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NOTES ON CARLITZ'S q-OPERATORS

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Abstract. In this paper, Carlitz's q-operator and the auxiliary ones are applied to prove q-Christoffel-Darboux formulas and some Carlitz type generating functions. In addition, the technique of exponential operator decomposition to deduce q-Mehler's formulas for Rogers-Szegö and Hahn polynomials are shown.

1. Introduction

One of the customary ways to define the Hermite polynomials is by the relation [14, p. 193]

$$H_n(x) = (-1)^n \exp\big(x^2\big) D^n \exp\big(-x^2\big), \quad D = \mathrm{d/dx}. \tag{1.1}$$

Burchnall [5] employed the operational formula

$$(D-2x)^n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} H_{n-k}(x) D^k \tag{1.2}$$

to prove the formula of Nielsen [21, p. 31]

$$H_{m+n}(x) = \sum_{k=0}^{\min\{m,n\}} (-2)^k \binom{m}{k} \binom{n}{k} k! H_{m-k}(x) H_{n-k}(x). \tag{1.3}$$

For more information about the classical Hermite polynomial and its operational formula, please refer to [1, 5, 9, 14, 15, 17, 20, 25].

The Rogers-Szegö polynomials [7, 23]

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$$h_n(x|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^k, \quad g_n(x|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{k(k-n)} x^k = h_n\big(x|q^{-1}\big), \tag{1.4}$$

which are in some respects the analogue of the Hermite polynomial (See [8]), are closely related to the continuous q-Hermite polynomials via [22]

$$H_n(\cos\theta|q) = e^{-in\theta} h_n\big(e^{2i\theta}|q\big). \tag{1.5}$$

The Hahn polynomials [3, 11, 24] are defined by

$$\begin{aligned} \phi_n^{(a)}(x|q) &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (a;q)_k x^k, \\ \psi_n^{(a)}(x|q) &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{k(k-n)} x^k (aq^{1-k};q)_k. \end{aligned} \tag{1.6}$$

Carlitz gave a clever q-analogue of Burchnall's method by defining the shifted operator $\mathbb{E}$ and $\triangle$ as [8, Eq. (4) and (5)]

$$\mathbb{E}^n f(x) = f(xq^n) \quad \text{and} \quad \triangle^n = (1-\mathbb{E})(q-\mathbb{E})\cdots(q^{n-1}-\mathbb{E}), \tag{1.7}$$

and obtained the following results

$$\mathbb{E}^n = \sum_{r=0}^{n} (-1)^r \begin{bmatrix} n \\ r \end{bmatrix} \triangle^r \quad \text{and} \quad (\mathbb{E}_x + x)^n = \sum_{r=0}^{n} (-1)^r \begin{bmatrix} n \\ r \end{bmatrix} h_{n-r}(x|q) \triangle^r, \tag{1.8}$$

by means of inverse series relations and the noncommutative q-analogue of binomial theorem (See Lemma 2.1 below).

Using mathematical induction, Carlitz obtained the general formula [8, Eq. (11)]

$$\triangle^r h_m(x|q) = \frac{(q;q)_m}{(q;q)_{m-r}} q^{\binom{r}{2}} x^r h_{m-r}(x|q), \tag{1.9}$$

and deduced the following linearization formulas for $h_n(x|q)$:

Proposition 1.1. ([7, Eq. (1.7) and (1.8)]). *For $m, n \in \mathbb{N}$, we have*

$$h_m(x|q) h_n(x|q) = \sum_{r=0}^{\min\{m,n\}} \begin{bmatrix} m \\ r \end{bmatrix} \begin{bmatrix} n \\ r \end{bmatrix} (q;q)_r x^r h_{m+n-2r}(x|q), \tag{1.10}$$

$$h_{m+n}(x|q) = \sum_{r=0}^{\min\{m,n\}} (-1)^r q^{\binom{r}{2}} \begin{bmatrix} m \\ r \end{bmatrix} \begin{bmatrix} n \\ r \end{bmatrix} (q;q)_r x^r h_{m-r}(x|q) h_{n-r}(x|q). \tag{1.11}$$

In this paper, we define the auxiliary operator of (1.8) as

$$\left(\mathbb{E}_x^{-1}+x\right)^n=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}q^{k(k-n)}x^{n-k}\mathbb{E}_x^{-k}, \tag{1.12}$$

and the inverse pairs

$$\mathbb{E}^{-k}=\sum_{r=0}^{k}\begin{bmatrix}k\\r\end{bmatrix}q^{r(r-k)}\delta^r \quad \text{and} \quad (-\delta)^k=\sum_{r=0}^{k}(-1)^r\begin{bmatrix}k\\r\end{bmatrix}q^{\binom{r}{2}-\binom{k}{2}}\mathbb{E}^{-r}, \tag{1.13}$$

then we obtain the following result and further deduce the linearization formulas for $g_n(x|q)$ (See Proposition 2.3 below).

Theorem 1.1. *For $r, m \in \mathbb{N}$, we have*

$$\delta^r g_m(x|q)=q^{-rm}\begin{bmatrix}m\\r\end{bmatrix}(q;q)_r x^r g_{m-r}(x|q). \tag{1.14}$$

In [8], Carlitz gave a clever proof of q-Mehler's formula for $h_n(x|q)$ (See Proposition 3.1 below) by relations among operators $\mathbb{E}_x$, $\mathbb{E}_y$ and $\mathbb{E}_t$.

In fact, we can deduce q-Mehler's formula for Rogers-Szegö polynomials by Carlitz's q-operators directly, the thought is decomposition, so the method may be called "exponential operator decomposition". See details in Sections 3 and 6.

The Christoffel-Darboux formula for Hermite polynomial reads that

Proposition 1.2. ([14, p. 193]).

$$\sum_{m=0}^{n}\frac{H_m(x)H_m(y)}{2^m m!}=\frac{H_{n+1}(x)H_n(y)-H_n(x)H_{n+1}(y)}{2^{n+1}n!(x-y)}, \quad n\in\mathbb{N}. \tag{1.15}$$

We give q-analogue of Christoffel-Darboux formula for Hermite polynomial as follows.

