TAIWANESE JOURNAL OF MATHEMATICS
Vol. 13, No. 3, pp. 1043-1052, June 2009
This paper is available online at http://www.tjm.nsysu.edu.tw/

COMMUTATORS OF FRACTIONAL INTEGRAL OPERATOR ASSOCIATED TO A NONDOUBLING MEASURES ON METRIC SPACES

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Abstract. The main purpose of this paper is to prove the boundedness of commutators of fractional integral operators associated to a measure on metric space satisfying just a mild growth condition.

1. INTRODUCTION

In classical harmonic analysis, a critical supposition is the Borel measure μ on metric (quasi-metric) space satisfying the so-called ”doubling condition”, which means that there exists a positive constant C, such that, for every ball $B(x, r)$ of center x and radius r,

$$\mu(B(x, 2r)) \leq C\mu(B(x, r)). \tag{1.1}$$

The commutators in these kinds of space have been studied by many authors for a long time. A well known result which was discovered by Coifman, Rochberg and Weiss ([1, 7, 15]) is that the commutators $[b, T]$ of singular integral operators are bounded on some $L^p(\mathbb{R}^n)(1 < p < \infty)$ if and only if $b \in BMO$, where $[b, T]$ is defined by

$$[b, T]f(x) = b(x)Tf(x) - T(bf)(x).$$

Let I_α be the standard fractional integral, that is,

$$I_\alpha f(x) = \int_{R^n} \frac{f(x)}{|x-y|^{n-\alpha}} dy$$

Received April 14, 2007, accepted October 26, 2007.
Communicated by Yongsheng Han.
2000 *Mathematics Subject Classification*: 42B20, 47B38.
Key words and phrases: Commutator, Fractional integral operator, Lipschitz space, Mild growth condition.
[1]Supported by Hunan Provincial Natural Science Foundation of China 06A0074.
[2]The work is partially supported by NSF of China (10771054).
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The result in [12] states that $[b, I_\alpha] : L^p \to L^q$ is bounded, $1 < p < n/\alpha$, $1/q = 1/p - \alpha/n$ when $b \in BMO$. And M. Paluszyński [8] proved that the commutators $[b, I_\alpha]$ is a bounded operator from $L^p(\mathbb{R}^n)(1 < p < \infty)$ to $L^r(\mathbb{R}^n)$ when $b \in \mathrm{Lip}_\beta$, here $1/p - 1/r = (\alpha+\beta)/n$, $0 < \beta < 1$ and $[b, I_\alpha]f(x) = b(x)I_\alpha f(x) - I_\alpha(bf)(x)$.

In recent years, many authors begin to develop the complete theory of Calderón-Zygmund operators and commutators in a separable metric space, on which the measure is nondoubling measure. A few articles pertaining to this line of research are [2, 4-6, 9-11, 13,14], and so on. In 2004, J.Garc´a and A. Eduardo investigated the behavior of the fractional integral I_α associated to an n-dimensional measure μ on a metric space with not assuming that the metric space is separable, and the other related operators K_α in [3]. Motivated by their works, we want to obtain the boundedness of commutators of I_α and K_α in these metric spaces.

2. The Boundedness of the Fractional Integral I_α

In this paper, (X, d, μ) will always be a metric measure space, where d is a distance on X and μ is a Borel measure on X, such that, for every ball

$$B(x, r) = \{y \in X : d(x, y) < r\}, x \in X, r > 0,$$

we have

$$\mu(B(x, r)) \leq Cr^n, \tag{2.1}$$

where n is some fixed positive real number and C is independent of x and r. Sometimes we shall refer to condition (2.1) by saying that the measure μ is n-dimensional. kB will denote the ball having the same center as B and radius k times that of B.

Before we give the results in our paper, we firstly state two lemmas which will be used all throughout the paper.

Lemma 2.1. ([3]). *For every* $\gamma > 0$,

$$\int_{B(x,r)} \frac{1}{d(x, y)^{n-\gamma}} d\mu(y) \leq Cr^\gamma. \tag{2.2}$$

Lemma 2.2. ([3]). *For every* $\gamma > 0$,

$$\int_{X \setminus B(x,r)} \frac{1}{d(x, y)^{n+\gamma}} d\mu(y) \leq Cr^{-\gamma}. \tag{2.3}$$

In order to discuss the boundedness of commutator of fractional integral , we recall two definitions now.

