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VARIATIONAL METHODS TO MIXED BOUNDARY VALUE PROBLEM FOR IMPULSIVE DIFFERENTIAL EQUATIONS WITH A PARAMETER

Yu Tian,* Jun Wang and Weigao Ge

Abstract. In this paper, we study mixed boundary value problem for second-order impulsive differential equations with a parameter. By using critical point theory, several new existence results are obtained. This is one of the first times that impulsive boundary value problems are studied by means of variational methods.

1. INTRODUCTION

In this paper, we study the following impulsive problem

$$\begin{cases} -u''(t) = \lambda u(t) + f(t, u(t)), \quad t \neq t_i,\, t \in [0, T], \\ -\Delta u'(t_i) = I_i(u(t_i)), \quad i = 1, 2, \cdots, l, \\ u'(0) = 0, \quad u(T) = 0, \end{cases} \tag{1.1}$$

where $\lambda \in R, 0 = t_0 < t_1 < \cdots < t_l < t_{l+1} = T$, $\Delta u'(t_i) = u'(t_i^+) - u'(t_i^-)$, (where $u'(t_i^+)$ (resp. $u'(t_i^-)$) denotes the right limit (resp. left limit) of $u'(t)$ at $t = t_i$, $u'(t_i^-) = u'(t_i)$, $I_i \in C(R, R), i = 1, 2, \cdots, l$, $f \in C([0, T] \times R, R)$.

By applying critical point theory to (1.1), several existence results are obtained when f is imposed some assumptions and λ lies in suitable interval.

Motivated by the wide applications in evolution process, impulsive differential equations are studied extensively, we refer the readers to the monographs and some

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*Corresponding author

recent contributions as [5, 7, 10-12, 16, 17]. Main results are obtained by using the tools such as fixed point theorems in cones [1, 4, 6, 8], the method of lower and upper solutions [3]. On the other hand, also critical point theory is a powerful tool to study differential and difference equations (see, for instance, [2, 9, 13-15, 18, 19]). However, besides [20] , there are only a few papers where impulsive differential equations are studied by means of critical point theory.

This paper is organized as follows. In Section 2, we present some preliminary results, which are necessary to Section 3. In Section 3, we establish several existence results of impulsive problem (1.1) by using critical point theory. Besides, some examples are presented to illustrate the results obtained.

2. Preliminaries

Let us recall some basic results in critical point theory.

Definition 2.1. Let E be a real Banach space. We say that $\varphi \in C^1(E, R)$ satisfies the Palais-Smale condition (PS) if any sequence $(u_k) \subset E$ for which $\varphi(u_k)$ is bounded and $\varphi'(u_k) \to 0$ as $k \to \infty$ possesses a convergent subsequence.

Theorem 2.1. (Theorem 2.2 [13]). *Let E be a real Banach space and $I \in C^1(E, R)$ satisfying (PS). Suppose $I(0) = 0$ and*

$(C1)$ *there are constants $\rho, \alpha > 0$ such that $I|_{\partial B_\rho} \geq \alpha$, and*

$(C2)$ *there is an $e \in E \setminus \overline{B}_\rho$ such that $I(e) \leq 0$.*

Then I possesses a critical value $c \geq \alpha$. Moreover, c can be characterized as

$$c = \inf_{g\in\Gamma} \max_{u\in g([0,1])} I(u),$$

where

$$\Gamma = \{g \in C([0, 1], E) : g(0) = 0, g(1) = e\}.$$

Theorem 2.2. (Theorem 5.3 [13]). *Let E be a real Banach space with $E = V \oplus X$, where V is finite dimensional. Suppose $I \in C^1(E, R)$, satisfies (PS), and*

$(C3)$ *there are constants $\rho, \alpha > 0$ such that $I|_{\partial B_\rho \cap X} \geq \alpha$, and*

$(C4)$ *there is an $e \in \partial B_1 \cap X$ and $R > \rho$ such that if $Q \equiv (\overline{B}_R \cap V) \oplus \{re : 0 < r < R\}$, then $I|_{\partial Q} \leq 0$.*

Then I possesses a critical value $c \geq \alpha$ which can be characterized as

$$c = \inf_{h\in\Gamma} \max_{u\in Q} I(h(u)),$$

where

$$\Gamma = \{h \in C(\overline{Q}, E) : h = id \,\text{on}\, \partial Q\}.$$

Theorem 2.3. (Theorem 9.1 [13]). *Let E be a real Banach space, $I \in C^1(E, R)$ with I even, bounded from below, and satisfying (PS). Suppose $I(0) = 0$, there is a set $K \subset E$ such that K is homeomorphic to S^{j-1} by an odd map, and $\sup_K I < 0$. Then I possesses at least j distinct pairs of critical points.*

Theorem 2.4. (Theorem 9.12 [13]). *Let E be an infinite dimensional Banach space and let $I \in C^1(E, R)$ be even, satisfy (PS), and $I(0) = 0$. If $E = V \oplus X$, where V is finite dimensional, and I satisfies*

$(C5)$ *there are constants $\rho, \alpha > 0$ such that $I_{\partial B_\rho \cap X} \geq \alpha$, and*

$(C6)$ *for each finite dimensional subspace $\widetilde{E} \subset E$, there is an $R = R(\widetilde{E})$ such that $I \leq 0$ on $\widetilde{E} \setminus B_{R(\widetilde{E})}$,*

then I possesses an unbounded sequence of critical values.