Theorem 1.2. *For $n \in \mathbb{N}$, we have*

$$\sum_{k=0}^{n}h_k(x/q|q)h_k(y|q)\frac{y^{n-k}q^k}{(q;q)_k}=\frac{h_{n+1}(x|q)h_n(y|q)-h_n(x|q)h_{n+1}(y|q)}{(x-y)(q;q)_n}. \tag{1.16}$$

Theorem 1.3. *For $n \in \mathbb{N}$, we have*

$$\begin{aligned}&\sum_{k=0}^{n}g_k(xq|q)g_k(y|q)\frac{(-y)^{n-k}q^{\binom{k}{2}-\binom{n+1}{2}}}{(q;q)_k}\\ &=\frac{g_{n+1}(x|q)g_n(y|q)-g_n(x|q)g_{n+1}(y|q)}{(x-y)(q;q)_n}.\end{aligned} \tag{1.17}$$

Corollary 1.1. *For $n \in \mathbb{N}$, we have*

$$\sum_{k=0}^{n} h_k(x/q|q)h_k(y|q)\frac{y^{n-k}q^k}{(q;q)_k} = \sum_{k=0}^{n} h_k(y/q|q)h_k(x|q)\frac{x^{n-k}q^k}{(q;q)_k}. \tag{1.18}$$

Corollary 1.2. *For $n \in \mathbb{N}$, we have*

$$\sum_{k=0}^{n} g_k(xq|q)g_k(y|q)\frac{(-y)^{n-k}q^{\binom{k}{2}}}{(q;q)_k} = \sum_{k=0}^{n} g_k(yq|q)g_k(x|q)\frac{(-x)^{n-k}q^{\binom{k}{2}}}{(q;q)_k}. \tag{1.19}$$

The structure of this paper is organized as follows. In Section 2, we prove Theorem 1.1 and the linearization formulas for $g_n(x|q)$. In Section 3, we show how to deduce Mehler's formula for Rogers-Szegö polynomials by Carlitz's q-operators. In Section 4, we give a new proof of Carlitz type Mehler's formulas for Rogers-Szegö polynomials and deduce Theorems 1.2 and 1.3. In Section 5, we deduce Mehler's formula for Hahn polynomials by Carlitz's q-operators. In Section 6, we give some results related to Carlitz's q-operators.

2. Notations and Proof of Theorem 1.1

In this paper, we follow the notations and terminology in [16] and suppose that $0 < q < 1$. The q-shifted and its compact factorials are defined by

$$(a;q)_0 = 1, \quad (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k), \quad (a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k)$$

and $(a_1, a_2, \ldots, a_m; q)_n = (a_1;q)_n(a_2;q)_n \ldots (a_m;q)_n$, respectively, where n is an integer or ∞. The operator $\mathbb{E}$ acting on the variable x will be denoted by $\mathbb{E}_x$. LHS (or RHS) means the left (or right) hand side of certain equality, and $\mathbb{N} = \{0, 1, 2, \cdots\}$.

The basic hypergeometric series ${}_r\phi_s$ is given by

$${}_r\phi_s\left[\begin{matrix} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix}; q, z\right] = \sum_{n=0}^{\infty}\frac{(a_1, a_2, \ldots, a_r;q)_n}{(q, b_1, \ldots, b_s;q)_n} z^n \left[(-1)^n q^{\binom{n}{2}}\right]^{s+1-r}, \tag{2.1}$$

for convergence of the infinite series in (2.1), $|q| < 1$ and $|z| < \infty$ when $r \le s$, or $|q| < 1$ and $|z| < 1$ when $r = s+1$, provided that no zeros appear in the denominator.

The q-Chu-Vandermonde formula [16, Eq. (II.6) and (II.7)] reads that

$${}_2\phi_1\left[\begin{matrix} q^{-n}, b \\ c \end{matrix}; q, q\right] = \frac{(c/b;q)_n}{(c;q)_n} b^n \quad \text{and} \quad {}_2\phi_1\left[\begin{matrix} q^{-n}, b \\ c \end{matrix}; q, \frac{cq^n}{b}\right] = \frac{(c/b;q)_n}{(c;q)_n}. \tag{2.2}$$

The noncommutative q-analogue of binomial theorem states that

Lemma 2.1. ([16, p. 28] or [13, Lem. 2.2]). *Let A and B be two noncommutative identerminates satisfying $BA = qAB$, then we have*

$$(A+B)^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} A^k B^{n-k}. \tag{2.3}$$

The linearization formulas for $g_n(x|q)$ are

Proposition 2.3. ([7, Eq. (4.18) and (4.19)]). *For $m, n \in \mathbb{N}$, we have*

$$\begin{aligned} g_n(x|q)g_m(x|q) = & \sum_{r=0}^{\min\{m,n\}} \begin{bmatrix} m \\ r \end{bmatrix} \begin{bmatrix} n \\ r \end{bmatrix} (q;q)_r q^{\binom{r}{2}+r(r-m-n)} \\ & \times (-x)^r g_{m+n-2r}(x|q), \end{aligned} \tag{2.4}$$

$$g_{m+n}(x|q) = \sum_{r=0}^{\min\{m,n\}} q^{r(r-m-n)} \begin{bmatrix} m \\ r \end{bmatrix} \begin{bmatrix} n \\ r \end{bmatrix} (q;q)_r x^r g_{m-r}(x|q) g_{n-r}(x|q). \tag{2.5}$$

