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ON $|\overline{N}, p_n; \delta|_k$ SUMMABILITY FACTORS OF INFINITE SERIES

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Abstract. In this paper a general theorem on $|\overline{N}, p_n; \delta|_k$ summability factors, which generalizes a theorem of Bor [3] on $|\overline{N}, p_n|_k$ summability factors, is proved.

1. Introduction

Let $\sum a_n$ be a given infinite series with partial sums (s_n). Let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad \text{as} \quad n \to \infty, \quad (P_{-i} = p_{-i} = 0, \ i \geq 1).$$

The sequence-to-sequence transformation

$$u_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v$$

defines the sequence (u_n) of the $(\overline{N}, p_n)$ means of the sequence (s_n), generated by the sequence of coefficients (p_n) (see [5]) .

The series $\sum a_n$ is said to be summable $|\overline{N}, p_n|_k, \ k \geq 1$, if (see [1])

$$\sum_{n=1}^{\infty} (P_n/p_n)^{k-1} |u_n - u_{n-1}|^k < \infty,$$

and it is said to be summable $|\overline{N}, p_n; \delta|_k, \ k \geq 1$ and $\delta \geq 0$, if (see [2])

$$\sum_{n=1}^{\infty} (P_n/p_n)^{\delta k+k-1} |u_n - u_{n-1}|^k < \infty.$$

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In the special case when $\delta = 0$ (resp. $p_n = 1$ for all values of n) $|\overline{N}, p_n; \delta|_k$ summability is the same as $|\overline{N}, p_n|_k$ (resp. $|C, 1; \delta|_k$) summability.

Quite recently Bor [3] proved the following theorem for $|\overline{N}, p_n|_k$ summability factors of infinite series.

Theorem A. *Let (p_n) be a sequence of positive numbers such that*

$$P_n = O(np_n) \quad \text{as} \quad n \to \infty. \tag{1}$$

Let (X_n) be a positive non-decreasing sequence and suppose that there exists sequences (λ_n) and (β_n) such that

$$|\Delta\lambda_n| \leq \beta_n; \tag{2}$$

$$\beta_n \to 0 \quad \text{as} \quad n \to \infty; \tag{3}$$

$$\sum_{n=1}^{\infty} n|\Delta\beta_n|X_n < \infty; \tag{4}$$

$$|\lambda_n|X_n = O(1) \quad \text{as} \quad n \to \infty. \tag{5}$$

If

$$\sum_{n=1}^{m} \frac{p_n}{P_n}|t_n|^k = O(X_m) \quad \text{as} \quad m \to \infty, \tag{6}$$

where

$$t_n = \frac{1}{n+1}\sum_{v=1}^{n} va_v, \tag{7}$$

then the series $\sum a_n\lambda_n$ is summable $|\overline{N}, p_n|_k$, $k \geq 1$.

The aim of this paper is to generalize Theorem A for $|\overline{N}, p_n; \delta|_k$ summability. Now, we shall prove the following theorem.

Theorem. *Let (X_n) be a positive non-decreasing sequence and the sequences (λ_n) and (β_n) be such that conditions (2)-(5) of Theorem A are satisfied. If (p_n) is a sequence such that condition (1) of Theorem A is satisfied and*

$$\sum_{n=v+1}^{\infty} \left(\frac{P_n}{p_n}\right)^{\delta k-1} \frac{1}{P_{n-1}} = O\left\{\left(\frac{P_v}{p_v}\right)^{\delta k} \frac{1}{P_v}\right\}, \tag{8}$$

(9)
$$\sum_{n=1}^{m} \left(\frac{P_n}{p_n}\right)^{\delta k-1} |t_n|^k = O(X_m) \quad \text{as} \quad m \to \infty,$$

where (t_n) is as in (7), then the series $\sum a_n \lambda_n$ is summable $|\overline{N}, p_n; \delta|_k$ for $k \geq 1$ and $0 \leq \delta < 1/k$.

Remark: It may be noted that, if we take $\delta = 0$ in this theorem, then we get Theorem A. In this case condition (9) reduces to condition (6) and condition (8) reduces to

$$\sum_{n=v+1}^{\infty} \frac{p_n}{P_n P_{n-1}} = O(1/P_v),$$

which always holds.

We need the following lemma for the proof of our theorem.

Lemma ([4]). *If the conditions* (2)-(5) *on (X_n), (β_n) and (λ_n) are satisfied, then*

(10)
$$n\beta_n X_n = O(1) \quad \text{as} \quad m \to \infty,$$

(11)
$$\sum_{n=1}^{\infty} \beta_n X_n < \infty.$$

2. Proof of the Theorem

Let (T_n) be the sequence of $(\overline{N}, p_n)$ means of the series $\sum a_n \lambda_n$. Then, by definition, we have

$$T_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v \sum_{i=0}^{v} a_i \lambda_i = \frac{1}{P_n} \sum_{v=0}^{n} (P_n - P_{v-1}) a_v \lambda_v.$$

Then, for $n \geq 1$, we have

$$T_n - T_{n-1} = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_{v-1} a_v \lambda_v = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} \frac{P_{v-1}\lambda_v}{v} v a_v.$$

Using Abel's transformation, we get

$$\begin{aligned}
T_n - T_{n-1} &= \frac{(n+1)}{nP_n} p_n t_n \lambda_n - \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} p_v t_v \lambda_v \frac{v+1}{v} \\
&\quad + \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v \Delta\lambda_v t_v \frac{v+1}{v} + \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v t_v \lambda_{v+1} \frac{1}{v} \\
&= T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4}, \text{ say.}
\end{aligned}$$

