TAIWANESE JOURNAL OF MATHEMATICS
Vol. 13, No. 6B, pp. 2037-2049, December 2009
This paper is available online at http://www.tjm.nsysu.edu.tw/

OSCILLATION AND NONOSCILLATION OF SOLUTIONS OF PDE WITH $p-$LAPLACIAN

Zhiting Xu

Abstract. Some necessary conditions are established for the nonoscillation of the following PDE with $p-$Laplacian

$$\text{div}(\|\nabla y\|^{p-2}\nabla y) + c(x)|y|^{p-2}y = 0.$$

Using these results, we obtain some oscillation criteria for the above equation.

1. INTRODUCTION

We are here concerned with the oscillatory behavior of solutions of the following partial differential equation (PDE) with $p-$Laplacian

$$\text{div}(\|\nabla y\|^{p-2}\nabla y) + c(x)|y|^{p-2}y = 0, \tag{1.1}$$

where $p > 1$ is the $p-$Laplacian, $x = (x_1, \cdots, x_N) \in \mathbb{R}^N$, $\|\cdot\|$ is the usual Euclidean norm in $\mathbb{R}$, ∇ denotes the usual nabla operator, $c \in \mathbf{C}^{\mu}_{loc}(\Omega(1), \mathbb{R})$, $\mu \in (0, 1)$, and the set $\Omega(a) := \{x \in \mathbb{R}^N : \|x\| \geq a\}$ for some $a > 0$.

The PDEs with $p-$Laplacian have applications in various physical and biological problems–in the study of non–Newtonian fluids, in the glaciology and slow diffusion problems. For more detailed discussion about applications of PDEs with $p-$Laplacian, see [1] and references therein.

By a solution of (1.1) we mean a function $y \in \mathbf{C}^{1+\mu}_{loc}(\Omega(1), \mathbb{R})$ with the property $\|\nabla y\|^{p-1}\nabla y \in \mathbf{C}^{1+\mu}_{loc}(\Omega(1), \mathbb{R})$ satisfies (1.1) for all $\Omega(1)$. Regarding the question of existence of solutions to (1.1) we refer the reader to the monograph [1]. A nontrivial solution $y(x)$ of (1.1) is called oscillatory if $y(x)$ has zero on $\Omega(a)$ for every $a \geq 1$; otherwise it is said to be nonoscillatory. (1.1) is oscillatory if all its solutions are oscillatory.

Received June 2, 2006, accepted March 10, 2008.
Communicated by Chiun-Chuan Chen.
2000 *Mathematics Subject Classification*: 35J60, 34C35, 34K25.
Key words and phrases: Oscillation, Nonoscillation, PDE with $p-$Laplacian, Riccati transformation.

The literature on the oscillation/or nonoscillation of (1.1) is voluminous, cf, [2, 5, 7-9, 11, 14-16]. Among these we would like to mention Usami's work, which seems to be the first paper to study the oscillation of (1.1) in some sense and gives the following Leighton-type oscillation criteria [6] for (1.1) [11, Theorem 4].

Theorem 1.1. *Equation (1.1) is oscillatory if there exists a positive* $\mathbf{C}^1-$ *function* ϱ *satisfying*

$$\int^{\infty}\frac{r^{N-1}|\varrho'(r)|}{\varrho^{p-1}(r)}dr<\infty,\quad \int^{\infty}\frac{dr}{[r^{N-1}\varrho(r)]^{1/(p-1)}}=\infty,$$

and

$$\int\limits_{\Omega(1)}\varrho(\|x\|)c(x)dx=\infty.$$

However, when the case that

$$\int\limits_{\Omega(1)}\varrho(\|x\|)c(x)dx<\infty$$

holds, the results in [11] do not provide any information concerning the oscillation/or nonoscillation of (1.1).

In this paper, by using the generalized Riccati transformation and following the ideas of Wong [13], we establish some necessary conditions for the nonoscillation of (1.1). Using these results, we obtain some oscillation criteria for (1.1).