Definition 2.1. Let $0 < \alpha < n$. The fractional integral I_α associated to the measure μ will be defined, for appropriate functions f on X as

$$I_\alpha f(x) = \int_X \frac{f(y)}{d(x,y)^{n-\alpha}}\, d\mu(y). \tag{2.4}$$

Definition 2.2. Given $0 < \beta < 1$. We shall say that the function $f : X \to C$ satisfies a Lipschitz condition of order β provided with

$$|f(x) - f(y)| \leq C d(x,y)^\beta \qquad \forall x, y \in X \tag{2.5}$$

and the infimum of constants C in (2.5) will be denoted by $\|f\|_{\mathrm{Lip}(\beta)}$.

It is easy to see that the linear space of all Lipschitz functions of order β, modulo constants, becomes, with the norm $\|\cdot\|_{\mathrm{Lip}(\beta)}$, a Banach space, which is denoted by $\mathrm{Lip}(\beta)$.

Theorem 2.1. *Let $0 < \beta < 1$ and $b \in \mathrm{Lip}(\beta)$. If $0 < \alpha+\beta < n$, $1 \leq p < \frac{n}{\alpha+\beta}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha+\beta}{n}$, then $[b,\, I_\alpha]$ is a bounded operator from $L^p(\mu)$ into Lorentz space $L^{q,\infty}(\mu)$, that is,*

$$\mu\left(\{x \in X : |[b,\, I_\alpha]f(x)| > \lambda\}\right) \leq \left(\frac{C\|f\|_{L^p(\mu)}}{\lambda}\right)^q. \tag{2.6}$$

Proof. By the definition of commutator,

$$\begin{aligned} |[b,\, I_\alpha]f(x)| &= |b(x) I_\alpha f(x) - I_\alpha(bf)(x)| \\ &\leq \int_X \frac{|b(x)-b(y)||f(y)|}{d(x,y)^{n-\alpha}}\, d\mu(y) \\ &\leq C\|b\|_{\mathrm{Lip}(\beta)} \int_X \frac{|f(y)|}{d(x,y)^{n-\alpha-\beta}}\, d\mu(y), \end{aligned}$$

since $b \in \mathrm{Lip}(\beta)$. Using the estimate in the proof of Theorem 3.2 in [3], we can easily prove Theorem 2.1. ∎

Applying Marcinkiewicz's interpolation theorem with induices, we have following result.

Corollary 2.1 For $0 < \beta < 1$ and $b \in \mathrm{Lip}(\beta)$, $1 < p < \frac{n}{\alpha+\beta}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha+\beta}{n}$, we have

$$\|[b,\, I_\alpha]f\|_{L^q(\mu)} \leq C\|f\|_{L^p(\mu)}. \tag{2.7}$$

3. The Boundedness of the Fractional Integral Operator K_α

In this section, we will concentrate on the behave of fractional integral operator.

Definition 3.1. Let $0 < \alpha < n$ and $0 < \varepsilon \leq 1$. A function $k_\alpha : X \times X \to C$ is said to be a fractional kernel of order α and regularity ε if it satisfies the following two conditions:

$$|k_\alpha(x,y)| \leq \frac{C}{d(x,y)^{n-\alpha}}, \quad for \quad all \quad x \neq y \tag{3.1}$$

and

$$|k_\alpha(x,y) - k_\alpha(x',y)| \leq C\frac{d(x,x')^\varepsilon}{d(x,y)^{n-\alpha+\varepsilon}}, \quad for \quad d(x,y) \geq 2d(x,x'). \tag{3.2}$$

The corresponding operator K_α, which is called a ”fractional integral operator”, is given by

$$K_\alpha(f)(x) = \int_X k_\alpha(x,y)f(y)d\mu(y). \tag{3.3}$$

From (3.1), we know that $K_\alpha(f)$ is well defined for $f \in L^p(\mu)$ and $1 \leq p < n/\alpha$. And (2.6), (2.7) are also hold on in this case.

Definition 3.2. Let k_α be a fractional kernel of order α and regularity ε. For $f \in L^p(\mu)$, We define

$$\widetilde{K}_\alpha(f)(x) = \int_X \{k_\alpha(x,y) - k_\alpha(x_0,y)\} f(y)d\mu(y), \tag{3.4}$$

where x_0 is some fixed point of X.