Proof of Theorem 1.1. In view of the fact that

$${}_2\phi_1 \left[\begin{matrix} q^{-n}, b \\ 0 \end{matrix}; q, q \right] = b^n, \tag{2.6}$$

then applying operator $\mathbb{E}_x^{-n}$ to the second formula in (1.4) and using Lemma 2.1 give

$$\begin{aligned} & \mathbb{E}_x^{-n} g_m(x|q) \\ & = \sum_{j=0}^{m} \begin{bmatrix} m \\ j \end{bmatrix} x^j q^{j(j-m-n)} \\ & = \sum_{j=0}^{m} \begin{bmatrix} m \\ j \end{bmatrix} x^j q^{j(j-m)} {}_2\phi_1 \left[\begin{matrix} q^{-n}, q^{-j} \\ 0 \end{matrix} ; q, q \right] \\ & = \sum_{j=0}^{m} \frac{x^j q^{j(j-m)} (q;q)_m}{(q;q)_{m-j}} \sum_{r=0}^{n} \begin{bmatrix} n \\ r \end{bmatrix} q^{r(r-n-j)} \frac{1}{(q;q)_{j-r}} \\ & = \sum_{r=0}^{n} \begin{bmatrix} n \\ r \end{bmatrix} q^{r(r-m-n)} \frac{(q;q)_m}{(q;q)_{m-r}} \sum_{j=r}^{m} \begin{bmatrix} m-r \\ j-r \end{bmatrix} q^{(j-r)(j-m)} x^j \\ & = \sum_{r=0}^{n} \begin{bmatrix} n \\ r \end{bmatrix} q^{r(r-m-n)} \frac{(q;q)_m}{(q;q)_{m-r}} x^r \sum_{j=0}^{m-r} \begin{bmatrix} m-r \\ j \end{bmatrix} q^{j(j-m+r)} x^j \\ & = \sum_{r=0}^{n} \begin{bmatrix} n \\ r \end{bmatrix} q^{r(r-n)} q^{-rm} \begin{bmatrix} m \\ r \end{bmatrix} (q;q)_r x^r g_{m-r}(x|q). \end{aligned} \tag{2.7}$$

Using the first formula in (1.13) yields

$$\mathbb{E}_x^{-n} g_m(x|q) = \sum_{r=0}^{n} \begin{bmatrix} n \\ r \end{bmatrix} q^{r(r-n)} \delta^r g_m(x|q). \tag{2.8}$$

Comparing the coefficient of (2.7) and (2.8), we have (1.14). The proof of Theorem is complete. ■

Proof of Proposition 2.3. Using (1.12), (1.13) and (1.14), LHS of (2.5) is equal to

$$\begin{aligned}\left(\mathbb{E}_x^{-1} + x\right)^n \{g_m(x|q)\} &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{k(k-n)} x^{n-k} \sum_{r=0}^{k} \begin{bmatrix} k \\ r \end{bmatrix} q^{r(r-k)} \delta^r \{g_m(x|q)\} \\ &= \sum_{r=0}^{n} \begin{bmatrix} n \\ r \end{bmatrix} q^{r(r-n)} g_{n-r}(x|q) \delta^r \{g_m(x|q)\},\end{aligned}$$

which is RHS of (2.5). The proof is complete.

Using formula (2.5), RHS of (2.4) is equal to

$$\begin{aligned}&\sum_{r=0}^{\min\{m,n\}} \begin{bmatrix} m \\ r \end{bmatrix} \begin{bmatrix} n \\ r \end{bmatrix} (q;q)_r q^{\binom{r}{2}+r(r-m-n)} (-x)^r \\ &\times \sum_{s=0}^{\infty} q^{s(s-m-n+2r)} \begin{bmatrix} m-r \\ s \end{bmatrix} \begin{bmatrix} n-r \\ s \end{bmatrix} (q;q)_s x^s g_{m-r-s}(x|q) g_{n-r-s}(x|q) \\ =& \sum_{k=0}^{\min\{m,n\}} \frac{(q;q)_m (q;q)_n q^{k(k-m-n)} x^k}{(q;q)_{m-k} (q;q)_{n-k} (q;q)_k} g_{m-k}(x|q) g_{n-k}(x|q) \sum_{r+s=k} \frac{(-1)^r q^{\binom{r}{2}} (q;q)_k}{(q;q)_r (q;q)_s} \\ =& \sum_{k=0}^{\min\{m,n\}} \frac{(q;q)_m (q;q)_n q^{k(k-m-n)} x^k}{(q;q)_{m-k} (q;q)_{n-k} (q;q)_k} g_{m-k}(x|q) g_{n-k}(x|q) \delta_{k,0},\end{aligned}$$

which is LHS of (2.4), where $\delta_{m,n}$ is the Kronecker delta. This achieves the proof. ■

3. q-Mehler's Formulas for Rogers-Szegö Polynomials

Carlitz [7] deduced the following q-Mehler's formulas by using the recurrence relations of Rogers-Szegö polynomials.

Proposition 3.1. ([7, Eq. (3.9)]). *For* $\max\{|t|, |xt|, |yt|, |xyt|\} < 1$, *we have*

$$\sum_{n=0}^{\infty} h_n(x|q) h_n(y|q) \frac{t^n}{(q;q)_n} = \frac{\left(xyt^2;q\right)_\infty}{(t, xt, yt, xyt; q)_\infty}. \tag{3.1}$$

Proposition 3.2. ([7, Eq. (3.13)]). *For* $\left|xyt^2/q\right| < 1$, *we have*

$$\sum_{n=0}^{\infty} g_n(x|q)g_n(y|q)\frac{(-1)^n q^{\binom{n}{2}} t^n}{(q;q)_n} = \frac{(t,xt,yt,xyt;q)_\infty}{\left(xyt^2/q;q\right)_\infty}. \tag{3.2}$$

Carlitz [10] gave another proofs of them by utilizing the transformation theory and the technique of operator. The authors [18] deduced them by the combinatorial method. Chen and Liu [12, 13] proved them by the method of parameter augmentation. For more information, please refer to [7, 10, 12, 13, 18].

In this section, we deduce Propositions 3.1 and 3.2 directly by the thought of exponential operator decomposition.