For simplicity to state our theorems, the following notations to be used throughout this paper. For any given function $\rho\in\mathbf{C}^1([1,\infty),\mathbb{R}^+)$ and constant $l>1$. Define

$$C_{\rho,l}(r):=\rho(r)\int\limits_{S_r}c(x)d\sigma-\frac{1}{l}\Big(\frac{l}{p}\Big)^p\omega_N r^{N-1}\rho^{1-p}(r)|\rho'(r)|^p,$$

and

$$h(r):=\frac{p-1}{l^*}(\omega_N r^{N-1}\rho(r))^{1/(1-p)},$$

where $S_r=\{x\in\mathbb{R}^N:\|x\|=r\}$, $d\sigma$ and ω_N denote the spherical integral element in $\mathbb{R}^N$ and the surface measure of unit sphere, respectively, and l^* is the conjugate number to l, i.e., $1/l+1/l^*=1$.

The main tool used for study of the nonoscillation of (1.1) is the generalized Riccati transformation. The special case of this transformation has been first introduced in [10]. A simple version of it, convenient for (1.1), has been developed in

[11]. The transformation is based on the fact that if $y = y(x)$ is a nonoscillatory solution of (1.1) then the vector function

$$W(x) = \frac{\|\nabla y\|^{p-2}\nabla y}{|y|^{p-1}}$$

satisfies the Riccati-type equation

$$\mathrm{div}W + c(x) + (p-1)\|W\|^q = 0, \tag{1.2}$$

where q is the conjugate number of p, i.e., $1/p + 1/q = 1$.

To prove our main results, the following two well-known inequalities [3, Theorems 27 and 41] are need.

Lemma 1.1. *If X and Y are nonnegative, then*

(1) $(X+Y)^\lambda \ge X^\lambda + Y^\lambda, \quad \lambda > 1$;

(2) $(X+Y)^\lambda \ge X^\lambda + \lambda X^{\lambda-1}Y, \quad \lambda > 1$.

2. Necessary Conditions of the Nonoscillation

In this section, we will establish some necessary conditions for the nonoscillation of (1.1).

Theorem 2.1. *If (1.1) is nonoscillatory, then there exist a constant $r_0 > 1$ and a function $Z \in \mathbf{C}^1([r_0,\infty),\mathbb{R})$ satisfy*

$$Z'(r) + C_{\rho,l}(r) + h(r)|Z(r)|^q \le 0, \quad r \ge r_0. \tag{2.1}$$

Proof. Let $y(x)$ be a nonoscillatory solution of (1.1). Without loss of generality, we may assume that $y(x) > 0$ on $\Omega(r_0)$ for some $r_0 \ge 1$. Put

$$Z(r) = \rho(r)\int_{S_r}\langle W(x),\nu(x)\rangle d\sigma \quad \text{for} \quad r \ge r_0, \tag{2.2}$$

where $\nu(x) = x/|x|$, $x \ne 0$, denotes the outward unit normal, and $\langle\cdot,\cdot\rangle$ denotes the scalar product. Then, by Green's formula in (2.2), and in view of (1.2), we have

$$Z'(r) = \frac{\rho'(r)}{\rho(r)}Z(r) - \rho(r)\int_{S_r} c(x)d\sigma - (p-1)\rho(r)\int_{S_r}\|W(x)\|^q d\sigma. \tag{2.3}$$

The Hölder inequality gives that

$$
\begin{aligned}
|Z(r)| &\le \rho(r)\Big(\int\limits_{S_r} \|W(x)\|^q d\sigma\Big)^{1/q}\Big(\int\limits_{S_r} \|\nu\|^p d\sigma\Big)^{1/p} \\
&= \rho(r)(\omega_N r^{N-1})^{1/p}\Big(\int\limits_{S_r} \|W(x)\|^q d\sigma\Big)^{1/q},
\end{aligned}
$$

or equivalently,

$$
\int\limits_{S_r} \|W(x)\|^q d\sigma \ge \rho^{-q}(r)(\omega_N r^{N-1})^{1/(1-p)}|Z(r)|^q. \tag{2.4}
$$

