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GAP FUNCTIONS FOR NONSMOOTH EQUILIBRIUM PROBLEMS

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Abstract. We consider equilibrium problems *(EP)* with directionally differentiable (not necessarily $\mathcal{C}^1$) bifunctions which are convex with respect to the second variable and we use a gap function approach to solve them. In the first part of the paper we establish a condition under which any stationary point of the gap function solves *(EP)* and we propose a solution method which uses descent directions of the gap function. In the final section we study the problem when this condition is not satisfied. In this case we use a family of gap functions depending on a parameter α which allows us to overcome the trouble due to the lack of a descent direction.

1. Introduction

Different kinds of competitive situations (see for example [2, 5] and references therein) can be formulated via general equilibrium model of this type

$$(EP) \qquad \text{find } \ \bar{x} \in C \ \text{ s.t. } \ f(\bar{x}, y) \geq 0, \qquad \forall y \in C,$$

where $C \subseteq \mathbb{R}^n$ is a compact convex set and the equilibrium bifunction $f \in \mathcal{A}$ where

$$\mathcal{A} = \{f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \, : \, f(\cdot, y) \text{ is directionally differentiable for all } y \in C, \\ f(x, \cdot) \text{ is convex for all } x \in C, \ f(z, z) = 0 \text{ for all } z \in C\}$$

Recently an increasing effort has been made to develop algorithms for computing equilibrium solutions. Some of them are based on the fact that *(EP)* can be reformulated as an optimization problem via gap functions. This has been proposed also in [1, 3, 6, 7, 8]. In this paper we focus on *(EP)* in which the bifunction f is only directionally differentiable and not $\mathcal{C}^1$. The scheme proposed follows the same line of [1].

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The following notation will be used in the paper. If $f \in \mathcal{A}$ we denote the directional derivative of $f(\cdot, y)$ at x along the direction u by

$$D_x f(x, y; u) = \lim_{t \to 0^+} \frac{f(x + tu, y) - f(x, y)}{t}.$$

Analogously $D_y f(x, y; v)$ indicates the directional derivative of $f(x, \cdot)$ at y along the direction v.

The following function

$$\varphi(x) = -\min_{y \in C} f(x, y)$$

was introduced in [7]. We prove, for the sake of completeness, that it is a gap function for (*EP*).

Theorem 1.1. *Let $f \in \mathcal{A}$ be given; then*

(*i*) $\varphi(x) \geq 0$ *for all* $x \in C$*;*

(*ii*) $\bar{x} \in C$ *verifies* $\varphi(\bar{x}) = 0$ *if and only if* $\bar{x}$ *solves (EP).*

Proof. For all $x \in C$ we have

$$\varphi(x) = -\min_{y \in C} f(x, y) \geq -f(x, x) = 0$$

proving statement *(i)*. If $0 = \varphi(\bar{x}) = -\min_{y \in C} f(\bar{x}, y)$ then $\bar{x} \in C$ solves (*EP*); the converse is trivial since $f(z, z) = 0$ for all $z \in C$ and this proves (*ii*). ■

Next sections are devoted to seek for descent directions of the gap function φ in order to minimize it.

2. Strictly Convex and Strictly D-Monotone Equilibrium Problems

In this section we suppose that $f \in \mathcal{A}$ is strictly convex with respect to y, for all $x \in C$. This assumption, together with the compactness of C, implies the existence of the unique minimizer $y(x) \in C$ such that

$$\varphi(x) = -f(x, y(x)). \tag{1}$$

The next result of Danskin [4] permits to compute the directional derivative of φ.

Theorem 2.1. *Let $f \in \mathcal{A}$ be strictly convex with respect to y; the function $x \mapsto y(x)$ is continuous and φ is directionally differentiable with*

$$D\varphi(x; v) = -D_x f(x, y(x); v)$$

for all $x \in C$ and $v \in \mathbb{R}^n$.

Since $f(z, z) = 0$, it is easy to show that the solution set of *(EP)* coincides with the set of the fixed points of the function $x \mapsto y(x)$, i.e. $\bar{x} \in C$ is a solution of *(EP)* if and only if $\bar{x} = y(\bar{x})$. When $\bar{x} \neq y(\bar{x})$, in order to establish whether $y(\bar{x}) - \bar{x}$ is a descent direction for φ, additional assumptions on f are usually assumed in literature (see for instance [3, 6, 8]). We will use the following.

