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VERTEX-COLORING EDGE-WEIGHTINGS OF GRAPHS

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Abstract. A k-*edge-weighting* of a graph G is a mapping $w : E(G) \to \{1, 2, \ldots, k\}$. An edge-weighting w induces a vertex coloring $f_w : V(G) \to \mathbb{N}$ defined by $f_w(v) = \sum_{v\in e} w(e)$. An edge-weighting w is *vertex-coloring* if $f_w(u) \neq f_w(v)$ for any edge uv. The current paper studies the parameter $\mu(G)$, which is the minimum k for which G has a vertex-coloring k-edge-weighting. Exact values of $\mu(G)$ are determined for several classes of graphs, including trees and r-regular bipartite graph with $r \geq 3$.

1. INTRODUCTION

A k-*edge-weighting* of a graph G is a mapping $w : E(G) \to \{1, 2, \ldots, k\}$. An edge-weighting w induces a vertex coloring $f_w : V(G) \to \mathbb{N}$ defined by $f_w(v) = \sum_{v\in e} w(e)$. An edge-weighting w is *vertex-coloring* (respectively, *vertex-injective*) if $f_w(u) \neq f_w(v)$ for any edge uv (respectively, every pair of distinct vertices u and v). Denote by $\mu(G)$ (respectively, $\mu^*(G)$) the minimum k for which G has a vertex-coloring (respectively, vertex-injective) k-edge-weighting. We refer a graph *non-trivial* if it contains no single edge as a component. Notice that $\mu(G) \leq \mu^*(G)$ for every non-trivial graph G.

An edge-weighting is *adjacent vertex-distinguishing* (respectively, *vertex-distinguishing*) if for any edge uv (respectively, every pair of distinct vertices u and v), the multi-set of weights appearing on edges incident to u is distinct from the multi-set of weights appearing on the edges incident to v. Denote by $\mu_m(G)$ (respectively, $\mu_m^*(G)$) the minimum k for which G has an adjacent vertex-distinguishing (respectively, vertex-distinguishing) k-edge-weighting. Notice that $\mu_m(G) \leq \mu_m^*(G)$ for every non-trivial graph G. Then, upper bounds for $\mu(G)$ (respectively, $\mu^*(G)$) provide upper bounds for $\mu_m(G)$ (respectively, $\mu_m^*(G)$).

It is clear that a vertex-coloring (respectively, vertex-injective) edge-weighting is adjacent vertex-distinguishing (respectively, vertex-distinguishing), but the converse

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is not necessarily true. Consequently, $\mu_m(G) \leq \mu(G)$ and $\mu_m^*(G) \leq \mu^*(G)$ for every non-trivial graph G.

Adjacent vertex-distinguishing edge-weighting and vertex-distinguishing edge-weighting have been studied by many researchers [4, 6, 5, 7]. Karoński, Luczak and Thomason [10] proved that $\mu_m(G) \leq 213$ for every non-trivial graph and that $\mu_m(G) \leq 30$ for every graph with minimum degree at least 10^{99}. Addario-Berry et al. [1] improved the results to $\mu_m(G) \leq 4$ for every non-trivial graph and $\mu_m(G) \leq 3$ for every graph of minimum degree at least 1000.

For vertex-coloring edge-weighting, Karoński, Luczak and Thomason [10] posed the following question:

Question. Does $\mu(G) \leq 3$ for every non-trivial graph G?

Karoński, Luczak and Thomason [10] showed that if G is a k-colorable graph with k odd then G admits a vertex-coloring k-edge-weighting. So, for the class of 3-colorable graphs, including bipartite graphs, the answer is affirmative. However, in general, this question is still open. The first constant bound was obtained by Addario-Berry et al. [2], who showed that $\mu(G) \leq 30$ for every non-trivial graph G. The bound is improved to $\mu(G) \leq 16$ in [3], to $\mu(G) \leq 13$ in [11], and to $\mu(G) \leq 5$ in [9].

Even we are still far from providing a positive answer to the question, actually $\mu(G) \leq 2$ for many graphs (in fact, experiments suggest (see [10]) that $\mu(G) \leq 2$ for almost all graphs). The current paper is devoted to study graphs with such a property. We determine $\mu(G)$ for some classes of graphs with this property, including trees and r-regular bipartite graphs with $r \geq 3$.