Definition 2.2. A function $u \in \big\{x \in C([0,T]) : x'(\cdot) \in C^1([0,T] \setminus \{t_1, t_2, \cdots, t_l\})\big\}$ is said to be a classical solution of problem (1.1) if u satisfies equation in (1.1) for $t \in [0,T] \setminus \{t_1, t_2, \cdots, t_l\}$ and impulsive condition and boundary condition of (1.1).

Define the space $Y = \{u \in C([0,T]) : u'(\cdot) \in L^2([0,T]), u(T) = 0\}$ with the inner product

$$(u, v) = \int_0^T u'(t)v'(t)dt$$

inducing the norm

$$\|u\| = \left(\int_0^T |u'(t)|^2 dt\right)^{\frac{1}{2}}.$$

We claim that

$$\left(\int_0^T u^2(t)dt\right)^{\frac{1}{2}} \leq \frac{1}{\sqrt{\lambda_1}} \left(\int_0^T |u'(t)|^2 dt\right)^{\frac{1}{2}}, \tag{2.1}$$

here $\lambda_1 = \frac{\pi^2}{(2T)^2}$ is the first nonzero eigenvalue of the problem

$$-u''(t) = \lambda u(t), t \in [0,T], \quad u'(0) = 0, u(T) = 0. \tag{2.2}$$

As is well known, (2.2) possesses a sequence of eigenvalues (λ_i) $\left(\lambda_i = \left(\frac{(2i-1)\pi}{2T}\right)^2\right)$ with

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_j < \cdots$$

In fact, let

$$u_1(t) = \begin{cases} u(t-4kT), & t \in [4kT, (4k+1)T], \\ u(-t+4kT), & t \in [(4k-1)T, 4kT], \\ -u(t-4kT-2T), & t \in [(4k+1)T, (4k+2)T], \\ -u(t-4kT+2T), & t \in [(4k-2)T, (4k-1)T]. \end{cases}$$

Then u_1 is a $4T-$periodic function on $(-\infty, +\infty)$. By the expression of Fourier expansion, we have

$$\int_0^T u^2(t)dt = \frac{1}{4}\int_{-2T}^{2T} |u_1(t)|^2 dt = \frac{1}{4}\int_{-2T}^{2T} \left(\sum_{k=1}^{\infty} a_k \cos\frac{k\pi t}{2T}\right)^2 dt = \frac{1}{4}\sum_{k=1}^{\infty} a_k^2 2T.$$

Besides, by the above equality and Parseval equality

$$\begin{aligned} \int_0^T |u'(t)|^2 dt &= \frac{1}{4}\int_{-2T}^{2T} |u_1'(t)|^2 dt \\ &= \frac{1}{4}\int_{-2T}^{2T} \left(\sum_{k=1}^{\infty} a_k \frac{k\pi}{2T}\sin\frac{k\pi t}{2T}\right)^2 dt \\ &= \frac{1}{4}\int_{-2T}^{2T} \sum_{k=1}^{\infty} a_k^2 \frac{k^2\pi^2}{4T^2}\sin^2\frac{k\pi t}{2T} dt \\ &= \frac{1}{4}\sum_{k=1}^{\infty} a_k^2 \frac{k^2\pi^2}{4T^2}\cdot 2T \geq \lambda_1 \int_0^T u^2(t)dt. \end{aligned}$$

So (2.1) holds. and $\lambda_j \to \infty$ as $j \to \infty$. (The number of times of an eigenvalue appears in the sequence equals to its multiplicity). The corresponding eigenfunctions are normalized so that $\|\varphi_j\| = 1 = \lambda_j \int_0^T |\varphi_j(t)|^2 dt$, here

$$\varphi_j(t) = \sqrt{\frac{2}{T\lambda_j}}\cos(\sqrt{\lambda_j}t), \quad j = 1, 2, \cdots. \tag{2.3}$$

Lemma 2.5. *If $\lambda < \lambda_1$, then $\left(\int_0^T (u'(t))^2 - \lambda u^2(t)dt\right)^{\frac{1}{2}}$ can be taken as a norm on Y.*

Proof. If $0 \leq \lambda < \lambda_1$, we have by (2.1)

$$\begin{aligned} \left(1 - \frac{\lambda}{\lambda_1}\right)\int_0^T (u'(t))^2 dt &= \int_0^T (u'(t))^2 dt - \frac{\lambda}{\lambda_1}\int_0^T (u'(t))^2 dt \\ &\leq \int_0^T (u'(t))^2 - \lambda u^2(t)dt \leq \int_0^T (u'(t))^2 dt. \end{aligned} \tag{2.4}$$

If $\lambda < 0$, we have by (2.1)

$$
\begin{aligned}
\int_0^T (u'(t))^2 dt &\le \int_0^T (u'(t))^2 - \lambda u^2(t) dt \\
&< \int_0^T (u'(t))^2 dt - \frac{\lambda}{\lambda_1} \int_0^T (u'(t))^2 dt \\
&= \left(1 - \frac{\lambda}{\lambda_1}\right) \int_0^T (u'(t))^2 dt.
\end{aligned}
\tag{2.5}
$$

The result follows from (2.4) (2.5). ■

Remark 2.1. By Lemma 2.5, if $\lambda < \lambda_1$, then there exist $\theta_1, \theta_2 > 0$ satisfying

$$\theta_1 \|u\|^2 \le \int_0^T (u'(t))^2 - \lambda u^2(t) dt \le \theta_2 \|u\|^2$$

for $u \in Y$.

Lemma 2.6. *If $u \in Y$, then*

$$\|u\|_0 \le T^{\frac{1}{2}} \|u\|,$$

where $\|u\|_0 = \max\limits_{t \in [0,T]} |u(t)|$.