Definition 2.1. A bifunction $g \in \mathcal{A}$ is called *strictly D-monotone* on C if

$$D_x g(x, y; y - x) > D_y g(x, y; x - y), \qquad \forall x, y \in C \text{ with } x \neq y. \tag{2}$$

If $g \in \mathcal{A}$ is continuously differentiable, the concept of strict D-monotonicity collapses with the concept of strict ∇-monotonicity introduced in [1]. It is easy to prove that the concept of strict D-monotonicity is not related to the classical concept of strict monotonicity. Several nice properties hold.

Theorem 2.2. *Suppose that $f \in \mathcal{A}$ is strictly convex with respect to y and strictly D-monotone on C, then*

$$D\varphi(x; y(x) - x) < 0, \qquad \forall x \in C \textit{ with } x \neq y(x). \tag{3}$$

Proof. From Theorem 2.1 and the strict D-monotonicity of f we deduce

$$D\varphi(x; y(x) - x) = -D_x f(x, y(x); y(x) - x) < -D_y f(x, y(x); x - y(x));$$

since $y(x)$ is a global minimum of $f(x, \cdot)$ over C, the first order necessary optimality condition implies

$$D_y f(x, y(x); x - y(x)) \geq 0$$

that concludes the proof. ∎

Strict D-monotonicity guarantees also the following "stationarity property" for the reformulation of *(EP)* as optimization problem through φ.

Theorem 2.3. *Suppose that $f \in \mathcal{A}$ is strictly convex with respect to y and strictly D-monotone on C; if $\bar{x}$ is a stationary point of φ over C, i.e.*

$$D\varphi(\bar{x}; y - \bar{x}) \geq 0, \qquad \forall y \in C, \tag{4}$$

then $\bar{x}$ solves (EP).

Proof. By contradiction, suppose that $\bar{x}$ does not solve ***(EP)*** and hence $y(\bar{x}) \neq \bar{x}$. Since $y(\bar{x})$ is a global minimum for the function $f(\bar{x}, \cdot)$ we deduce

$$D_y f(\bar{x}, y(\bar{x}); \bar{x} - y(\bar{x})) \geq 0. \tag{5}$$

Moreover, from (4) valued at $y(\bar{x})$ and Theorem 2.1 we obtain

$$D_x f(\bar{x}, y(\bar{x})); y(\bar{x}) - \bar{x}) \leq 0. \tag{6}$$

But (5) and (6) contradict the assumption of strict D-monotonicity of f. ■

The results proved in Theorem 2.2 and Theorem 2.3 give us a solution method for solving ***(EP)***. In fact, we have a descent direction (Theorem 2.2), a stopping rule (Theorem 2.3), and we can propose an exact linesearch rule to find the stepsize. The iterative sequence of the solution method is given

$$x^{k+1} = x^k + t_k d^k$$

where $d^k = y(x^k) - x^k$ and $t_k \in [0, 1]$ minimizes $\theta(t) = \varphi(x^k + td^k)$ over $[0, 1]$. Since d^k depends with continuity upon x^k, convergence to a stationary point of φ is achieved via Zangwill's Theorem.

3. Convex and Strictly D-Monotone Equilibrium Problems

When $f(x, \cdot)$ is not strictly convex, we have not the uniqueness of the minimum point $y(x)$. For this reason we consider a continuously differentiable bifunction $h : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ such that

(a) $h(x, y) \geq 0$ for all $x, y \in C$ and $h(z, z) = 0$ for all $z \in C$,

(b) $h(x, \cdot)$ is strictly convex for all $x \in C$.

As immediate consequence of (a) and (b) we have that $\nabla_y h(z, z) = 0$, for all $z \in C$.

We define the bifunction $F = f + h$ and we introduce the following *auxiliary equilibrium problem*

$$(AEP) \qquad \text{find} \quad \bar{x} \in C \quad \text{s.t.} \quad F(\bar{x}, y) \geq 0, \qquad \forall y \in C.$$

The next result shows the equivalence between (*EP*) and (*AEP*).