Proof of Proposition 3.1. LHS of (3.1) equals

$$\begin{aligned}
&\sum_{n=0}^{\infty} h_n(y|q)\frac{t^n}{(q;q)_n}\big(\mathbb{E}_x+x\big)^n\{1\} = \frac{1}{\Big(\big(\mathbb{E}_x+x\big)t, \big(\mathbb{E}_x+x\big)yt;q\Big)_\infty}\{1\}\\
&= \frac{1}{\Big(\big(\mathbb{E}_x+x\big)t;q\Big)_\infty}\left\{\sum_{k=0}^{\infty}\frac{(yt)^k}{(q;q)_k}\big(\mathbb{E}_x+x\big)^k\{1\}\right\}\\
&= \frac{1}{\Big(\big(\mathbb{E}_x+x\big)t;q\Big)_\infty}\left\{\frac{1}{(yt,xyt;q)_\infty}\right\}\\
&= \frac{1}{(yt;q)_\infty}\sum_{k=0}^{\infty}\frac{t^k}{(q;q)_k}\big(\mathbb{E}_x+x\big)^k\left\{\frac{1}{(xyt;q)_\infty}\right\}\\
&= \frac{1}{(yt;q)_\infty}\sum_{k=0}^{\infty}\frac{t^k}{(q;q)_k}\sum_{s=0}^{k}\begin{bmatrix}k\\s\end{bmatrix}x^{k-s}\frac{1}{(xytq^s;q)_\infty}\\
&= \frac{1}{(yt,xyt;q)_\infty}\sum_{s=0}^{\infty}\frac{t^s(xyt;q)_s}{(q;q)_s}\sum_{k=s}^{\infty}\frac{(xt)^{k-s}}{(q;q)_{k-s}},
\end{aligned}$$

which is the RHS of (3.1). This completes the proof. ∎

Proof of Proposition 3.2. LHS of (3.2) is equal to

$$\begin{aligned}
&\left(\left(\mathbb{E}_x^{-1}+x\right)t, \left(\mathbb{E}_x^{-1}+x\right)yt;q\right)_\infty\{1\}\\
&= \sum_{n=0}^{\infty}(-1)^n q^{\binom{n}{2}}\frac{t^n}{(q;q)_n}\left(\mathbb{E}_x^{-1}+x\right)^n\{(yt,xyt;q)_\infty\}\\
&= (yt;q)_\infty\sum_{n=0}^{\infty}(-1)^n q^{\binom{n}{2}}\frac{t^n}{(q;q)_n}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}q^{k(k-n)}x^{n-k}(xytq^{-k};q)_\infty\\
&= (yt,xyt;q)_\infty\sum_{k=0}^{\infty}\frac{(xytq^{-k};q)_k}{(q;q)_k}\sum_{n=k}^{\infty}\frac{(-1)^n q^{\binom{n}{2}} t^n q^{k(k-n)} x^{n-k}}{(q;q)_{n-k}}
\end{aligned}$$

$$= (yt, xyt; q)_\infty \sum_{k=0}^{\infty} \frac{q^{-\binom{k+1}{2}}(xyt)^k (q/(xyt); q)_k}{(q;q)_k} \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n+k}{2}} t^{n+k} q^{-nk} x^n}{(q;q)_n}$$
$$= (yt, xyt; q)_\infty \sum_{k=0}^{\infty} \frac{(xyt^2/q)^k (q/(xyt); q)_k}{(q;q)_k} \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}} (xt)^n}{(q;q)_n},$$

which is equivalent to RHS of (3.2). This achieve the proof. ■

4. q-CHRISTOFFEL-DARBOUX FORMULAS

The following Carlitz type generating functions for $h_n(x|q)$ is deduced by

Proposition 4.3. ([11, Eq. (4.1)]). *For* $m \in \mathbb{N}$*, we have*

$$\begin{aligned}\sum_{n=0}^{\infty} h_{m+n}(x|q) h_n(y|q) \frac{t^n}{(q;q)_n} \\ = \frac{x^m (xyt^2; q)_\infty}{(t, xt, yt, xyt; q)_\infty} {}_3\phi_1 \left[\begin{matrix} q^{-m}, xt, xyt \\ xyt^2 \end{matrix} ; q, \frac{q^m}{x} \right],\end{aligned} \tag{4.1}$$

where $\max\{|t|, |xt|, |yt|, |xyt|\} < 1$.

The auxiliary ones is given by

Proposition 4.4. ([6, Eq. (4.3)]). *For* $m \in \mathbb{N}$ *and* $\left|xyt^2/q\right| < 1$*, we have*

$$\begin{aligned}\sum_{n=0}^{\infty} g_{m+n}(x|q) g_n(y|q) (-1)^n q^{\binom{n}{2}} \frac{t^n}{(q;q)_n} \\ = \frac{x^m (t, xt, yt, xyt; q)_\infty}{(xyt^2/q; q)_\infty} {}_3\phi_2 \left[\begin{matrix} q^{-m}, q/(xt), q/(xyt) \\ 0, q^2/(xyt^2) \end{matrix} ; q, q \right].\end{aligned} \tag{4.2}$$

There are many proofs of above Propositions. Al-Salam and Ismail [2] gave the proof of Proposition 4.3 by using the transformation theory, while Srivastava and Jain [24] obtained it by the technique of generating function. The author [6] utilized the method of parameter augmentation [12, 13] to deduce above two Propositions. For more information, please refer to [2, 6, 11, 12, 13, 24].

In this section, we will use Carlitz's q-operators to derive Propositions 4.3 and 4.4, then we give the proof of Theorems 1.2 and 1.3.