Combining (2.3) and (2.4), we get

$$
\begin{aligned}
&Z'(r) \\
&\le -\rho(r)\int\limits_{S_r} c(x)d\sigma + \frac{|\rho'(r)|}{\rho(r)}|Z(r)| - (p-1)(\omega_N r^{N-1}\rho(r))^{1/(1-p)}|Z(r)|^q \\
&= -\rho(r)\int\limits_{S_r} c(x)d\sigma + \frac{|\rho'(r)|}{\rho(r)}|Z(r)| - \frac{p-1}{l}(\omega_N r^{N-1}\rho(r))^{1/(1-p)}|Z(r)|^q \\
&\quad - \frac{p-1}{l^*}(\omega_N r^{N-1}\rho(r))^{1/(1-p)}|Z(r)|^q.
\end{aligned} \tag{2.5}
$$

The Young inequality implies that

$$
\begin{aligned}
&\frac{|\rho'(r)|}{\rho(r)}|Z(r)| - \frac{p-1}{l}\left(\omega_N r^{N-1}\rho(r)\right)^{1/(1-p)}|Z(r)|^q \\
&\le \frac{1}{l}\left(\frac{l}{p}\right)^p \omega_N r^{N-1}\rho^{1-p}(r)|\rho'(r)|^p.
\end{aligned} \tag{2.6}
$$

From (2.5) and (2.6) we find

$$
\begin{aligned}
Z'(r) \le& -\rho(r)\int\limits_{S_r} c(x)d\sigma + \frac{1}{l}(\frac{l}{p})^p \omega_N r^{N-1}\rho^{1-p}(r)|\rho'(r)|^p \\
&- \frac{p-1}{l^*}(\omega_N r^{N-1}\rho(r))^{1/(1-p)}|Z(r)|^q \\
=& -C_{\rho,l}(r) - h(r)|Z(r)|^q.
\end{aligned}
$$

This inequality is equivalent to (2.1). ■

The following Theorem 2.1 is an extension of Leghton's Theorem [6] to (1.1) and improves Theorem 1.1 [11, Theorem 4].

Theorem 2.2. *If*

$$\int^{\infty} C_{\rho,l}(s)ds = \infty \tag{2.7}$$

and

$$\int^{\infty} h(s)\,ds = \infty, \tag{2.8}$$

then (1.1) is oscillatory.

Proof. Assume to the contrary, that (1.1) is nonoscillatory. It follows from Theorem 2.1 that there exist a constant $r_0 > 1$ and a function $Z \in \mathbf{C}^1([r_0,\infty),\mathbb{R})$ satisfy (2.1). Integrating (2.1) over $[r,b]$, $b \geq r \geq r_1$, we obtain

$$Z(b) \leq Z(r) - \int_r^b C_{\rho,l}(s)ds - \int_r^b h(s)|Z(s)|^q\,ds. \tag{2.9}$$

It follows from (2.7) that there exists a $r_1 \geq r \geq 1$ such that

$$Z(r) - \int_r^b C_{\rho,l}(s)ds \leq 0 \quad \text{for} \quad b \geq r_1.$$

Hence

$$Z(b) \leq -H(b) =: \int_r^b h(s)|Z(s)|^q ds \quad for\;\; b \geq r_1.$$

Thus

$$H'(b) = -h(b)|Z(b)|^q \geq h(b)|H(b)|^q,$$

which follows that

$$H^{-q}(b)H'(b) \geq h(b) \quad \text{for} \quad b \geq r_1.$$

Integrating the above inequality over the interval $[r_1,\infty)$ gives a convergent integral on the left-hand side and a divergent integral on the right-hand side of this inequality, by virtue of (2.8). This contradiction completes the proof. ∎

