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A GLOBAL CONVERGENCE THEOREM IN BOOLEAN ALGEBRA

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Dedicated to the Memory of Professor Sen-Yen Shaw

Abstract. Robert has established a global convergence theorem in $\{0,1\}^n$: If a map $\hat{F}$ from $\{0,1\}^n$ to itself is contracting relative to the boolean vector distance d, then there exists a positive integer $p \leq n$ such that $\hat{F}^p$ is constant. In other words, $\hat{F}$ has a unique fixed point ξ such that for any x in $\{0,1\}^n$, we have $\hat{F}^p(x) = \xi$. The structure $(\{0,1\}, +, \cdot, -, 0, 1)$ may be regarded as the two-element boolean algebra. In this paper, this result is extended to any map F from the product X of n finite boolean algebras to itself.

1. Introduction

In the course of Robert's analysis of boolean contraction and applications, he introduced the boolean vector distance, the discrete incidence matrix for the maps from $\{0,1\}^n$ to itself and the notion of spectra of boolean matrices [2, 4, 7]. In the first place, he proved the following characterizations for boolean contraction [4]:

Theorem 1.1. The following conditions are mutually equivalent:

(1) The map $\hat{F}$ from $\{0,1\}^n$ to itself is contracting to the boolean vector distance.
(2) $\rho(B(\hat{F})) = 0$ (the boolean spectra radius of the incidence matrix of $\hat{F}$ is zero).
(3) There exists a positive integer $p \leq n$, such that $(B(\hat{F}))^n = 0$.
(4) There exists an $n \times n$ permutation matrix P such that $P^T B(\hat{F}) P$ is strictly lower triangular.

This implies the Robert's global convergence theorem:

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Theorem 1.2. *If a map $\hat{F}$ from $\{0,1\}^n$ to itself is contracting relative to the boolean vector distance, then there exists a positive integer $p \leq n$ such that $\hat{F}^p$ is constant. Stated differently, $\hat{F}$ has a unique fixed point ξ, such that for any x in $\{0,1\}^n$, we have $\hat{F}^p(x) = \xi$.*

The spectral condition "$\rho(B(F)) = 0$" implies the conclusions, one is the existence of the unique fixed point, the other one concludes that this fixed point is a global attractor for the boolean network: $\hat{F}^p(x^r) = x^{r+1}$ $(r = 0, 1, \ldots)$ $[3, 4, 5, 6]$. We can obtain these conclusions by the weaker conditions which has been studied in $[9, 11]$. Also we can obtain the first conclusion alone by the condition with discrete Jacobian matrix which has been studied in the finite discrete case and the combinatorial boolean case $[1, 10]$. In this paper, we attempted to extend Theorem 1.2 in the finite boolean algebra case.

In order to extend this theorem from the $\{0,1\}$ case to the finite boolean algebra case, we will adopt notations from [8] to introduce here, the incidence matrix for maps from the product X of n finite boolean algebras to itself which generalizes Robert's incidence matrix for maps from $\{0,1\}^n$ to itself. Let us recall that a *boolean algebra* is a structure $(A, +, \cdot, -, 0, 1)$ with two binary operations $+$ and $\cdot$, a unary operation $-$, and two distinguished elements 0 and 1, such that the associativity, commutativity, absorption, distributivity and complementation laws hold. For example, let X be any set and $P(X)$ its power set. The structure $(P(X), \cup, \cap, -, \phi, X)$ where $-D$ is the complement $X \setminus D$ of D with respect to X, is a boolean algebra. $P(X)$ is called the *power set algebra* of X. A boolean algebra $(A, +, \cdot, -, 0, 1)$ is *finite* if its underlying set A is finite. Then, we present and prove the extension of the Robert's global convergence theorem: if the boolean spectra radius of the incidence matrix of a map F from the product X of n finite boolean algebras to itself is zero, then there exists a positive integer $p \leq n$ and a unique fixed point ξ such that, for any x in X, $F^p(x) = \xi$.

This paper ends with an example to illustrate the contraction of F is sufficient, but not necessary condition for the map F^n to be constant.

2. Incidence Matrix

In this section, we state some notions needed to formulate and to prove the main result.

