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EXTENSION OF ISOMETRIES ON UNIT SPHERE OF L^∞

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Abstract. We prove that every surjective isometry between unit spheres of $L^\infty(\Sigma, \Omega, \mu)$ and a Banach space F can be linearly and isometrically extended to the whole space, which means that if the unit sphere of a Banach space F is isometric to the unit sphere of $L^\infty(\Sigma, \Omega, \mu)$, then F is linearly isometric to $L^\infty(\Sigma, \Omega, \mu)$.

1. Introduction

Let (Ω, Σ, μ) be a σ-finite measure space. By $L^\infty(\Omega, \Sigma, \mu)$ we denote the space of all measurable essentially bounded functions f with the essential supremum norm

$$\|f\| = \text{ess. } \sup_{t\in\Omega}|f(t)|.$$

Throughout this paper, we shall denote $L^\infty(\Sigma, \Omega, \mu)$ by L^∞. As usual, for any Banach space F, its unit sphere is denoted by $S(F)$.

The classical Mazur-Ulam theorem [13] stated that any surjective isometry V between two real normed spaces with $V(0) = 0$ must be linear. Mankiewicz [12] extended this result by showing that every surjective isometry between open connected subsets of two normed spaces E and F can be extended to an affine isometry from E onto F. These two results demonstrate that the linear structure of a normed space is completely determined by its unit ball as a metric space. A very natural set which one feels may determine the space is the unit sphere. In 1987, Tingley raised the following problem in [15]:

Problem. *Let E and F be normed spaces with unit spheres $S(E)$ and $S(F)$ respectively. Suppose that $V : S(E) \to S(F)$ is a surjective isometry. Is there a linear isometry $\widetilde{V} : E \to F$ such that $\widetilde{V}|_{S(E)} = V$.*

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It is called isometric extension problem or Tingley's problem. To this problem, we always consider the real case. It is because the answer is obviously negative in the complex case. For example, $E = F = \mathbb{C}$ and $V(x) = \bar{x}$ for all $x \in \mathbb{C}$ with $|x| = 1$. Since there is no linear or even metrically convex structure on unit spheres, it is hard to answer this problem. So far, it is still open in the general case.

During the past decade, Ding and his students have been working on this topic (see [7] for its history) and have obtained many important results. For the surjective isometries between unit spheres of classical Banach spaces Tingley's problem has been almost solved in the positive way (see [1–6, 8–11, 14, 16–17]).

Recently, for every isometry V between the unit spheres of L^∞ and a Banach space F, Li and Ren [9] gave some sufficient conditions as follows:

(i) For any x_1, x_2 in $S(L^\infty)$ and real numbers λ_1, λ_2 in $\mathbb{R}$, $\|\lambda_1 V(x_1)+\lambda_2 V(x_2)\| = 1$ implies that $\lambda_1 V(x_1) + \lambda_2 V(x_2) \in V[S(L^\infty)]$.

(ii) For any mutual disjoint subsets $\{A_1, \cdots A_n\}$ of Ω with $\mu(A_k) > 0 (1 \leq k \leq n)$, x in $S(L^\infty)$ and real numbers $\{\lambda_1, \cdots, \lambda_n\}$, $V(x) = \sum_{k=1}^n \lambda_k V(\chi_{A_k})$ implies that there exist $x_0 \in S(L^\infty)$ and real numbers $\{\lambda'_1, \cdots, \lambda'_n\}$ such that $x = \sum_{k=1}^n \lambda'_k \chi_{A_k} + x_0$ where supp.$x_0 \subset (\cup_{k=1}^n A_k)^c$.

(ii)$'$ For any disjoint elements x_1 and x_2 in $S(L^\infty)$, we have

$$\text{dim.span} V[S(\text{span}\{x_1, x_2\})] = 2.$$

Li and Ren [9] proved that V satisfying (i) (ii) or (i) (ii)' can be linearly extended to the whole space. These conditions are similar to those given by Ding [5] for $\mathcal{L}^\infty(\Gamma)$–type spaces. It is easy to see that if V is surjective, then (i) is satisfied. However, we shall point out that in fact the conditions (ii) and (ii)' can be removed. Although this can be inferred from one of the main results of Liu [11] who established that for every bijective-ϵ-isometry T between unit spheres of two Banach spaces E, F, if E has property (m), then T can be extended to a bijective 5ϵ-isometry between their closed unit balls (it is also shown in [11] by the knowledge of vector lattices that L^∞ has property (m)), the proof here is a direct and simple method which is quite distinct from that of [11]. We feel it may be worth noting in the literature.