Of course the function just defined depends on the point x_0. But the difference between any two functions obtained in (3.4) for different choice of x_0 is just a constant. In fact, let $\widetilde{K}_\alpha^{x_0}(f)(x) = \int_X \{k_\alpha(x,y) - k_\alpha(x_0,y)\} f(y)d\mu(y)$ and $\widetilde{K}_\alpha^{x_1}(f)(x) = \int_X \{k_\alpha(x,y) - k_\alpha(x_1,y)\} f(y)d\mu(y)$. We know

$$\begin{aligned}
&\widetilde{K}_\alpha^{x_1}(f)(x) - \widetilde{K}_\alpha^{x_0}(f)(x) \\
&= \int_X \{k_\alpha(x,y) - k_\alpha(x_1,y)\} f(y)d\mu(y) - \int_X \{k_\alpha(x,y) - k_\alpha(x_0,y)\} f(y)d\mu(y) \\
&= \int_X \{k_\alpha(x_0,y) - k_\alpha(x_1,y)\} f(y)d\mu(y)).
\end{aligned}$$

and this is independent on x.

Making use of the above fact, we can discuss the boundedness of commutators of $\widetilde{K}_\alpha(f)(x)$.

Theorem 3.1. *Let* k_α *be a fractional kernel with regularity* ε *and* $b \in \mathrm{Lip}(\beta)$*,* $0 < \beta < 1$. *If* $1 < \dfrac{n}{\alpha+\beta} < p < \infty$ *and* $\alpha + \beta - \frac{n}{p} < \varepsilon$*, then* $[b,\, \widetilde{K}_\alpha]$ *maps* $L^p(\mu)$ *boundedly into* $\mathrm{Lip}(\alpha + \beta - \frac{\mathrm{n}}{\mathrm{p}})$.

Proof. Suppose $p < \infty$. Assume $x \neq y$. Let B be the ball with center x and radius $r = d(x, y)$. Since the difference between any two functioned defined in (3.4) for different elections of x_0 is just a constant, we can choose $x_0 \in 2B$. Write

$$\begin{aligned}
[b,\, \widetilde{K}_\alpha]f(x) &= b(x)\widetilde{K}_\alpha(f)(x) - \widetilde{K}_\alpha(bf)(x) \\
&= b(x)\int_X \{k_\alpha(x,y) - k_\alpha(x_0,y)\}\, f(y)d\mu(y) \\
&\quad - \int_X \{k_\alpha(x,y) - k_\alpha(x_0,y)\}\, b(y)f(y)d\mu(y) \\
&= \int_X k_\alpha(x,y)[b(x) - b(y)]f(y)d\mu(y) \\
&\quad - \int_X k_\alpha(x_0,y)[b(x_0) - b(y)]f(y)d\mu(y) \\
&\quad + [b(x_0) - b(x)]\int_X k_\alpha(x_0,y)f(y)d\mu(y).
\end{aligned}$$

Then, we have

$$\begin{aligned}
|[b,\, \widetilde{K}_\alpha]f(x) - [b,\, \widetilde{K}_\alpha]f(y)| &= \bigg| \int_X (b(x) - b(z))\, k_\alpha(x,z)f(z)d\mu(z) \\
&\quad - \int_X (b(y) - b(z))\, k_\alpha(y,z)f(z)d\mu(z) \\
&\quad + \int_X (b(y) - b(x))\, k_\alpha(x_0,z)f(z)d\mu(z) \bigg| \\
&\le \int_{2B(x,r)} |b(x) - b(z)|\, |k_\alpha(x,z)f(z)|\, d\mu(z) \\
&\quad + \int_{2B(x,r)} |b(y) - b(z)|\, |k_\alpha(y,z)f(z)|\, d\mu(z)
\end{aligned}$$

$$
\begin{aligned}
&+\int_{2B(x,r)}|b(y)-b(x)|\,|k_\alpha(x_0,z)f(z)|d\mu(z)\\
&+\int_{X\setminus 2B(x,r)}|b(x)-b(z)|\,|k_\alpha(x,z)-k_\alpha(x_0,z)|\,|f(z)|d\mu(z)\\
&+\int_{X\setminus 2B(x,r)}|b(y)-b(z)|\,|k_\alpha(x_0,z)-k_\alpha(y,z)|\,|f(z)|d\mu(z)\\
=:&\ \mathrm{I}+\mathrm{II}+\mathrm{III}+\mathrm{IV}+\mathrm{V}.
\end{aligned}
$$

We shall estimate each of these five terms separately.

