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# KINDS OF VECTOR INVEX

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**Abstract.** Necessary Lagrangian conditions for a constrained minimum become sufficient under generalized convex assumptions, in particular *invex,* and duality results follow. Many classes of vector functions with properties related to *invex* have been studied, but It has not been clear how far these classes are distinct. Various inclusions between these classes are now established Some modifications of invex can be regarded as perturbations of invex. There is a stability criterion for when the invex property is preserved under small perturbations. Some results extend to nondifferentiable (Lipschitz) functions.

## 1. INTRODUCTION

For a differentiable constrained optimization problem, necessary Lagrangian conditions for a minimum become also sufficient under convex assumptions, and various duality results follow. It is well known that *convex* can be weakened to various kinds of *generalized convex.* Many classes of generalized convex functions have been defined and studied. To what extent are these classes significantly different? Many are variants of *invex* vector functions, for which the vector function $F(\cdot)$ satisfies $F(x)-F(p) \geq F'(p)\eta(x,p)$, with a *scale function* $\eta(\cdot,\cdot)$ replacing $x-p$ for convex functions, and relate closely to duality questions for optimization problems.

Invexity for a vector function is much more restrictive than invexity of the components, Various published result apply only to invex scalar functions, but do not hold for vector functions. with possibly different scale functions for each.

Zălinescu (2008) has discussed eight recent papers on generalizations of invexity, criticizing various imprecise definitions and trivial results. These include various theorems that assume "Condition C". However, the present paper deals with different classes of generalized invex functions, which are well defined, but the relations between them (and to Lagrangian conditions in optimization) need some study.

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Some generalized convexity properties are known to coincide, as e.g. in Capari (2003). However, those results relate to real-valued functions.and must be extended to vector-valued functions, to comply with the requirement that all component functions in the optimization problem are invex with respect to the same *scale function* $\eta$. But not all results for real-valued functions extend to vector functions. However, Capari's result that $(\mathcal{F}, \rho)$-invex coincides with $\rho$-invex extends to the vector case, under an additional hypothesis that holds in the known applications. But, for example, while Li, Dong and Liu (1997) show that a class of generalized-convex real-valued functions called *B-vex* coincides with the class of quasiconvex functions, this does not extend to vector-value functions, because vector-quasiconvex is not the same as quasiconvex for each component.

Several other modifications of invex mean that a certain perturbation of a vector function is invex. However, a slightly perturbed invex function may still be invex. A stability criterion is given (in Section 6) for this to happen. Some of the results extend to nondifferentiable (Lipschitz) functions (see Section 7).

The case of finitely many variables and components is presented. Similar result will hold for abstract spaces and cone constraints.

## 2. Basic Ideas

Consider the differentiable vector minimization problem:

$$(P1) \qquad \text{MIN } f(x) \text{ subject to } g(x) \leq 0,$$

with differentiable vector functions $f$ and $g$. For now, assume $f$ has one component, $g$ has $m$ components, and inequalities apply to each component. *Assume that a constraint qualification holds.* Then necessary Karush-Kuhn-Tucker conditions for a minimum are:

$$(KKT) \qquad f'(p) + \lambda g'(p) = 0,\ \lambda g(p) = 0,\ \lambda \geq 0, \quad g(p) \leq 0.$$

These necessary conditions become sufficient under some additional hypothesis. In particular, it suffices if $F := (f, g)$ is *invex.* Here $F$ is *invex at* $p$ if, for some *scale function* $\eta$:

$$(\forall x) F(x) - F(p) \geq F'(p)\eta(x,p). \qquad (invex);$$

and *invex* if it is invex at each $p$. Also $F$ is called invex at $p$ on $s$ domain $A$ if $F$ if the definition holds with $x$ restricted to $A$.

However, the additional hypothesis need not restrict $f$ when $f(x) < f(p)$, nor $g$ when $g(x) \not\leq 0$. This holds exactly with the well-known additional conditions that, for some (vector) *scale function* $\eta(\cdot,\cdot)$,

$$f(x) < f(p) \Rightarrow f'(p)\eta(x,p) < 0 \qquad \text{pseudoinvex;}$$

$$g'(p)\eta(x,p) \not\leq 0 \Rightarrow g(x) \not\leq g(p) \qquad \text{quasiinvex.}$$

The function $F$ my be called *pqvex* if $f$ and $g$ satisfy these conditions.