2. Main Results

The following lemma has been proved in [5]. We give its proof just for completeness.

Lemma 2.1. (See [5, Lemma 2]). *Let Y be a normed space, and let $\{y_i\}_{i=1}^n$ be a sequence in the unit sphere $S(Y)$. If for all signs $\theta_k = \pm 1$ $(1 \leq k \leq n)$ and for every $1 \leq m \leq n$,*

$$\|\theta_1 y_1 + \cdots + \theta_m y_m\| = 1, \tag{2.1}$$

then for all $\{\lambda_k\}_{k=1}^n \subset \mathbb{R}$*,*

$$\|\lambda_1 y_1 + \cdots + \lambda_n y_n\| = \max\{|\lambda_k| : 1 \leq k \leq n\}.$$

Proof. We may assume that $\max\{|\lambda_i| : 1 \leq i \leq n\} = |\lambda_1|$. By the assumption (2.1) and the Hahn-Banach theorem, there exists a functional $f \in Y^*$ with $\|f\| = 1$ such that

$$f(y_1) = 1, f(y_k) = 0 \quad \text{for } 2 \leq k \leq n.$$

Thus we have $\|\lambda_1 y_1 + \cdots + \lambda_n y_n\| \geq |f(\sum_{i=1}^n \lambda_i y_i)| = |\lambda_1|$. On the other hand, notice that every normed space Y can be embedded linearly and isometrically into a $C(\Omega)$ space with Ω being a compact subset of the unit ball of Y^*. Thus we can consider Y as a linear subspace of $C(\Omega)$. Then by (2.1),

$$\sum_{i=1}^n |y_i(t)| \leq 1 \quad \forall\, t \in \Omega.$$

Therefore,

$$\left| \left(\sum_{i=1}^n \lambda_i y_i \right)(t) \right| \leq \sum_{i=1}^n |\lambda_i y_i(t)| \leq |\lambda_1|, \quad \forall\, t \in \Omega,$$

which leads to

$$\left\| \sum_{i=1}^n \lambda_i y_i \right\| \leq |\lambda_1|, \quad \forall\, t \in \Omega.$$

Thus the proof is complete. ■

Lemma 2.2. *(See [10, Corollary 1] or [8, Corollary 2.2]). Let* E*,* F *be Banach spaces, and let* $V : S(E) \to S(F)$ *be a surjective isometry. Then for all* $x, y \in S(E)$*,*

$$\|V(x) + V(y)\| = 2$$

if and only if $\|x + y\| = 2$*.*

To derive our main result, we need a simple basic fact in L^∞ described as follows.

Lemma 2.3. *Let* f *be in* L^∞*, and let* $r > 0$*. For every* $A \in \Sigma$*, if* $\mu(\{t \in A, |f(t)| < r\}) > 0$*, then there is an* $n_0 \in \mathbb{N}$ *such that* $\mu(\{t \in A, |f(t)| < r - 1/n_0\}) > 0$*.*

The following lemma can also be seen in [9]. Our proof here simplifies the original one.

Lemma 2.4. *Let F be a Banach space. Suppose that $V : S(L^\infty) \to S(F)$ be a surjective isometry. Then for every $A \in \Sigma$ with $\mu(A) > 0$,*

$$V(-\chi_A) = -V(\chi_A),$$

where χ_A is the characteristic function of the set A.

