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ON SOME INEQUALITY FOR THE LANDAU CONSTANTS

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Abstract. We improve several results recently established by Dejun Zhao for the Landau's constants.

1. Introduction

The constants of Landau are defined for all integer $n \geq 0$ by

$$G_n = \sum_{k=0}^{n} \frac{1}{16^k} \binom{2k}{k}^2.$$

They occur and play an important role in the complex analysis.

There is a rich literature concerning the Landau's constants. H. Alzer, [2], proved the following inequalities for the constants G_n in terms of the ψ function:

$$c_0 + \frac{1}{\pi}\psi(n+\alpha) < G_n < c_0 + \frac{1}{\pi}\psi(n+\beta), \tag{1}$$

where $c_0 = \dfrac{1}{\pi}(\gamma + 4\ln 2)$, $\alpha = \dfrac{5}{4}$, $\beta = \psi^{-1}\big(\pi(1-c_0)\big)$ and γ is the Euler constant.

D. Zhao [10] has observed that (1) do not imply the Watson asymptotic formula (see [8]):

$$G_n = \frac{1}{\pi}\ln(n+1) + c_0 - \frac{1}{4\pi(n+1)} + O\big(\frac{1}{n^2}\big), \tag{2}$$

when $n \to \infty$ and it was proved (see [10]) some sharp inequalities:

$$\frac{1}{\pi}\ln(n+1) + c_0 - \frac{1}{4\pi(n+1)} + \frac{5}{192\pi(n+1)^2} < G_n$$

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$$< \frac{1}{\pi}\ln(n+1) + c_0 - \frac{1}{4\pi(n+1)} + \frac{5}{192\pi(n+1)^2} + \frac{3}{128\pi(n+1)^3}, \tag{3}$$

thereby it follows the Watson asymptotic formula.

In [7] an improvement for the left-hand inequality of Zhao is given

$$\frac{1}{\pi}\ln(n+1)+c_0-\frac{1}{4\pi(n+1)}+\frac{5}{192\pi(n+1)^2}+\frac{17}{256\pi}\Big(\zeta(4)-\sum_{k=1}^{n+1}\frac{1}{k^4}\Big) < G_n \tag{4}$$

where

$$\zeta(s) = \sum_{k=1}^{\infty}\frac{1}{k^s}, \quad s \geq 1,$$

is the Riemann zeta function.

In this paper we will establish an improvement for the right-hand inequality of (3) in terms of the Riemann zeta function.

Theorem 1. *We have for all integers* $n \geq 1$

$$G_n < \frac{1}{\pi}\ln(n+1) + c_0 - \frac{1}{4\pi(n+1)} + \frac{5}{192\pi(n+1)^2}$$

$$+\frac{9}{128\pi}\Big(\zeta(4)-\sum_{k=1}^{n}\frac{1}{k^4}\Big) - \frac{2263}{61440\pi(n+1)^4}. \tag{5}$$

Proof. We denote $p_n = \dfrac{(2n)!}{4^n(n!)^2}$ and we have (see [9])

$$\frac{1}{\sqrt{\pi n\left(1+\dfrac{1}{4n\left(1-\dfrac{1}{8n+3}\right)}\right)}} < p_n$$

for all integers $n \geq 1$, thence

$$\frac{32n+8}{32n^2+16n+3} < \pi p_n^2.$$

But

$$G_n - G_{n-1} = p_n^2$$

and therefore

$$\pi(G_n - G_{n-1}) > \frac{32n+8}{32n^2+16n+3}. \tag{6}$$

Let be now

$$x_n = G_n - \frac{1}{\pi}\ln(n+1) - c_0 + \frac{1}{4\pi(n+1)} - \frac{5}{192\pi(n+1)^2} - \frac{A}{\pi}\Big(\zeta(4) - \sum_{k=1}^{n}\frac{1}{k^4}\Big) + \frac{B}{\pi(n+1)^4}$$

with undetermined constants A and B.

