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ON LATIF'S FIXED POINT THEOREMS

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Abstract. In this paper, we prove that Latif's fixed point statements for w-distance or other weak distances are indeed equivalent with some weak distance variants of Caristi's fixed point statement, Ekeland's variational principle statement and Takahashi's nonconvex minimization statement.

1. Introduction and Preliminaries

Let (X, d) be a metric space. Recall that an extended real valued function $f : X \to (-\infty, \infty]$ on X is said to be proper if $f \not\equiv \infty$. A function $p : X \times X \to [0, \infty)$ is called a w-$distance$ on X, introduced and studied by Kada, Suzuki and Takahashi [7, 9, 11, 17], if the following are satisfied:

$(W1)$ $p(x, z) \leq p(x, y) + p(y, z)$ for any $x, y, z \in X$;

$(W2)$ for any $x \in X$, $p(x, \cdot) : X \to [0, \infty)$ is lower semicontinuous;

$(W3)$ for any $\varepsilon > 0$, there exists $\delta > 0$ such that $p(z, x) \leq \delta$ and $p(z, y) \leq \delta$ imply $d(x, y) \leq \varepsilon$.

It is well known that any metric d is a w-distance. Many useful examples of w-distance were stated in [7, 17].

The following Lemma is crucial in this paper.

Lemma K. (7, Lemma 1). Let (X, d) be a metric space and p a w-distance on X. Let $\{x_n\}$ and $\{y_n\}$ be sequences in X and $x, y, z \in X$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[0, \infty)$ converging to 0. If $p(x_n, y) \leq \alpha_n$ and $p(x_n, z) \leq \beta_n$ for any $n \in N$, then $y = z$. In particular, if $p(x, y) = 0$ and $p(x, z) = 0$, then $y = z$.

In what follows, unless otherwise specified, we always assume that (X, d) is a metric space, p a w-distance on X and $f : X \to (-\infty, \infty]$ a proper lower semicontinuous and bounded below function. We shall use the following statements.

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Statement COM. X is complete.

Statement CFP. (Caristi's fixed point statement). Suppose that $T : X \to X$ is a single-valued map satisfying

$$d(x, Tx) \leq f(x) - f(Tx)$$

for each $x \in X$. Then T has a fixed point in X.

Statement EVP. (Ekeland's variational principle statement). Suppose that there exists $u \in X$ such that $f(u) < \infty$. Then for every $\varepsilon > 0$, there exists $v \in X$ such that

(i) $\varepsilon d(u, v) \leq f(u) - f(v)$;
(ii) $\varepsilon d(v, x) > f(v) - f(x)$ for all $x \in X$ with $x \neq v$.

Statement TNM. (Takahashi's nonconvex minimization statement). Suppose that for any $x \in X$ with $f(x) > \inf_{z \in X} f(z)$, there exists $y_x \in X$ with $y_x \neq x$ such that

$$d(x, y_x) \leq f(x) - f(y_x).$$

Then there exists $v \in X$ such that $f(v) = \inf_{x \in X} f(x)$.

In 1976, Caristi [4] proved the following famous fixed point theorem.

Theorem CFP. (Caristi). Statement COM $\Longrightarrow$ Statement CFP.

The primitive Ekeland's variational principle [6] and Takahashi's nonconvex minimization theorem [16] can be declared respectively as follows.

Theorem EVP. (Ekeland). Statement COM $\Longrightarrow$ Statement EVP.

Theorem TNM. (Takahashi). Statement COM $\Longrightarrow$ Statement TNM.

It is well known (see, e.g., [5, 7, 10-14, 16, 17]) that

Statement COM $\Longleftrightarrow$ Statement CFP $\Longleftrightarrow$ Statement EVP $\Longleftrightarrow$ Statement TNM.

A number of generalizations in various different directions of these results in metric (or quasi-metric) spaces and more general in topological vector spaces have been investigated by several authors in the past; see [1, 2, 5, 7-15, 17] and references therein. It is interesting to mention that some generalizations were real logical equivalent with the original theorems; see e.g. [3, 5, 15].

In this paper, motivated by the recent Latif's results in [9], we will demonstrate that Latif's fixed point statements (see Statements L1-L6 below) for w-distance or other weak distances are equivalent with some weak distance variants of Caristi's fixed point statement, Ekeland's variational principle statement and Takahashi's nonconvex minimization statement.

2. Some Equivalences

In this section, we use the following statements.

