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# ON PERTURBATION OF $\alpha$-TIMES INTEGRATED $C$-SEMIGROUPS

Chung-Cheng Kuo

**Abstract.** Let $\alpha \geq 0$, and $C$ be a bounded linear injection on a complex Banach space $X$. We first show that if $A$ generates an exponentially bounded nondegenerate $\alpha$-times integrated $C$-semigroup $S_\alpha(\cdot)$ on $X$, $B$ is a bounded linear operator on $\overline{D(A)}$ such that $BC = CB$ on $\overline{D(A)}$ and $BA \subset AB$, then $A + B$ generates an exponentially bounded nondegenerate $\alpha$-times integrated $C$-semigroup $T_\alpha(\cdot)$ on $X$. Moreover, $T_\alpha(\cdot)$ is also exponentially Lipschitz continuous or norm continuous if $S_\alpha(\cdot)$ is. We show that the exponential boundedness of $T_\alpha(\cdot)$ can be deleted and $\alpha$-times integrated $C$-semigroups can be extended to the context of local $\alpha$-times integrated $C$-semigroups when $R(C) \subset \overline{D(A)}$ and $BS_\alpha(\cdot) = S_\alpha(\cdot)B$ on $\overline{D(A)}$ both are added. Moreover, $T_\alpha(\cdot)$ is also locally Lipschitz continuous or norm continuous if $S_\alpha(\cdot)$ is. We show that $A + B$ generates a nondegenerate local $\alpha$-times integrated $C$-semigroup $T_\alpha(\cdot)$ on $X$ if $A$ generates a nondegenerate local $\alpha$-times integrated $C$-semigroup $S_\alpha(\cdot)$ on $X$ and $B$ is a bounded linear operator on $X$ such that either $BC = CB$, $BS_\alpha = S_\alpha B$ on $X$; or $BC = CB$ on $\overline{D(A)}$ and $BA \subset AB$. Moreover, $T_\alpha(\cdot)$ is also locally Lipschitz continuous, (norm continuous, exponentially bounded or exponentially Lipschitz continuous) if $S_\alpha(\cdot)$ is.

## 1. INTRODUCTION

Let $X$ be a complex Banach space with norm $\|\cdot\|$, and $B(X)$ denote the set of all bounded linear operators from $X$ into itself. For each $\alpha > 0$, $0 < T_0 \leq \infty$ and $C \in B(X)$, a family $S(\cdot)(= \{S(t)|0 \leq t < T_0\})$ in $B(X)$ is called a local $\alpha$-times integrated $C$-semigroup on $X$ if it is strongly continuous, $S(\cdot)C = CS(\cdot)$ and satisfies

$$S(t)S(s)x = \frac{1}{\Gamma(\alpha)}[\int_0^{t+s} - \int_0^t - \int_0^s](t+s-r)^{\alpha-1}S(r)Cxdr \tag{1.1}$$

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for all $x \in X$ and $0 \leq t, s \leq t+s < T_0$ (see [1-6,8-9,11-15,19-20]); or called a local (0-times integrated) $C$-semigroup on $X$ if it is strongly continuous, $S(\cdot)C = CS(\cdot)$ and satisfies

$$S(t)S(s)x = S(t+s)Cx \tag{1.2}$$

for all $x \in X$ and $0 \leq t, s \leq t+s < T_0$ (see [2,17,18]), where $\Gamma(\cdot)$ denotes the Gamma function. Moreover, we say that $S(\cdot)$ is

(i) exponentially bounded if there exist constants $K, \omega \geq 0$ such that

$$\|S(t)\| \leq Ke^{\omega t} \quad \text{for all } t \geq 0; \tag{1.3}$$

(ii) exponentially Lipschitz continuous if there exist constants $K, \omega \geq 0$ such that

$$\|S(t+h) - S(t)\| \leq Khe^{\omega(t+h)} \quad \text{for all } t, h \geq 0; \tag{1.4}$$

(iii) locally Lipschitz continuous if for each $0 < t_0 < T_0$ there exists a $K_{t_0} > 0$ such that

$$\|S(t+h) - S(t)\| \leq K_{t_0}h \quad \text{for all } 0 \leq t, h \leq t+h \leq t_0; \tag{1.5}$$

(iv) nondegenerate if $x = 0$ whenever $S(t)x = 0$ for all $0 \leq t < T_0$. In this case, the (integral) generator of $S(\cdot)$ is a closed linear operator $A : D(A) \subset X \to X$ defined by $D(A) = \{x | x, y_x \in X$ and $S(t)x - \frac{t^\alpha}{\Gamma(\alpha+1)}Cx = \int_0^t S(r)y_x dr$ for all $0 \leq t < T_0\}$ and $Ax = y_x$ for all $x \in D(A)$. It is known that the following properties hold (see [6, 11, 17]):

$$C \text{ is injective and } C^{-1}AC = A; \tag{1.6}$$

$$S(0) = 0 \text{ on } X \text{ if } \alpha > 0, \text{ and } S(0) = C \text{ on } X \text{ if } \alpha = 0; \tag{1.7}$$

$$S(t)x \in D(A) \text{ and } S(t)Ax = AS(t)x \text{ for all } x \in D(A) \text{ and } 0 \leq t < T_0; \tag{1.8}$$

$$\int_0^t S(r)xdr \in D(A) \text{ and } A\int_0^t S(r)xdr = S(t)x - \frac{t^\alpha}{\Gamma(\alpha+1)}Cx \quad \text{for all } x \in X \text{ and } 0 \leq t < T_0. \tag{1.9}$$