Corollary 2.1. [Leighton-type Theorem]. *Let* $p \geq N$. *If*

$$\int_{\Omega(1)} c(x)dx = \infty, \tag{2.10}$$

then (1.1) *is oscillatory.*

Proof. Follows from Theorem 2.2 for $\rho(r) \equiv 1$. ■

Corollary 2.2. *If there exists* $m > 1$ *such that*

$$\int_{\Omega(1)} \left[\|x\|^{p-N} c(x) - m \left| \frac{p-N}{p} \right|^p \|x\|^{-N} \right] dx = \infty, \tag{2.11}$$

then (1.1) *is oscillatory.*

Proof. Follows from Theorem 2.2 for $\rho(r) \equiv r^{p-N}$ and $m = l^{p-1}$. ■

It is clear that Theorem 2.2 cannot be applied in the following case:

$$\int^{\infty} C_{\rho,l}(s)\, ds < \infty. \tag{2.12}$$

Next, we will discuss the behavior of solutions of Eq.(1.1) satisfying (2.12). For this case, we shall start with a useful theorem which is similar to Hartman's Lemma [4, p. 365] for second order linear ordinary differential equation.

Theorem 2.3. *Let (2.8) and (2.12) hold, and define*

$$\Theta(r) := \int_r^{\infty} C_{\rho,l}(s) ds < \infty, \quad r \geq 1. \tag{2.13}$$

If (1.1) is nonoscillatory, then there exist a constant $r_0 > 1$ *and a function* $Z \in \mathbf{C}([r_0, \infty), \mathbb{R})$ *such that*

$$Z(r) \geq \Theta(r) + \int_r^{\infty} h(s)|Z(s)|^q ds, \quad r \geq r_0. \tag{2.14}$$

Proof. As in the proof Theorem 2.2, there exist a constant $r_0 > 1$ and a function $Z \in \mathbf{C}^1([r_0, \infty), \mathbb{R})$ satisfy (2.9) for $b \geq r \geq r_0$. Now, we claim that

$$\int_r^{\infty} h(s)|Z(s)|^q ds < \infty. \tag{2.15}$$

Otherwise,

$$\int_r^{\infty} h(s)|Z(s)|^q ds = \infty. \tag{2.16}$$

Note that (2.9), (2.13) and (2.16), there is a $r_1 \geq 1$ such that

$$Z(b) \leq -\int_{r_1}^{b} h(s)|Z(s)|^q ds, \quad b \geq r_1.$$

As in the proof of Theorem 2.2, we obtain $\int_{r_1}^{\infty} h(s)ds < \infty$, which contradicts (2.8). Thus (2.15) holds. It follows from (2.9), (2.13) and (2.15) that

$$Z(r) \geq \limsup_{b\to\infty} Z(b) + \int_r^{\infty} C_{\rho,l}(s)ds + \int_r^{\infty} h(s)|Z(s)|^q ds, \quad r \geq 1. \tag{2.17}$$

If $\limsup_{b\to\infty} Z(b) < 0$, then there exist two numbers $\delta < 0$ and $r_2 \geq r_1$ such that $Z(b) < \delta$ for $b \geq r_2$. It follows from (2.8) that

$$\int_r^{\infty} h(s)|Z(s)|^q ds \geq \delta^q \int_r^{\infty} h(s)ds = \infty,$$

which contradicts (2.15). Thus, $\limsup_{b\to\infty} Z(b) \geq 0$. It follows from (2.17) that (2.14) holds for $r \geq r_1$. This completes the proof. ∎

In what follows, we assume further the following condition holds.

$$\Theta(r) \geq 0 \quad \text{for all sufficient large} \quad r \geq 1. \tag{2.18}$$

Following the ideas of Wong [13], we may establish the following two theorems.