Statement GE. (Generalized Ekeland's variational principle statement). Let $\varepsilon > 0$. Suppose that there exists $u \in X$ such that $f(u) < \infty$ and $p(u, u) = 0$. Then there exists $v \in X$ such that

(a) $\varepsilon p(u, v) \leq f(u) - f(v)$;

(b) $\varepsilon p(v, x) > f(v) - f(x)$ for all $x \in X$ with $x \neq v$.

Statement GCCM. (Generalized Caristi's common fixed point statement for a family of multivalued maps). Let I be an index set. For each $i \in I$, let $T_i : X \to 2^X$ be a multivalued map with nonempty values such that for each $x \in X$, there exists $y = y(x, i) \in T_i(x)$ with

$$p(x, y) \leq f(x) - f(y).$$

Then there exists $v \in X$ such that $v \in \bigcap_{i \in I} T_i(v)$ (that is, the family of multivalued maps $\{T_i\}_{i \in I}$ has a common fixed point v in X), $f(v) < \infty$ and $p(v, v) = 0$.

Statement GCM. (Generalized Caristi's fixed point statement for a multivalued map). Let $T : X \to 2^X$ be a multivalued map with nonempty values such that for each $x \in X$, there exists $y \in T(x)$ with

$$p(x, y) \leq f(x) - f(y).$$

Then there exists $v \in X$ such that $v \in T(v)$, $f(v) < \infty$ and $p(v, v) = 0$.

Statement GCCS. (Generalized Caristi's common fixed point statement for a family of single-valued maps). Let I be an index set. For each $i \in I$, suppose that $T_i : X \to X$ is a single-valued map satisfying

$$p(x, T_i x) \leq f(x) - f(T_i x)$$

for each $x \in X$. Then there exists $v \in X$ such that $T_i v = v$ for all $i \in I$, $f(v) < \infty$ and $p(v, v) = 0$.

Statement GCS. (Generalized Caristi's fixed point statement for a single-valued map). Suppose that $T : X \to X$ is a single-valued map satisfying

$$p(x, Tx) \leq f(x) - f(Tx)$$

for each $x \in X$. Then there exists $v \in X$ such that $Tv = v$, $f(v) < \infty$ and $p(v, v) = 0$.

Statement GT. (Generalized Takahashi's nonconvex minimization statement). Suppose that for any $x \in X$ with $f(x) > \inf_{z \in X} f(z)$, there exists $y_x \in X$ with $y_x \neq x$ such that

$$p(x, y_x) \leq f(x) - f(y_x).$$

Then there exists $v \in X$ such that $f(v) = \inf_{x \in X} f(x)$.

The following generalizations of Ekeland's variational principle, Caristi's (common) fixed point theorem and Takahashi's nonconvex minimization theorem had been established.

Theorem GE. (Generalized Ekeland's variational principle) [7, 10-14, 17].

Statement COM $\Longrightarrow$ Statement GE.

Theorem GCCM. (Generalized Caristi's common fixed point theorem for a family of multivalued maps) [10, 12].

Statement COM $\Longrightarrow$ Statement GCCM.

Theorem GCM. (Generalized Caristi's fixed point theorem for a multivalued map) [7, 10-14, 17].

Statement COM $\Longrightarrow$ Statement GCM.

Theorem GCCS. (Generalized Caristi's common fixed point theorem for a family of single-valued maps) [10, 12].

Statement COM $\Longrightarrow$ Statement GCCS.

Theorem GCS. (Generalized Caristi's fixed point theorem for a single-valued map) [7, 9-14, 17].

Statement COM $\Longrightarrow$ Statement GCS.

Theorem GT. (Generalized Takahashi's nonconvex minimization theorem) [7, 10-14, 17].

Statement COM $\Longrightarrow$ Statement GT.

Parts of the following equivalence were established in [10] or [12], but we give the proof for the sake of completeness and the readers convenience.

Theorem D1. Statement GE, Statement GCCM, Statement GCM, Statement GCCS, Statement GCS and Statement GT are equivalent.

Proof.