Proof. By Hölder inequality, for $u \in Y$,

$$|u(t)| = \left| u(T) - \int_t^T u'(s) ds \right| \le T^{\frac{1}{2}} \left(\int_0^T |u'(s)|^2 ds \right)^{\frac{1}{2}} = T^{\frac{1}{2}} \|u\|.$$

Define $F(t, u) = \int_0^u f(t, s) ds$. Now we consider the functional $E : Y \to R$ defined by

$$E(u) = \frac{1}{2} \int_0^T (u'(t))^2 - \lambda u^2(t) dt - \int_0^T F(t, u(t)) dt - \sum_{i=1}^{l} \int_0^{u(t_i)} I_i(s) ds.$$

Clearly E is a Gâteaux differentiable functional whose Gâteaux derivative at the point $u \in Y$ is the functional $E'(u) \in Y^* = Y$, given by

$$\langle E'(u), v \rangle = \int_0^T u'(t) v'(t) - \lambda u(t) v(t) dt - \int_0^T f(t, u(t)) v(t) dt - \sum_{i=1}^{l} I_i(u(t_i)) v(t_i).$$

Definition 2.3. A function $u \in Y$ is said to be a weak solution of (1.1), if u satisfies $\langle E'(u), v\rangle = 0$ for all $v \in Y$.

Lemma 2.7. *If $u \in Y$ is a weak solution of* (1.1), *then u is a classical solution of* (1.1).

Proof. By Definition 2.3, if u is a weak solution of (1.1), then $\langle E'(u), v\rangle = 0$ holds for all $v \in Y$, i.e.

$$\int_0^T u'(t)v'(t) - \lambda u(t)v(t)dt - \int_0^T f(t,u(t))v(t)dt - \sum_{i=1}^{l} I_i(u(t_i))v(t_i) = 0.$$

By integrating by parts, we have

$$\begin{aligned}
&\int_0^T u'(t)v'(t) - \lambda u(t)v(t)dt - \int_0^T f(t,u(t))v(t)dt - \sum_{i=1}^{l} I_i(u(t_i))v(t_i)\\
&= \sum_{i=0}^{l} u'(t)v(t)\Big|_{t=t_i^+}^{t_{i+1}} + \int_0^T [-u''(t) - \lambda u(t) - f(t,u(t))]v(t)dt - \sum_{i=1}^{l} I_i(u(t_i))v(t_i)\\
&= \int_0^T [-u''(t) - \lambda u(t) - f(t,u(t))]v(t)dt\\
&\quad - \sum_{i=1}^{l}[\Delta u'(t_i) + I_i(u(t_i))]v(t_i) - u'(0)v(0) + u'(T)v(T)\\
&= \int_0^T [-u''(t) - \lambda u(t) - f(t,u(t))]v(t)dt\\
&\quad - \sum_{i=1}^{l}[\Delta u'(t_i) + I_i(u(t_i))]v(t_i) - u'(0)v(0).
\end{aligned}$$

Thus

$$\begin{aligned}
&\int_0^T [-u''(t) - \lambda u(t) - f(t,u(t))]v(t)dt\\
&- \sum_{i=1}^{l}[\Delta u'(t_i) + I_i(u(t_i))]v(t_i) - u'(0)v(0) = 0
\end{aligned} \tag{2.6}$$

holds for all $v \in Y$. Without loss of generality, we assume that $v \in C_0^\infty(t_i, t_{i+1}), v(t) \equiv 0, t \in [0, t_i] \cup [t_{i+1}, T]$, then substituting v into (2.6) we get

$$-u''(t) - \lambda u(t) - f(t,u(t)) = 0, \quad t \in (t_i, t_{i+1}).$$

Thus u satisfies the equation in (1.1). So (2.6) becomes

$$-\sum_{i=1}^{l}[\Delta u'(t_i) + I_i(u(t_i))]v(t_i) - u'(0)v(0) = 0. \tag{2.7}$$

Now we will show that u satisfies impulsive condition in (1.1). If not, without loss of generality, we assume that there exists $i \in \{1, 2, \cdots, l\}$ such that

$$\Delta u'(t_i) + I_i(u(t_i)) \neq 0.$$

Let $v(t) = \prod_{j=0, j\neq i}^{l+1}(t - t_j)$, then

$$-\sum_{i=1}^{l}[\Delta u'(t_i) + I_i(u(t_i))]v(t_i) - u'(0)v(0) = -[\Delta u'(t_i) + I_i(u(t_i))] \times v(t_i) \neq 0,$$

which contradicts to (2.7). So u satisfies impulsive condition in (1.1), which yields that u satisfies boundary condition. Therefore, u is a solution of (1.1). ∎

3. Main Results

In this section, we will establish the existence of solutions for (1.1).

We will use the following assumptions

$(H1)$ $f(t, x) = \circ(|x|), I_i(x) = \circ(|x|)$ as $|x| \to 0$, $i = 1, 2, \cdots, l$.

$(H2)$ There exist constants $\mu > 2$ and $r \geq 0$ such that for $|\xi| \geq r$,

$$0 < \mu F(t, \xi) \leq \xi f(t, \xi), \quad 0 < \mu \int_0^{\xi} I_i(s)ds \leq \xi I_i(\xi), i = 1, 2, \cdots, l.$$

$(H3)$ $f(t, u), I_i(u)$ are odd in u.