Proof. By the hypothesis on V, for every $A \in \Sigma$ with $\mu(A) > 0$ there is an $f \in S(L^\infty)$ such that $V(f) = -V(\chi_A)$. Then we assert that $|f(t)| = 1$ a.e on A. Indeed if this is not true, then there is a measurable subset $C_0 \subset A$ with $\mu(C_0) > 0$ such that $|f(t)| < 1$ for every $t \in C_0$. By Lemma 2.3 we may find an $n_0 \geq 1$ and a subset C_1 of C_0 with $\mu(C_1) > 0$ such that $|f(t)| < 1 - 1/n_0$ for every $t \in C_1$. Thus by Lemma 2.2,

$$2 - \frac{1}{n_0} \geq \|f - \chi_{C_1}\| = \| - V(\chi_A) - V(\chi_{C_1})\| = \|\chi_A + \chi_{C_1}\| = 2.$$

A contradiction thus proves the assertion. Moreover, for every $A_0 CA$ with $\mu(A_0) > 0$ by $\|f - \chi_{A_0}\| = \|V(f) - V(\chi_{A_0})\| = 2$ we see in fact that $f(t) = -1$ a.e on A.

Now for every measurable $B \subset \Omega \setminus A$ with $\mu(B) > 0$, we may also find $f_1, f_2 \in S(L^\infty)$ such that $V(f_1) = -V(\chi_B)$ and $V(f_2) = -V(-\chi_B)$. Analogous to the above argument we see that

$$f_1(t) = -1 \quad \text{and} \quad f_2(t) = 1 \quad \text{a.e on } B.$$

Thus the equations

$$\|f - f_i\| = \|V(f) - V(f_i)\| = 1 \text{ for } i = 1, 2$$

allow us to conclude that $f(t) = 0$ a.e on B. This finally yields $f = -\chi_A$ and completes the proof. ■

Theorem 2.5. *Let F be a Banach space. Then every surjective isometry V from $S(L^\infty)$ onto $S(F)$ can be extended to be a linear isometry on the whole space L^∞.*

Proof. For any disjoint members $A_1, A_2 \in \Sigma$ with $\mu(A_i) > 0\,(i = 1, 2)$ and $a_1, a_2 \in \mathbb{R} \setminus \{0\}$ with $\max\{|a_i| : i = 1, 2\} = 1$, note first from Lemma 2.4 that $\|V(\chi_{A_1}) \pm V(\chi_{A_2})\| = 1$. Thus we obtain from Lemma 2.1 that

$$\|a_1 V(\chi_{A_1}) + a_2 V(\chi_{A_2})\| = 1.$$

This guarantees that there is an $f \in S(L^\infty)$ such that

$$V(f) = a_1 V(\chi_{A_1}) + a_2 V(\chi_{A_2}).$$

For each $i \in \{1, 2\}$, we apply Lemma 2.1 and Lemma 2.4 again to obtain that

$$\|f + \text{sign}(a_i)\chi_{A_i}\| = \|V(f) + \text{sign}(a_i)V(\chi_{A_i})\| = 1 + |a_i|$$

and

$$\|f - \text{sign}(a_i)\chi_{A_i}\| = \|V(f) - \text{sign}(a_i)V(\chi_{A_i})\| = \max_{j \neq i}\{1 - |a_i|, |a_j|\} \leq 1.$$

Hence we conclude that

$$\text{sign}(f(t)) \cdot \text{sign}(a_i) \geq 0 \quad a.e. \text{ on } A_i. \tag{2.2}$$

and

$$|f(t)| \leq |a_i| \qquad a.e. \text{ on } A_i. \tag{2.3}$$

Now for every measurable subset $B \subset \Omega \setminus (A_1 \cup A_2)$ with $\mu(B) > 0$, we can also find a $g \in S(L^\infty)$ such that

$$V(g) = a_2 V(\chi_{A_2}) + V(\chi_B).$$

An observation that $\|g + \chi_B\| = \|V(g) + V(\chi_B)\| = 2$ ensures the existence of a measurable set $B_0 \subset B$ with $\mu(B_0) > 0$ such that

$$\text{ess. sup}_{t \in B_0} |g(t)| = 1 \quad \text{and} \quad g(t) \geq 0 \;\; a.e. \text{ on } B_0.$$