In general, a local $\alpha$-times integrated C-semigroup is also called an $\alpha$-times integrated C-semigroup if $T_0 = \infty$, an $\alpha$-times integrated C-semigroup may not be exponentially bounded, and the generator of a nondegenerate local $\alpha$-times integrated C-semigroup may not be densely defined. Moreover, a local $\alpha$-times integrated $I_X$-semigroup on $X$ is also called a local $\alpha$-times integrated semigroup on $X$, where $I_X$

denotes the identity operator on $X$. Using Hille-Yosida type theorems to obtain some additive perturbation results concerning exponentially bounded $n$-times integrated $C$-semigroup (for $n \in \mathbb{N} \cup \{0\}$) or $\alpha$-times integrated semigroup (for $\alpha > 0$) have been extensively studied by many authors (see [6, 14, 15, 18, 21] and [10, 17, 19], respectively). Some interesting applications of this topic are also illustrated in [3, 7, 19]. In particular, Xiao and Liang [19, Theorem 1.3.5] show that $A + B$ generates an exponentially bounded nondegenerate $\alpha$-times integrated semigroup on $X$ if $A$ generates an exponentially bounded nondegenerate $\alpha$-times integrated semigroup on $X$ and $B$ is a bounded linear operator on $X$ such that $BA \subset AB$, and Li and Shaw [10] have obtained an additive perturbation theorem which shows that if $A$ generates a nondegenerate $\alpha$-times integrated $C$-semigroup $S_\alpha(\cdot)$ on $X$ and $B$ is a bounded linear operator on $X$ which commutes with $S_\alpha(\cdot)$ and $C$ on $X$, then $A+B$ generates a nondegenerate $\alpha$-times integrated $C$-semigroup $T_\alpha(\cdot)$ on $X$. These two results are also extended to the context of local $\alpha$-times integrated $C$-semigroups here by another method (see Theorems 2.11 and 2.13 below). We also have to investigate some additive perturbation theorems concerning (local) $\alpha$-times integrated $C$-semigroups which may be done by using a Hille-Yosida type theorem (see Theorem 2.4 below) concerning exponentially bounded nondegenerate $\alpha$-times integrated $C$-semigroups on $X$ as results in [9] for the case $\alpha \in \mathbb{N} \cup \{0\}$, in [1] for the case $C = I_X$ and in [19] for the general case $\alpha > 0$. We first extend the perturbation result of Xiao and Liang [19, Theorem 1.3.5]. In Theorem 2.8, we show that if $A$ generates an exponentially bounded nondegenerate $\alpha$-times integrated $C$-semigroup $S_\alpha(\cdot)$ on $X$, $B$ is a bounded linear operator on $\overline{D(A)}$ which commutes with $C$ on $\overline{D(A)}$ and $BA \subset AB$, then $A+B$ generates an exponentially bounded nondegenerate $\alpha$-times integrated $C$-semigroup $T_\alpha(\cdot)$ on $X$. Moreover, $T_\alpha(\cdot)$ is exponentially Lipschitz continuous or norm continuous if $S_\alpha(\cdot)$ is. We show that the exponential boundedness of $T_\alpha(\cdot)$ in Theorem 2.8 can be deleted and the conclusion of Theorem 2.8 can be extended to the context of local $\alpha$-times integrated $C$-semigroups when $S_\alpha(\cdot)$ is a nondegenerate local $\alpha$-times integrated $C$-semigroup on $X$ with generator $A$ and $R(C) \subset \overline{D(A)}$ and $BS_\alpha(\cdot) = S_\alpha(\cdot)B$ on $\overline{D(A)}$ both are added (see Theorem 2.9 below). Moreover, $T_\alpha(\cdot)$ is also locally Lipschitz continuous or norm continuous if $S_\alpha(\cdot)$ is. A simple illustrative example of these results is presented in the final part of this paper.

## 2. Perturbation Theorems

From now on, we always assume that $C \in B(X)$ and $A : D(A) \subset X \to X$ is a closed liner operator with domain $D(A)$ and range $R(A)$.

**Lemma 2.1.** (see [6, 17]). *Let $S(\cdot)$ be a strongly continuous family in $B(X)$ satisfying (1.3). For $\lambda > \omega$ and $x \in X$, we define $R_\lambda x = \lambda^\alpha \int_0^\infty e^{-\lambda t} S(t)x dt$.*

*Then (1.1) holds if and only if $\{R_\lambda|\lambda > \omega\}$ is a $C$-pseudoresolvent. That is, $R_\lambda C - R_\mu C = (\mu-\lambda)R_\lambda R_\mu$ on $X$ for all $\lambda, \mu > \omega$.*

**Theorem 2.2.** (see [6, 17]). *Let $\alpha \geq 0$, and $C \in B(X)$ be injective. A strongly continuous family $S(\cdot)$ in $B(X)$ satisfying (1.3) is a nondegenerate $\alpha$-times integrated $C$-semigroup on $X$ with generator $A$ if and only if $CS(\cdot) = S(\cdot)C$, $C^{-1}AC = A$, $\lambda - A$ is injective, $R(C) \subset R(\lambda - A)$ and $\lambda^\alpha L_\lambda(\lambda - A) \subset \lambda^\alpha(\lambda - A)L_\lambda = C$ for all $\lambda > \omega$. Here $L_\lambda x = \int_0^\infty e^{-\lambda t}S(t)xdt$ for $x \in X$.*