Proof of Proposition 4.3. Formula (3.1) can be written as

$$\sum_{n=0}^{\infty} h_n(x|q) h_n(y|q) \frac{t^n}{(q;q)_n} = \frac{1}{(t;q)_\infty} \sum_{n=0}^{\infty} \frac{(t;q)_n t^n}{(q;q)_n} \frac{x^n}{(xt;q)_\infty} \frac{y^n}{(yt;q)_\infty}. \tag{4.3}$$

Applying $\left(\mathbb{E}_x + x\right)^m$ to both sides of (4.3) gives

$$\begin{aligned}
&\sum_{n=0}^{\infty} h_{m+n}(x|q) h_n(y|q) \frac{t^n}{(q;q)_n} \\
&= \frac{1}{(t;q)_\infty} \sum_{n=0}^{\infty} \frac{(t;q)_n t^n}{(q;q)_n} \frac{y^n}{(yt;q)_\infty} \left(\mathbb{E}_x + x\right)^m \left\{ \frac{x^n}{(xt;q)_\infty} \right\} \\
&= \frac{1}{(t;q)_\infty} \sum_{n=0}^{\infty} \frac{(t;q)_n t^n}{(q;q)_n} \frac{y^n}{(yt;q)_\infty} \sum_{k=0}^{\infty} \frac{t^k}{(q;q)_k} \sum_{s=0}^{m} \begin{bmatrix} m \\ s \end{bmatrix} x^{m-s+n+k} q^{(k+n)s} \\
&= \frac{1}{(t, yt;q)_\infty} \sum_{s=0}^{m} \begin{bmatrix} m \\ s \end{bmatrix} x^{m-s} \sum_{n=0}^{\infty} \frac{(t;q)_n (xytq^s)^n}{(q;q)_n} \sum_{k=0}^{\infty} \frac{(xtq^s)^k}{(q;q)_k},
\end{aligned}$$

which is the RHS of (4.1). The proof is complete. ■

Proof of Proposition 4.4. We can rewrite (3.2) as

$$\begin{aligned}
&\sum_{n=0}^{\infty} (-1)^n q^{\binom{n}{2}} g_n(x|q) g_n(y|q) \frac{t^n}{(q;q)_n} \\
&= (t;q)_\infty \sum_{n=0}^{\infty} \frac{(t^2/q)^n (q/t;q)_n}{(q;q)_n} x^n (xt;q)_\infty y^n (yt;q)_\infty .
\end{aligned} \tag{4.4}$$

Similar to the proof of (4.1), utilizing operator $\left(\mathbb{E}_x^{-1} + x\right)^m$ to both sides of (4.4), we obtain the proof of Proposition 4.4. ■

Proof of Theorem 1.2. For $m = 1$, formula (4.1) becomes

$$\sum_{n=0}^{\infty} h_{n+1}(x|q) h_n(y|q) \frac{t^n}{(q;q)_n} = \frac{(xyt^2;q)_\infty}{(t, xt, yt, xyt;q)_\infty} \frac{1 + x - xt - xyt}{1 - xyt^2}. \tag{4.5}$$

Replacing x by y in (4.5), then differencing between them gives

$$\sum_{n=0}^{\infty} \frac{h_{n+1}(x|q) h_n(y|q) - h_n(x|q) h_{n+1}(y|q)}{(x-y)(q;q)_n} t^n = \frac{(xyt^2 q;q)_\infty}{(tq, xt, yt, xyt;q)_\infty}. \tag{4.6}$$

By virtue of formula (3.1), RHS of (4.6) equals

$$\begin{aligned}
&\frac{(xyt^2 q;q)_\infty}{(tq, xt, ytq, xyt;q)_\infty} \frac{1}{1 - yt} \\
&= \sum_{k=0}^{\infty} (yt)^k \sum_{n=k}^{\infty} h_{n-k}(x/q|q) h_{n-k}(y|q) \frac{(tq)^{n-k}}{(q;q)_{n-k}} \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{n} h_{n-k}(x/q|q) h_{n-k}(y|q) \frac{y^k q^{n-k} t^n}{(q;q)_{n-k}}.
\end{aligned} \tag{4.7}$$

Comparing the coefficient of (4.6) and (4.7) gives the proof of Theorem 1.2. ∎

Proof of Theorem 1.3. Similar to (4.6), by (4.2), we have

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\frac{g_{n+1}(x|q)g_n(y|q)-g_n(x|q)g_{n+1}(y|q)}{(x-y)(q;q)_n}t^n(-1)^nq^{\binom{n}{2}}\\
&=\frac{(t/q,xt,yt,xyt;q)_\infty}{(xyt^2/q^2;q)_\infty}\\
&=\frac{(t/q,xt,yt/q,xyt;q)_\infty}{(xyt^2/q^2;q)_\infty}\frac{1}{1-yt/q}\\
&=\sum_{k=0}^{\infty}\left(\frac{yt}{q}\right)^k\sum_{n=k}^{\infty}g_{n-k}(xq|q)g_{n-k}(y|q)\frac{(-t/q)^{n-k}q^{\binom{n-k}{2}}}{(q;q)_{n-k}}\\
&=\sum_{n=0}^{\infty}\sum_{k=0}^{n}g_{n-k}(xq|q)g_{n-k}(y|q)\frac{(-1)^{n-k}q^{\binom{n-k}{2}}y^kt^n}{q^n(q;q)_{n-k}}.
\end{aligned}
\tag{4.8}
$$

Equating the coefficient of t on both sides of (4.8) yields the proof of Theorem 1.3. ∎

5. q-Mehler's Formula for Hahn Polynomials

Al-Salam and Carlitz [3] gave the following two bilinear generating functions by the transformation theory. For more information, please refer to [3, 19].

Proposition 5.1. ([3, Eq. (1.17)]). *If* $\max\{|z|,|xz|,|yz|,|xyz|\}<1$, *we have*

$$
\sum_{n=0}^{\infty}\phi_n^{(a)}(x|q)\phi_n^{(b)}(y|q)\frac{z^n}{(q;q)_n}=\frac{(axz,byz;q)_\infty}{(z,xz,yz;q)_\infty}{}_3\phi_2\left[\begin{matrix}a,b,z\\axz,byz\end{matrix};q,xyz\right].
\tag{5.1}
$$

Proposition 5.2. ([3, Eq. (1.18)]). *If* $\max\{|qaxz|,|qbyz|\}<1$, *we have*

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\psi_n^{(a)}(x|q)\psi_n^{(b)}(y|q)\frac{(-1)^nq^{\binom{n+1}{2}}z^n}{(q;q)_n}\\
&=\frac{(qz,qxz,qyz;q)_\infty}{(qaxz,qbyz;q)_\infty}{}_3\phi_2\left[\begin{matrix}\frac{1}{a},\frac{1}{b},\frac{1}{z}\\\frac{1}{axz},\frac{1}{byz}\end{matrix};q,q\right].
\end{aligned}
\tag{5.2}
$$

In this section, we will deduce Propositions 5.1 and 5.2 directly from q-Mehler's formula for Rogers-Szegö polynomials by Carlitz's q-operators.