Then we can deduce from this and $\|f - g\| = \|V(f) - V(g)\| = 1$ that there is a measurable subset B_1 of B_0 with $\mu(B_1) > 0$ such that

$$f(t) \geq 0 \quad \text{for every } t \in B_1.$$

And much more, since B is arbitrary, we see in fact that

$$f(t) \geq 0 \quad a.e. \text{ on } \quad \Omega \setminus (A_1 \cup A_2).$$

Considering the element $a_2 V(\chi_{A_2}) - V(\chi_B)$ in the same way, we can obtain that

$$f(t) \leq 0 \quad a.e. \quad \text{on} \quad \Omega \setminus (A_1 \cup A_2).$$

The two possibilities therefore yield

$$f(t) = 0 \quad a.e. \quad \text{on} \quad \Omega \setminus (A_1 \cup A_2). \tag{2.4}$$

For every measurable set $A_1^0 \subset A_1$ with $\mu(A_1^0) > 0$, let $f_0 \in S(L^\infty)$ satisfy $V(f_0) = -\text{sign}(a_1)V(\chi_{A_1^0}) + a_2V(\chi_{A_2})$. Then similar to the previous argument, we know that

$$\text{sign}(f_0(t)) \cdot \text{sign}(a_2) \geq 0 \quad a.e. \quad \text{on } A_2 \tag{2.5}$$

and

$$f_0(t) = 0 \quad a.e. \quad \text{on} \quad \Omega \setminus (A_0 \cup A_2). \tag{2.6}$$

From (2.2) (2.5), (2.4) and (2.6), it is easily verified that

$$\begin{aligned} \|f - f_0\| = \max\{&\text{ess.sup}_{t\in A_1^0}|f(t) - f_0(t)|, \\ &\text{ess.sup}_{t\in A_2}|f(t) - f_0(t)|, \text{ess.sup}_{t\in A_1\setminus A_1^0}|f(t)|\} \\ \leq \max\{&\text{ess.sup}_{t\in A_1^0}|f(t) - f_0(t)|, 1\}. \end{aligned}$$

The fact that V is an isometry therefore implies that

$$\begin{aligned} \max\{\text{ess.sup}_{t\in A_1^0}|f(t) - f_0(t)|, 1\} &\geq \||a_1|V(\chi_{A_1}) + V(\chi_{A_1^0})\| \\ &\geq \|V(\chi_{A_1}) + V(\chi_{A_1^0})\| - (1 - |a_1|) \\ &= 2 - (1 - |a_1|) = 1 + |a_1|. \end{aligned}$$

Noticing $f_0(t) \leq 1$ a.e. and inequality (2.3) we see that ess. $\sup_{t\in A_1^0}|f(t)| = |a_1|$. Inferring from relation (2.2) and using the arbitrariness of A_1^0 and Lemma 2.3 again we are sure that

$$f(t) = a_1 \quad a.e. \quad \text{on } A_1.$$

Similarly we can obtain that $f(t) = a_2$ $a.e.$ on A_2. To sum up we have established that

$$V(a_1\chi_{A_1} + a_2\chi_{A_2}) = a_1V(\chi_{A_1}) + a_2V(\chi_{A_2}).$$

With this in hand we are able to apply induction to prove that

$$V\left(\sum_{i=1}^{n} \lambda_i\chi_{B_i}\right) = \sum_{i=1}^{n} \lambda_i V(\chi_{B_i}) \tag{2.7}$$

for any finite sequence $\{B_1, \cdots, B_n\}$ of mutual disjoint members of Σ with $\mu(B_i) > 0$ and $\{\lambda_1 \cdots \lambda_n\} \subset \mathbb{R}$ with $\max\{|\lambda_i| : 1 \leq i \leq n\} = 1$. Indeed, assume that equation (2.7) holds for $n \leq k - 1$. Then for all signs $\theta_i = \pm 1$ $(1 \leq i \leq k)$ and $1 \leq m \leq k$,

$$\|\theta_1 V(\chi_{B_1}) + \cdots + \theta_m V(\chi_{B_m})\| = \left\| V\left(\sum_{i=1}^{m-1} \theta_i \chi_{B_i}\right) + \theta_m V(\chi_{B_m})\right\| = 1.$$

It follows from this and Lemma 2.1 that

$$\|\lambda_1 V(\chi_{B_1}) + \cdots + \lambda_k V(\chi_{B_k})\| = 1.$$

Take $h \in S(L^\infty)$ such that

$$V(h) = \sum_{i=1}^{k} \lambda_i V(\chi_{B_i}).$$

Then proceeding in a similar manner as above case where $n = 2$ implies that $h = \sum_{i=1}^{k} \lambda_i \chi_{B_i}$. This finishes the proof of equation (2.7).