**Lemma 2.3.** (see [1,Theorem 2.4.1] or [19,Theorem 1.2.1]). *Let $0 < \theta \leq 1$, and $r : (\omega, \infty) \to X$ be an infinitely differentiable function for some $\omega \geq 0$. Then the following are equivalent:*

$(i)$ *There exists a constant $K \geq 0$ such that*

$$\|(\lambda-\omega)^{k+1}r^{(k)}(\lambda)/k!\| \leq K$$

*for all $\lambda > \omega$ and $k \in \mathbb{N}\cup\{0\}$, where $r^{(k)}(\lambda)$ denotes the $k$th order derivative of $r$ at $\lambda$;*

$(ii)$ *There exist $K_\theta \geq 0$ and $F_\theta : [0, \infty) \to X$ such that $F_\theta(0) = 0$, $\|F_\theta(t+h) - F_\theta(t)\| \leq K_\theta h^\theta e^{w(t+h)}$ for all $t, h \geq 0$, and $r(\lambda) = \lambda^\theta \int_0^\infty e^{-\lambda t}F_\theta(t)dt$ for all $\lambda > \omega$.*

Combining Lemma 2.1 with Lemma 2.3, we can obtain the next Hille-Yosida type theorem concerning exponentially Lipschitz continuous $(\alpha+1)$-times integrated $C$-semigroups which has been presented in [19] and in [1] for the case $C = I_X$.

**Theorem 2.4.** *Let $\alpha, \omega \geq 0$, $0 < \theta \leq 1$ and $R(\cdot) = \{R(\lambda)|\lambda > \omega\} \subset B(X)$. Then the following are equivalent :*

$(i)$ *$R(\cdot)$ is a $C$-pesudoresolvent and there exists a $K \geq 0$ such that*

$$\|\frac{(\lambda-\omega)^{k+1}}{k!}\frac{d^k}{d\lambda^k}R(\lambda)/\lambda^\alpha\| \leq K$$

*for all $\lambda > \omega$ and $k \in \mathbb{N}\cup\{0\}$;*

$(ii)$ *There exists an $(\alpha+\theta)$-times integrated $C$-semigroup $S(\cdot)$ on $X$ such that*

$$\|S(t+h) - S(t)\| \leq K_\theta h^\theta e^{\omega(t+h)}$$

*for all $t, h \geq 0$ and for some fixed $K_\theta \geq 0$, and $R(\lambda)x = \lambda^{\alpha+\theta}\int_0^\infty e^{-\lambda t}S(t)xdt$ for all $\lambda > \omega$ and $x \in X$.*

Applying Theorems 2.2 and 2.4 we also obtain the following two results. Their proofs are almost the same with those in [19, Theorem 1.3.3] and in [9, Proposition 6.1 and Theorem 6.2] for the case $\alpha \in \mathbb{N} \cup \{0\}$, and so are omitted.

**Theorem 2.5.** *Let $\alpha \geq 0$, and $A$ be the generator of a nondegenerate $(\alpha+1)$-times integrated $C$-semigroup $S(\cdot)$ on $X$ satisfying (1.4). Then*

$(i)$ *$R(C) \subset R((\lambda - A)^k)$ and $\frac{d^{k-1}}{d\lambda^{k-1}}(\lambda - A)^{-1}Cx = (-1)^{k-1}(k-1)!(\lambda - A)^{-k}Cx$ for all $x \in X$, $k \in \mathbb{N}$ and $\lambda > \omega$;*

$(ii)$
$$\left\| \frac{(\lambda-\omega)^k}{(k-1)!} \frac{d^{k-1}}{d\lambda^{k-1}}(\lambda - A)^{-1}C/\lambda^\alpha \right\| \leq K$$

*for all $k \in \mathbb{N}$ and $\lambda > \omega$.*

**Theorem 2.6.** *Let $\alpha \geq 0$, $C \in B(X)$ be injective and $C^{-1}AC = A$. Then the following are equivalent:*

$(i)$ *$A$ generates a nondegenerate $(\alpha+1)$-times integrated $C$-semigroup $S(\cdot)$ on $X$ satisfying (1.4);*

$(ii)$ *$\lambda - A$ is injective, $R(C) \subset R((\lambda - A)^k)$ and $\|\frac{(\lambda-\omega)^{k+1}}{k!}\frac{d^k}{d\lambda^k}(\lambda - A)^{-1}C/\lambda^\alpha\| \leq K$ for all $\lambda > \omega$ and $k \in \mathbb{N} \cup \{0\}$ and for some fixed $K \geq 0$.*

**Lemma 2.7.** *Let $\alpha \geq 0$, and $A$ be the generator of a nondegenerate $\alpha$-times integrated $C$-semigroup $S(\cdot)$ on $X$ satisfying (1.3). Then for $\lambda > \omega$ and $k \in \mathbb{N}$, we have*

$$(\lambda - A)^{-k}Cx = \sum_{j=0}^{k-1} \binom{\alpha}{j} \lambda^{\alpha-j} \frac{(-1)^j}{(k-1-j)!} \int_0^\infty e^{-\lambda t} t^{k-1-j} S(t)x dt$$