Proof of Proposition 5.1. We first prove that

$$\begin{aligned}
&\sum_{m=0}^{\infty}\frac{a^m}{(q;q)_m}\big(\mathbb{E}_x+x\big)^m\left\{\frac{x^n}{(xz;q)_\infty}\right\}\\
&=\sum_{k=0}^{\infty}\frac{z^k}{(q;q)_k}\sum_{m=0}^{\infty}\frac{a^m}{(q;q)_m}\big(\mathbb{E}_x+x\big)^m\{x^{n+k}\}\\
&=\sum_{k=0}^{\infty}\frac{z^k}{(q;q)_k}\sum_{m=0}^{\infty}\frac{a^m}{(q;q)_m}\sum_{j=0}^{m}\begin{bmatrix}m\\ j\end{bmatrix}x^{m-j}\mathbb{E}_x^j\{x^{n+k}\}\\
&=x^n\sum_{k=0}^{\infty}\frac{(xz)^k}{(q;q)_k}\sum_{j=0}^{\infty}\frac{a^jq^{j(n+k)}}{(q;q)_j}\sum_{m=j}^{\infty}\frac{(ax)^{m-j}}{(q;q)_{m-j}}\\
&=\frac{x^n(a;q)_n(axz;q)_\infty}{(axz;q)_n(a,ax,xz;q)_\infty}.
\end{aligned}\tag{5.3}$$

Similarly, we have

$$\sum_{m=0}^{\infty}\frac{b^m}{(q;q)_m}\big(\mathbb{E}_y+y\big)^m\left\{\frac{y^n}{(yz;q)_\infty}\right\}=\frac{y^n(b;q)_n(byz;q)_\infty}{(byz;q)_n(a,by,yz;q)_\infty}.\tag{5.4}$$

A little computation shows that [11, Eq. (3.3)]

$$\sum_{m=0}^{\infty}\frac{a^m}{(q;q)_m}\big(\mathbb{E}_x+x\big)^m\{h_n(x|q)\}=\frac{1}{(a,ax;q)_\infty}\phi_n^{(a)}(x|q)\tag{5.5}$$

and

$$\sum_{m=0}^{\infty}\frac{b^m}{(q;q)_m}\big(\mathbb{E}_y+y\big)^m\{h_n(y|q)\}=\frac{1}{(b,by;q)_\infty}\phi_n^{(b)}(y|q).\tag{5.6}$$

Now, we applying operators

$$\frac{1}{\Big(a\big(\mathbb{E}_x+x\big);q\Big)_\infty}\quad\text{and}\quad\frac{1}{\Big(b\big(\mathbb{E}_y+y\big);q\Big)_\infty}\tag{5.7}$$

to both sides of (4.3), then combining (5.3)-(5.6) yield

$$\begin{aligned}
&\frac{1}{(a,ax,b,by;q)_\infty}\sum_{n=0}^{\infty}\phi_n^{(a)}(x|q)\phi_n^{(b)}(y|q)\frac{z^n}{(q;q)_n}\\
&=\frac{(axz,byz;q)_\infty}{(a,ax,xz,b,by,yz,z;q)_\infty}\sum_{n=0}^{\infty}\frac{(z;q)_nz^n}{(q;q)_n}\frac{x^n(a;q)_n}{(axz;q)_n}\frac{y^n(b;q)_n}{(byz;q)_n},
\end{aligned}\tag{5.8}$$

which equals RHS of (5.1). The proof is complete. ∎

Proof of Proposition 5.2. From (1.12), we have

$$\begin{aligned}
&\sum_{m=0}^{\infty}\frac{a^m}{(q;q)_m}(-1)^m q^{\binom{m+1}{2}}\left(\mathbb{E}_x^{-1}+x\right)^m\{x^n(xz;q)_\infty\}\\
&=\sum_{k=0}^{\infty}\frac{(-1)^k q^{\binom{k}{2}}z^k}{(q;q)_k}\\
&\quad\times\sum_{m=0}^{\infty}\frac{a^m}{(q;q)_m}(-1)^m q^{\binom{m+1}{2}}\sum_{j=0}^{m}\begin{bmatrix}m\\ j\end{bmatrix}q^{j(j-m-n-k)}x^{m-j+n+k}\\
&=(-1)^n q^{-\binom{n}{2}}(ax)^n(1/a;q)_n\frac{(aq,axq,xz;q)_\infty}{(axzq^{-n};q)_\infty}.
\end{aligned}\tag{5.9}$$

Similarly

$$\begin{aligned}
&\sum_{m=0}^{\infty}\frac{b^m}{(q;q)_m}(-1)^m q^{\binom{m+1}{2}}\left(\mathbb{E}_y^{-1}+y\right)^m\{y^n(yz;q)_\infty\}\\
&=(-1)^n q^{-\binom{n}{2}}(by)^n(1/b;q)_n\frac{(bq,byq,yz;q)_\infty}{(byzq^{-n};q)_\infty}.
\end{aligned}\tag{5.10}$$