By using the inequality (6), we obtain

$$\begin{aligned}\pi(x_n - x_{n+1}) > {} & \frac{32n+8}{32n^2+16n+3} - \ln\Big(1+\frac{1}{n}\Big) + \frac{1}{4}\Big(\frac{1}{n+1} - \frac{1}{n}\Big)\\ & - \frac{5}{192}\Big(\frac{1}{(n+1)^2} - \frac{1}{n^2}\Big) + \frac{A}{n^4} + B\Big(\frac{1}{(n+1)^4} - \frac{1}{n^4}\Big).\end{aligned} \tag{7}$$

Next we consider the function $f(x)$, $x \geq 1$, defined by

$$\begin{aligned}f(x) = {} & \frac{32x+8}{32x^2+16x+3} - \ln\Big(1+\frac{1}{x}\Big) + \frac{1}{4}\Big(\frac{1}{x+1} - \frac{1}{x}\Big)\\ & - \frac{5}{192}\Big(\frac{1}{(x+1)^2} - \frac{1}{x^2}\Big) + \frac{A}{x^4} + B\Big(\frac{1}{(x+1)^4} - \frac{1}{x^4}\Big).\end{aligned}$$

One has

$$\begin{aligned}f'(x) = {} & \frac{32^2x^2+512x+32}{(32x^2+16x+3)^2} + \frac{1}{x(x+1)} + \frac{1}{4}\Big(\frac{1}{x^2} - \frac{1}{(x+1)^2}\Big)\\ & - \frac{5}{192}\Big(\frac{2}{x^3} - \frac{2}{(x+1)^3}\Big) - \frac{4A}{x^5} + B\Big(\frac{4}{x^5} - \frac{4}{(x+1)^5}\Big)\\ = {} & \frac{F(x)}{96x^5(x+1)^5(32x^2+16x+3)^2},\end{aligned}$$

where

$$\begin{aligned}F(x) = {} & 4\cdot 32^2\Big(-96A + \frac{27}{4}\Big)x^9 + 32(61440B - 2263)x^8\\ & + (130512 + 4\cdot 96\cdot 32^2\cdot 15B - 4\cdot 96\cdot 32\cdot 494A)x^7\\ & + (91201 + 4\cdot 96\cdot 22720B - 4\cdot 96\cdot 22816A)x^6\\ & + (32163 + 4\cdot 96\cdot 20320B - 4\cdot 96\cdot 20329A)x^5\\ & + \big[6502 + 32\cdot 139548(B-A)\big]x^4 + \Big[\frac{1857}{2} + 4\cdot 96\cdot 4314(B-A)\Big]x^3\\ & + \Big[\frac{153}{2} + 4\cdot 96\cdot 1018(B-A)\Big]x^2\\ & + 54144(B-A)x + 3456(B-A).\end{aligned}$$

For $A = \dfrac{9}{128}$ and $B = \dfrac{2263}{61440}$, we obtain a polynomial $F(x)$ with negative coefficients and $\deg F(x) = 7$.

Thus $f'(x) < 0$ for $x \geq 1$ and $\lim\limits_{x\to\infty} f(x) = 0$.

It implies $f(x) \geq 0$ for any $x \geq 1$. Consequently, we conclude that $x_n \underset{n}{\longrightarrow} 0$ such that $x_n < 0$, which implies the inequality from statement. ∎

Remark 1. The inequality (5) is an improvement of the right-hand inequality of (3) because one has the following result:

Corollary 1. *For any* $n \geq 16$, *we have*

$$\frac{9}{128\pi}\left(\zeta(4) - \sum_{k=1}^{n} \frac{1}{k^4}\right) - \frac{2263}{61440\pi(n+1)^4} < \frac{3}{128\pi(n+1)^3}. \tag{8}$$

Proof. Let

$$x_n = \zeta(4) - \sum_{k=1}^{n} \frac{1}{k^4} - \frac{2263}{4320(n+1)^4} - \frac{1}{3(n+1)^3}$$

be a sequence for which $\lim\limits_{n} x_n = 0$.

We have

$$\begin{aligned} x_n - x_{n-1} &= -\frac{1}{n^4} - \frac{2263}{4320}\Big(\frac{1}{(n+1)^4} - \frac{1}{n^4}\Big) - \frac{1}{3}\Big(\frac{1}{(n+1)^3} - \frac{1}{n^3}\Big) \\ &= \frac{412n^3 - 6582n^2 - 6788n - 2057}{4320n^4(n+1)^4}. \end{aligned}$$