$(H4)$ There exist $x_1 > 0, r > \lambda_k$, λ_k is the k-th eigenvalue of (2.2) such that

$$rx_1 + f(t, x_1) \leq 0, t \in [0, T], \quad I_i(x_1) \leq 0, i = 1, 2, \cdots, l.$$

Remark 3.1. By (H2), there exist $b_1, b_2, c_i, d_i > 0, i = 1, 2, \cdots, l$ such that

$$F(t, \xi) \geq b_1|\xi|^{\mu} - b_2, \quad \int_0^{\xi} I_i(s)ds \geq c_i|\xi|^{\mu} - d_i$$

for all $t \in [0, T], \xi \in R$.

Theorem 3.1. *Suppose that (H1) (H2) hold. Then for all $\lambda \in R$, problem* (1.1) *has at least one nontrivial solution.*

Proof. We will finish the proof by two cases (i) $\lambda < \lambda_1$, (ii) $\lambda \geq \lambda_1$.

(i) $\lambda < \lambda_1$. We will apply Theorem 2.1 to show the existence of solutions. Clearly $E \in C^1(Y, R)$ and $E(0) = 0$. By Lemma 2.5, there exists $\theta_1 > 0$ such that

$$\int_0^T (u'(t))^2 - \lambda u^2(t)dt \geq \theta_1 \|u\|^2.$$

By (H1), for any $\varepsilon > 0$, there exists a $\delta > 0$ such that $|\xi| \leq \delta$ implies

$$|F(t,\xi)| \leq \frac{1}{2}\varepsilon|\xi|^2, \quad \sum_{i=1}^{l} \int_0^{\xi} I_i(s)ds \leq \frac{1}{2}\varepsilon|\xi|^2$$

for all $t \in [0, T]$. Consequently, by Lemma 2.6, for $\|u\| \leq \frac{\delta}{\sqrt{T}}$

$$\int_0^T F(t, u(t))dt + \sum_{i=1}^{l} \int_0^{u(t_i)} I_i(s)ds \leq \frac{1}{2}\varepsilon T(T+1)\|u\|^2. \tag{3.1}$$

Since ε is arbitrary, (3.1) shows

$$\int_0^T F(t, u(t))dt + \sum_{i=1}^{l} \int_0^{u(t_i)} I_i(s)ds = \circ(\|u\|^2)$$

as $u \to 0$. Therefore,

$$\begin{aligned} E(u) &= \frac{1}{2}\int_0^T (u'(t))^2 - \lambda u^2(t)dt - \int_0^T F(t,u(t))dt - \sum_{i=1}^{l}\int_0^{u(t_i)} I_i(s)ds \\ &\geq \frac{\theta_1}{2}\|u\|^2 + \circ(\|u\|^2), \end{aligned} \tag{3.2}$$

as $u \to 0$. So (C1) holds.

In order to verify (C2), we choose $e(t) = \varphi_1(t) \in Y, \gamma \in R$. Then by Lemma 2.5, (H2), Remark 2.1, Remark 3.1,

$$\begin{aligned} &E(\gamma e) \\ &= \frac{\gamma^2}{2}\int_0^T \left[(e'(t))^2 - \lambda e^2(t)\right] dt - \int_0^T F(t,\gamma e(t))dt - \sum_{i=1}^{l}\int_0^{\gamma e(t_i)} I_i(s)ds \\ &\leq \frac{\theta_2\gamma^2}{2}\|e\|^2 - \int_0^T (b_1|\gamma|^\mu|e(t)|^\mu - b_2)\,dt - \sum_{i=1}^{l}(c_i|\gamma|^\mu|e(t_i)|^\mu - d_i)). \end{aligned} \tag{3.3}$$

By computation,

$$\|e\|^2 = \|\varphi_1\|^2 = 1. \tag{3.4}$$

By Hölder inequality, we have

$$\int_0^T |e(t)|^\mu dt \geq \left[\int_0^T e^2(t)dtT^{\frac{2-\mu}{\mu}}\right]^{\frac{\mu}{2}} = \left(\frac{1}{\lambda_1}\right)^{\frac{\mu}{2}} T^{\frac{2-\mu}{2}} = \left(\frac{2T}{\pi}\right)^\mu T^{\frac{2-\mu}{2}}. \tag{3.5}$$

Substituting (3.4) (3.5) into (3.3), we have

$$E(\gamma e) \leq \frac{\theta_2\gamma^2}{2} - b_1|\gamma|^\mu \left(\frac{2T}{\pi}\right)^\mu T^{\frac{2-\mu}{2}} + b_2T + \sum_{i=1}^{l} d_i \to -\infty$$

as $|\gamma| \to +\infty$. Hence (C2) holds.

It remains to check that E satisfies (PS) or that $|E(u_m)| \leq M$ and $|E'(u_m)| \to 0$ imply (u_m) strongly converges to u in Y up to a subsequence. By the definitions of E, E', Lemma 2.5 and (H2), one has $\beta = \frac{1}{\mu}$

$$\begin{aligned} M + \beta\|u_m\| \ &\geq \ E(u_m) - \frac{1}{\mu}\langle E'(u_m), u_m\rangle \\ &= \left(\frac{1}{2} - \frac{1}{\mu}\right)\int_0^T (u_m^{'}(t))^2 - \lambda u_m^2(t)dt \\ &\quad - \int_0^T \left[F(t, u_m(t)) - \frac{1}{\mu}f(t, u_m(t))u_m(t)\right]dt \\ &\quad - \left[\sum_{i=1}^{l}\int_0^{u_m(t_i)} I_i(s)ds - \frac{1}{\mu}\sum_{i=1}^{l} I_i(u_m(t_i))u_m(t_i)\right] \\ &\geq \left(\frac{1}{2} - \frac{1}{\mu}\right)\theta_1\|u_m\|^2 + c, \end{aligned}$$

for some constant c. So (u_m) is bounded in Y.