For the first term, using the condition (3.1), Lemma 2.1 and Hölder inequality, we obtain that

$$
\begin{aligned}
\mathrm{I}&\leq C\|b\|_{\mathrm{Lip}(\beta)}\int_{2B(x,r)}\frac{d(x,z)^\beta}{d(x,z)^{n-\alpha}}|f(z)|d\mu(z)\\
&\leq C\|b\|_{\mathrm{Lip}(\beta)}\|f\|_{L^p(\mu)}\left(\int_{2B(x,r)}\frac{d\mu(z)}{d(x,z)^{(n-\alpha-\beta)p'}}\right)^{1/p'}\\
&\leq C\|b\|_{\mathrm{Lip}(\beta)}\|f\|_{L^p(\mu)}(2r)^{\alpha+\beta-\frac{n}{p}},
\end{aligned}
$$

here $(n-\alpha-\beta)p'=n-p'(\alpha+\beta-\frac{n}{p})$ and $\alpha+\beta-\frac{n}{p}>0$.

The second term and the third term are estimated in a similar way after noticing that $2B\subset B(y,3r)$ and $2B\subset B(x_0,4r)$.

Now, we turn to the last two parts. By the regularity condition of fractional kernel (3.2) and Lemma 2.2, we obtain

$$
\begin{aligned}
\mathrm{IV}&\leq\int_{X\setminus 2B(x,r)}|b(x)-b(z)|\,|k_\alpha(x,z)-k_\alpha(x_0,z)||f(z)|d\mu(z)\\
&\leq C\|b\|_{\mathrm{Lip}(\beta)}\int_{X\setminus 2B(x,r)}d(x,z)^\beta\frac{d(x,x_0)^\varepsilon}{d(x,z)^{n-\alpha+\varepsilon}}|f(z)|d\mu(z)\\
&\leq C\|b\|_{\mathrm{Lip}(\beta)}d(x,y)^\varepsilon\|f\|_{L^p(\mu)}\left(\int_{X\setminus 2B(x,r)}\frac{d\mu(z)}{d(x,z)^{(n-\alpha-\beta+\varepsilon)p'}}\right)^{1/p'}\\
&\leq C\|b\|_{\mathrm{Lip}(\beta)}\|f\|_{L^p(\mu)}d(x,y)^{\alpha+\beta-\frac{n}{p}},
\end{aligned}
$$

here $(n-\alpha-\beta+\varepsilon)p'=n+p'(\frac{n}{p}+\varepsilon-\alpha-\beta)$ and $\frac{n}{p}+\varepsilon-\alpha-\beta>0$.

For V, since $r=d(x,y)$, we have $B(y,r)\subset B(x,2r)\subset B(y,3r)$, and $X\setminus B(y,r)\supset X\setminus 2B(x,r)$. Using the size condition (3.1) and Lemma 2.2, similar to IV, we have

$$\begin{aligned}
\mathrm{V} &\leq \int_{X\setminus 2B(x,r)} |b(y)-b(z)|\,|k_\alpha(x_0,z)-k_\alpha(y,z)||f(z)|d\mu(z)\\
&\leq C\|b\|_{\mathrm{Lip}(\beta)} \int_{X\setminus 2B(x,r)} d(y,z)^\beta \frac{d(y,x_0)^\varepsilon}{d(y,z)^{n-\alpha+\varepsilon}}|f(z)|d\mu(z)\\
&\leq C\|b\|_{\mathrm{Lip}(\beta)} d(y,x_0)^\varepsilon \int_{X\setminus B(y,r)} \frac{d(y,z)^\beta}{d(y,z)^{n-\alpha+\varepsilon}}|f(z)|d\mu(z)\\
&\leq C\|b\|_{\mathrm{Lip}(\beta)} d(x,y)^\varepsilon \|f\|_{L^p(\mu)} \left(\int_{X\setminus B(y,r)} \frac{d\mu(z)}{d(x,z)^{(n-\alpha-\beta+\varepsilon)p'}}\right)^{1/p'}\\
&\leq C\|b\|_{\mathrm{Lip}(\beta)}\|f\|_{L^p(\mu)} d(x,y)^{\alpha+\beta-\frac{n}{p}},
\end{aligned}$$

Here $d(y,x_0) \leq d(x,y)+d(x,x_0) \leq 3d(x,y)$.

Combining with all the estimates above, we obtain that

$$|[b,\,\widetilde{K}_\alpha]f(x)-[b,\,\widetilde{K}_\alpha]f(y)| \leq C\|b\|_{\mathrm{Lip}(\beta)}\|f\|_{L^p(\mu)} d(x,y)^{\alpha+\beta-\frac{n}{p}}.$$

This finishes the proof of Theorem 3.1. ■

Furthermore, we can consider the boundedness of commutator in Lipschitz spaces.