Since *pseudoconvex* is a special case of *pseudoinvex*, and *quasiconvex* is a special case of *quasiinvex*, these cases need not be discussed here.

Note that pseudoconveity of a vector function $f$ is not the same as pseudoconvexity of each component of $f$.

The function $F$ is *V-invex* if, for some scale function $\eta$ and positive scalars $\beta_i(x,p)$ :

$$(\forall x)(\forall i) F_i(x) - F_i(p) \geq F_i'(p)\beta_i(x,p)\eta(x,p).$$

These concepts may be applied, not only to sufficiency of (KKT), but also to the relation between (P1) and several possible *dual problems*, including:

$(P2)$ Wolfe dual: MAX $f(u) + vg(u)$ subject to $f'(u) + vg'(u) = 0,\ v \geq 0;$

$(P3)$ Mond-Weir dual: MAX $f(u)$ subject to $f'(u)+vg'(u)=0,\ v\geq 0; vg(u)\geq 0;$

$(P4)$ Lagrangian dual: MAX $_{v\geq 0}\mathrm{MIN}_u f(u) + vg(u).$

The additional hypothesis required for duality is different for the different duals, so the different kinds of invex are not all equivalent.

Th following theorem is Theorem 1 of Craven (2002). (Theorem 2.1 of Craven (2005) gives a generalization for cone-constraints.)

**Theorem 1.** (Characterization). *The differentiable function $F : \mathbf{R}^n \to \mathbf{R}^k$ is invex at $p$ if and only if:*

$$[0 \neq \alpha \geq 0,\ \alpha F'(p) = 0] \Rightarrow [0 \neq \alpha \geq 0,\ \alpha(F(x) - F(p)) \geq 0].$$

**Remark.** Since $[0 \neq \alpha \geq 0,\ \alpha(F(x) - F(p)) \geq 0] \Leftrightarrow F(x) - F(p) \not< 0$, Theorem 1 states that a weak stationary point of the vector function $F(\cdot)$ is a weak minimum exactly when $F(\cdot)$ is invex. This result needs modification when invex is generalized for a nonsmooth function (see Section 7).

The following theorem is a consequence of Theorem 1 (see Craven 2002, Theorem 3). The additional hypotheses on gradients exclude the case of a zero multiplier for $f$.

**Theorem 2.** *For the minimization problem (P), assume that the gradient $g'(\cdot)$ has full rank (omitting inactive constraints), and $f'(\cdot) \neq 0$.) Then the Wolfe dual and the Lagrangian dual are equivalent if and only if $(f, g)$ is invex.*

## 3. A Counter-example

Consider the problem (for fixed $b > 0$) :

$$\text{MIN } \phi(x) := 1 - \frac{1}{1+b+x} \text{ subject to } -x \leq 0. \qquad (P5).$$

The minimum is reached at $x = 0$, with Lagrange multiplier $\lambda = (1+b)^{-2}$. The problem is clearly pqvex and V-invex. If (P5) is invex, then:

$$(\exists \eta)\ \phi(x) - \phi(0) \geq \phi'(0)\eta \text{ and } -x + 0 \geq \eta.$$

Hence:

$$0 \leq \phi(x) - \phi(0) - \phi'(0)x = \frac{1}{2}\phi''(0)x^2 + o(x^2) = -(1+b)^{-}3x^2 + o(z^2) < 0,$$

a contradiction. The Wolfe dual has the form:

$$\text{MIN } \phi(u) + v(-u) \text{ subject to } v \geq 0,$$

for which $\phi(u) + \phi'(u)u$ is maximized when $u = 0$.

But the Lagrangian dual gives:

$$\text{MAX }_{y\geq 0} \text{ MIN }_{u\geq 0}\phi(u) = vu = \text{MAX }_{v\geq 0}\ 1 - v^{1/2} - v(v^{1/2} - 1 - p),$$

but this is maximized at $v = 0$, rather than at $v = (1 = b)^{-2}$.

To sum up, (P5) is pqvex and V-invex, but not invex. Wolfe duality and Mond-Weir duality work for (P5)¡ but not Lagrangian duality. For another counter-example, see Jeyakumar and Mond (1992).