Let $(A, +, \cdot, -, 0, 1)$ be a finite boolean algebra. Define $a \in A$ to be an *atom* of A if $0 < a$ but there is no x in A satisfying $0 < x < a$. We denoted by $At(A)$ the set of atoms of A. We say A is *atomic* if for each positive element x of A , there is some atom a such that $a \leq x$. We say A is *complete* if the least upper bound and the greatest lower bound of D belong to A for each $D \subseteq A$. Write the cardinality of $At(A)$ by $\#At(A)$ and the power set algebra of $At(A)$ by $P(At\,(A))$. Remark that for every boolean algebra A, the map φ^* from A to the power set algebra $P(At\,(A))$ defined by

$$\varphi^*(x) = \{a \in At(A) : a \leq x\}$$

is a homomorphism. It is an embedding if A is atomic, and f is an epimorphism if A is complete.(see [8, Proposition 2.6])

Given a finite boolean algebra $(A, +, \cdot, -, 0, 1)$ with $At(A) = \{a_1, \ldots, a_m\}$, consider a positive integer n, such that $X_i = A$ $(i = 1, \ldots, n)$. Then $X = X_1 \times \ldots \times X_n$ is a product of n finite boolean algebras. For $x = (x_1, \ldots, x_n) \in X$, we denoted by $\tilde{x}^{j(k)}$ the $j(k)$th *neighbour* of x ($j = 1, \ldots, n$; $k = 1, \ldots, m$) which is defined to be an element in X such that

if $i \neq j$, then $\tilde{x}_i^{j(k)} = x_i$ $\;(i = 1, \ldots, n)$,

if $i = j$ and $a_k \notin \varphi^*(x_j)$, then set $\tilde{x}_i^{j(k)}$ with $\varphi^*(\tilde{x}_j^{j(k)}) = \varphi^*(x_j) \cup \{a_k\}$,

if $i = j$ and $a_k \in \varphi^*(x_j)$, then set $\tilde{x}_i^{j(k)}$ with $\varphi^*(\tilde{x}_j^{j(k)}) = \varphi^*(x_j) \setminus \{a_k\}$.

Furthermore, the element γ_k in $\{0, 1\}^m$ is the m-tuple whose kth component is 1 and whose other components are 0. Therefore, if $m = n$ then it is the kth unit vector e_k of $\{0, 1\}^n$. Define the map η^* from the power set algebra $P(At(A))$ to$\{0, 1\}^m$ by

$$\eta^*(D) = \begin{cases} \mathbf{0} \text{ (zero vector)} & \text{if } D = \phi, \\ \gamma_k & \text{if } D = \{a_k\}, \\ \displaystyle\sum_{j \in \{k : a_k \in D\}} \gamma_j & \text{otherwise.} \end{cases}$$

For a map $F = (f_1, \ldots, f_n)$ from the product X of n finite boolean algebras to itself. Define a map $\bar{F} = (\bar{f}_{1(1)}, \ldots, \bar{f}_{1(m)}, \bar{f}_{2(1)}, \ldots, \bar{f}_{2(m)}, \ldots, \bar{f}_{n(1)}, \ldots, \bar{f}_{n(m)})$ from X to $\{0, 1\}^{nm}$ by

$$\bar{f}_{i(k)}(x) = [\eta^*(\varphi^*(f_i(x)))]_k \; (i = 1, \ldots, n; k = 1, \ldots, m).$$

Now, it is in position to introduce the notion of incidence matrix. Given a map $F = (f_1, \ldots, f_n)$ from X to itself. We denoted by

$$B(F) = \begin{pmatrix} b_{1(1)1(1)} \cdots b_{1(1)1(m)} & \cdots & b_{1(1)n(1)} \cdots b_{1(1)n(m)} \\ \vdots \ddots \vdots & \vdots \ddots \vdots & \vdots \ddots \vdots \\ b_{1(m)1(1)} \cdots b_{1(m)1(m)} & \cdots & b_{1(m)n(1)} \cdots b_{1(m)n(m)} \\ \cdots & \cdots & \cdots \\ \vdots \ddots \vdots & \vdots \ddots \vdots & \vdots \ddots \vdots \\ \cdots & \cdots & \cdots \\ b_{n(1)1(1)} \cdots b_{n(1)1(m)} & \cdots & b_{n(1)n(1)} \cdots b_{n(1)n(m)} \\ \vdots \ddots \vdots & \vdots \ddots \vdots & \vdots \ddots \vdots \\ b_{n(m)1(1)} \cdots b_{n(m)1(m)} & \cdots & b_{n(m)n(1)} \cdots b_{n(m)n(m)} \end{pmatrix}$$

$$= (b_{i(k_1)j(k_2)})$$

the *incidence matrix* of F. It is the $nm \times nm$ matrix over $\{0,1\}$ defined by

$$b_{i(k_1)j(k_2)} = \begin{cases} 0 & \text{if } \ \bar{f}_{i(k_1)}(x) = \bar{f}_{i(k_1)}(\tilde{x}^{j(k_2)}) \ \text{ for all } x \in X, \\ 1 & \text{otherwise.} \end{cases}$$

