x_i}$, $i \in I$.

For the convex function $f:(0,\infty)\to\mathbb{R}$, $f(t):=t\ln(t)$, consider the functional

$$S(I,p,x;\Psi):=D(\cdot\ln(\cdot),I,p,x;\Psi):=P_I\Psi\left[\ln\left(\frac{G\left(p,x^{x},I\right)}{A(p,x,I)^{A(p,x,I)}}\right)\right]. \tag{4.8}$$

We can state the following.

Proposition 2. *Let $f\in Conv(C,\mathbb{R}), I\in P_f(\mathbb{N})$ and $x\in J_*(C)$. Assume that $\Psi:[0,\infty)\to\mathbb{R}$ is monotonic nonincreasing and convex where it is defined.*

(i) If $p,q\in J^{+}(\mathbb{R})$, then

$$S(I,p+q,x;\Psi)\le S(I,p,x;\Psi)+S(I,q,x;\Psi), \tag{4.9}$$

i.e., S is subadditive as a function of weights.

(ii) If $I,H\in P_f(\mathbb{N})$ with $I\cap H=\varnothing$, then

$$S(I\cup H,p,x;\Psi)\le S(I,p,x;\Psi)+S(H,p,x;\Psi), \tag{4.10}$$

i.e., S is subadditive as an index set function on $P_f(\mathbb{N})$.

Remark 1. For the function $\Psi:[0,\infty)\to(0,\infty)$ defined by $\Psi(t)=\exp(-t)$ we can consider the functional

$$S_e(I,p,x):=S(I,p,x;\exp(-\cdot))=\frac{P_I A(p,x,I)^{A(p,x,I)}}{G\left(p,x^{x},I\right)}. \tag{4.11}$$

By the above Proposition 2 we have that S_e is both additive as a weights and index set functional.

For the function $\Psi:(0,\infty)\to(0,\infty)$ defined by $\Psi(t)=t^{-\alpha}$ with $\alpha>0$, we can also consider the functional

$$Z_{\ln,\alpha}(I,p,x):=S\left(I,p,x;(\cdot)^{-\alpha}\right)=P_I\left[\ln\left(\frac{G\left(p,x^{x},I\right)}{A(p,x,I)^{A(p,x,I)}}\right)\right]^{-\alpha}. \tag{4.12}$$

By the above Proposition 2 we have that $Z_{\ln,\alpha}$ is both additive as a weights and index set functional.

The interested reader can consider other examples of functions f and Ψ and derive functionals that are associated with the Ky Fan, triangle or other inequalities that can be obtained from the Jensen result. However, the details are not presented here.

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S. S. Dragomir
Mathematics School of Engineering & Science
Victoria University
P. O. Box 14428
Melbourne City, MC 8001
Australia
and
School of Computational and Applied Mathematics
University of the Witwatersrand
Private Bag-3
Wits-2050, Johannesburg
South Africa
E-mail: sever.dragomir@vu.edu.au

Y. J. Cho
Department of Mathematical Education and the RINS
Gyeongsang National University
Chinju 660-701
Korea
E-mail: yjcho@nongae.gsnu.ac.kr

J. K. Kim
Department of Mathematics Education
Kyungnam University
Masan 631-701
Korea
E-mail: jungkyuk@kyungnam.ac.kr