For $n \geq 17$, one has

$$412n^3 - 6582n^2 - 6788n - 2057 > 0$$

and hence $x_n \to 0$, $x_n < 0$ for $n \geq 16$. ∎

Remark 2. The constants of Lebesgue are defined by

$$L_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|\frac{\sin\left(nt + \frac{1}{2}t\right)}{\sin\left(\frac{1}{2}t\right)}\right| dt$$

and the connection with the Landau constants comes from the following inequalities (see [10]):

$$\begin{aligned} L_{n/2} - G_n < {} & \frac{4-\pi}{\pi^2}\ln(n+1) + (c_1 - c_0) \\ & + \frac{1}{4\pi(n+1)} + \frac{e_1}{(n+1)^2} - \frac{d_1}{(n+1)^4} + \frac{d_2}{(n+1)^6}, \end{aligned} \tag{9}$$

$$
\begin{aligned}
L_{n/2} - G_n > \frac{4-\pi}{\pi^2}\ln(n+1) + (c_1 - c_0) \\
+\frac{1}{4\pi(n+1)} + \frac{e_1}{(n+1)^2} - \frac{e_2}{(n+1)^3} - \frac{d_1}{(n+1)^4},
\end{aligned} \tag{10}
$$

where

$$
\begin{aligned}
c_1 &= \frac{8}{\pi^2}\sum_{k=1}^{\infty}\frac{\ln k}{4k^2-1} + \frac{4}{\pi^2}(\gamma + 2\ln 2),\\
e_1 &= \frac{2}{3\pi^2} - \frac{5}{192\pi} - \frac{1}{18}, \quad e_2 = \frac{3}{128\pi},\\
d_1 &= \frac{7}{120\pi^2}\Big(8 - \frac{2\pi^2}{3} - \frac{\pi^4}{90}\Big), \quad d_2 = \frac{1}{16\pi^2}\Big(32 - \frac{8\pi^2}{3} - \frac{2\pi^4}{45} - \frac{\pi^6}{945}\Big).
\end{aligned}
$$

By using (4) and (5) and the following results of [10]

$$
\begin{aligned}
\frac{4}{\pi^2}\ln(n+1) + c_1 + \frac{d_0}{(n+1)^2} - \frac{d_1}{(n+1)^4} < L_{n/2} \\
< \frac{4}{\pi^2}\ln(n+1) + c_1 + \frac{d_0}{(n+1)^2} - \frac{d_1}{(n+1)^4} + \frac{d_2}{(n+1)^6}
\end{aligned} \tag{11}
$$

we obtain:

Corollary 2. *We have, for all integers* $n \geq 1$,

$$
\begin{aligned}
L_{n/2}-G_n < \frac{4-\pi}{\pi^2}\ln(n+1)+(c_1-c_0)+\frac{1}{4\pi(n+1)}+\frac{e_1}{(n+1)^2}-\frac{d_1}{(n+1)^4}\\
+\frac{d_2}{(n+1)^6} - \frac{17}{256\pi}\Big(\zeta(4) - \sum_{k=1}^{n+1}\frac{1}{k^4}\Big).
\end{aligned} \tag{12}
$$

$$
\begin{aligned}
L_{n/2} - G_n > \frac{4-\pi}{\pi^2}\ln(n+1) + (c_1 - c_0) + \frac{1}{4\pi(n+1)} + \frac{e_1}{(n+1)^2}\\
-\frac{9}{128\pi}\Big(\zeta(4) - \sum_{k=1}^{n+1}\frac{1}{k^4}\Big) + \Big(\frac{2263}{61440\pi} - d_1\Big)\frac{1}{(n+1)^4}.
\end{aligned} \tag{13}
$$

By means of Corollary 1, the above result is an improvement of (9) and (10).

REFERENCES

1. H. Alzer, On some inequalities for the Gamma and Psi function, *Math. Comput.*, **66** (1997), 373-389.
2. H. Alzer, Inequalities for the constants of Landau and Lebesgue, *J. Comput. Appl. Math.*, **139** (2002), 215-230.

3. L. Brutman, A sharp estimate of the Landau constants, *J. Approx. Theory*, **34** (1992), 217-220.

4. D. Cvijović and J. Klinowski, Inequalities for the Landau constants, *Math. Slovaca*, **50** (2000), 159-164.

5. D. Cvijović and H. M. Srivastava, Asymptotics of the Landau constants and their relationships with hypergeometric functions, *Taiwanese J. Math.*, **13** (2009), 855-870.

6. L. P. Falaleev, Inequalities for the Landau constants, *Siberian Math. J.*, **32** (1991), 896-897.

7. E. C. Popa, Note on the constants of Landau, *Gen. Math.*, **18(1)** (2010), 119-122.

8. G. N. Watson, The constants of Landau and Lebesgue, *Quart. J. Math. Oxford, Ser (2)*, **1** (1930), 310-318.

9. G. M. Zhang, The upper bounds and lower bounds on Wallis's inequality, *Math. Practice Theory*, **37(5)** (2007), 111-116, (in China).

10. D. Zhao, Some sharp estimates of the constants of Landau and Lebesgue, *J. Math. Anal. Appl.*, **349** (2009), 68-73.

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