$(i, j = 1, \ldots, n; k_1, k_2 = 1, \ldots, m)$

Note that this incidence matrix is the Robert's $n \times n$ boolean incidence matrix [4] when $A = \{0,1\}$. We will state it in the next section.

Throughout this paper, a *boolean matrix* is meant to be a matrix over $\{0,1\}$. Here the incidence matrices are the boolean matrices. Boolean matrix multiplication and addition are the same as in the case of complex matrices but the concerned products of entries are boolean. A non-zero element $u \in \{0,1\}^n$ is called the (*boolean*) *eigenvector* of a boolean matrix M if there exists an λ in $\{0,1\}$ such that $Mu = \lambda u$; λ is called the (*boolean*) *eigenvalue* associated with eigenvector. For any boolean matrix M, the symbol $\sigma(M)$ denotes the (*boolean*) *spectrum* of M, it is the set of all eigenvalues of M, so that $\sigma(M) \subset \{0,1\}$. The (*boolean*) spectral radius of M, which is denoted by $\rho(M)$, is defined to be the largest eigenvalue of M.

3. Main Result

We shall establish the following theorem.

Theorem 3.1. *Given a finite boolean algebra* $(A, +, \cdot, -, 0, 1)$ *with* $At(A) = \{a_1, \ldots, a_m\}$. *Let* X *be the product of* n *finite boolean algebras with* $X_i = A$ $(i = 1, \ldots, n)$. *If a map* F *from* X *to itself is such that* $\rho(B(F)) = 0$, *then it has a unique fixed point* ξ *and there exists a positive integer* $p \leq nm$ *such that, for any* x *in* X, *we have* $F^p(x) = \xi$.

Let $A = \{0,1\}$. Its operations are given by the table below

a	b	$a+b$	$a \cdot b$	$-a$
0	0	0	0	1
0	1	1	0	1
1	0	1	0	0
1	1	1	1	0

Obviously, this structure $(\{0,1\}, +, \cdot, -, 0, 1)$ is a boolean algebra, it is called the two-element boolean algebra. Then $X = \{0,1\}^n$; hence the map F from X to itself is a boolean network.

The order "$\leq$" in $\{0,1\}$ is given by $0 \leq 0 \leq 1 \leq 1$. Then $At(\{0,1\}) = \{1\} = \{a_1\}$; hence $m = 1$. By the definitions of the maps φ^* and η^*, we obtain

$a \in A$	$\varphi^*(a)$	$\eta^*(\varphi^*(a))$
0	ϕ	0
1	$\{1\}$	1

,

which shows that $\bar{f}_{i(1)}(x) = (f_i(x))_1 = f_i(x)(i = 1, \ldots, n)$. Hence $\bar{F} = (\bar{f}_{1(1)}, \bar{f}_{2(1)}, \ldots, \bar{f}_{n(1)}) = (f_1, \ldots, f_n) = F$. Note also that

$\tilde{x}_j^{j(1)}$	$\varphi^*(\tilde{x}_j^{j(1)})$	$\varphi^*(x_j)$	x_j
0	ϕ	$\{1\}$	1
1	$\{1\}$	ϕ	0

,

hence $\tilde{x}_j^{j(1)} = -x_j$, and then $\tilde{x}^{j(1)} = (x_1, \ldots, -x_j, \ldots, x_n) = \tilde{x}^j$, which is the jth neighbor of x in $\{0,1\}^n$ [4], so that now the incidence matrix of F is the matrix over $\{0,1\}$ defined by

$$b_{i(1)j(1)} = \begin{cases} 0 & \text{if} \quad f_i(x) = f_i(\tilde{x}^j) \ \text{ for all } x \in \{0,1\}^n, \\ 1 & \text{otherwise.} \end{cases}$$

$(i, j = 1, \ldots, n)$. it is the Robert's $n \times n$ boolean incidence matrix. Hence, for $A = \{0,1\}$, this theorem is equivalent to the Robert's global convergence theorem (Theorem 1.2).