Theorem 3.1. *The point $\bar{x}$ solves (EP) if and only if $\bar{x}$ solves (AEP).*

Proof. Trivially, every solution of (*EP*) solves (*AEP*). Vice versa let $\bar{x}$ be a solution of (*AEP*) and suppose, by contradiction, there exists $\bar{y} \in C$ such that $f(\bar{x}, \bar{y}) < 0$. Since h is continuously differentiable, $h(\bar{x}, \bar{x}) = 0$ and $\nabla_y h(\bar{x}, \bar{x}) = 0$, we have

$$h(\bar{x}, \bar{x} + t(\bar{y} - \bar{x})) = \|\bar{x} - \bar{y}\| t\omega(t), \qquad \forall t \in (0, 1]$$

where $\omega(t)$ tends to 0 for $t \to 0^+$. Therefore, from the convexity of $f(\bar{x}, \cdot)$ and since $f(\bar{x}, \bar{x}) = 0$, we deduce

$$\begin{aligned} 0 &\le F(\bar{x}, \bar{x} + t(\bar{y} - \bar{x})) \\ &= f(\bar{x}, \bar{x} + t(\bar{y} - \bar{x})) + h(\bar{x}, \bar{x} + t(\bar{y} - \bar{x})) \\ &\le (1 - t) f(\bar{x}, \bar{x}) + t f(\bar{x}, \bar{y}) + \|\bar{x} - \bar{y}\| t\omega(t) \\ &= t[f(\bar{x}, \bar{y}) + \|\bar{x} - \bar{y}\|\omega(t)]. \end{aligned}$$

Since $f(\bar{x},\, \bar{y}) + \|\bar{x} - \bar{y}\|\omega(t) < 0$ for t sufficiently small, we achieve the contradiction. ■

Theorem 3.1 allows us to apply the results of the previous section to the gap function associated to the bifunction F and for this we need the strict D-monotonicity of F.

Definition 3.1. A bifunction $g \in \mathcal{A}$ is called D-*monotone* on C if

$$D_x g(x, y; y - x) \ge D_y g(x, y; x - y), \qquad \forall x, y \in C. \tag{7}$$

If g is continuously differentiable the concept of D-monotonicity collapses with the concept of ∇-monotonicity defined in [1], i.e.

$$\langle \nabla_x g(x, y) + \nabla_y g(x, y), y - x \rangle \ge 0, \qquad \forall x, y \in C.$$

All the usually used bifunctions h as, for instance, the square of the euclidean norm $h(x, y) = \|x - y\|^2$ are D-monotone but not strictly D-monotone.

It is easy to show that if f is strictly D-monotone and h is D-monotone then F is strictly D-monotone. So we can adapt to this case the solution method presented at the end of Section 2.

4. A Larger Class of Equilibrium Problems

Since strict D-monotonicity is not always verified by the bifunction f, we now analyse the case when f doesn't satisfy this condition but it is only D-monotone. In this case (even if $f(x, \cdot)$ is strictly convex) Theorem 2.2 can not be applied. In fact, if we substitute the assumption of strict D-monotonicity with D-monotonicity, it is possible to show that $y(x) - x$ is not always a descent direction (see [1, Example 2.5]). For this reason we substitute the auxiliary bifunction F with a family of bifunctions $F_\alpha = f + \alpha h$, with $\alpha > 0$ and we denote by φ_α the associated gap function. We will show that the parameter α allows us to overcome the troubles due to the lack of the "stationarity property". Anyway, we will require the following additional assumption on f

$$f(x, y) + D_x f(x, y; y - x) \geq 0, \qquad \forall x, y \in C. \tag{8}$$

It is possible to prove that condition (8) is stronger than D-monotonicity.

Theorem 4.1. *If the bifunction $f \in \mathcal{A}$ satisfies (8) then it is D-monotone.*

Proof. From the convexity of $f(x, \cdot)$ we have

$$0 = f(x, x) \geq f(x, y) + D_y f(x, y; x - y), \qquad \forall x, y \in C;$$

therefore, the above inequality and (8) guarantee

$$\begin{aligned} & D_x f(x, y; y - x) - D_y f(x, y; x - y) \\ = & [f(x, y) + D_x f(x, y; y - x)] - [f(x, y) + D_y f(x, y; x - y)] \geq 0 \end{aligned}$$

for all $x, y \in C$ and thus f is D-monotone. ∎

Some examples presented in [1] show that no relationship exists between condition (8) and the strict D-monotonicity and, moreover, that the stationarity property is not guaranteed for a fixed gap function φ_α. When condition (8) holds, we can overcome the trouble of finding a descent direction by eventually modifying the parameter α and therefore by changing the considered gap function.