It's easily to verify that [11, Eq. (8.5)]

$$\sum_{m=0}^{\infty}\frac{a^m}{(q;q)_m}(-1)^m q^{\binom{m+1}{2}}\left(\mathbb{E}_x^{-1}+x\right)^m\{g_n(x|q)\}=(aq,axq;q)_\infty\psi_n^{(a)}(x|q)\tag{5.11}$$

and

$$\sum_{m=0}^{\infty}\frac{b^m}{(q;q)_m}(-1)^m q^{\binom{m+1}{2}}\left(\mathbb{E}_y^{-1}+y\right)^m\{g_n(y|q)\}=(bq,byq;q)_\infty\psi_n^{(b)}(y|q).\tag{5.12}$$

Applying operators $\left(a\left(\mathbb{E}_x^{-1}+x\right);q\right)$ and $\left(b\left(\mathbb{E}_y^{-1}+y\right);q\right)$ to both sides of (4.4) yields

$$\begin{aligned}
&(aq,axq,bq,byq;q)_\infty\sum_{n=0}^{\infty}\psi_n^{(a)}(x|q)\psi_n^{(b)}(y|q)\frac{(-1)^n q^{\binom{n}{2}}z^n}{(q;q)_n}\\
&=\frac{(z,aq,axq,xz,bq,byq,yz;q)_\infty}{(axz,byz;q)_\infty}\\
&\quad\times\sum_{n=0}^{\infty}\frac{(z^2/q)^n(q/z;q)_n}{(q;q)_n}q^{-n^2+n}\frac{(abxy)^n(1/a,1/b;q)_n}{(axzq^{-n},byzq^{-n};q)_n},
\end{aligned}\tag{5.13}$$

replacing z by qz gives the proof. ∎

6. Some Results Related to Carlitz's q-Operators

The generalized Rogers-Szegö polynomials are defined by

$$h_n(x,y|q)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}x^{n-k}y^k \quad \text{and} \quad g_n(x,y|q)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}q^{k(k-n)}x^{n-k}y^k. \tag{6.1}$$

In this section, using Carlitz's q-operators, we first deduce their q-Mehler's formulas as follows.

Proposition 6.1. *For* $\max\{|xtu|,|xtv|,|ytu|,|ytv|\}<1$, *we have*

$$\sum_{n=0}^{\infty}h_n(x,y|q)h_n(u,v|q)\frac{t^n}{(q;q)_n}=\frac{(xyuvt^2;q)_\infty}{(xtu,xtv,ytu,ytv;q)_\infty}. \tag{6.2}$$

Proposition 6.2. *For* $\left|xyuvt^2/q\right|<1$, *we have*

$$\sum_{n=0}^{\infty}g_n(x,y|q)g_n(u,v|q)\frac{(-1)^nq^{\binom{n}{2}}t^n}{(q;q)_n}=\frac{(xut,xvt,yut,yvt;q)_\infty}{(xyuvt^2/q;q)_\infty}. \tag{6.3}$$

Remark 1. *Comparing* (1.4) *and* (6.1), *we find that* $h_n(x,1|q)=h_n(x|q)$ *and* $g_n(x,1|q)=g_n(x|q)$. *So when* $y=v=1$, *Propositions 6.1 and 6.2 reduce to Propositions 3.1 and 3.2 respectively.*

In addition, we derive the following q-analogue of binomial theorem.

Proposition 6.3. ([16, p. 20]). *For* $n\in\mathbb{N}$, *we have*

$$(xy;q)_n=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}(x;q)_k(y;q)_{n-k}y^k. \tag{6.4}$$

Proof of Proposition 6.1. By formula (1.8) and Proposition 2.1, we get

$$\begin{aligned}(y\mathbb{E}_x+x)^n&=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}x^{n-k}y^k\sum_{r=0}^{k}(-1)^r\begin{bmatrix}k\\r\end{bmatrix}\triangle^r\\&=\sum_{r=0}^{n}(-1)^r\begin{bmatrix}n\\r\end{bmatrix}\sum_{k=r}^{n}\begin{bmatrix}n-r\\k-r\end{bmatrix}x^{n-k}y^k\triangle^r\\&=\sum_{r=0}^{n}(-1)^r\begin{bmatrix}n\\r\end{bmatrix}y^r\sum_{k=0}^{n-r}\begin{bmatrix}n-r\\k\end{bmatrix}x^{n-k-r}y^k\triangle^r\\&=\sum_{r=0}^{n}(-1)^r\begin{bmatrix}n\\r\end{bmatrix}y^rh_{n-r}(x,y|q)\triangle^r,\end{aligned}$$

where $\triangle$ defined by (1.7), so we have $\big(y\mathbb{E}_x+x\big)^n\{1\}=h_n(x,y|q)$. We can verified that

$$\sum_{n=0}^{\infty}h_n(x,y|q)\frac{t^n}{(q;q)_n}=\frac{1}{(xt,yt;q)_\infty}. \tag{6.5}$$

Using the technique of exponential operator decomposition, LHS of (6.2) equals

$$\begin{aligned}&\frac{1}{\Big(ut\big(y\mathbb{E}_x+x\big),vt\big(y\mathbb{E}_x+x\big);q\Big)_\infty}\{1\}\\ &=\frac{1}{(yvt;q)_\infty}\frac{1}{\Big(ut\big(y\mathbb{E}_x+x\big);q\Big)_\infty}\left\{\frac{1}{(xvt;q)_\infty}\right\},\end{aligned} \tag{6.6}$$

which is RHS of (6.2) after some computation. The proof is complete. ∎

Proof of Proposition 6.2. First we can deduce that

$$\sum_{n=0}^{\infty}(-1)^n q^{\binom{n}{2}}g_n(x,y|q)\frac{t^n}{(q;q)_n}=(xt,yt;q)_\infty, \tag{6.7}$$

and $\Big(y\mathbb{E}_x^{-1}+x\Big)^n\{1\}=g_n(x,y|q)$. Similar to (6.6), LHS of (6.3) is equivalent to

$$\begin{aligned}&\Big(ut\Big(y\mathbb{E}_x^{-1}+x\Big),vt\Big(y\mathbb{E}_x^{-1}+x\Big);q\Big)_\infty\{1\}\\ &=(yvt;q)_\infty\Big(ut\Big(y\mathbb{E}_x^{-1}+x\Big);q\Big)_\infty\{(xvt;q)_\infty\},\end{aligned} \tag{6.8}$$