(1) We first prove that "Statement GE $\Rightarrow$ Statement GCCM $\Rightarrow$ Statement GCM $\Rightarrow$ Statement GCS". By Statement GE, there exists $v \in X$ such that $p(v, x) > f(v) - f(x)$ for all $x \in X$ with $x \neq v$. Clearly, $f(v) < \infty$. We want to prove that $v \in \bigcap_{i \in I} T_i(v)$ and $p(v, v) = 0$. By our hypothesis, for each $i \in I$, there exists $y(v, i) \in T_i(v)$ such that $p(v, y(v, i)) \leq f(v) - f(y(v, i))$. Then $y(v, i) = v$ for each $i \in I$. Indeed, if $y(v, i_0) \neq v$ for some $i_0 \in I$, then we have $p(v, y(v, i_0)) \leq f(v) - f(y(v, i_0))) < p(v, y(v, i_0))$. This is a contradiction. Hence $v = y(v, i) \in T_i(v)$ for all $i \in I$. Since $p(v, v) \leq f(v) - f(v) = 0$, we obtain $p(v, v) = 0$. Hence Statement GCCM holds. It is clear that Statement GCCM $\Rightarrow$ Statement GCM $\Rightarrow$ Statement GCS.

(2) "Statement GCCM $\Rightarrow$ Statement GCCS $\Rightarrow$ Statement GCS" are obvious.

(3) Let us prove that "Statement GCS $\Rightarrow$ Statement GE". Suppose that for each $x \in X$, there exists $y_x \in X$ with $y_x \neq x$ such that $p(x, y_x) \leq f(x) - f(y_x)$. Then we can define a single-valued map $T : X \to X$ by

$$T(x) = y_x.$$

Then, T satisfies $p(x, Tx) \leq f(x) - f(Tx)$ for each $x \in X$. But by Statement GCS, T have a fixed point $x_0 \in X$, which contradicts with $T(x_0) \neq x_0$ and hence Statement GE holds.

(4) Finally, we prove that "Statement GE $\Longleftrightarrow$ Statement GT".

Let us prove $(\Rightarrow)$. By Statement GE, there exists $v \in X$ such that $p(v, x) > f(v) - f(x)$ for all $x \in X$ with $x \neq v$. We claim that $f(v) = \inf_{x \in X} f(x)$. Suppose to the contrary that $f(v) > \inf_{x \in X} f(x)$. By our assumption, there exists $y_v \in X$ with $y_v \neq v$ such that

$$p(v, y_v) \leq f(v) - f(y_v) < p(v, y_v),$$

which leads to a contradiction. Hence $f(v) = \inf_{x \in X} f(x)$.

Let us prove $(\Leftarrow)$. Suppose that for each $x \in X$, there exists $y \in X$ with $y \neq x$ such that $p(x, y) \leq f(x) - f(y)$. Then, by Statement GT, there exists $\hat{v} \in X$ such that $f(\hat{v}) = \inf_{x \in X} f(x)$. By our hypothesis, there exists $w \in X$ with $w \neq \hat{v}$ such that $p(\hat{v}, w) \leq f(\hat{v}) - f(w) \leq 0$. Hence $p(\hat{v}, w) = 0$ and $f(\hat{v}) = f(w) = \inf_{x \in X} f(x)$. Also, there exists $z \in X$ with $z \neq w$ such that $p(w, z) \leq f(w) - f(z) \leq 0$. So $p(w, z) = 0$. Since $p(\hat{v}, z) \leq p(\hat{v}, w) + p(w, z) = 0$, $p(\hat{v}, z) = 0$. By Lemma K, we have $w = z$, a contradiction. Therefore Statement GE is true. ∎

Remark 1. Review the proof of Theorem D1. Then, we can point out an important fact that there exists $v \in X$ such that

(a) $\varepsilon p(u, v) \leq f(u) - f(v)$ and $\varepsilon p(v, x) > f(v) - f(x)$ for all $x \in X$ with $x \neq v$;

(b) $v \in T_i(v)$ (T_i is defined as in Statement GCCM) or $T_i(v) = v$ (T_i is defined as in Statement GCCS) for all $i \in I$;

(c) $f(v) = \inf_{x \in X} f(x)$.

Very recently, Latif [9] gave some generalized Caristi's fixed point theorems for w-distances. In [9], Latif stated the following.

Statement L1. Let $g : X \to (0, \infty)$ be any function such that for some $r > 0$,

$$\sup\{g(x) : x \in X, f(x) \leq \inf_{z \in X} f(z) + r\} < \infty.$$

let $T : X \to 2^X$ be a multivalued map such that for each $x \in X$, there exists $y \in T(x)$ satisfying
$$p(x, y) \leq g(x)(f(x) - f(y)).$$
Then T has a fixed point $x_0 \in X$ such that $p(x_0, x_0) = 0$.