In the rest of this section, we fix some notation. For $n \geq 1$, the *n-path P_n* is the graph with vertex set $\{v_i : 1 \leq i \leq n\}$ and edge set $\{v_iv_{i+1} : 1 \leq i \leq n-1\}$. For $n \geq 3$, the *n-cycle C_n* is the graph with vertex set $\{v_i : 1 \leq i \leq n\}$ and edge set $\{v_iv_{i+1} : 1 \leq i \leq n\}$, where $v_{n+1} = v_1$. The *complete graph K_n* is the graph with vertex set $\{v_i : 1 \leq i \leq n\}$ and edge set $\{v_iv_j : 1 \leq i < j \leq n\}$. The *complete bipartite graph $K_{m,n}$* is the graph with vertex set $\{u_i : 1 \leq i \leq m\} \cup \{v_j : 1 \leq j \leq n\}$ and edge set $\{u_iv_j : 1 \leq i \leq m, 1 \leq j \leq n\}$. The *neighborhood* of a vertex v is the set $N(v) = \{u : uv \in E(G)\}$, and the *closed neighborhood* is $N[v] = N(v) \cup \{v\}$. The *degree* of a vertex v is $d(v) = |N(v)|$. We use $\delta(G)$ to denote the minimum degree of a vertex in a graph G.

2. $\mu(G)$ for Some Classes of Graphs

This section establishes values of $\mu(G)$ for some classes of graphs, including paths, cycles, complete graphs and complete bipartite graphs.

Fact 1. *For every non-trivial graph G, $\mu(G) = 1$ if and only if G has no adjacent vertices with the same degree.*

Fact 2. $\mu(P_3) = 1$ *and* $\mu(P_n) = 2$ *for* $n \geq 4$.

Proof. This follows from Fact 1 and the fact that the following mapping w is a vertex-coloring 2-edge-weighting: $w(v_iv_{i+1}) = 1$ for $i \equiv 1, 2$ (mod 4) and $w(v_iv_{i+1}) = 2$ for $i \equiv 3, 4$ (mod 4). ■

Proposition 3. $\mu(C_n) = 2$ *for* $n \equiv 0 \pmod 4$ *and* $\mu(C_n) = 3$ *for* $n \not\equiv 0 \pmod 4$.

Proof. First, $\mu(C_n) \geq 2$ by Fact 1. For the case when $n \equiv 0 \pmod 4$, $\mu(C_n) = 2$ follows from that the following mapping w is a vertex-coloring 2-edge-weighting: $w(v_iv_{i+1}) = 1$ for $i \equiv 1, 2$ (mod 4) and $w(v_iv_{i+1}) = 2$ for $i \equiv 3, 4$ (mod 4).

For the case $n = 4k + r$, $1 \leq r \leq 3$, $\mu(C_n) \leq 3$ follows from that the following mapping w is a vertex-coloring 3-edge-weighting: $w(v_iv_{i+1}) = 1$ for $i \equiv 1, 2$ (mod 4) and $w(v_iv_{i+1}) = 2$ for $i \equiv 3, 4$ (mod 4) with the modifications that $w(v_{4k+1}v_{4k+2}) = w(v_{4k+2}v_{4k+3}) = 3$ and $w(v_{4k+3}v_{4k+4}) = 2$. On the other hand, we claim that $\mu(C_n) \neq 2$. Suppose to the contrary that C_n has a vertex-coloring 2-edge-weighting w. Then, $f_w(v_{i+1}) \neq f_w(v_{i+2})$ implies $w(v_iv_{i+1}) \neq w(v_{i+2}v_{i+3})$ and so $w(v_iv_{i+1}) = w(v_{i+4}v_{i+5})$, where the indices are taken modulo 4. These in turn imply that $w(v_iv_{i+1}) \neq w(v_{i+4j+2}v_{i+4j+3})$. This is a contradiction since $v_i = v_{i+n} = v_{i+4j+2}$ when $r = 2$ with $j = \frac{n-2}{4}$ and $v_i = v_{i+2n} = v_{i+4j+2}$ when $r = 1, 3$ with $j = \frac{n-1}{2}$. ■

Proposition 4. *If* $n \geq 3$, *then* $\mu(K_n) = 3$.