Theorem 3.2 *Let k_α be a fractional kernel with regularity ε, $\alpha,\gamma>0$ and $0<\beta<1$ such that $\alpha+\beta+\gamma<\varepsilon$. Suppose that $b\in \mathrm{Lip}(\beta)$. Then $[b,\,\widetilde{K}_\alpha]$ is bounded mapping from $\mathrm{Lip}(\gamma)$ to $\mathrm{Lip}(\alpha+\beta+\gamma)$ if and only if $[b,\,\widetilde{K}_\alpha](1)=C$.*

Proof. Noticing that the continuity of the operator $[b,\,\widetilde{K}_\alpha]$ implies that $[b,\,\widetilde{K}_\alpha](1)=C$, the necessity of Theorem 3.2 is obvious.

To prove the sufficiency, let $x\neq y$, $r=d(x,y)$, $B=B(x,r)$ and choose $x_0\in 2B$. We want to estimate $|[b,\,\widetilde{K}_\alpha](f)(x)-[b,\,\widetilde{K}_\alpha](f)(y)|$. Observe that $[b,\,\widetilde{K}_\alpha](1)=C$, therefore, $[b,\,\widetilde{K}_\alpha](1)(x)-[b,\,\widetilde{K}_\alpha](1)(y)=0$, that is,

$$\begin{aligned}
&\int_X (b(x)-b(z))\,k_\alpha(x,z)d\mu(z)-\int_X (b(y)-b(z))\,k_\alpha(y,z)d\mu(z)\\
&\quad+\int_X (b(y)-b(x))\,k_\alpha(x_0,z)d\mu(z)=0.
\end{aligned}$$

As the proof in Theorem 3.1, we can write

$$\begin{aligned}
&|[b,\,\widetilde{K}_\alpha]f(x)-[b,\,\widetilde{K}_\alpha]f(y)|\\
&=\left|\int_X (b(x)-b(z))\,k_\alpha(x,z)f(z)d\mu(z)-\int_X (b(y)-b(z))\,k_\alpha(y,z)f(z)d\mu(z)\right.\\
&\quad\left.+\int_X (b(y)-b(x))\,k_\alpha(x_0,z)f(z)d\mu(z)\right|
\end{aligned}$$

$$
\begin{aligned}
&= \Big| \int_X (b(x)-b(z))\, k_\alpha(x,z)(f(z)-f(x))d\mu(z) \\
&\quad - \int_X (b(y)-b(z))\, k_\alpha(y,z)(f(z)-f(x))d\mu(z) \\
&\quad + \int_X (b(y)-b(x))\, k_\alpha(x_0,z)(f(z)-f(x))d\mu(z) \Big| \\
&\leq \int_{2B} |b(x)-b(z)|\,|k_\alpha(x,z)||f(z)-f(x)|d\mu(z) \\
&\quad + \int_{2B} |b(y)-b(z)|\,|k_\alpha(y,z)||f(z)-f(x)|d\mu(z) \\
&\quad + \int_{2B} |b(y)-b(x)|\,|k_\alpha(x_0,z)||f(z)-f(x)|d\mu(z) \\
&\quad + \int_{X\setminus 2B} |b(x)-b(z)|\,|k_\alpha(x,z)-k_\alpha(x_0,z)|\,|f(z)-f(x)|d\mu(z) \\
&\quad + \int_{X\setminus 2B} (b(y)-b(z))\,|k_\alpha(x_0,z)-k_\alpha(y,z)|\,|f(z)-f(x)|d\mu(z) \\
&=: \mathrm{I}+\mathrm{II}+\mathrm{III}+\mathrm{IV}+\mathrm{V}.
\end{aligned}
$$

For the first part, using the size condition of the kernel k_α and Lemma 2.1, we have

$$
\begin{aligned}
\mathrm{I} &\leq C\|b\|_{\mathrm{Lip}(\beta)}\|f\|_{\mathrm{Lip}(\gamma)} \int_{2B} \frac{d(x,z)^\beta d(x,z)^\gamma}{d(x,z)^{n-\alpha}} d\mu(z) \\
&\leq C\|b\|_{\mathrm{Lip}(\beta)}\|f\|_{\mathrm{Lip}(\gamma)} d(x,y)^{\alpha+\beta+\gamma}.
\end{aligned}
$$