The following lemma will play a prominent role in the proof of the principal theorem.

Lemma 3.1. *Given a finite boolean algebra ($A, +, \cdot, -, 0, 1$) with $At(A) = \{a_1, \ldots, a_m\}$. Let X be the product of n finite boolean algebras with $X_i = A$ ($i = 1, \ldots, n$). For a map F from X to itself, there is a map $\hat{F}$ from $\{0,1\}^{nm}$ to itself and two isomorphisms η and φ such that*

$$F = (\eta\varphi)^{-1}\,\hat{F}\eta\varphi$$

Proof. Define the map φ from X into $[P\,(At(A))]^n$ by

$$\begin{aligned}\varphi\,(x) &= \varphi\,(x_1, ..., x_n)\\ &= (\varphi^*\,(x_1)\,, ..., \varphi^*\,(x_n)),\end{aligned}$$

where φ^* is defined in the section 2. Since A is a finite boolean algebra, $At(A)$ is finite and A is both complete and atomic. Hence φ^* is an isomorphism (see [8, Corollary 2.7]). Then φ is also an isomorphism.

Also we had defined the map η^* from the power set algebra $P(At(A))$ to $\{0,1\}^m$ in the section 2. For $y = (y_1, \ldots, y_m) \in \{0,1\}^m$, define $-y = (-y_1, \ldots, -y_m)$. Obviously, η^* is a bijection. For any $D_1,\ D_2 \in P(At(A))$

$$\begin{aligned}&\eta^*\,(D_1 \cup D_2)\\ &= \sum_{j \in \{k : a_k \in D_1 \cup D_2\}} e_j = \sum_{j \in \{k : a_k \in D_1\}} e_j + \sum_{j \in \{k : a_k \in D_2\}} e_j \ \text{(boolean sum)}\\ &= \eta^*\,(D_1) + \eta^*\,(D_2)\\ &\eta^*\,(D_1 \cap D_2)\end{aligned}$$

$$= \sum_{j\in\{k:a_k\in D_1\cap D_2\}} e_j = \sum_{j\in\{k:a_k\in D_1\}} e_j \cdot \sum_{j\in\{k:a_k\in D_2\}} e_j \text{(boolean vector product)}$$

$$= \eta^*(D_1)\cdot\eta^*(D_2)$$

$$\eta^*(\phi) = (0,...,0)$$

$$\eta^*(At(A)) = (1,...,1)$$

$$\eta^*(-D) = \sum_{j\in\{k:a_k\in -D\}} e_j = \sum_{j\in\{k:a_k\notin D\}} e_j = -\left(\sum_{j\in\{k:a_k\in D\}} e_j\right) = -\eta^*(D)\,.$$

Hence η^* is a homomorphism(see [8]), and so η^* is an isomorphism.

Define the map η from $[P(At(A))]^n$ to $\{0,1\}^{nm}$ by

$$\begin{aligned}\eta(D) &= \eta(D_1,...,D_n)\\ &= (\eta^*(D_1),...,\eta^*(D_n))\end{aligned}$$

Then η is also an isomorphism. For any $y \in \{0,1\}^{nm}$, throughout this paper, we will set

$$y = (y_{1(1)},\ldots,y_{1(m)},y_{2(1)},\ldots,y_{2(m)},\ldots,y_{n(1)},\ldots,y_{n(m)}),$$

and

$$\eta^{-1}(y) = ((\eta^*)^{-1}\left(y_{1(1)},\ldots,y_{1(m)}\right),...,(\eta^*)^{-1}\left(y_{n(1)},\ldots,y_{n(m)}\right)).$$

Now we can define a map

$$\hat{F} = (\hat{f}_{1(1)},\ldots,\hat{f}_{1(m)},\hat{f}_{2(1)},\ldots,\hat{f}_{2(m)},\ldots,\hat{f}_{n(1)},\ldots,\hat{f}_{n(m)})$$

from $\{0,1\}^{nm}$ to itself by

$$\hat{f}_{i(k)}(y) = [\eta^*(\varphi^*(f_i(\varphi^{-1}(\eta^{-1}(y)))))]_k \ (i=1,\ldots,n; k=1,\ldots,m).$$

$$\begin{array}{ccccc} X & \xrightarrow{\varphi} & [P(At(A))]^n & \xrightarrow{\eta} & \{0,1\}^{nm} \\ f_j\downarrow & & & & \downarrow\ \hat{f}_j \\ A & \xrightarrow{\varphi^*} & P(At(A)) & \xrightarrow{\eta^*} & \{0,1\}^m \end{array}$$