Proof. We first consider the following mapping w: $w(v_iv_j) = 1$ for $i + j \leq n$, $w(v_iv_n) = 3$ for $\lfloor \frac{n+2}{2} \rfloor \leq i \leq n - 1$, and $w(v_iv_j) = 2$ for all other edges. It is straightforward to check that $f_w(v_i) = n - 1 + i$ for $1 \leq i \leq n - 1$ and $f_w(v_n) = \lfloor \frac{5n-5}{2} \rfloor$. Hence, f_w is vertex-coloring and so $\mu(K_n) \leq 3$.

On the other hand, we claim that $\mu(K_n) \neq 2$. Suppose to the contrary that K_n has a vertex-coloring 2-edge-weighting w. Then, each $f_w(v_i)$ is one of the n possible values in $\{n-1, n, \ldots, 2n-2\}$. So, there is exactly one v_i (resp. v_j) with $f_w(v_i) = n-1$ (resp. $f_w(v_j) = 2n-2$). The first equation implies that $w(v_iv_j) = 1$ while the second one implies that $v(v_jv_i) = 2$, a contradiction. Thus, $\mu(K_n) = 3$. ■

Proposition 5. $\mu(K_{m,n}) = 1$ *when* $m \neq n$ *and* $\mu(K_{m,n}) = 2$ *when* $m = n \geq 2$.

Proof. The former case follows from Fact 1. The latter case follows from that for $m = n \geq 2$ the following mapping w is a vertex-coloring 2-edge-weighting: $w(u_iv_j) = 1$ and $w(u_mv_j) = 2$ for $1 \leq i \leq m - 1$ and $1 \leq j \leq n$. ■

The *theta graph* $\theta(\ell_1, \ell_2, \ldots, \ell_r)$ is the graph obtained from r disjoint paths of lengths $\ell_1, \ell_2, \ldots, \ell_r$, respectively, by identifying their end-vertices called the *roots* of the graph. Notice that $\theta(\ell_1) = P_{1+\ell_1}$ and $\theta(\ell_1, \ell_2) = C_{\ell_1+\ell_2}$. In the following we only consider the case $r \geq 3$ and assume that $\ell_1 \leq \ell_2 \leq \ldots \leq \ell_r$. We also assume that $\ell_1 = 1$ implies $\ell_2 > 1$. In other words, we only consider simple graphs.

Proposition 6. *Let $G = \theta(\ell_1, \ell_2, \ldots, \ell_r)$ with $r \geq 3$. Then $\mu(G) = 1$ when $\ell_i = 2$ for all i; $\mu(G) = 3$ when $\ell_1 = 1$ and $\ell_i \equiv 1 \pmod 4$ for all $i \neq 1$; and $\mu(G) = 2$ otherwise.*

Proof. The first equality follows from Proposition 1 and that any two adjacent vertices have different degrees if and only if all $\ell_i = 2$.

For the case when $\ell_1 = 1$ with all $\ell_i \equiv 1 \pmod 4$, we claim that $\mu(G) \geq 3$. Suppose, to the contrary that the graph admits a vertex-coloring 2-edge-weighting w. Then, in each path the kth edge must have the different weight from the $(k+2)$th edge, and has the same weight with the $(k+4)$th edge. Consequently, the first edge has the same weight with the last edge in each path of the theta graph. Then, $f_w(u) = f_w(v)$ for the two roots u and v, however, this is impossible as they are adjacent. On the other hand, the following mapping w is a vertex-coloring 3-edge-weighting: for each path of the theta graph, assign the weights $1, 1, 2, 2$ periodically except the last edge assigned with 3.

For the remaining case, we may construct a vertex-coloring 2-edge-weighting as follows. Notice that for a periodical weight assignment $\ldots 1, 1, 2, 2 \ldots$ of a path with first edge e_i and last edge e'_i, we may properly choose the starting weight such that $w(e_i) = w(e'_i) = 2$ (respectively, $w(e_i) \neq w(e'_i)$) when $\ell_i \not\equiv 3$ (mod 4) (respectively, $\ell_1 \not\equiv 1$ (mod 4)). We then may properly arrange the weights on edges to make a vertex-coloring 2-edge-weighting. ∎

3. $\boldsymbol{\mu(G)}$ for Bipartite Graphs

In this section, we consider $\mu(G)$ for a bipartite graph G. We use $G = (A, B, E)$ to denote a bipartite graph with vertex bipartition (A, B) and edge set E.