The fact (u_m) is bounded in Y means that $u_m \rightharpoonup u$. Following we will show (u_m) strongly converges to u in Y. By the definition of E', Lemma 2.5, Remark 2.1, one has

$$\begin{aligned} 0 \leftarrow &\langle E'(u_m) - E'(u), u_m - u\rangle \\ = &\int_0^T (u_m^{'}(t) - u'(t))^2 - \lambda(u_m(t) - u(t))^2 dt \\ &- \int_0^T [f(t, u_m(t)) - f(t, u(t))][u_m(t) - u(t)]dt \\ &- \sum_{i=1}^{l}[I_i(u_m(t_i)) - I_i(u(t_i))][u_m(t_i) - u(t_i)] \end{aligned} \tag{3.6}$$

$$
\begin{aligned}
&\leq \theta_2 \|u_m - u\|^2 - \int_0^T [f(t, u_m(t)) - f(t, u(t))][u_m(t) - u(t)]dt \\
&\quad - \sum_{i=1}^{l} [I_i(u_m(t_i)) - I_i(u(t_i))][u_m(t_i) - u(t_i)].
\end{aligned}
$$

Similar to the proof of Proposition 1.2 in [9], $u_m \rightharpoonup u$ implies (u_m) uniformly converges to u in $C([0, T])$. So

$$
\begin{cases}
\displaystyle\int_0^T [f(t, u_m(t)) - f(t, u(t))][u_m(t) - u(t)]dt \to 0 \\
\displaystyle\sum_{i=1}^{l} [I_i(u_m(t_i)) - I_i(u(t_i))][u_m(t_i) - u(t_i)] \to 0
\end{cases}
\tag{3.7}
$$

as $n \to \infty$. (3.6) (3.7) mean that (u_m) strongly converges to u in Y. Therefore, E satisfies (PS). Applying Theorem 2.1, the result follows.

(ii) $\lambda \geq \lambda_1$, i.e. $\lambda \in [\lambda_k, \lambda_{k+1})$ for some $k > 0$. We will show that E satisfies the hypotheses of Theorem 2.2. Clearly $E \in C^1(Y, R)$. Set $V = span\{\varphi_1, \varphi_2, \cdots, \varphi_k\}$ and $X = V^{\perp}$, where $\varphi_j^{'} s$ are the eigenfunctions defined in (2.3). For $u \in X$, $u = \sum\limits_{i=k+1}^{\infty} a_i \varphi_i$.

$$
\begin{aligned}
\int_0^T (u'(t))^2 - \lambda u^2(t)dt &= \int_0^T \left(\sum_{i=k+1}^{\infty} a_i \varphi_i^{'}(t) \right)^2 - \lambda \left(\sum_{i=k+1}^{\infty} a_i \varphi_i(t) \right)^2 dt \\
&= \sum_{i=k+1}^{\infty} a_i^2 \int_0^T (\varphi_i^{'}(t))^2 dt - \lambda \sum_{i=k+1}^{\infty} a_i^2 \int_0^T \varphi_i^2(t)dt \\
&= \sum_{i=k+1}^{\infty} a_i^2 \left(1 - \frac{\lambda}{\lambda_i}\right) \geq \left(1 - \frac{\lambda}{\lambda_{k+1}}\right) \|u\|^2.
\end{aligned}
\tag{3.8}
$$

By (H1), we have for $u \in X$

$$
\int_0^T F(t, u(t))dt + \sum_{i=1}^{l} \int_0^{u(t_i)} I_i(s)ds = \circ(\|u\|^2)
$$

as $u \to 0$. So E satisfies (C3) in Theorem 2.2.

Now we will check (C4). For $u = \sigma + re$, where $\sigma \in V, e = \varphi_{k+1}$, we have

$$
\begin{aligned}
E(u) &= \frac{1}{2}\int_0^T (u'(t))^2 - \lambda u^2(t)dt \\
&\quad - \int_0^T F(t,u(t))dt - \sum_{i=1}^{l}\int_0^{u(t_i)} I_i(s)ds \\
&= \frac{1}{2}\int_0^T (\sigma'(t))^2 - \lambda\sigma^2(t)dt + \frac{r^2}{2}\int_0^T (e'(t))^2 - \lambda e^2(t)dt \\
&\quad - \int_0^T F(t,u(t))dt - \sum_{i=1}^{l}\int_0^{u(t_i)} I_i(s)ds.
\end{aligned} \tag{3.9}
$$

Since $\sigma \in V$, we assume $\sigma = \sum\limits_{i=1}^{k} a_i\varphi_i$, where $\varphi_i (i = 1, 2, \cdots, k)$ is defined in (2.3). Thus

$$\int_0^T (\sigma'(t))^2 - \lambda\sigma^2(t)dt \le \left(1 - \frac{\lambda}{\lambda_k}\right)\sum_{i=1}^{k} a_i^2 = \left(1 - \frac{\lambda}{\lambda_k}\right)\|\sigma\|^2 \le 0. \tag{3.10}$$

$$r^2\int_0^T (e'(t))^2 - \lambda e^2(t)dt = r^2\left(1 - \frac{\lambda}{\lambda_{k+1}}\right) \le r^2. \tag{3.11}$$

Substituting (3.10) (3.11) into (3.9), noticing Remark 3.1, we have

$$E(u) \le \frac{r^2}{2} - \int_0^T (b_1|u(t)|^\mu - b_2)\, dt - \sum_{i=1}^{l}(c_i|u(t_i)|^\mu - d_i). \tag{3.12}$$