For $y \in \{0,1\}^{nm}$, we obtain

$$\hat{F}(x) = \begin{pmatrix} \hat{f}_{1(1)} \\ \vdots \\ \hat{f}_{n(m)} \end{pmatrix}(y) = \begin{pmatrix} [\hat{f}_{1(1)}(y),\cdots,\hat{f}_{1(m)}(y)]^T \\ \vdots \\ [\hat{f}_{n(1)}(y),\cdots,\hat{f}_{n(m)}(y)]^T \end{pmatrix}$$

$$= \begin{pmatrix} [\ [\eta^*\varphi^* f_1(\eta\varphi)^{-1}(y)]_1,\cdots,[\eta^*\varphi^* f_1(\eta\varphi)^{-1}(y)]_m]^T \\ \vdots \\ [\ [\eta^*\varphi^* f_n(\eta\varphi)^{-1}(y)]_1,\cdots,[\eta^*\varphi^* f_n(\eta\varphi)^{-1}(y)]_m]^T \end{pmatrix}$$

$$
= \begin{pmatrix} [\eta^*\varphi^* f_1\,(\eta\varphi)^{-1}\,(y)]^T \\ \vdots \\ [\eta^*\varphi^* f_n\,(\eta\varphi)^{-1}\,(y)]^T \end{pmatrix}
$$

$$
= \eta\varphi \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} (\eta\varphi)^{-1}\,(y) = \eta\varphi F\,(\eta\varphi)^{-1}\,(y)\,.
$$

Therefore, $\hat{F}$ is a map from $\{0,1\}^{nm}$ to itself such that $F = (\eta\varphi)^{-1}\,\hat{F}\eta\varphi$. ■

4. Proof of Theorem 3.1

Given a finite boolean algebra $(A, +, \cdot, -, 0, 1)$ with $At(A) = \{a_1, \ldots, a_m\}$. Let X be the product of n finite boolean algebras with $X_i = A$ ($i = 1, \ldots, n$). For a map F from X to itself is such that $\rho(B(F)) = 0$. By Lemma 3.1, there is a map $\hat{F}$ from $\{0,1\}^{nm}$ to itself and two isomorphisms η and φ such that $F = (\eta\varphi)^{-1}\,\hat{F}\eta\varphi$.

Let $B(F) = (b_{i(k_1)j(k_2)})$ be the incidence matrix of F. For the map

$$
\hat{F} = (\hat{f}_{1(1)}, \ldots, \hat{f}_{1(m)}, \hat{f}_{2(1)}, \ldots, \hat{f}_{2(m)}, \ldots, \hat{f}_{n(1)}, \ldots, \hat{f}_{n(m)}),
$$

here we quote the definition of the incidence matrix of $\hat{F}$ from [4]. It is the $nm \times nm$ matrix $B(\hat{F}) = (d_{i(k_1)j(k_2)})$ over $\{0,1\}$ defined by

$$
d_{i(k_1)j(k_2)} = \begin{cases} 0 & \text{if} \quad \hat{f}_{i(k_1)}(y) = \hat{f}_{i(k_1)}(\tilde{y}^{j(k_2)}) \quad \text{for all } y \in \{0,1\}^{nm}, \\ 1 & \text{otherwise.} \end{cases}
$$

$(i, j = 1, \ldots, n; k_1, k_2 = 1, \ldots, m)$, where

$$
\tilde{y}^{j(k_2)} = (y_{1(1)}, \ldots, y_{1(m)}, \ldots, -y_{j(k_2)}, \ldots, y_{n(1)}, \ldots, y_{n(m)})
$$