*for all $x \in X$. Here $\binom{\alpha}{j} = \alpha(\alpha-1)\cdots(\alpha-j+1)/j!$.*

*Proof.* Since $(\lambda - A)^{-1}Cx = \lambda^\alpha \int_0^\infty e^{-\lambda t}S(t)x dt$, we have

$$\begin{aligned}
&(\lambda - A)^{-k}Cx \\
&= \frac{(-1)^{k-1}}{(k-1)!}\frac{d^{k-1}}{d\lambda^{k-1}}(\lambda - A)^{-1}Cx \\
&= \frac{(-1)^{k-1}}{(k-1)!}\int_0^\infty \frac{\partial^{k-1}}{\partial\lambda^{k-1}}\lambda^\alpha e^{-\lambda t}S(t)x dt \\
&= \frac{(-1)^{k-1}}{(k-1)!}\int_0^\infty \sum_{j=0}^{k-1}\binom{k-1}{j}\alpha(\alpha-1)\cdots(\alpha-j+1)\lambda^{\alpha-j}(-1)^{k-1-j}t^{k-1-j}S(t)x dt
\end{aligned}$$

$$= \frac{(-1)^{k-1}}{(k-1)!}\sum_{j=0}^{k-1}\frac{(k-1)!}{(k-1-j)!j!}\binom{\alpha}{j}j!\lambda^{\alpha-j}(-1)^{k-1-j}\int_0^\infty e^{-\lambda t}t^{k-1-j}S(t)xdt$$
$$= \sum_{j=0}^{k-1}\binom{\alpha}{j}\frac{(-1)^j}{(k-1-j)!}\lambda^{\alpha-j}\int_0^\infty e^{-\lambda t}t^{k-1-j}S(t)xdt.$$

Applying Lemma 2.7 and Theorem 2.2, we can extend the perturbation theorem of Xiao and Liang in [19, Theorem 1.3.5] concerning exponentially bounded $\alpha$-times integrated $C$-semigroups in which $B$ is a bounded linear operator on $X$ and $C = I_X$.

**Theorem 2.8.** *Let $\alpha \geq 0$, and $A$ be the generator of an exponentially bounded nondegenerate $\alpha$-times integrated $C$-semigroup $S_\alpha(\cdot)$ on $X$. Assume that $B \in B(\overline{D(A)}), BC = CB$ on $\overline{D(A)}$ and $BA \subset AB$. Then $A + B$ generates an exponentially bounded nondegenerate $\alpha$-times integrated $C$-semigroup $T_\alpha(\cdot)$ on $X$ given by*

$$T_\alpha(t)x = \sum_{k=0}^{\infty}\binom{\alpha}{k}(-1)^k B^k T^k N_\alpha(t)x \quad \textit{for all } x \in X \textit{ and } t \geq 0. \tag{2.1}$$

*Here $T^0 f(t) = f(t)$, $(Tf)(t) = \int_0^t f(s)ds$, $(T^k f)(t) = T(T^{k-1}f)(t)$ and $N_\alpha(t) = e^{tB}S_\alpha(t)$ for all $t \geq 0$ and $k \in \mathbb{N}$. Moreover, $T_\alpha(\cdot)$ is also exponentially Lipschitz continuous or norm continuous if $S_\alpha(\cdot)$ is.*

*Proof.* For simplicity we may assume that $\|S_\alpha(t)\| \leq Ke^{\omega t}$ for all $t \geq 0$ and for some fixed $K, \omega \geq 0$. By induction, we have $\|T^k N_\alpha(t)\| \leq Ke^{(\|B\|+\omega)t}/(\|B\| + \omega)^k$ for all $k \in \mathbb{N}\cup\{0\}$ and $t \geq 0$, and so $\|T_\alpha(t)\| \leq \sum_{k=0}^{\infty}|\binom{\alpha}{k}|\frac{\|B\|^k}{(\|B\|+\omega)^k}Ke^{(\|B\|+\omega)t} < \infty$ for all $t \geq 0$. Hence

$$\begin{aligned}
&\|T^k N_\alpha(t)x - T^k N_\alpha(s)x\| \\
=&\|T[T^{k-1}N_\alpha](t)x - T[T^{k-1}N_\alpha](s)x\| \\
=&\|\int_s^t T^{k-1}N_\alpha(r)xdr\| \\
\leq&\int_s^t K\frac{e^{(\|B\|+\omega)r}}{(\|B\|+\omega)^{k-1}}dr\|x\| \\
\leq&(t-s)K\frac{e^{(\|B\|+\omega)t}}{(\|B\|+\omega)^{k-1}}\|x\|
\end{aligned} \tag{2.2}$$

for all $k \in \mathbb{N}$, $x \in X$ and $t \geq s \geq 0$. Since

$$\begin{aligned} &\|N_\alpha(t)x - N_\alpha(s)x\| \\ =&\|e^{tB}S_\alpha(t)x - e^{sB}S_\alpha(s)x\| \\ =&\|(e^{tB} - e^{sB})S_\alpha(t)x + e^{sB}(S_\alpha(t)x - S_\alpha(s)x)\| \\ \leq&(t-s)e^{t\|B\|}Ke^{\omega t}\|x\| + e^{t\|B\|}\|S_\alpha(t)x - S_\alpha(s)x\| \end{aligned} \tag{2.3}$$