For any finite dimensional subspace $V_1 \subset Y$, the norms $\|\cdot\|$ and $\|\cdot\|_{V_1}$ are equivalent. So there exists $\overline{c} > 0$ satisfying

$$\|u\| \le \overline{c}\|u\|_{L^\mu} \tag{3.13}$$

for $u \in V_1$. Thus

$$\int_0^T b_1|u(t)|^\mu dt \ge b_1\overline{c}^{-\mu}\|u\|^\mu = b_1\overline{c}^{-\mu}(\|\sigma\|^2 + \|re\|^2)^{\frac{\mu}{2}} \ge b_1\overline{c}^{-\mu}(\|\sigma\|^\mu + r^\mu). \tag{3.14}$$

By (3.12) (3.13) (3.14), one has

$$E(u) \le \frac{r^2}{2} - b_1\overline{c}^{-\mu}(\|\sigma\|^\mu + r^\mu) + b_2T + \sum_{i=1}^{k} d_i. \tag{3.15}$$

Let

$$g_1(r) = \frac{r^2}{2} - b_1\overline{c}^{-\mu}r^\mu + b_2T + \sum_{i=1}^{l} d_i, \quad g_2(r) = -b_1\overline{c}^{-\mu}r^\mu.$$

Then $\lim_{r\to+\infty} g_1(r) = \lim_{r\to+\infty} g_2(r) = -\infty$, and $g_1(r), g_2(r)$ are bounded from above. Thus there exists $R_1 > 0$ such that $E(u) \le 0$ for all $u \in \partial Q$, where $Q = (\overline{B}_{R_1} \cap V) \oplus \{re : 0 < r < R_1\}$.

It remains to check that E satisfies (PS), i.e. $|E(u_m)| \le M$ and $|E'(u_m)| \to 0$ imply that (u_m) strongly converges to u in Y.

First we show (u_m) is bounded in Y. Choose $\beta \in \left(\frac{1}{\mu}, \frac{1}{2}\right)$. Then for m sufficiently large, by (H2) and definitions of E, E', we have

$$\begin{aligned}
&M + \beta\|u_m\| \ge E(u_m) - \beta\langle E'(u_m), u_m\rangle \\
&= \left(\frac{1}{2} - \beta\right)\int_0^T (u_m^{'}(t))^2 - \lambda u_m^2(t)dt \\
&\quad - \int_0^T [F(t, u_m(t)) - \beta f(t, u_m(t))u_m(t)]dt \\
&\quad - \left[\sum_{i=1}^{l}\int_0^{u_m(t_i)} I_i(s)ds - \beta\sum_{i=1}^{l} I_i(u_m(t_i))u_m(t_i)\right] \\
&\ge \left(\frac{1}{2} - \beta\right)\|u_m\|^2 - \left(\frac{1}{2} - \beta\right)\lambda\|u_m\|_{L^2}^2 + (\beta\mu - 1)\int_0^T F(t, u_m(t))dt \\
&\quad + (\beta\mu - 1)\sum_{i=1}^{l}\int_0^{u_m(t_i)} I_i(s)ds \\
&\ge \left(\frac{1}{2} - \beta\right)\|u_m\|^2 - \left(\frac{1}{2} - \beta\right)\lambda\|u_m\|_{L^2}^2 \\
&\quad + (\beta\mu - 1)\int_0^T \left[b_1|u_m(t)|^\mu - b_2\right]dt + (\beta\mu - 1)\sum_{i=1}^{l}\left[c_i|u_m(t_i)|^\mu - d_i\right].
\end{aligned} \tag{3.16}$$

By Hölder inequality,

$$\|u\|_{L^\mu}^\mu \ge \left(\|u\|_{L^2}^2 T^{\frac{2-\mu}{\mu}}\right)^{\frac{\mu}{2}} = \|u\|_{L^2}^\mu T^{\frac{2-\mu}{2}}. \tag{3.17}$$

Substituting (3.17) into (3.16), we have

$$M_1 + M_2\|u_m\| + M_3\|u_m\|_{L^2}^2 \ge \left(\frac{1}{2} - \beta\right)\|u_m\|^2 + M_4\|u_m\|_{L^2}^\mu$$

for $M_1, M_2, M_3, M_4 > 0$. Therefore, (u_m) is bounded in Y and $L^2([0,T])$. The fact (u_m) is bounded in Y means that $u_m \rightharpoonup u$. Following we will show (u_m)

strongly converges to u. By the definition of E', one has

$$
\begin{aligned}
& 0 \leftarrow \langle E'(u_m) - E'(u), u_m - u\rangle \\
&= \int_0^T (u_m'(t) - u'(t))^2 - \lambda(u_m(t) - u(t))^2 dt \\
&\quad - \int_0^T [f(t, u_m(t)) - f(t, u(t))][u_m(t) - u(t)]dt \\
&\quad - \sum_{i=1}^{l} [I_i(u_m(t_i)) - I_i(u(t_i))][u_m(t_i) - u(t_i)] \\
&= \|u_m - u\|^2 - \lambda \int_0^T (u_m(t) - u(t))^2 dt \\
&\quad - \int_0^T [f(t, u_m(t)) - f(t, u(t))][u_m(t) - u(t)]dt \\
&\quad - \sum_{i=1}^{l} [I_i(u_m(t_i)) - I_i(u(t_i))][u_m(t_i) - u(t_i)].
\end{aligned} \tag{3.18}
$$