Now the required extension mapping $\widetilde{V} : L^\infty \to F$ is defined by

$$\widetilde{V}(f) = \begin{cases} \|f\| V(\frac{f}{\|f\|}) & \text{if } f \neq 0, \\ 0 & \text{if } f = 0. \end{cases}$$

Then by (2.7) and its definition, $\widetilde{V}$ is a linear isometry on the subspace X consisting of all simple functions of L^∞. Since X is dense in L^∞, it follows that $\widetilde{V}$ must be a linear isometry on the whole space L^∞, and its restriction to the unit sphere is just V. The proof is complete. ∎

The following conclusion improves the main results in [9] by showing that the conditions (Δ_2) in [9, Theorem 3.1] and (Δ_3) in [9, Theorem 3.2] can be removed.

Corollary 2.6. *Let F be a Banach space. Then every isometry $V : S(L^\infty) \to S(F)$ can be extended to a linear isometry if and only if the following condition holds:* (Δ) *For any x_1, x_2 in $S(L^\infty)$ and real numbers λ_1, λ_2 in $\mathbb{R}$, $\|\lambda_1 V(x_1) + \lambda_2 V(x_2)\| = 1$ implies that $\lambda_1 V(x_1) + \lambda_2 V(x_2) \in V[S(L^\infty)]$.*

Proof. It is obvious that if V can be linearly extended, then condition (Δ) is satisfied. For the converse, note that condition (Δ) implies that there is a linear subspace F_0 of F such that its unit sphere $S(F_0)$ is just $V[S(L^\infty)]$. Indeed $F_0 = \bigcup_{r \geq 0} r \cdot V[S(L^\infty)]$, and it is clear that F_0 is a Banach space. Thus the desired conclusion follows immediately from Theorem 2.5. ∎

Remark 2.7. Given a nonempty index set Γ, recall that a normed space E is called an $\mathcal{L}^\infty(\Gamma)$–type space if it is a subspace of $\ell^\infty(\Gamma)$ such that $\{e_\gamma\}_{\gamma \in \Gamma} \subset E$. Such as, $c_{00}(\Gamma)$, $c_0(\Gamma)$, $\ell^\infty(\Gamma)$ (in particular, c_{00}, c_0, ℓ^∞) are all $\mathcal{L}^\infty(\Gamma)$–type spaces. It is easy to see from the proof of Theorem 2.5 that the statement of Corollary 2.6 remains valid if we replace L^∞ by $\mathcal{L}^\infty(\Gamma)$–type spaces. This generalizes the main result [5, Theorem 1] by dropping the assumption (ii) of Theorem 1.

Remark 2.8. An analogous result of Theorem 2.5 holds for $L^1(\mu)$ (see [6]). However, both cases do not extend to general "into" isometries with respect to "linear" extension. For example, define $V : S(\ell^\infty_{(2)}) \to S(\ell^\infty_{(3)})$ by

$$V(a_1, a_2) = (a_1, a_2, f(a_2)),$$

where $f(t) : \mathbb{R} \to \mathbb{R}$ is a non-linear Lipschitz map with $f(0) = 0$ whose Lipschitz-constant is ≤ 1. Then it is apparent that V is an isometry but cannot be "linearly" extended to $\ell^\infty_{(2)}$. For the $L^1(\mu)$-case, let $T : S(\ell^1_{(2)}) \to S(\ell^1_{(3)})$ be defined by

$$T(a_1, a_2) = \begin{cases} (1/8a_1, 7/8a_1, a_2) & \text{if } a_1 \geq 0, \\ (1/2a_1, 1/2a_1, a_2) & \text{otherwise.} \end{cases}$$

Then V is an isometry but $T(-1, 0) \neq -T(1, 0)$, and thus cannot be "linearly" extended.

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