for all $x \in X$ and $t \geq s \geq 0$, we have

$$\begin{aligned} &\|T_\alpha(t)x - T_\alpha(s)x\| \\ \leq&\sum_{k=1}^{\infty} |\left(\begin{smallmatrix}\alpha \\ k\end{smallmatrix}\right)| \frac{\|B\|^k}{(\|B\|+\omega)^{k-1}}(t-s)Ke^{(\|B\|+\omega)t}\|x\| \\ &+ (t-s)Ke^{(\|B\|+\omega)t}\|x\| + e^{\|B\|t}\|S_\alpha(t)x - S_\alpha(s)x\| \end{aligned} \tag{2.4}$$

for all $x \in X$ and $t \geq s \geq 0$, which implies that $T_\alpha(\cdot)$ is a strongly continuous family in $B(X)$, and is also exponentially Lipschitz continuous or norm continuous if $S_\alpha(\cdot)$ is. Since $S_\alpha(t)X \subset \overline{D(A)}$ for all $t \geq 0$, $CB = BC$ on $\overline{D(A)}$ and $S_\alpha(\cdot)C = CS_\alpha(\cdot)$ on $X$, we have $T_\alpha(\cdot)C = CT_\alpha(\cdot)$ on $X$. Applying Lemma 2.7, we also have

$$\begin{aligned} \|(\lambda - A)^{-k}C\| &\leq \sum_{j=0}^{k-1} |\left(\begin{smallmatrix}\alpha \\ j\end{smallmatrix}\right)| \lambda^{\alpha-j}\frac{K}{(k-1-j)!}\int_0^\infty e^{-(\lambda-\omega)t}t^{k-1-j}dt \\ &= \sum_{j=0}^{k-1} |\left(\begin{smallmatrix}\alpha \\ j\end{smallmatrix}\right)| \lambda^{\alpha-j}\frac{K}{(\lambda-\omega)^{k-j}} \\ &= \sum_{j=0}^{k-1} |\left(\begin{smallmatrix}\alpha \\ j\end{smallmatrix}\right)| (\frac{\lambda-\omega}{\lambda})^j K\lambda^\alpha/(\lambda-\omega)^k \\ &= K_\alpha/(\lambda-\omega)^k \end{aligned}$$

for all $k \in \mathbb{N}$. Here $K_\alpha = \sum\limits_{j=0}^{\infty} |\left(\begin{smallmatrix}\alpha \\ j\end{smallmatrix}\right)| (\frac{\lambda-\omega}{\lambda})^j K\lambda^\alpha$. Now let $R_\lambda = \sum\limits_{k=0}^{\infty} B^k(\lambda - A)^{-k-1}C$ for $\lambda > \|B\| + \omega$, then

$$\begin{aligned} \sum_{k=0}^{\infty} \|B^k(\lambda-A)^{-k-1}C\| &\leq K_\alpha \sum_{k=0}^{\infty} \|B\|^k/(\lambda-\omega)^{k+1} \\ &= K_\alpha/(\lambda - \omega - \|B\|) \end{aligned}$$

for all $\lambda > \|B\|+\omega$. Combining this and the closedness of $A+B : D(A) \subset X \to X$ with the assumption $BA \subset AB$, we have $(\lambda - A - B)R_\lambda = C$ on $\overline{X}$ for all $\lambda > \|B\| + \omega$, which together with the assumption $CB = BC$ on $\overline{D(A)}$ implies that $R_\lambda(\lambda - A - B) = C$ on $D(A)$ for all $\lambda > \|B\| + \omega$. By hypothesis, we have $C^{-1}(A + B)C = C^{-1}AC + C^{-1}BC = A + B$. Applying Lemma 2.7 again, we have

$$\begin{aligned}
R_\lambda x &= \sum_{k=0}^{\infty} B^k \sum_{j=0}^{k} \binom{\alpha}{j} \frac{(-1)^j}{(k-j)!} \lambda^{\alpha-j} \int_0^\infty e^{-\lambda t} t^{k-j} S_\alpha(t)x dt \\
&= \sum_{j=0}^{\infty} \binom{\alpha}{j} \lambda^{\alpha-j} (-1)^j \sum_{k=j}^{\infty} \frac{B^k}{(k-j)!} \int_0^\infty e^{-\lambda t} t^{k-j} S_\alpha(t)x dt \\
&= \sum_{j=0}^{\infty} \binom{\alpha}{j} \lambda^{\alpha-j} (-1)^j \sum_{k=0}^{\infty} \frac{B^{k+j}}{k!} \int_0^\infty e^{-\lambda t} t^{k} S_\alpha(t)x dt \\
&= \sum_{j=0}^{\infty} \binom{\alpha}{j} \lambda^{\alpha-j} (-1)^j B^j \int_0^\infty e^{-\lambda t} e^{Bt} S_\alpha(t)x dt \\
&= \sum_{j=0}^{\infty} \binom{\alpha}{j} \lambda^{\alpha} (-1)^j B^j \int_0^\infty e^{-\lambda t} T^j N_\alpha(t)x dt \\
&= \lambda^\alpha \int_0^\infty e^{-\lambda t} \sum_{j=0}^{\infty} \binom{\alpha}{j} (-1)^j B^j T^j N_\alpha(t)x dt \\
&= \lambda^\alpha \int_0^\infty e^{-\lambda t} T_\alpha(t)x dt
\end{aligned}$$

for all $x \in X$ and $\lambda > \|B\| + \omega$. We obtain from Theorem 2.2 that $T_\alpha(\cdot)$ is an exponentially bounded nondegenerate $\alpha$-times integrated $C$-semigroup on $X$ with generator $A+B$, and is also exponentially Lipschitz continuous or norm continuous if $S_\alpha(\cdot)$ is.