Similar to the proof of Proposition 1.2 in [9], $u_m \rightharpoonup u$ implies (u_m) uniformly converges to u in $C([0, T])$. So

$$
\begin{cases}
\int_0^T (u_m(t) - u(t))^2 dt \to 0, \\
\int_0^T [f(t, u_m(t)) - f(t, u(t))][u_m(t) - u(t)]dt \to 0 \\
\sum_{i=1}^{l} [I_i(u_m(t_i)) - I_i(u(t_i))][u_m(t_i) - u(t_i)] \to 0
\end{cases} \tag{3.19}
$$

as $n \to \infty$. By (3.18) (3.19), (u_m) strongly converges to u in Y. Therefore, E satisfies (PS). By Theorem 2.2, E possesses critical value $c > 0$, where

$$c = \inf_{h \in \Gamma} \max_{u \in Q} E(h(u)),$$

$$\Gamma = \{h \in C(\overline{Q}, Y) : h|_{\partial Q} = id\}.$$

Let $x \in Y$ be a critical point associated to the critical value c of E, i.e. $E(x) = c$. $c > 0$ implies that $x \not\equiv 0$. Lemma 2.7 means that problem (1.1) has at least one nontrivial solution. ∎

Theorem 3.2. *For $k \in Z$. Assume that (H1) (H3) (H4) hold. Then for $\lambda \in (\lambda_k, r]$, problem* (1.1) *has at least k distinct pairs of solutions.*

Proof. Set

$$q(\lambda,t,x)=\begin{cases}\lambda x+f(t,x), & x\in[-x_1,x_1],\\ \lambda x_1+f(t,x_1), & x\in[x_1,\infty),\\ -\lambda x_1+f(t,-x_1), & x\in(-\infty,-x_1],\end{cases}\qquad J_i(x)=\begin{cases}I_i(x), & x\in[-x_1,x_1],\\ I_i(x_1), & x\in[x_1,\infty),\\ I_i(-x_1), & x\in(-\infty,-x_1].\end{cases}$$

Consider

$$\begin{cases}-u''(t)=q(\lambda,t,u(t)), \ \ t\in[0,T], t\neq t_i,\\ -\Delta u'(t_i)=J_i(u(t_i)), \ \ i=1,2,\cdots,l,\\ u'(0)=0, \ \ u(T)=0.\end{cases}\tag{3.20}$$

We claim any solution of (3.20) is a solution of (1.1), that is, any solution u of (3.20) satisfies $u(t)\in[-x_1,x_1], t\in[0,T]$.

For this, let $\mathcal{B}_1=\{t\in(a_1,b_1)\subseteq[0,T]: u(t)>x_1\}$. By the definitions of $q(\lambda,t,x)$ and $J_i(x)$, (3.20) is reduced to

$$\begin{cases}-u''(t)=q(\lambda,t,x_1)=\lambda x_1+f(t,x_1)\\ \le rx_1+f(t,x_1)\le 0, \quad t\in(a_1,b_1), t\neq t_i,\\ -\Delta u'(t_i)=J_i(u(t_i))=I_i(x_1)\le 0, \quad i=1,2,\cdots,l,\\ u(a_1)=u(b_1)=x_1.\end{cases}\tag{3.21}$$

The solution $u(t)$ of (3.21) satisfies $u(t)\le x_1, t\in(a_1,b_1)$. So $\mathcal{B}_1=\emptyset$ and $u(t)\le x_1$.

Let $\mathcal{B}_2=\{t\in(a_2,b_2)\subseteq[0,T]: u(t)<-x_1\}$. By the definitions of $q(\lambda,t,x)$ and $J_i(x)$, (3.20) is reduced to

$$\begin{cases}-u''(t)=q(\lambda,t,-x_1)=-\lambda x_1+f(t,-x_1)\\ \ge -rx_1-f(t,x_1)\ge 0, \quad t\in(a_2,b_2), t\neq t_i,\\ -\Delta u'(t_i)=J_i(u(t_i))=-I_i(x_1)\ge 0, \quad i=1,2,\cdots,l,\\ u(a_2)=u(b_2)=-x_1.\end{cases}\tag{3.22}$$

The solution $u(t)$ of (3.22) satisfies $u(t)\ge -x_1, t\in(a_2,b_2)$. So $\mathcal{B}_2=\emptyset$ and $u(t)\ge -x_1$.

Therefore, any solution of (3.20) is a solution of (1.1). Hence to prove Theorem 3.2, it suffices to produce at least k distinct pairs of critical points of

$$E(u)=\frac{1}{2}\int_0^T|u'(t)|^2dt-\int_0^T Q(\lambda,t,u(t))dt-\sum_{i=1}^{l}\int_0^{u(t_i)}J_i(s)ds,$$

where $Q(\lambda, t, x) = \int_0^x q(\lambda, t, s)ds$. We will apply Theorem 2.3 to finish the proof. Clearly $E \in C^1(Y, R)$ is even and $E(0) = 0$. Since $q(\lambda, t, x), J_i(x)$ are bounded functions, $E(u)$ is bounded from below and (PS) hold as in the proof of Theorem 3.1. Now set $K = \left\{ \sum_{i=1}^{k} a_i\varphi_i : \sum_{i=1}^{k} a_i^2 = a^2 \right\}$, where φ_i is defined in (2.3). It is clear that K is homeomorphic to S^{k-1} by an odd map for any $a > 0$. We claim $E|_K < 0$ if a is sufficiently small.