## 4. Restricted Directions

Let $f$ be pseudoinvex. Let $f(x) - f(p) < 0$. Since then s$f'(p)\eta(x,p) < 0$ there is $\delta > 0$ so that $f'(p)\eta(x,p) + w < 0$ whenever $\|w\| < \delta$. For $\epsilon = \delta/\|f(x) - f(p)\|$, there follows (as in Craven (2005), Theorem .1):

$$f'(p) - \epsilon^{-1}(f(x) - f(p)) < 0.$$

Setting the scalar $\kappa(x,p) := \epsilon^{-1}$,

$$f(x) - f(p) \geq \kappa(x,p)f'(p)\eta(x,p) = f'(p)\kappa(x,p)\eta(x,p).$$

If the constraints inactive at $p$ are omitted; then the minimum is not affected, and $g(p) = 0$. If $g$ is quasiinvex, snd $g(x) \leq 0$, then $g'(p)\eta(x,p) < 0$. But when does the same $\eta$ apply to both $f$ and $g$?

Consider the property that, in the domain defined by $f(x) < f(u),\ g(x) \leq 0$,

$$\kappa(x,p)^{-1}(f(x) - f(p) \geq f'(p)\eta(x,p), /g'(p)\eta(x,p) \leq 0. \qquad \text{(dinvex)}$$

Applying Theorem 1 to $(f, 0)$ instead of to $(f, g)$ then shows that *dinvex* is equivalent to:

$$[0 \neq (\tau, \lambda) \geq 0, \tau f'(p) + \lambda g'(p) = 0] \Rightarrow \kappa(x.p)\tau(f(x) - f(p)) + \lambda(0) \geq 0$$

$$\Leftrightarrow \tau(f(x) - f(p) \geq 0 \qquad \Leftrightarrow \qquad f(x) - f(p) \not< 0$$

This has proved:

**Theorem 3.** *Subject to a constraint qualification, dinvex is equivalent to sufficient KKT.*

## 5. Relations

The following implications are thus established for differentiable problems. Note that each $\Rightarrow$ is not reversible.

Lagrange $\Leftrightarrow$ invex $\Rightarrow$ V-invex $\Rightarrow$ Wolfe, Mond-Weir

Wolfe, MW $\Leftarrow$ invex $\Rightarrow$ pqvex $\Rightarrow$ Wolfe, Mond-Weir

## 6. Perturbed Invex

A (vector) function $F(\cdot)$ is invex at $p$, thus $F(\cdot) - F(p) \geq F'(p)\eta(\cdot, p)$, exactly when $\tilde{F}(\cdot) := F(\cdot) - F(p)$ is invex at $p$, since $\tilde{F}'(p) = F'(p)$.

Let the vector function $F = (f_1, f_2, \ldots)$ be V-invex at $p$. Then:

$$(\forall x)(\forall j)\ \gamma_j(x,p)\tilde{f}_j(x) \geq f_j'(p)\eta(x,p),$$

where $\gamma_j(x,p) := \beta_j(x,p)^{-1}$.

**Theorem 4.** *The vector function $F = (f_1, f_2, \ldots)$ is V-invex at $p$ if and only if $\gamma \# F := (\gamma_1 f_1, \gamma_2 f_2, \ldots)$ is invex at $p$.*

*Proof.* Using $\gamma_j'(p)\tilde{f}_j(p) = 0$,

$$(\gamma_i \tilde{f}_j)(x) := \gamma_i(x)\tilde{f}_j)(x) \geq (\gamma_j \tilde{f}_j)'(p)\kappa_j \eta(x,p),$$

where $\kappa_j = \gamma_j(p,p)^{-1}$. Let $h(x) := (\gamma_i \tilde{f}_j)(x)$. Assume (without loss of generality) that $\eta(x,p) = x - p + \mathbf{o}(\|x-p\|)$. Setting $x - p = \alpha d$ with $\|d\| = 1$ and $\alpha > 0$,

$$h'(p)\alpha d + \mathbf{o}(\|x-p\|) \geq \kappa_j h'(p)\alpha d + \mathbf{o}(\|x-p\|),$$

hence

$$(\forall d)\ h'(p)(1-\kappa_j)\eta(x,p) \geq 0,\ \text{hence } = 0.$$

Hence $\kappa_j$ can be replaced by $1.$ Thus the vector function $\gamma\#F$ is invex at $p.$

**Remark.** A (vector) function $F(\cdot)$ is $\rho$*-invex* at $p$ if:

$$F(.) - F(p) \geq F'(p)\eta(\cdot,p) + \psi(\cdot)\rho,$$

where $\rho$ is a constant vector, and $\psi(x) > 0$ when $x \neq p$, $\psi(p) = 0$, and $\psi'(p) = 0$; in particular, $\psi(\cdot) = \|\cdot - p\|^2$. Then $F$ is $\rho$*-invex* at $p$ exactly when $F(\cdot) - \psi(\cdot)\rho$ is invex at $p$. Given a nonzero multiplier (row) vector $\zeta \geq 0$, it follows that a sufficient condition for $\zeta F(\cdot)$ to be invex at $p$ is that $\zeta\rho \geq 0$, for then $\zeta\psi(\cdot)\rho \geq 0$.