Next we deduce a new perturbation theorem concerning local $\alpha$-times integrated $C$-semigroups. In particular, the exponential boundedness of $T_\alpha(\cdot)$ in Theorem 2.8 can be deleted when $R(C) \subset \overline{D(A)}$ and $BS_\alpha(\cdot) = S_\alpha(\cdot)B$ on $\overline{D(A)}$ both are added.

**Theorem 2.9.** *Let $\alpha \geq 0$, and $A$ be the generator of a nondegenerate local $\alpha$-times integrated $C$-semigroup $S_\alpha(\cdot)$ on $X$. Assume that $R(C) \subset \overline{D(A)}$, $B$ is a bounded linear operator on $\overline{D(A)}$ which commutes with $S_\alpha(\cdot)$ and $C$ on $\overline{D(A)}$ and $BA \subset AB$. Then $A + B$ generates a nondegenerate local $\alpha$-times integrated $C$-semigroup $T_\alpha(\cdot)$ on $X$ which is given as in (2.1). Moreover, $T_\alpha(\cdot)$ is also locally Lipschitz continuous or norm continuous if $S_\alpha(\cdot)$ is.*

*Proof.* Clearly, $j_\alpha(\cdot)I_{\overline{D(A)}}$ is an exponentially bounded $\alpha$-times integrated semigroup on $\overline{D(A)}$ with generator $0$. It follows from Theorem 2.8 that $\sum_{k=0}^{\infty}\binom{\alpha}{k}(-1)^k B^k T^k M_\alpha(\cdot)$ is an exponentially bounded nondegenerate $\alpha$-times integrated semigroup on $\overline{D(A)}$ with generator $B$. Here $M_\alpha(t) = e^{tB}j_\alpha(t)I_{\overline{D(A)}}$ and $j_\alpha(t) = \frac{t^\alpha}{\Gamma(\alpha+1)}$ for all $t \geq 0$. Combining this and (1.9) with the assumption $R(C) \subset \overline{D(A)}(= D(B))$, we have

$$BT\sum_{k=0}^{\infty}\binom{\alpha}{k}(-1)^k B^k T^k M_\alpha(t)Cx - \sum_{k=0}^{\infty}\binom{\alpha}{k}(-1)^k B^k T^k M_\alpha(t)Cx = -j_\alpha(t)Cx$$

for all $x \in X$ and $0 \leq t < T_0$. Just as in the proof of Theorem 2.8, it is easy to see from the boundedness of $\{\|S_\alpha(t)\| | 0 \leq t \leq t_0\}$ for all $0 < t_0 < T_0$ and the strong continuity of $S_\alpha(\cdot)$ that we have the following inequalities:

$$\|T^k N_\alpha(t)\| \leq K_t e^{\|B\|t}\frac{t^k}{k!} \quad \text{for all } k \in \mathbb{N}\cup\{0\} \text{ and } 0 \leq t < T_0; \tag{2.5}$$

$$\|T_\alpha(t)\| \leq \sum_{k=0}^{\infty}\left|\binom{\alpha}{k}\right|\frac{\|B\|^k t^k}{k!}K_t e^{\|B\|t} < \infty \quad \text{for all } 0 \leq t < T_0; \tag{2.6}$$

$$\begin{aligned}&\|T^k N_\alpha(t)x - T^k N_\alpha(s)x\| \leq (t-s)K_t e^{\|B\|t}\frac{t^{k-1}}{(k-1)!}\|x\|\\ &\text{for all } k \in \mathbb{N}, x \in X \text{ and } 0 \leq s \leq t < T_0;\end{aligned} \tag{2.7}$$

$$\begin{aligned}&\|N_\alpha(t)x - N_\alpha(s)x\| \leq (t-s)e^{t\|B\|}K_t\|x\| + e^{t\|B\|}\|S_\alpha(t)x - S_\alpha(s)x\|\\ &\text{for all } x \in X \text{ and } 0 \leq s \leq t < T_0;\end{aligned} \tag{2.8}$$

$$\begin{aligned}\|T_\alpha(t)x - T_\alpha(s)x\| &\leq \sum_{k=1}^{\infty}\left|\binom{\alpha}{k}\right|\frac{\|B\|^k t^{k-1}}{(k-1)!}(t-s)K_t e^{\|B\|t}\|x\|\\ &\quad + (t-s)K_t e^{\|B\|t}\|x\| + e^{\|B\|t}\|S_\alpha(t)x - S_\alpha(s)x\|\\ &\text{for all } x \in X \text{ and } 0 \leq s \leq t < T_0.\end{aligned} \tag{2.9}$$