For any $u \in K$, $u = \sum_{i=1}^{k} a_i\varphi_i$. By (H1) (3.9),

$$\begin{aligned}
E(u) &= \frac{1}{2}\int_0^T \left[\left(\sum_{i=1}^{k} a_i\varphi_i(t)\right)'\right]^2 dt - \int_0^T Q(\lambda, t, u(t))dt - \sum_{i=1}^{l}\int_0^{u(t_i)} J_i(s)ds \\
&= \frac{1}{2}\sum_{i=1}^{k} a_i^2\left(1 - \frac{\lambda}{\lambda_i}\right) - \int_0^T F(t, u(t))dt - \sum_{i=1}^{l}\int_0^{u(t_i)} J_i(s)ds \\
&\le \frac{1}{2}\left(1 - \frac{\lambda}{\lambda_k}\right) a^2 + \circ(a^2) + \circ(a^2)
\end{aligned}$$

for small $a > 0$. Since $\lambda \in (\lambda_k, r]$, $E(u) < 0$ and the proof is complete.

Theorem 3.3. *Suppose that* (*H1*) (*H2*) (*H3*) *hold. Then for* $\lambda < \lambda_1$, *problem* (1.1) *possesses infinitely many solutions.*

Proof. We will apply Theorem 2.4 to finish the proof. Clearly $E \in C^1(Y, R)$ is even and $E(0) = 0$. The arguments of Theorem 3.1 show that E satisfies (PS) and (C5) in Theorem 2.4. To verify (C6), set $\widetilde{E} = span\{\varphi_1, \varphi_2, \cdots, \varphi_k\}$. For any $u \in \widetilde{E}$, then $u = \sum_{i=1}^{k} a_i\varphi_i$. By Lemma 2.5, Remark 2.1 and (H2),

$$\begin{aligned}
E(u) &= \frac{1}{2}\int_0^T (u'(t))^2 - \lambda u^2(t)dt - \int_0^T F(t, u(t))dt - \sum_{i=1}^{l}\int_0^{u(t_i)} I_i(s)ds \\
&\le \frac{1}{2}\theta_2\|u\|^2 - \int_0^T (b_1|u(t)|^\mu - b_2)\, dt - \sum_{i=1}^{l} (c_i|u(t_i)|^\mu - d_i)\,.
\end{aligned} \tag{3.23}$$

Similar to (3.13),

$$E(u) \le \frac{1}{2}\theta_2\|u\|^2 - k_5\|u\|^\mu + k_6$$

for $k_5, k_6 > 0$. So $E(u) \to -\infty$ as $\|u\| \to +\infty$. That is, there exists $R > 0$ such that $E(u) < 0$ for $u \in \widetilde{E} \setminus B_{R(\widetilde{E})}$. The proof is complete. ∎

Remark 3.2. Similar to the above process, the existence results for problem

$$\begin{cases} -u''(t) = \lambda u(t) + f(t, u(t)), \quad t \neq t_i,\, t \in [0, T], \\ -\Delta u'(t_i) = I_i(u(t_i)), \quad i = 1, 2, \cdots, l, \\ u(0) = 0, \quad u'(T) = 0, \end{cases} \tag{3.24}$$

are established. Theorems 3.1, 3.2, 3.3 are applicable to (3.24).

Example 3.1. Let $T > 0, t_i \in (0, T)$, $a, b_i \in C([0, T], R^+), i = 1, 2, \cdots, l$. Consider mixed boundary value problem with impulse

$$\begin{cases} -u''(t) = \lambda u(t) + a(t)u^3(t), t \neq t_i, t \in [0, T], \\ -\Delta u'(t_i) = b_i(t)u^3(t_i), i = 1, 2, \cdots, l, \\ u'(0) = 0, \quad u(T) = 0. \end{cases} \tag{3.25}$$

Compared with (1.1), $f(t, u) = a(t)u^3, I_i(u) = b_i(t)u^3$.

The conditions (H1) (H2) are satisfied. Applying Theorem 3.1, problem (3.25) has at least one nontrivial solution for $\lambda \in R$.

The condition (H3) is satisfied. Applying Theorem 3.3, problem (3.25) has infinitely many solutions for $\lambda < \lambda_1 = \frac{\pi^2}{(2T)^2}$.

Example 3.2. Let $T = \frac{\pi}{2}, t_i \in (0, \frac{\pi}{2}), i = 1, 2, \cdots, l$. Consider mixed boundary value problem with impulse

$$\begin{cases} -u''(t) = \lambda u(t) - (t + 1)u^3(t), t \neq t_i, t \in [0, T], \\ -\Delta u'(t_i) = -10u^3(t_i), i = 1, 2, \cdots, l, \\ u'(0) = 0, \quad u(T) = 0. \end{cases} \tag{3.26}$$

Compared with (1.1), $f(t, u) = -(t+1)u^3, I_i(u) = -10u^3$. Clearly (H1) (H3) are satisfied. Let $x_1 = 10$, then (H4) is satisfied with $r = 100$. Applying Theorem 3.2, for $\lambda \in (\lambda_k, 100] = ((2k-1)^2, 100]$, $k = 1, 2, 3, 4, 5$, problem (3.26) has at least k distinct pairs of solutions.

Remark 3.3. In Theorem 3.1, the assumption of $\lambda \in R$ is very weak, which we have not seen such results in the literatures. Theorems 3.2, 3.3 can not be obtained by using classical methods, such as fixed point theory in cones and methods of lower and upper solutions.

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Yu Tian
School of Science,
Beijing University of Posts and Telecommunications,
Beijing 100876,
P. R. China
E-mail: tianyu2992@163.com

Jun Wang
School of Mathematical Sciences,
Fudan University,
Shanghai 200433,
P. R. China

Weigao Ge
Department of Applied Mathematics,
Beijing Institute of Technology,
Beijing 100081,
P. R. China