A (vector) function $F$ is *(*$\mathcal{F},\rho$*)-convex* at $p$ if:

$$(\forall x)\ F(x)) - F(p) \geq \mathcal{F}(x,p,F'(p)) + \psi(x)\rho,$$

where $\mathcal{F}(x,p,\cdot)$ i s sublinear increasing, and zero at $p$.

**Theorem 5.** *(*$\mathcal{F},\rho$*)-convex inplies* $\rho-$*invex.*

*Proof.* Define $\eta(x,p)$ as the gradient at $0$ of $\mathcal{F}(x,p,\cdot)$ at $0$. Then, from sublinear,

$$F(x) - F(p) \geq F'(p)\eta(x,p) + \psi(\cdot)\rho;$$

thus $F(\cdot)$ is $\rho-$invex at $p$.

Gulati and Gupta (2007) define *higher-order* $\mathcal{F}$*-convex* by (in the present notation):

$$F(x) - F(p) \geq \mathcal{F}(x,p,F'(u) + h_q(p,q)) + h(p,q) + q^T h_q(p,q,$$

where $h$ is an auxiliary function and $h_q$ denotes partial derivative. If $\mathcal{F}(x,p,\cdot)$ is sublinear, with gradient at $0$ denoted by $\eta(x,p)$, then the stated property implies a generalized invex property

$$F(x) - F(p) \geq (F'(u) + h_q(p,q))\eta(x,p) + h(p,q) + q^T h_q(p,q),$$

with $\eta(\cdot,\cdot)$ as scale function.

A *generalized B-vex* real function $f$ (Craven1995), generalizing *B-vex* of Weir and Mond (1988), satisfies

$$f(p + \Omega(\lambda, x-p) \leq (1 - bf(p) + bf(x),$$

in which $b = b(p,x,\lambda) \in (0,1)$. (While a B-vex real function is quasiconvex, this does not extend to vector functions.) If, .as $\lambda \downarrow 0$, there hold:

$$\Omega(\lambda,x,p) = \lambda\eta(x,p) + \mathbf{o}(\lambda) \text{ and } b(p,x,\lambda) = \lambda\theta(x,p) + \mathbf{o}(\lambda),$$

then

$$\theta(x,p)[f(x)-f(p)] \geq f'(p)\eta(x,p).$$

Thus, if each component $t$ of a vector function $F$ is generalized B-vex, with the same $\Omega$ for each (so $\eta$ is the same), but different $b$, and if each $\theta(\cdot,\cdot) > 0$, then $F$ is V-invex.

Given an invex vector function $F$, how far can $F$ be perturbed, while retaining the invex property? Assume that $x$ has $n$ components, $F$ has $m$ components, Consider a perturbation of $F(x)$ to $\hat{F}(x,q)$ where $q$ is a perturbation parameter, and $\tilde{F}(x,0) = F(x)$. Suppose, for given $x$ and $p$, that $F(x) - F(p) - F'(p)\eta \geq 0$, and $\tilde{F}(\cdot,\cdot)$ is $C^1$. When is there a solution $\xi$ to the inequality $\tilde{F}(x,q) - \tilde{F}(p,q) \geq M_q\xi$, where $M_q$ is the gradient of $\tilde{F}(x,q)$ with respect to $x$, and the perturbation parameter $q$ is small? According to Robinson's stability theorem (Robinson, 1976), if $\phi(\cdot,\cdot)$ is $C^1$,, $\phi(z.0) \leq 0$ has a solution $z = p$, and the stability condition holds:

$$0 \in \text{int}[\phi(p,0) + \text{range } \phi_z(\cdot,0) + \mathbf{R}^m_+],$$

then $\phi(z,q) \leq 0$ has a solution $z = z(q)$ when $\|q\|$ is sufficiently small, and, for some positive $\gamma$. the distance:

$$d(p, \{z : \phi(z,q) \leq 0\}) < \gamma d(0, \phi(p,q) + \mathbf{R}^r_+]).$$

Applying this to the linear system

$$c := F(x,q) - F(p,q) \geq M_q z,$$

where $F(x,0) = F(x)$, and $M_q = F_x(x,q)$, with a pertubation parameter $q$, it follows that a solution $z$ exists, when $\|q\|$ is small, if a stability condition holds. This has proved:

**Theorem 6.** *If $F(x) - F(p) \geq F'(p)\eta(x,p)$ for some $\eta$, and the stability condition holds:*

$$0 \in \textit{ int } [(F(x) - F(p)) + \textit{ range } (M_0) + \mathbf{R}^m_+],$$

*then $F(x,q) - F(p,q) \geq M_q\zeta(x,p)$ has a solution for sufficiently small $\|q\|$. Moreover, the distance from $\eta$ to the solution set of the inequality is less than a constsnt multiple of the distance from $F(x,q) - F(p,q) - M_q\zeta$ to $\mathbf{R}^m_+$.*

(Note that $M$ is not generally surjective).

## 7. Invex for Nonsmooth Vector Functions

In Craven (1988), a locally Lipschitz function $F : \mathbf{R}^n \to \mathbf{R}^k$ is called invex at $p$ if, for some scale function $\eta$:

$$(\forall x)\ F(x) - F(p) \geq F^o(p, \eta(x,p)),$$

where $F^o(p, \cdot)$ is the Clarke generalized gradient. An equivalent property is:

$$(\forall x)(\forall i)(\forall \xi \in \partial F_i(p))\ F(x) - F(p) \geq \xi\eta(x,p),$$

where $\partial F_i(p))$ is the Clarke generalized subdifferential. For another approach to nondifferentiable invex, see Craven (1995 and 1999).

For each $i$, choose $\xi_i \in \partial F_i(p))$; denote by $M$ the matrix with rows $\xi_1, \xi_2, \ldots$. If $F(\cdot)$ is invex, then the weaker property:

$$(\forall x)F(x) - F(p) \geq M\eta(x,p) \qquad \text{(partial invex)}$$

holds. If $M$ is substituted for $F'(p)$ in Theorem 1, then there follows:

$$[0 \neq \alpha \geq 0,\ \alpha M) = 0] \Rightarrow [0 \neq \alpha \geq 0,\ \alpha(F(x) - F(p)) \geq 0] \Leftrightarrow F(x) = F(p) \not< 0.$$

Subject to a constraint qualification, necessary conditions for a minimum at $x = p$ of $F_1(x)$ subject to $F_(x) \leq 0, F_3(x) \leq 0, \ldots$ are that there exist $\xi_i \in \partial F_i(p)$ and multipliers $\lambda_i \geq 0$, not all zero, such that:

$$\sum_i \lambda_i \xi_i = 0.$$

Under a constraint qualification, the multiplier $\lambda_1 \neq 0$. Hence there holds:

**Theorem 7.** *Subject to a constraint qualification, partial invex holds for some $\xi_i \in \partial F_i(p)$ if and only if the Wolfe dual and Lagrangian dual are equivalent.*

The perturbation discussion of Section 6 may be applied to *partial invex* for the npnsmooth case, with $F_x(x,q)$ replaced by a matrix $M_q$, constructed as $M$ above from the generalized gradient, but now depending on the parsmeter $q$.

The $(\mathcal{F}, \rho)$-convex property generalizes to the following;

$$(\forall x)(\forall \theta \in \partial F(p))\ F(x)) - F(p) \geq \mathcal{F}(x, p, \theta) + \psi(x)\rho,$$

where $\partial F(p)$ is the Clarke generalized differential.

**Theorem 8.** *Assume that $\mathcal{F}(x, p, \cdot)$ is sublinear and differentiable. Then the generalized $(\mathcal{F}, \rho)$-convex property implies $\rho$-invex, in terms of the Clarke generalized gradient.*

*Proof.* Denote by $\eta(x,p)$ the gradient at 0 of $\mathcal{F}(x, p, \cdot)$. Then

$$(\forall \theta \in \partial F(p))\ F(x) - F(p) \geq \theta\eta(x,p) + \psi(x)\rho.$$

Hence

$$F(x) - F(p) \geq F^0(p; \eta(x,p) + \psi(x)\rho.$$

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