Theorem 4.2. *Suppose $f \in \mathcal{A}$ satisfies (8) and assume that*

$$\begin{aligned} & \lim_{y' \to y} \lim_{t \to 0^+} \frac{f(x + t(y' - x), y') - f(x, y')}{t} \\ = & \lim_{t \to 0^+} \lim_{y' \to y} \frac{f(x + t(y' - x), y') - f(x, y')}{t}, \end{aligned} \tag{9}$$

for all $x, y \in C$. If $x \in C$ is not a solution of (EP), then there exists $\bar{\alpha}$ such that $y_\alpha(x) - x$ is a descent direction at x for all positive $\alpha \leq \bar{\alpha}$.

Proof. Suppose, by contradiction, there exists a sequence $\{\alpha_k\} \downarrow 0$ such that

$$D\varphi_{\alpha_k}(x; y_{\alpha_k}(x) - x) \geq 0 \tag{10}$$

Since C is compact, we can suppose that the sequence $\{y_{\alpha_k}(x)\}$ converges to $y \in C$. By assumption

$$f_{\alpha_k}(x, y_{\alpha_k}(x)) = -\varphi_{\alpha_k}(x) < 0$$

and therefore, since $f(x, \cdot)$ is continuous, taking the limit for $k \to \infty$, we deduce that

$$f(x, y) = \lim_{k \to \infty} f_{\alpha_k}(x, y_{\alpha_k}(x)) \leq 0.$$

On the other hand y_{α_k} minimizes $f_{\alpha_k}(x, \cdot)$ over C, then

$$D_y f_{\alpha_k}(x, y_{\alpha_k}(x); x - y_{\alpha_k}(x)) \geq 0.$$

Let $a > D_y f(x, y; x - y)$ then there exists $t_0 \in (0, 1)$ such that $y + t(x - y) \in C$ and

$$\frac{f(x, y + t(x - y)) - f(x, y)}{t} < a$$

for all $t \in (0, t_0)$. Moreover $f(x, y_{\alpha_k}(x) + t(x - y_{\alpha_k}(x)))$ tends to $f(x, y + t(x - y))$ and $f(x, y_{\alpha_k}(x))$ tends to $f(x, y)$ for $k \to \infty$. Hence, for k sufficiently large, we have

$$\frac{f(x, y_{\alpha_k}(x) + t(x - y_{\alpha_k}(x))) - f(x, y_{\alpha_k}(x))}{t} < a.$$

Since

$$D_y f(x, y_{\alpha_k}(x); x - y_{\alpha_k}(x)) \leq \frac{f(x, y_{\alpha_k}(x) + t(x - y_{\alpha_k}(x))) - f(x, y_{\alpha_k}(x))}{t}$$

it follows that

$$\limsup_{k \to \infty} D_y f(x, y_{\alpha_k}(x); x - y_{\alpha_k}(x)) \leq a$$

This is true for any $a > D_y f(x, y; x - y)$ and then

$$\begin{aligned} D_y f(x, y; x - y) &\geq \limsup_{k \to \infty} D_y f(x, y_{\alpha_k}(x); x - y_{\alpha_k}(x)) \\ &= \limsup_{k \to \infty} D_y f_{\alpha_k}(x, y_{\alpha_k}(x); x - y_{\alpha_k}(x)) \geq 0. \end{aligned}$$

Therefore, from Theorem 4.1, we deduce that

$$D_x f(x, y; y - x) \geq 0.$$

Condition (10) can be written

$$D_x f_{\alpha_k}(x, y_{\alpha_k}(x); y_{\alpha_k}(x) - x) \leq 0$$

and taking the limit for $k \to \infty$ and using condition (9) we have the converse inequality

$$D_x f(x, y; y - x) \leq 0$$

and therefore

$$D_x f(x, y; y - x) = 0.$$

Since condition (8) holds, we have $f(x, y) \geq 0$ and therefore we deduce $f(x, y) = 0$. Moreover $f_{\alpha_k}(x, y_{\alpha_k}(x)) \leq f_{\alpha_k}(x, y')$ for all $y' \in C$, hence, taking the limit again,

$$0 = f(x, y) \leq f(x, y'), \qquad \forall y' \in C.$$

This implies that x solves *(EP)* in contradiction with the assumption. ■

When f is continuously differentiable, condition (9) is trivially satisfied. The above result provides the key idea for the solution method for D-monotone bifunctions: decrease the value of α whenever $y_\alpha(x) - x$ isn't any longer a descent direction for φ_α and apply the scheme presented in Section 2.

Nevertheless, in order to device a new kind of solution method more efficient from the computational point of view, we have to implement an Armijo-type rule for the stepsize. If we adopt this kind of rule, we need the following theorem.