Statement L2. Let $T : X \to 2^X$ be a multivalued map such that for each $x \in X$, there exists $y \in T(x)$ satisfying

$$p(x, y) \leq \max\{c(f(x), c(f(y)\}(f(x) - f(y)),$$

where $c : [0, \infty) \to (0, \infty)$ is an upper semicontinuous function from the right. Then T has a fixed point $x_0 \in X$ such that $p(x_0, x_0) = 0$.

Statement L3. Let $T : X \to 2^X$ be a multivalued map such that for each $x \in X$, there exists $y \in T(x)$ satisfying

$$p(x, y) \leq c(f(x))(f(x) - f(y)),$$

where $c : [0, \infty) \to (0, \infty)$ is a nondecreasing function. Then T has a fixed point $x_0 \in X$ such that $p(x_0, x_0) = 0$.

Statement L4. Let $T : X \to 2^X$ be a multivalued map such that for each $x \in X$, there exists $y \in T(x)$ satisfying

$$p(x, y) \leq c(f(y))(f(x) - f(y)),$$

where $c : [0, \infty) \to (0, \infty)$ is a nondecreasing function. Then T has a fixed point $x_0 \in X$ such that $p(x_0, x_0) = 0$.

Statement L5. Let $T : X \to 2^X$ be a multivalued map such that for each $x \in X$, there exists $y \in T(x)$ satisfying $p(x, y) \leq f(x)$ and

$$p(x, y) \leq \eta(p(x, y))(f(x) - f(y)),$$

where $\eta : [0, \infty) \to (0, \infty)$ is an upper semicontinuous function. Then T has a fixed point $x_0 \in X$ such that $p(x_0, x_0) = 0$.

Statement L6. Let $\phi : [0, \infty) \to [0, \infty)$ be a lower semicontinuous function such that
$$\limsup_{t \to 0+} \frac{t}{\phi(t)} < \infty.$$
Let $T : X \to 2^X$ be a multivalued map such that for each $x \in X$, there exists $y \in T(x)$ satisfying $p(x, y) \leq f(x)$ and

$$\phi(p(x, y)) \leq f(x) - f(y),$$

Then T has a fixed point $x_0 \in X$ such that $p(x_0, x_0) = 0$.

The following results were presented by Latif in [9].

Theorem L1. [9, Theorem 2.1].

$$\text{Statement COM} \Longrightarrow \text{Statement L1.}$$

Theorem L2. [9, Theorem 2.2].

$$\text{Statement COM} \Longrightarrow \text{Statement L2.}$$

Theorem L3. [9, Theorem 2.3].

$$\text{Statement COM} \Longrightarrow \text{Statement L3.}$$

Theorem L4. [9, Corollary 2.4].

$$\text{Statement COM} \Longrightarrow \text{Statement L4.}$$

Theorem L5. [9, Theorem 2.5].

$$\text{Statement COM} \Longrightarrow \text{Statement L5.}$$

Theorem L6. [9, Corollary 2.6].

$$\text{Statement COM} \Longrightarrow \text{Statement L6.}$$

In [9], Latif had shown the following.

- Statement GCS $\Rightarrow$ Statement L1;
- Statement L1 $\Rightarrow$ Statement L2;
- Statement L1 $\Rightarrow$ Statement L3 $\Rightarrow$ Statement L4;
- Statement L1 $\Rightarrow$ Statement L3 $\Rightarrow$ Statement L5 $\Rightarrow$ Statement L6.

It is obvious that each of Statements L1-L6 implies Statement GCM, so we can obtain the following very important result.

Theorem D2. Statement GE, Statement GCCM, Statement GCM, Statement GCCS, Statement GCS and Statements L1-L6 are equivalent.

Remark 2.

(a) By the same argument as above, we can prove that Latif's fixed point statements for other weak distances (τ-distances [13, 14] or τ-functions [10, 12] and so on) are equivalent with some weak distance variants of Caristi's fixed point statement, Ekeland's variational principle statement and Takahashi's nonconvex minimization statement.

(b) Some authors studied weak distances in quasi-metric space (X, d) (that is, d is not necessarily symmetric) and gave some generalizations of Caristi's fixed point theorem, Ekeland's variational principle and Takahashi's nonconvex minimization theorem by using their weak distances. But it is worth to mention that they may first define a rational condition for a Cauchy sequence and a convergent sequence to avoid making some problems. For example, does $x_n \to x$ mean $d(x_n, x) \to 0$ or $d(x, x_n) \to 0$ as $n \to \infty$?

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