Theorem 2.4. *Let (2.8) and (2.18) hold. If (1.1) is nonoscillatory, then there exist a constant* $r_0 \geq 1$ *and a function* $u \in \mathbf{C}([r_0,\infty),\mathbb{R})$ *such that*

$$u(r) \geq \Theta_1(r) + \int_r^{\infty} h(s)u^q(s)\, ds, \quad r \geq r_0, \tag{2.19}$$

where $\Theta_1(r) = \int_r^{\infty} h(s)\Theta^q(s)ds.$

Proof. Since (1.1) is nonoscillatory, it follows from Theorem 2.3 that there exist a constant $r_0 > 1$ and a function $Z \in \mathbf{C}([r_0,\infty),\mathbb{R})$ satisfy (2.14) for $r \geq r_0$. Note that (2.18), one find that $Z(r) \geq 0$ and $Z(r) \geq \Theta(r) + u(r)$ for $r \geq r_0$, here

$u(r) := \int_r^\infty h(s)Z^q(s)ds$, which satisfies $u'(r) = -h(r)Z^q(r)$. Then, by Lemma 1.1(1), we have

$$u'(r) \leq -h(r)[\Theta(r) + u(r)]^q \leq -h(r)[\Theta^q(r) + u^q(r)]. \tag{2.20}$$

Integrating (2.20) follows that

$$u(r) \geq u(b) + \int_r^b h(s)\Theta^q(s)ds + \int_r^b h(s)u^q(s)ds.$$

Now, we may replace b in the above inequality by ∞ and readily obtain (2.19). ■

Applying the technique to (2.19) instead of (2.14), we may obtain the analogue of Theorem 2.4 as following, whose proof is omitted.

Theorem 2.5. *Let* (2.8) *and* (2.18) *hold. If* (1.1) *is nonoscillatory, then there exists a number* $r_0 > 1$ *and a function* $v \in \mathbf{C}([r_0, \infty), \mathbb{R})$ *such that*

$$v(r) \geq \Theta_2(r) + \int_r^\infty h(s)v^q(s)ds, \quad r \geq r_0, \tag{2.21}$$

where $\Theta_2(r) = \int_r^\infty h(s)\Theta_1^2(s)ds$, *and* $\Theta_1(r)$ *is as in Theorem 2.4.*

It is clear from the proofs of Theorems 2.4 and 2.5 that the process of generating higher-order iterated Riccati integral equations may be continued if we assume further that iterated integrals $\Theta(r)$, $\Theta_1(r), \cdots$, are integrable.

As the immediate consequences of Theorem 2.4 and 2.5, we have

Corollary 2.3. *Let* (2.8) *and* (2.18) *hold. If one of the following conditions holds, then (1.1) is oscillatory.*

$$(1) \quad \int_1^\infty h(r)\Theta^q(r)dr = \infty; \tag{2.22}$$

$$(2) \quad \int_1^\infty h(r)\Theta_1^q(r)dr = \infty. \tag{2.23}$$

Example 2.1. Consider the equation

$$\text{div}(\|\nabla y\|\nabla y) + \frac{\gamma}{\|x\|^3}|y|y = 0, \tag{2.24}$$

where $x \in \Omega(1)$, $p = 3$, $N = 2$, $c(x) = \gamma/\|x\|^3$, and $\gamma > 0$.

Here no matter how we choose ϱ, it is impossible to ensure that the conditions of Theorem 1.1 [11, Theorem 4] are satisfied. Thus, the results in [11] cannot apply to (2.24). But, by Corollary 2.2, it is easy to show that if there exists a constant $m > 1$ such that $\gamma > m/27$ then (2.24) is oscillatory.