which equals RHS of (6.3) after some computation. The proof is ended. ∎

Proof of Proposition 6.3. We consider the following type of Carlitz's q-operator

$$\Big(y(1-x)\mathbb{E}_x+(1-y)\mathbb{E}_y\Big)^n \tag{6.9}$$

and find the fact that

$$\begin{aligned}&\Big(y(1-x)\mathbb{E}_x+(1-y)\mathbb{E}_y\Big)\{1\}=y(1-x)+1-y=1-xy,\\ &\Big(y(1-x)\mathbb{E}_x+(1-y)\mathbb{E}_y\Big)^2\{1\}=\Big(y(1-x)\mathbb{E}_x+(1-y)\mathbb{E}_y\Big)\{1-xy\}\\ &=y(1-x)(1-xyq)+(1-y)(1-xyq)=(1-xy)(1-xyq)=(xy;q)_2.\end{aligned}$$

Generally we have

$$\left(y(1-x)\mathbb{E}_x + (1-y)\mathbb{E}_y\right)^n\{1\} = (xy;q)_n. \tag{6.10}$$

Similarly, we get

$$\left((1-y)\mathbb{E}_y\right)^k\{1\} = (y;q)_k \quad \text{and} \quad \left(y(1-x)\mathbb{E}_x\right)^k\{1\} = y^k(x;q)_k. \tag{6.11}$$

From Proposition 2.1, we gain

$$\left(y(1-x)\mathbb{E}_x + (1-y)\mathbb{E}_y\right)^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \left(y(1-x)\mathbb{E}_x\right)^k \left((1-y)\mathbb{E}_y\right)^{n-k}. \tag{6.12}$$

Combining (6.10), (6.11) and (6.12), we conclude the proof. ■

References

1. W. A. Al-Salam, Operational representations for the Laguerre and other polynomials, *Duke Math. J.*, **31** (1964), 127-142.
2. W. A. Al-Salam and M. E. H. Ismail, q-beta integrals and the q-hermite polynomials, *Pacific J. Math.*, **135** (1988), 209-221.
3. W. A. Al-Salam and L. Carlitz, Some orthogonal q-polynomials, *Math. Nachr.*, **30** (1965), 47-61.
4. D. Bowman, q-difference operator, orthogonal polynomials, and symmetric expansions, *Mem. Amer. Soc.*, **159** (2002).
5. J. L. Burchnall, A note on the polynomials of Hermite, *Quart. J. Math. Oxford Ser.*, **12** (1941), 9-11.
6. J. Cao, New proofs of generating functions for Rogers-Szegő polynomials, *Appl. Math. Comp.*, **207** (2009), 486-492.
7. L. Carlitz, Some polynomials related to theta functions, *Annali di Matematica Pura ed Applicata*, **41** (1956), 359-373.
8. L. Carlitz, Some polynomials related to theta functions, *Duke Math. Jour.*, **24** (1957), 521-527.
9. L. Carlitz, A note on the Laguerre polynomials, *Michigan Math. Jour.*, **7** (1960), 219-223.
10. L. Carlitz, A q-identity, *Monatsh. für. Math.*, **67** (1963), 305-310.
11. L. Carlitz, Generating functions for certain q-orthogonal polynomials, *Collectanea Math.*, **23** (1972), 91-104.

12. W. Y. C. Chen and Z.-G. Liu, Parameter augmentation for basic hypergeometric series II, *J. Combin. Theory Ser. A*, **80** (1997), 175-195.

13. W. Y. C. Chen and Z.-G. Liu, Parameter augmentation for basic hypergeometric series, I, in: *Mathematical Essays in honor of Gian-Carlo Rota*, B. E. Sagan and R. P. Stanley (Eds.), Birkäuser, Basel, 1998, pp. 111-129.

14. A. Erdélyi, W. Magnus, F. Oberhettinger and F. G. Tricomi, *Higher Transcendental Functions* II, McGraw-Hill, New York, (1953).

15. G. Dattoli, A. Torre and M. Carpanese, Operational rules and arbitrary order Hermite generating functions, *J. Math. Anal. Appl.*, **227** (1998), 98-111.

16. G. Gasper and M. Rahman, *Basic Hypergeometric Series*, Cambridge Univ. Press, Cambridge, MA, 1990.

17. H. W. Gould and A. T. Hopper, Operational formulas connected with two generalizations of Hermite polynomials, *Duke Math. J.*, **29** (1962), 51-64.

18. M. E. H. Ismail, D. Stanton and G. Viennot, The combinatorics of q-Hermite polynomials and the Askey-Wilson integral, *Europ. J. Combinatorics*, **8** (1987), 379-392.

19. Z.-G. Liu, q-Hermite polynomials and a q-Beta integral, *Northeast Math. J.*, **13** (1997), 361-366.

20. Z.-G. Liu, Some identiites of differential operator and Hermite polynomials, *J. Math. Res. Exp.*, **18** (1998), 412-416, (in Chinese).

21. N. Nielsen, Recherches sur les polynomes d'Hermite, *Kgl. Danske Vidensk. Selskat. mat.-fys. Meddelelser (I)*, **6** (1918), 1-78.

22. L. J. Rogers, On the expansion of some infinite products, *Proc. London Math. Soc.*, **24** (1893), 337-352.

23. G. Szegö, Ein Beitrag zur Theorie der Thetafunktionen, *Sitz. Preuss. Akad. Wiss., Phys. Math. Klasse*, **19** (1926), 242-252.

24. H. M. Srivastava and V. K. Jain, Some multilinear generating functions for q-hermite polynomials, *J. Math. Anal. Appl.*, **144** (1989), 147-157.

25. H. M. Srivastava and H. L. manocha, *A treatise on generating functions*, Ellis Horwood, New York, 1984.

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