is the $j(k_2)$th neighbor of y [4]. Then

$$
\begin{aligned}
& b_{i(k_1)j(k_2)} = 0 \quad (i, j = 1, \ldots, n; k_1, k_2 = 1, \ldots, m) \\
\Leftrightarrow\ & \bar{f}_{i(k_1)}(x) = \bar{f}_{i(k_1)}(\tilde{x}^{j(k_2)}) \quad \text{for all } x \in X \\
\Leftrightarrow\ & [\eta^*(\varphi^*(f_i(x)))]_{k_1} = [\eta^*(\varphi^*(f_i(\tilde{x}^{j(k_2)})))]_{k_1} \quad \text{for all } x \in X \\
\Leftrightarrow\ & \hat{f}_{i(k_1)}(\eta^{-1}(\varphi^{-1}(x))) = \hat{f}_{i(k_1)}(\eta^{-1}(\varphi^{-1}(\tilde{x}^{j(k_2)}))) \quad \text{for all } x \in X \\
\Leftrightarrow\ & \hat{f}_{i(k_1)}(y) = \hat{f}_{i(k_1)}(\tilde{y}^{j(k_2)}) \quad \text{for all } y \in \{0,1\}^{nm} \\
\Leftrightarrow\ & d_{i(k_1)j(k_2)} = 0 \ (i, j = 1, \ldots, n; k_1, k_2 = 1, \ldots, m).
\end{aligned}
$$

So that $B(F) = B(\hat{F})$, we obtain $\rho(B(\hat{F})) = 0$. Combining Theorems 1.1 and 1.2, we see that $\hat{F}$ has a unique fixed point c.

Put $\xi = (\eta\varphi)^{-1}(c)$. Then $\xi \in X$ and we have

$$\begin{aligned} &\hat{F}(c) = c \\ \Rightarrow\ &\eta\varphi F(\eta\varphi)^{-1}(c) = c \\ \Rightarrow\ &F(\eta\varphi)^{-1}(c) = (\eta\varphi)^{-1}(c) \\ \Rightarrow\ &F(\xi) = \xi. \end{aligned}$$

Since $(\eta\varphi)^{-1}$ is an isomorphism, ξ is a unique fixed point of F.

Note also that there is a positive integer $p (\leq nm)$ such that $\hat{F}^p(y) = c$ for any $y \in \{0,1\}^{nm}$. For any $x \in X$, there exists an element y in $\{0,1\}^{nm}$ such that $x = (\eta\varphi)^{-1}(y)$. Then

$$\begin{aligned} &\hat{F}^p(y) = c \\ \Rightarrow\ &\eta\varphi F^p(\eta\varphi)^{-1}(y) = c \\ \Rightarrow\ &F^p(\eta\varphi)^{-1}(y) = (\eta\varphi)^{-1}(c) \\ \Rightarrow\ &F^p(x) = \xi. \end{aligned}$$

Hence p is also the positive integer such that $F^p(x) = \xi$ for any $x \in X$. This completes the proof of Theorem 3.1

5. Remarks

If F is a map from the product of n finite boolean algebras to itself. Then the spectral condition "$\rho(B(F)) = 0$" implies that F has a unique fixed point ξ, and it also implies there exists a positive integer $p \leq nm$ such that, for any x in X, we have $F^p(x) = \xi$. But this condition is not necessary for a map to obtain the result, as shown by the following example.

Example 1. If $A = \{0, a_1, a_2, 1\}$ with $0 < a_1 < 1$ and $0 < a_2 < 1$, such that this structure $(A, +, \cdot, -, 0, 1)$ is a boolean algebra. Then it is finite and $At(A) = \{a_1, a_2\}$.

Let $X = X_1 \times X_2$ with $X_i = A$ $(i = 1, 2)$. Let the map $F : X \to X$ be defined by

x	$(0,*)$	$(*,1)$	$(a_1,0)$	(a_1,a_1)	(a_1,a_2)
$F(x)$	$(1,1)$	$(1,1)$	$(1,a_1)$	$(1,1)$	$(1,a_1)$

x	$(a_2,0)$	(a_2,a_1)	(a_2,a_2)	$(1,0)$	$(1,a_1)$	$(1,a_2)$
$F(x)$	$(1,1)$	$(a_2,1)$	$(1,1)$	$(1,1)$	$(a_2,1)$	$(1,1)$

where $*$ is any element in A, Then F has a unique fixed point $(1,1)$. Choose $p = 3$ so that $p \leq nm = 4$ and for any x in X, we have $F^p(x) = (1,1)$. Now we consider the incidence matrix of F

$$B(F) = \begin{pmatrix} b_{1(1)1(1)} & b_{1(1)1(2)} & b_{1(1)2(1)} & b_{1(1)2(2)} \\ b_{1(2)1(1)} & b_{1(2)1(2)} & b_{1(2)2(1)} & b_{1(2)2(2)} \\ b_{2(1)1(1)} & b_{2(1)1(2)} & b_{2(1)2(1)} & b_{2(1)2(2)} \\ b_{2(2)1(1)} & b_{2(2)1(2)} & b_{2(2)2(1)} & b_{2(2)2(2)} \end{pmatrix}.$$