Example 2.2. Consider the equation

$$\operatorname{div}(\|\nabla y\|^2 \nabla y) + \frac{1}{\|x\|^{7/4}}|y|^2 y = 0, \tag{2.25}$$

where $x \in \Omega(1)$, $p = 4$, $N = 2$, $c(x) = 1/\|x\|^{7/4}$. Let $l = 2$, $\rho(r) = r^{-1}$. By a direct calculation, we get

$$C_{\rho,l}(r) = \frac{2\pi}{r^{7/4}}\Big(1 - \frac{1}{32 r^{9/4}}\Big), \quad h(r) = \frac{3}{2}(2\pi)^{-1/3}.$$

Then

$$\Theta(r) = \int_r^\infty C_{\rho,l}(s)ds = \frac{8\pi}{3}\frac{1}{r^{3/4}}\Big(1 - \frac{1}{128 r^{9/4}}\Big).$$

So

$$\int_r^\infty h(s)\Theta^q(s)ds = 4\pi(\frac{4}{3})^{1/3}\int_1^\infty \frac{1}{s}\Big(1 - \frac{1}{128 s^{9/4}}\Big)^{4/3} ds = \infty.$$

Hence, by Corollary 2.3(1), (2.25) is oscillatory.

3. Oscillation Criteria

In this section, under the assumptions (2.8) and (2.18), we will obtain some oscillation criteria for (1.1) based upon the necessary conditions of the nonoscillation established in section 2, which extend Wong's theorems [13, Theorems 7, 8, 9] to (1.1).

Theorem 3.1. *If there exists a nonnegative function $g(r) \not\equiv 0$ such that*

$$\int_r^\infty h(s)\Theta^q(s)ds \geq \alpha\Theta(r) + g(r), \tag{3.1}$$

and

$$\int_r^\infty h(s)g(s)\Theta^{q-1}(s)ds \geq \beta g(r), \tag{3.2}$$

where $\alpha = (q-1)q^{-q}$, and $\beta \geq q^{-q}$, then (1.1) is oscillatory.

Proof. Assume that (1.1) is nonoscillatory, then by Theorem 2.3 that there exist a constant $r_0 > 1$ and a function $Z \in \mathbf{C}([r_0, \infty), \mathbb{R})$ satisfy (2.14), which follows $Z(r) \geq a_0\Theta(r) \geq 0$ with $a_0 = 1$. Substituting this into (2.14), we find

$$Z(r) \geq a_1\Theta(r) + b_1 g(r) \quad a_1 = 1 + \alpha, \quad b_1 = 1. \tag{3.3}$$

Substituting (3.3) again into (2.14), and in view of Lemma 1.1(2) and (3.2), we obtain

$$\begin{aligned} Z(r) &\geq \Theta(r) + \int_r^\infty h(s)[a_1\Theta(s) + b_1 g(s)]^q ds \\ &\geq \Theta(r) + \int_r^\infty h(s)[a_1^q\Theta^q(s) + qb_1a_1^{q-1}\Theta^{q-1}(s)g(s)]ds \\ &\geq (1 + \alpha\, a_1^q)\Theta(r) + [a_1^q + b_1 q\beta a_1^{q-1}]g(r) \\ &= a_2\Theta(r) + b_2 g(r), \quad a_2 = 1 + \alpha a_1^q, \quad b_2 = a_1^q + b_1 q\beta a_1^{q-1}. \end{aligned} \tag{3.4}$$

Using (3.4) and an easy induction, we can show in general that

$$Z(r) \geq a_n\Theta(r) + b_n F(r), \quad a_n = 1 + \alpha\alpha_{n-1}^q, \quad b_n = a_{n-1}^q + b_{n-1}q\beta a_{n-1}^{q-1}. \tag{3.5}$$

From the recurrence relation given by (3.5) and the fact that $a_2 \geq a_1$, $b_2 \geq b_1$. One readily has that $a_n \geq a_{n-1}$ and $b_n \geq b_{n-1}$ hold. Furthermore, it is easy to show from (3.5) that $\lim_{n\to\infty} a_n = q$. Now, if $\lim_{n\to\infty} b_n$ is finite, then we can show from (3.5) that $q^q\beta < 1$, contrary to the given hypothesis. Since $g(r) \not\equiv 0$, and $b_n \to \infty$ as $n \to \infty$, the desired contradiction follows from (3.5). This completes the proof. ∎

Applying the same argument to inequalities (2.19) and (2.21), we obtain respectively.