Theorem 4.3. *Suppose that $f \in \mathcal{A}$ satisfies condition (8) and h is ∇-monotone then*

$$\begin{aligned} & D\varphi_\alpha(x; y_\alpha(x) - x) \\ & \leq f(x, y_\alpha(x)) - \alpha\langle \nabla_x h(x, y_\alpha(x)), y_\alpha(x) - x\rangle \leq 0, \qquad \forall x \in C. \end{aligned} \tag{11}$$

Proof. The first inequality in (11) descends immediately from condition (8) since

$$\begin{aligned} D\varphi_\alpha(x; y_\alpha(x) - x) &= -D_x f_\alpha(x, y_\alpha(x); y_\alpha(x) - x) \\ &= -D_x f(x, y_\alpha(x); y_\alpha(x) - x) - \alpha\langle \nabla_x h(x, y_\alpha(x)), y_\alpha(x) - x\rangle \\ &\leq f(x, y_\alpha(x)) - \alpha\langle \nabla_x h(x, y_\alpha(x)), y_\alpha(x) - x\rangle. \end{aligned}$$

For the second inequality in (11), since $y_\alpha(x)$ is a global minimum for $f_\alpha(x, \cdot)$, the first order necessary optimality condition implies

$$\begin{aligned} & 0 \leq D_y f_\alpha(x, y_\alpha(x); x - y_\alpha(x)) \\ & = D_y f(x, y_\alpha(x); x - y_\alpha(x)) + \alpha\langle \nabla_y h(x, y_\alpha(x)), x - y_\alpha(x)\rangle. \end{aligned}$$

Moreover h is ∇-monotone then

$$\begin{aligned} & D_y f(x, y_\alpha(x); x - y_\alpha(x)) \\ & \geq \alpha\langle \nabla_y h(x, y_\alpha(x)), y_\alpha(x) - x\rangle \geq -\alpha\langle \nabla_x h(x, y_\alpha(x)), y_\alpha(x) - x\rangle. \end{aligned} \tag{12}$$

From the convexity of $f(x,\cdot)$ we obtain

$$0 = f(x,x) \geq f(x, y_\alpha(x)) + D_y f(x, y_\alpha(x); x - y_\alpha(x)),$$

and hence

$$D_y f(x, y_\alpha(x); x - y_\alpha(x)) \leq -f(x, y_\alpha(x)). \tag{13}$$

Comparing (12) and (13) we deduce the required second inequality. ■

Theorem 4.3 gives us an upper estimate of the directional derivative of the gap function. This is a fundamental result in order to obtain a globally convergent algorithm as we have seen in the continuously differentiable case [1]. In fact exploiting (11) we can force the gap function to have a decrease which is large enough. In particular the direction will be accepted when the inequality

$$-\varphi_\alpha(x) - \alpha(\langle \nabla_x h(x, y_\alpha(x) - x\rangle + h(x, y_\alpha(x))) < -\frac{1}{2}\varphi_\alpha(x) \tag{14}$$

holds. Naturally we can work decreasing the parameter α. In fact, if x is not a solution of *(EP)* condition (14) ensures that the direction is a descent direction (see Theorem 4.3).

References

1. G. Bigi, M. Castellani and M. Pappalardo, A new solution method for equilibrium problems, *Optimization Methods and Software*, to appear. DOI: 10.1080/10556780902855620.

2. E. Blum and W. Oettli, From optimization and variational inequalities to equilibrium problems, *The Mathematics Student*, **63** (1993), 1-23.

3. O. Chadli, I. V. Konnov and J. C. Yao, Descent methods for equilibrium problems in a Banach space, *Computers and Mathematics with Applications*, **48** (2004), 609-616.

4. J. M. Danskin, The theory of min-max with applications, *SIAM Journal on Applied Mathematics*, **14** (1966), 641-664.

5. A. Iusem and W. Sosa, New existence results for equilibrium problems, *Nonlinear Analisys*, **52** (2003), 621-635.

6. I. V. Konnov and M. S. S. Ali, Descent methods for monotone equilibrium problems in Banach spaces, *Journal of Computational and Applied Mathematics*, **188** (2006), 165-179.

7. G. Mastroeni, On auxiliary principle for equilibrium problems, in: *Equilibrium problems and variational models*, P. Daniele, F. Giannessi and A. Maugeri (eds.), Kluwer Academic Publishers, Dordrecht, 2003, pp. 289-298.

8. G. Mastroeni, Gap functions for equilibrium problems, *Journal of Global Optimization*, **27** (2003), 411-426.

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