Here $K_t = \sup_{0\leq r\leq t}\|S_\alpha(r)\|$. In particular, $T_\alpha(\cdot)$ is a strongly continuous family in $B(X)$, and is also locally Lipschitz continuous or norm continuous if $S_\alpha(\cdot)$ is. Since $BA \subset AB$ and $TS_\alpha(t)x \in D(A)$ for all $x \in X$ and $0 \leq t < T_0$, we have

$$
\begin{aligned}
&ATN_\alpha(t)x\\
&= A\int_0^t e^{sB}S_\alpha(s)xds\\
&= A[e^{tB}TS_\alpha(t)x - B\int_0^t e^{sB}TS_\alpha(s)xds]\\
&= Ae^{tB}TS_\alpha(t)x - AB\int_0^t e^{sB}TS_\alpha(s)xds\\
&= e^{tB}ATS_\alpha(t)x - \int_0^t Be^{sB}ATS_\alpha(s)xds\\
&= e^{tB}[S_\alpha(t)x - j_\alpha(t)Cx] - B\int_0^t e^{sB}[S_\alpha(s) - j_\alpha(t)Cx]ds\\
&= N_\alpha(t)x - M_\alpha(t)Cx - B[TN_\alpha(t)x - TM_\alpha(t)Cx]
\end{aligned} \tag{2.10}
$$

for all $x \in X$ and $0 \le t < T_0$. This implies that

$$
\begin{aligned}
&ATT_\alpha(t)x\\
&= A\sum_{k=0}^{\infty} \binom{\alpha}{k} (-1)^k B^k T^{k+1} N_\alpha(t)x\\
&= \sum_{k=0}^{\infty} \binom{\alpha}{k} (-1)^k AB^k T^{k+1} N_\alpha(t)x\\
&= \sum_{k=0}^{\infty} \binom{\alpha}{k} (-1)^k B^k AT^{k+1} N_\alpha(t)x\\
&= \sum_{k=0}^{\infty} \binom{\alpha}{k} (-1)^k B^k T^k ATN_\alpha(t)x\\
&= \sum_{k=0}^{\infty} \binom{\alpha}{k} (-1)^k B^k T^k ([N_\alpha(t)x - M_\alpha(t)Cx] - B[TN_\alpha(t)x - TM_\alpha(t)Cx])\\
&= \sum_{k=0}^{\infty} \binom{\alpha}{k} (-1)^k B^k T^k N_\alpha(t)x - BT\sum_{k=0}^{\infty} \binom{\alpha}{k} (-1)^k B^k T^k N_\alpha(t)x +\\
&\quad BT\sum_{k=0}^{\infty} \binom{\alpha}{k} (-1)^k B^k T^k M_\alpha(t)Cx - \sum_{k=0}^{\infty} \binom{\alpha}{k} (-1)^k B^k T^k M_\alpha(t)Cx\\
&= T_\alpha(t)x - BTT_\alpha(t)x - j_\alpha(t)Cx
\end{aligned} \tag{2.11}
$$

for all $x \in X$ and $0 \le t < T_0$ or equivalently, $(A+B)TT_\alpha(t)x = T_\alpha(t)x - j_\alpha(t)Cx$ for all $x \in X$ and $0 \le t < T_0$. To show that $T_\alpha(\cdot)$ is a nondegenerate local $\alpha$-times integrated $C$-semigroup on $X$ with generator $A+B$ it suffices to show that the abstract Cauchy problem $ACP(A+B, 0, 0)$ $u' = (A+B)u$ on $[0, T_0)$ and

$u(0) = 0$, has only the zero solution in $C^1([0, T_0), X) \cap C([0, T_0), [D(A+B)])$(see [7, Theorem 2.3] or [11, Theorem 5.1]). Here $[D(A + B)]$ denotes the Banach space $D(A + B) = D(A)$ with norm $|\cdot|_{A+B}$ defined by $|x|_{A+B} = \|x\| + \|(A+B)x\|$ for all $x \in D(A + B)$. Indeed, if $u$ is a solution of $ACP(A + B, 0, 0)$ in $C^1([0, T_0), X) \cap C([0, T_0), [D(A + B)])$. Applying the closedness of $A + B$ and the assumption $u(0) = 0$, we have

$$
\begin{aligned}
T_\alpha * u &= T(T_\alpha * u)' \\
&= T(T_\alpha * u') \\
&= TT_\alpha * (A + B)u \\
&= T(A + B)T_\alpha * u \\
&= (A + B)T(T_\alpha * u) \\
&= T_\alpha * u - Cj_\alpha * u
\end{aligned}
$$

on $[0, T_0)$, and so $Cj_\alpha * u = 0$ on $[0, T_0)$. Hence $u = 0$ on $[0, T_0)$. Consequently, $T_\alpha(\cdot)$ is a nondegenerate local $\alpha$-times integrated $C$-semigroup on $X$ with generator $A + B$, and is also locally Lipschitz continuous or norm continuous if $S_\alpha(\cdot)$ is.

**Corollary 2.10.** *Let $A$ be the generator of a (nondegenerate) local $C$-semigroup $S(\cdot)$ on $X$. Assume that $B$ is a bounded linear operator on $\overline{D(A)}$ which commutes with $S(\cdot)$ on $\overline{D(A)}$ and $BA \subset AB$. Then $A + B$ generates a (nondegenerate) local $C$-semigroup $T(\cdot)$ on $X$. Moreover, $T(\cdot)$ is also locally Lipschitz continuous or norm continuous if $S(\cdot)$ is.*

By slightly modifying the proof of Theorem 2.9 we also obtain the next perturbation theorem concerning local $\alpha$-times integrated $C$-semigroups which has been deduced by Li and Shaw in [10] when $T_0 = \infty$.