Theorem 3.2. *If there exists a nonnegative function $g_1(r) \not\equiv 0$ such that*

$$\int_r^\infty h(s)\Theta_1^q(s)\, ds \geq \alpha\Theta_1(r) + g_1(r), \tag{3.6}$$

and

$$\int_r^\infty h(s)g(s)\Theta_1^{q-1}(s)ds \geq \beta g_1(r), \tag{3.7}$$

where $\Theta_1(r)$, and α, β are the same as in Theorem 2.4 and Theorem 3.1, respectively, then (1.1) is oscillatory.

Theorem 3.3. *If there exists a nonnegative function* $g_2(r) \not\equiv 0$ *such that*

$$\int_r^\infty h(s)\Theta_2^q(s)ds \geq \alpha\Theta_2(r) + g_2(r), \tag{3.8}$$

and

$$\int_r^\infty h(s)g(s)\Theta_2^{q-1}(s)ds \geq \beta g_2(r), \tag{3.9}$$

where $\Theta_2(r)$, *and* α, β *are the same as in Theorem 2.5 and Theorem 3.1, respectively, then (1.1) is oscillatory.*

Let $\varepsilon > \max\{1, q-1\}$ be some constant. Taking

$$g(r) = (\varepsilon + 1 - q)q^{-q}\Theta(r) \quad \text{and} \quad g_i(r) = (\varepsilon + 1 - q)q^{-q}\Theta_i(r), \quad i = 1, 2,$$

in Theorems 3.1-3.3, we obtain the following corollary.

Corollary 3.4. *Let* $\varepsilon > \max\{1, q-1\}$ *be some constant. If one of the following conditions holds, then (1.1) is oscillatory.*

$$(1) \quad \int_r^\infty h(s)\Theta^q(s)ds \geq \varepsilon q^{-q}\Theta(r); \tag{3.10}$$

$$(2) \quad \int_r^\infty h(s)\Theta_1^q(s)ds \geq \varepsilon q^{-q}\Theta_1(r); \tag{3.11}$$

$$(3) \quad \int_r^\infty h(s)\Theta_2^q(s)ds \geq \varepsilon q^{-q}\Theta_2(r). \tag{3.12}$$

The following corollary extend Willett's oscillation criterion [12, Theorem 1.5] to (1.1).

Corollary 3.5. *Let* $\varepsilon > \max\{1, q-1\}$ *be some constant. If one of the following conditions holds, then (1.1) is oscillatory.*

$$(1) \quad \Theta_1(s) \geq \frac{\varepsilon}{q}\Theta(r); \tag{3.13}$$

$$(2) \quad \Theta_2(s) \geq \frac{\varepsilon}{q}\Theta_1(r). \tag{3.14}$$

Proof. The proofs of (1) and (2) are similar, we only show the first one. Indeed, by (3.13), we have

$$\int_r^\infty h(s)\Theta_1^q(s)ds \geq \left(\frac{\varepsilon}{q}\right)^q \int_r^\infty h(s)\Theta^q(s)ds = \left(\frac{\varepsilon}{q}\right)^q \Theta_1(r) \geq \varepsilon q^{-q}\Theta_1(r).$$

It follows from Corollary 3.1 (2) that (1.1) is oscillatory. ■

Example 3.1. Consider the equation

$$\mathrm{div}(\|\nabla y\|^2\nabla y) + \frac{\gamma[\,2(2-\cos\|x\|) - \|x\|\sin\|x\|\,]}{\|x\|^4}|y|^2 y = 0, \tag{3.15}$$

where $x \in \Omega(1)$, $p = 4$, $N = 2$, $c(x) = \gamma[\,(2-\cos\|x\|) - \|x\|\sin\|x\|\,]/\|x\|^4$, and $\gamma > 3^7/2^5$. Let $\rho(r) = 1$ and $l^* = 3$. By a direct calculation, for $r > 1$, we get

$$C_{\rho,\,l}(r) = \frac{2\pi\gamma[2(2-\cos r) - r\sin r]}{r^3}, \quad h(r) = (2\pi\, r)^{-1/3}.$$