Theorem 7. *Every non-trivial connected bipartite graph $G = (A, B, E)$ with $|A|$ even admits a vertex-coloring 2-edge-weighting w such that $f_w(u)$ is odd for $u \in A$ and $f_w(v)$ is even for $v \in B$. Consequently, $\mu(G) \leq 2$.*

Proof. Assume that $A = \{a_1, a_2, \ldots, a_{2r}\}$. Let P_i be a path from a_i to a_{r+i} for $1 \leq i \leq r$. For each edge e, denote $k(e)$ the number of such paths containing e; and for each vertex u, denote $m(u)$ the sum of $k(e)$ of all edges e incident to u. Then $m(u)$ is odd for $u \in A$ and $m(v)$ is even for $v \in B$. Now, let $w(e) = 1$ for any edge e with $k(e)$ odd and $w(e) = 2$ for any edge e with $k(e)$ even. Since $w(e)$ has the same parity as $k(e)$ for each edge e, the color $f_w(u)$ of a vertex u satisfies $f_w(u) \equiv m(u)$ (mod 2) for $u \in A \cup B$. Consequently, $f_w(u)$ is odd for $u \in A$ and $f_w(v)$ is even for $v \in B$. Hence, w is a vertex-coloring 2-edge-weighting of G. ∎

Theorem 8. *$\mu(G) \leq 2$ for every non-trivial connected bipartite graph $G = (A, B, E)$ with $\delta(G) = 1$.*

Proof. By Theorem 7, we may assume that both of $|A|$ and $|B|$ are odd. Without loss of generality, assume that $d(x) = 1$ for some vertex x in A, and that x is adjacent to a vertex y in B. Then $G - x = (A \setminus \{x\}, B, E \setminus \{xy\})$ is a non-trivial connected bipartite graph with $|A - \{x\}|$ even. By Theorem 7, $G - x$ has a 2-edge-weighting w' so that $f_{w'}(u)$ is odd for $u \in A \setminus \{x\}$ and $f_{w'}(v)$ is even for $v \in B$. Now, extend w' to w for G by assigning $w(xy) = 2$. This gives a vertex-coloring 2-edge-weighting with $f_w(x) = 2$, $f_w(u)$ odd for $u \in A \setminus \{x\}$, $f_w(v)$ even for $v \in B$ and $f_w(y) > 2$. ■

Corollary 9. *If T is a tree of at least three vertices, then $\mu(T) \leq 2$.*

Theorem 10. *$\mu(G) \leq 2$ for every non-trivial connected bipartite graph $G = (A, B, E)$ if $\lfloor d(u)/2 \rfloor + 1 \neq d(v)$ for any edge $uv \in E(G)$.*

Proof. By Theorem 7, we may assume that both of $|A|$ and $|B|$ are odd. We need a claim first.

Claim. There exists a vertex x, say $x \in B$, such that the vertices of $G - N[x]$ in A are all in a same component of $G - N[x]$.

Choose a vertex x such that the size of a *maximum* component of $G - N[x]$ becomes as large as possible. Without loss of generality, we assume that $x \in B$. Suppose that besides a maximum component $G_1 = (A_1, B_1, E_1)$ the graph $G-N[x]$ has another component $G_2 = (A_2, B_2, E_2)$, where A_1 and A_2 are nonempty subsets of A. Choose $x' \in A_2$. Since G is connected, $N(x)$ has a vertex adjacent to a vertex in B_1. Then, G_1 together with $N[x]$ are in a same component of $G - N[x']$, and then the size of a maximum component of $G - N[x']$ is larger than that of x, a contradiction to the choice of x.

From the claim, we see that $G - N[x]$ has a component $G_1 = (A_1, B_1, E_1)$ with $A_1 = A \setminus N(x)$ and all other components are isolated vertices in B. Now we consider two cases.