**Theorem 2.11.** *Let $\alpha \geq 0$, and $A$ be the generator of a nondegenerate local $\alpha$-times integrated $C$-semigroup $S_\alpha(\cdot)$ on $X$. Assume that $B$ is a bounded linear operator on $X$ which commutes with $S_\alpha(\cdot)$ and $C$ on $X$. Then $A + B$ generates a nondegenerate local $\alpha$-times integrated $C$-semigroup $T_\alpha(\cdot)$ on $X$ which is given as in (2.1). Moreover, $T_\alpha(\cdot)$ is also locally Lipschitz continuous (, norm continuous, exponentially bounded or exponentially Lipschitz continuous) if $S_\alpha(\cdot)$ is.*

*Proof.* Just as in the proof of Theorem 2.9, the assumption $R(C) \subset \overline{D(A)}$ is only used to show that $M_\alpha(\cdot)C$ is well-defined and (2.11) holds but both are automatically satisfied when $B \in B(X)$, and the assumption $BA \subset AB$ is only used to show that (2.10) holds but this is automatically satisfied if it is replaced by assuming that $B$ is a bounded linear operator which commutes with $S_\alpha(\cdot)$ and $C$ on $X$. Therefore, the conclusion of this theorem is true.

**Corollary 2.12.** *Let $A$ be the generator of a (nondegenerate) local $C$-semigroup $S(\cdot)$ on $X$. Assume that $B$ is a bounded linear operator on $X$ which commutes with $S(\cdot)$ on $X$. Then $A+B$ generates a (nondegenerate) local $C$-semigroup $T(\cdot)$ on $X$. Moreover, $T(\cdot)$ is also locally Lipschitz continuous(, norm continuous, exponentially bounded or exponentially Lipschitz continuous) if $S(\cdot)$ is.*

Similarly, the conclusion of Theorem 2.8 can be extended to the context of local $\alpha$-times integrated $C$-semigroups when $B$ is a bounded linear operator on $X$.

**Theorem 2.13.** *Let $\alpha \geq 0$, and $A$ be the generator of a nondegenerate local $\alpha$-times integrated $C$-semigroup $S_\alpha(\cdot)$ on $X$. Assume that $B$ is a bounded linear operator on $X$ such that $BC = CB$ on $\overline{D(A)}$ and $BA \subset AB$. Then $A+B$ generates a nondegenerate local $\alpha$-times integrated $C$-semigroup $T_\alpha(\cdot)$ on $X$ which is given as in (2.1). Moreover, $T_\alpha(\cdot)$ is also locally Lipschitz continuous (, norm continuous, exponentially bounded or exponentially Lipschitz continuous) if $S_\alpha(\cdot)$ is.*

**Corollary 2.14.** *Let $A$ be the generator of a (nondegenerate) local $C$-semigroup $S(\cdot)$ on $X$. Assume that $B$ is a bounded linear operator on $X$ such that $BC = CB$ on $\overline{D(A)}$ and $BA \subset AB$. Then $A+B$ generates a (nondegenerate) local $C$-semigroup $T(\cdot)$ on $X$. Moreover, $T(\cdot)$ is also locally Lipschitz continuous(, norm continuous, exponentially bounded or exponentially Lipschitz continuous) if $S(\cdot)$ is.*

We end this paper with a simple illustrative example. Let $X = C_b(\mathbb{R})$( or $L^\infty(\mathbb{R})$), and $A$ be the maximal differential operator in $X$ defined by $Au = \sum_{j=0}^{k} a_j D^j u$ on $\mathbb{R}$ for all $u \in D(A)$, then $Y= UC_b(\mathbb{R})$( or $C_0(\mathbb{R})$)= $\overline{D(A)}$. Here $a_0, a_1, \cdots, a_k \in \mathbb{C}$ and $D^j u(x) = u^{(j)}(x)$ for all $x \in \mathbb{R}$. It is shown in [12] that for each $\alpha > \frac{1}{2}$, $A$ generates an exponentially bounded, norm continuous $\alpha$-times integrated semigroup $S_\alpha(\cdot)$ on $X$ which is defined by $(S_\alpha(t)f)(x) = \frac{1}{\sqrt{2\pi}}(\widetilde{\phi_{\alpha,t}} * f)(t)$ for all $f \in X$ and $t \geq 0$ if the polynomial $p(x) = \sum_{j=0}^{k} a_j (ix)^j$ satisfies $\sup_{x\in\mathbb{R}} Re(p(x)) < \infty$. Here $\widetilde{\phi_{\alpha,t}}$ denotes the inverse Fourier transform of $\phi_{\alpha,t}$ with $\phi_{\alpha,t}(x) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} e^{p(x)s} ds$. An application of Theorem 2.8 shows that for each $B \in B(Y)$ and $BA \subset AB$, $A+B$ generates an exponentially bounded, norm continuous $\alpha$-times integrated semigroup $T_\alpha(\cdot)$ on $X$ which satisfies (2.1).

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Chung-Cheng Kuo
Department of Mathematics,
Fu Jen University,
Taipei 24205,
Taiwan
E-mail: cckuo@math.fju.edu.tw