Since $|T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4}|^k \leq 4^k \left(|T_{n,1}|^k + |T_{n,2}|^k + |T_{n,3}|^k + |T_{n,4}|^k \right)$, to complete the proof of the theorem, it is sufficient to show that

$$\sum_{n=1}^{\infty} (P_n/p_n)^{\delta k+k-1} |T_{n,r}|^k < \infty, \quad \text{for} \;\; r = 1, 2, 3, 4.$$

Since $\lambda_n = O(1/X_n) = O(1)$, by (5), we get that

$$\begin{aligned}
\sum_{n=2}^{m+1} (P_n/p_n)^{\delta K+k-1} |T_{n,1}|^k &= O(1) \sum_{n=1}^{m} |\lambda_n|^{k-1} |\lambda_n| (P_n/p_n)^{\delta k-1} |t_n|^k \\
&= O(1) \sum_{n=1}^{m} |\lambda_n| \left(\frac{P_n}{p_n}\right)^{\delta k-1} |t_n|^k \\
&= O(1) \sum_{n=1}^{m-1} \Delta|\lambda_n| \sum_{v=1}^{n} \left(\frac{P_v}{p_v}\right)^{\delta k-1} |t_v|^k \\
&\quad + O(1)|\lambda_m| \sum_{n=1}^{m} \left(\frac{P_n}{p_n}\right)^{\delta k-1} |t_n|^k \\
&= O(1) \sum_{n=1}^{m-1} |\Delta\lambda_n| X_n + O(1)|\lambda_m| X_m \\
&= O(1) \sum_{n=1}^{m-1} \beta_n X_n + O(1)|\lambda_m| X_m \\
&= O(1) \;\; \text{as} \;\; m \to \infty, \;\text{by}(2), (9) \text{ and } (11).
\end{aligned}$$

Now, applying Hölder's inequality with indices k and $\acute{k}$, where $\frac{1}{k} + \frac{1}{\acute{k}} = 1$, as in $T_{n,1}$, we have that

$$\begin{aligned}
&\sum_{n=2}^{m+1} (P_n/p_n)^{\delta k+k-1} |T_{n,2}|^k \\
&= O(1) \sum_{n=2}^{m+1} (P_n/p_n)^{\delta k-1} \frac{1}{P_{n-1}} \left\{ \sum_{v=1}^{n-1} p_v |t_v|^k |\lambda_v|^k \right\} \times \left\{ \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} p_v \right\}^{k-1} \\
&= O(1) \sum_{v=1}^{m} p_v |\lambda_v|^{k-1} |\lambda_v| \; |t_v|^k \sum_{n=v+1}^{m+1} (P_n/p_n)^{\delta k-1} \frac{1}{P_{n-1}} \\
&= O(1) \sum_{v=1}^{m} |\lambda_v| (P_v/p_v)^{\delta k-1} |t_v|^k = O \text{ as } \; m \to \infty.
\end{aligned}$$

Using the fact that $P_v = O(vp_v)$, by (1), and $n\beta_n = O(1/X_n) = O(1)$, by (10), we have that

$$\begin{aligned}
&\sum_{n=2}^{m+1} (P_n/p_n)^{\delta k+k-1} |T_{n,3}|^k \\
&= O(1) \sum_{n=2}^{m+1} (P_n/p_n)^{\delta k-1} \frac{1}{P_{n-1}} \left\{ \sum_{v=1}^{n-1} (v\beta_v)^k p_v |t_v|^k \right\} \times \left\{ \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} p_v \right\}^{k-1} \\
&= O(1) \sum_{v=1}^{m} (v\beta_v)(v\beta_v)^{k-1} p_v |t_v|^k \sum_{n=v+1}^{\infty} (P_n/p_n)^{\delta k-1} \frac{1}{P_{n-1}} \\
&= O(1) \sum_{v=1}^{m} (v\beta_v) \left(\frac{P_v}{p_v} \right)^{\delta k-1} |t_v|^k \\
&= O(1) \sum_{v=1}^{m-1} \Delta(v\beta_v) \sum_{i=1}^{v} (P_i/p_i)^{\delta k-1} |t_i|^k + O(1) m\beta_m \sum_{v=1}^{m} (P_v/p_v)^{\delta k-1} |t_v|^k \\
&= O(1) \sum_{v=1}^{m-1} |\Delta(v\beta_v)| X_v + O(1) m\beta_m X_m \\
&= O(1) \sum_{v=1}^{m-1} vX_v |\Delta\beta_v| + O(1) \sum_{v=1}^{m-1} \beta_{v+1} X_v + O(1) m\beta_m X_m \\
&= O(1) \text{ as } \; m \to \infty,
\end{aligned}$$

by (2), (4), (8), (9), (10) and (11).

Finally, using the fact that $P_v = O(vp_v)$, by (1), as in $T_{n,1}$ and $T_{n,2}$, we have that

$$\begin{aligned}
\sum_{n=1}^{m} (P_n/p_n)^{\delta k+k-1} |T_{n,4}|^k &= O(1) \sum_{v=1}^{m} |\lambda_{v+1}| (P_v/p_v)^{\delta k-1} |t_v|^k \\
&= O(1) \text{ as } m \to \infty.
\end{aligned}$$

Therefore, we get that

$$\sum_{n=1}^{m}(P_n/p_n)^{\delta k+k-1}|T_{n,r}|^k = O(1) \text{ as } m \to \infty, \text{ for } r = 1, 2, 3, 4.$$

This completes the proof of the theorem. If we take $p_n = 1$ for all values of n in this theorem, then we get a result concerning the $|C, 1; \delta|_k$ summability methods.

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