Case 1. $d(x)$ is odd. In this case, $|A_1|$ is even. According to Theorem 7, G_1 has a 2-edge-weighting w' such that $f_{w'}(u)$ is odd for $u \in A_1$ and $f_{w'}(v)$ is even for $v \in B_1$. We then extend w' to w for G by assigning the edges incident to x with weight 1 and the remaining edges with weight 2. Then, $f_w(u)$ is odd for $u \in A$ and $f_w(v)$ is even for $v \in B \setminus \{x\}$. Notice that $f_w(x) = d(x)$ and $f_w(u) = 2d(u) - 1$ for all $u \in N(x)$. These imply $f_w(x) \neq f_w(u)$ by hypothesis. Therefore, w is a vertex-coloring 2-edge-weighting of G.

Case 2. $d(x)$ is even. In this case, $|A_1|$ is odd. Notice that there is a vertex $u^* \in N(x)$ adjacent to some vertex $v^* \in B_1$. Let G' be the graph obtained from

G_1 by adding the vertex u^* and the edge u^*v^*. According to Theorem 7, G' has a 2-edge-weighting w' so that $f_{w'}(u)$ is odd for $u \in A_1 \cup \{u^*\}$ and $f_{w'}(v)$ is even for $v \in B_1$. We may extend w' to w for G by assigning the edges incident to x, except xu^*, with weight 1 and the remaining edges with weight 2. Then, $f_w(u)$ is odd for $u \in A$ and $f_w(v)$ is even for $v \in B$ except x. Notice that $f_w(x) = d(x)+1$ and $f_w(u) = 2d(u) - 1$ for all $u \in N(x) - u^*$. These imply $f_w(x) \neq f_w(u)$ by hypothesis. Therefore, w is a vertex-coloring 2-edge-weighting of G. ■

Consequently, we have the following result which is in fact our first thought.

Corollary 11. $\mu(G) = 2$ *for every* r*-regular bipartite graph* G *with* $r \geq 3$.

Notice that the theta graph $G = \theta(\ell_1, \ell_2, \ldots, \ell_r)$ with $\ell_1 = 1$ and all $\ell_i \equiv 1 \pmod 4$ is a bipartite graph with $\mu(G) = 3$. In particular, $\mu(C_{4k+2}) = 3$, which shows that the condition $r \geq 3$ in Corollary 11 is necessary.

We conclude the paper by posing the following problem.

Problem. Characterize bipartite graphs with vertex-coloring 2-edge-weighting.

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References

1. L. Addario-Berry, R. E. L. Aldred, K. Dalal and B. A. Reed, Vertex colouring edge partitions, *J. Combin. Theory, Series B*, **94** (2005), 237-244.

2. L. Addario-Berry, K. Dalal, C. Mcdiarmid, B. A. Reed and A. Thomason, Vertex-colouring edge-weightings, *Combinatorica*, **27** (2007), 1-12.

3. L. Addario-Berry, K. Dalal and B. A. Reed, Degree constrainted subgraphs, *Discrete Applied Math.*, **156** (2007), 1168-1174.

4. M. Aigner, E. Triesch and Zs. Tuza, Irregular assignments and vertex-distinguishing edge-colorings of graphs, *Combinatorics 90*, A. Barlotti et al., (eds), Elsevier Science Pub., New York, 1992, pp. 1-9.

5. P. N. Balister, O. M. Riordan and R. H. Schelp, Vertex-distinguishing edge colorings of graphs, *J. Graph Theory*, **42** (2003), 95-109.

6. A. C. Burris and R. H. Schelp, Vertex-distinguishing proper edge colorings of graphs, *J. Graph Theory*, **26** (1997), 73-82.

7. K. Edwards, The harmonious chromatic number of bounded degree graphs, *J. London Math. Soc.*, **55** (1997), 435-447.

8. A. Frieze, R. J. Gould, M. Karoński and F. Pfender, On graph irregularity strength, *J. Graph Theory*, **41** (2002), 120-137.

9. M. Kalkowski, M. Karoński and F. Pfender, Vertex-coloring edge-weighting: toward the 1-2-3-conjecture, *J. Combin. Theory, Series B*, **100** (2010), 347-349.

10. M. Karoński, T. Luczak and A. Thomason, Edge weights and vertex colours, *J. Combin. Theory, Series B*, **91** (2004), 151-157.

11. T. Wang and Q. Yu, On vertex-coloring 13-edge-weighting, *Front. Math. China*, **3** (2008), 1-7.

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