Since $2B(x,r)\subset B(y,3r)$ and $2B\subset B(x_0,4r)$,similar to the estimation of part I, we obtain

$$
\begin{aligned}
\mathrm{II} &\leq \|b\|_{\mathrm{Lip}(\beta)}\|f\|_{\mathrm{Lip}(\gamma)} \int_{B(y,3r)} \frac{d(y,z)^\beta d(x,z)^\gamma}{d(y,z)^{n-\alpha}} d\mu(z) \\
&\leq C\|b\|_{\mathrm{Lip}(\beta)}\|f\|_{\mathrm{Lip}(\gamma)} r^{\alpha+\beta+\gamma},
\end{aligned}
$$

and

$$
\begin{aligned}
\mathrm{III} &\leq \|b\|_{\mathrm{Lip}(\beta)}\|f\|_{\mathrm{Lip}(\gamma)} \int_{B(x_0,4r)} \frac{d(x,y)^\beta d(x,z)^\gamma}{d(x_0,z)^{n-\alpha}} d\mu(z) \\
&\leq C\|b\|_{\mathrm{Lip}(\beta)}\|f\|_{\mathrm{Lip}(\gamma)} r^{\alpha+\beta+\gamma},
\end{aligned}
$$

Now, we turn to estimate the last two parts. Similar to the proof of Theorem 3.1, by the regular condition (3.2) and Lemma 2.2, we have

$$\begin{aligned}
\mathrm{IV} &\leq \int_{X\setminus 2B} |b(x)-b(z)|\,|k_\alpha(x,z)-k_\alpha(x_0,z)|\,|f(z)-f(x)|d\mu(z)\\
&\leq C\|b\|_{\mathrm{Lip}(\beta)}\|f\|_{\mathrm{Lip}(\gamma)}\int_{X\setminus 2B} d(x,z)^\beta|k_\alpha(x,z)-k_\alpha(x_0,z)|d(x,z)^\gamma d\mu(z)\\
&\leq C\|b\|_{\mathrm{Lip}(\beta)}\|f\|_{\mathrm{Lip}(\gamma)}\int_{X\setminus 2B} d(x,z)^{\beta+\gamma}\frac{d(x,x_0)^\varepsilon}{d(x,z)^{n-\alpha+\varepsilon}}d\mu(z)\\
&\leq C\|b\|_{\mathrm{Lip}(\beta)}\|f\|_{\mathrm{Lip}(\gamma)}d(x,y)^\varepsilon\int_{X\setminus 2B}\frac{1}{d(x,z)^{n+\varepsilon-\alpha-\beta-\gamma}}d\mu(z)\\
&\leq C\|b\|_{\mathrm{Lip}(\beta)}\|f\|_{\mathrm{Lip}(\gamma)}d(x,y)^{\alpha+\beta+\gamma},
\end{aligned}$$

here we used $\alpha+\beta+\gamma<\varepsilon$.

For V, as the above proof, since $d(x,z)\leq d(x,y)+d(y,z)\leq 2d(y,z)$, we obtain

$$\begin{aligned}
\mathrm{V} &\leq \int_{X\setminus 2B} |b(y)-b(z)|\,|k_\alpha(y,z)-k_\alpha(x_0,z)|\,|f(z)-f(x)|d\mu(z)\\
&\leq C\|b\|_{\mathrm{Lip}(\beta)}\|f\|_{\mathrm{Lip}(\gamma)}\int_{X\setminus 2B} d(y,z)^\beta|k_\alpha(y,z)-k_\alpha(x_0,z)|d(x,z)^\gamma d\mu(z)\\
&\leq C\|b\|_{\mathrm{Lip}(\beta)}\|f\|_{\mathrm{Lip}(\gamma)}\int_{X\setminus B(y,r)} d(y,z)^{\beta+\gamma}\frac{d(y,x_0)^\varepsilon}{d(y,z)^{n-\alpha+\varepsilon}}d\mu(z)\\
&\leq C\|b\|_{\mathrm{Lip}(\beta)}\|f\|_{\mathrm{Lip}(\gamma)}d(x,y)^\varepsilon\int_{X\setminus B(y,r)}\frac{1}{d(y,z)^{n+\varepsilon-\alpha-\beta-\gamma}}d\mu(z)\\
&\leq C\|b\|_{\mathrm{Lip}(\beta)}\|f\|_{\mathrm{Lip}(\gamma)}d(x,y)^{\alpha+\beta+\gamma},
\end{aligned}$$

Therefore, Theorem 3.2 is proved. ■

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