Then

$$\Theta(r) = \int_r^\infty C_{\rho,\,l}(s)ds = \frac{2\pi\gamma(2-\cos r)}{r^2}.$$

So

$$\frac{2\pi\gamma}{r^2} \leq \Theta(r) \leq \frac{6\pi\gamma}{r^2},$$

and

$$\int_r^\infty h(s)\Theta^q(s)ds \geq 2\pi\gamma^{4/3}\int_r^\infty \frac{1}{s^3}ds = \frac{\pi\gamma^{4/3}}{r^2} \geq \varepsilon\, q^{-q}\Theta(r) \quad \text{for some} \quad \varepsilon > 1.$$

Hence, by Corollary 3.1(1), (3.15) is oscillatory.

Acknowledgment

The author would like to express his great appreciation to the referees for their careful reading of my manuscript and for their helpful suggestions.

References

1. J. I. Díaz, *Nonlinear Partial Differential Equations and Free Boundaries*, Vol. I. Elliptic Equations, Pitman, London, 1985.

2. O. Doslý and R. Marík, Nonexistence of positive solutions of PDE's with $p-$Laplacian, *Acta. Math. Hungar.*, **90(1-2)** (2001), 89-107.

3. G. H. Hardy, J. E. Littlewood and G. Polya, Inequalities, 2nd, Cambridge Univ Press, Cambridge, UK, 1999.

4. P. Hartman, *Ordinary Differential Equations*, New York, Wiley, 1982.

5. T. Kusano, J. Jaros and N. Yoshida, A Picone-type identity and Sturmian comparison and oscillation theorems for a class of half-linear partial differential equations of second order, *Nonl. Anal.*, **40** (2003), 381-395.

6. W. Leighton, The detection of the oscillation of solutions of a second order linear differential equations, *Duck Math. J.*, **17** (1950), 57-61.

7. R. Marík, Oscillation criteria for PDE with $p-$Laplacian via the Riccati technique, *J. Math. Anal. Appl.*, **248** (2000), 290-308.

8. R. Marík, Hartman-Wintner type theorem for PDE with $p-$Laplacian, *EJQTDE. Proc. 6th Coll. QTDE.*, **18** (2000), 1-7.

9. R. Marík, Integral averagings and oscillation criteria for half-linear partial differential equation, *Appl. Math. Comput.*, **150** (2004), 69-87.

10. E. S. Noussair and C. A. Swanson, Oscillation of semilinear elliptic inequalities by Riccati transformation, *Canad. J. Math.*, **32(4)** (1980), 908-923.

11. H. Usami, Some oscillation theorems for a class of quasilinear elliptic equations, *Ann. Math. Pura. Appl.*, **175** (1998), 277-283.

12. D. Willett, On the oscillatory behavior of the solutions of second order linear differential equations, *Ann. Polon. Math.*, **21** (1969), 175-194.

13. J. S. W. Wong, Oscillation and nonoscillation of solution of second order linear differential equations with integrable coefficients, *Trans. Amer. Math. Soc.*, **114** (1969), 197-215.

14. Z. Xu and H. Xing, Oscillation criteria of Kamenev-type for PDE with $p-$Laplacian, *Appl. Math. Comput.*, **145** (2003), 735-745.

15. Z. Xu, B. Jia and S. Xu, Averaging techniques and oscillation of quasilinear elliptic equations, *Ann. Polon. Math.*, **81(1)** (2004), 45-54.

16. Z. Xu and H. Xing, Oscillation criteria for PDE with $p-$Laplacian involving general means, *Annali. Math.*, **184(3)** (2005), 395-406.

Zhiting Xu
School of Mathematics Sciences,
South China Normal University,
Guangzhou, 510631,
P. R. China
E-mail: xuzhit@126.com