From

$a \in A$	$\varphi^*(a)$	$\eta^*(\varphi^*(a))$
0	ϕ	$(0,0)$
a_1	$\{a_1\}$	$(1,0)$
a_2	$\{a_2\}$	$(0,1)$
1	$\{a_1,a_2\}$	$(1,1)$

we have, for any $x = (x_1, x_2) \in X$,this map

$$\bar{f}_{i(k)}(x) = [\eta^*(\varphi^*(f_i(x)))]_k \quad (i = 1,2;\ k = 1,2).$$

is given by tables below.

x	$(0,*)$	$(*,1)$	$(a_1,0)$	(a_1,a_1)	(a_1,a_2)
$\bar{f}_{1(1)}(x)$	1	1	1	1	1
$\bar{f}_{1(2)}(x)$	1	1	1	1	1
$\bar{f}_{2(1)}(x)$	1	1	1	1	1
$\bar{f}_{2(2)}(x)$	1	1	0	1	0

x	$(a_2,0)$	(a_2,a_1)	(a_2,a_2)	$(1,0)$	$(1,a_1)$	$(1,a_2)$
$\bar{f}_{1(1)}(x)$	1	0	1	1	0	1
$\bar{f}_{1(2)}(x)$	1	1	1	1	1	1
$\bar{f}_{2(1)}(x)$	1	1	1	1	1	1
$\bar{f}_{2(2)}(x)$	1	1	1	1	1	1

Furthermore, the $j(k)$th neighbor of x is given by tables below.

x	$(0,0)$	$(0,a_1)$	$(0,a_2)$	$(0,1)$	$(a_1,0)$	(a_1,a_1)	(a_1,a_2)	$(a_1,1)$
$\tilde{x}^{1(1)}$	$(a_1,0)$	(a_1,a_1)	(a_1,a_2)	$(a_1,1)$	$(0,0)$	$(0,a_1)$	$(0,a_2)$	$(0,1)$
$\tilde{x}^{1(2)}$	$(a_2,0)$	(a_2,a_1)	(a_2,a_2)	$(a_2,1)$	$(1,0)$	$(1,a_1)$	$(1,a_2)$	$(1,1)$
$\tilde{x}^{2(1)}$	$(0,a_1)$	$(0,0)$	$(0,1)$	$(0,a_2)$	(a_1,a_1)	$(a_1,0)$	$(a_1,1)$	(a_1,a_2)
$\tilde{x}^{2(2)}$	$(0,a_2)$	$(0,1)$	$(0,0)$	$(0,a_1)$	(a_1,a_2)	$(a_1,1)$	$(a_1,0)$	(a_1,a_1)

x	$(a_2,0)$	(a_2,a_1)	(a_2,a_2)	$(a_2,1)$	$(1,0)$	$(1,a_1)$	$(1,a_2)$	$(1,1)$
$\tilde{x}^{1(1)}$	$(1,0)$	$(1,a_1)$	$(1,a_2)$	$(1,1)$	$(a_2,0)$	(a_2,a_1)	(a_2,a_2)	$(a_2,1)$
$\tilde{x}^{1(2)}$	$(0,0)$	$(0,a_1)$	$(0,a_2)$	$(0,1)$	$(a_1,0)$	(a_1,a_1)	(a_1,a_2)	$(a_1,1)$
$\tilde{x}^{2(1)}$	(a_2,a_1)	$(a_2,0)$	$(a_2,1)$	(a_2,a_2)	$(1,a_1)$	$(1,0)$	$(1,1)$	$(1,a_2)$
$\tilde{x}^{2(2)}$	(a_2,a_2)	$(a_2,1)$	$(a_2,0)$	(a_2,a_1)	$(1,a_2)$	$(1,1)$	$(1,0)$	$(1,a_1)$

Thus

$$B(F) = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix},$$

and this implies $\rho(B(F)) = 1$; hence the spectral condition "$\rho(B(F)) = 0$" fails to hold in this case. ■

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