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ON SUBORDINATIONS FOR CERTAIN ANALYTIC FUNCTIONS ASSOCIATED WITH THE DZIOK-SRIVASTAVA LINEAR OPERATOR

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Abstract. By making use of the method of differential subordination, we investigate some interesting properties of certain analytic functions associated with the Dziok-Srivastava linear operator.

1. Introduction

Let $A(p)$ denote the class of functions of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \quad (p \in N = \{1, 2, 3, \cdots\}), \tag{1.1}$$

which are analytic in the open unit disk $U = \{z : z \in C \text{ and } |z| < 1\}$. When $p = 1$,we denote by C the class of univalent convex function in U. Also let the Hadamard product (or convolution) $(f_1 * f_2)(z)$ of two functions

$$f_j(z) = z^p + \sum_{n=p+1}^{\infty} a_{n,j} z^n \in A(p) \quad (j = 1, 2) \tag{1.2}$$

be given by

$$(f_1 * f_2)(z) = z^p + \sum_{n=p+1}^{\infty} a_{n,1} a_{n,2} z^n. \tag{1.3}$$

Given two functions $f(z)$ and $g(z)$, which are analytic in U, we say that the function $g(z)$ is subordinate to $f(z)$ and write $g \prec f$, if there exists a Schwarz

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function $w(z)$ with $w(0) = 0$ and $|w(z)| < 1$ $(z \in U)$ such that $g(z) = f(w(z))$ $(z \in U)$. In particular, if $f(z)$ is univalent in U, we have the following equivalence

$$g(z) \prec f(z) \quad (z \in U) \Leftrightarrow g(0) = f(0) \quad \textit{and} \quad g(U) \subset f(U). \tag{1.4}$$

For each A and B such that $-1 \leq B < A \leq 1$, we define the function

$$h(A, B; z) = \frac{1 + Az}{1 + Bz} \quad (z \in U). \tag{1.5}$$

It is well known that $h(A, B; z)$ for $-1 \leq B \leq 1$ is the conformal map of the unit disk onto the disk symmetrical with respect to the real axis having the center $(1 - AB)/(1 - B^2)$ and the radius $(A - B)/(1 - B^2)$. The boundary circle cuts the real axis at the points $(1 - A)/(1 - B)$ and $(1 + A)/(1 + B)$.

For complex parameters $\alpha_1, \cdots, \alpha_q$ and $\beta_1, \cdots, \beta_s$ $(\beta_j \neq 0, -1, -2, \cdots; j = 1, \cdots, s)$, we define the generalized hypergeometric function ${}_qF_s(\alpha_1, \cdots, \alpha_q; \beta_1, \cdots, \beta_s; z)$ by

$$\begin{gathered} {}_qF_s(\alpha_1, \cdots, \alpha_q; \beta_1, \cdots, \beta_s; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_q)_n}{(\beta_1)_n \cdots (\beta_s)_n} \frac{z^n}{n!} \\ (q \leq s + 1; q, s \in N_0 = N \cup \{0\}; z \in U), \end{gathered} \tag{1.6}$$

where $(\lambda)_n$ is the Pochhammer symbol defined, in terms of the Gamma function Γ, by

$$(\lambda)_n = \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} = \begin{cases} 1 & (n = 0), \\ \lambda(\lambda + 1) \cdots (\lambda + n - 1) & (n \in N). \end{cases} \tag{1.7}$$

Corresponding to a function $h_p(\alpha_1, \cdots, \alpha_q; \beta_1, \cdots, \beta_s; z)$ defined by

$$h_p(\alpha_1, \cdots, \alpha_q; \beta_1, \cdots, \beta_s; z) = z^p {}_qF_s(\alpha_1, \cdots, \alpha_q; \beta_1, \cdots, \beta_s; z), \tag{1.8}$$

we consider a linear operator

$$H_p(\alpha_1, \cdots, \alpha_q; \beta_1, \cdots, \beta_s) : A(p) \to A(p),$$

defined by the convolution

$$H_p(\alpha_1, \cdots, \alpha_q; \beta_1, \cdots, \beta_s)f(z) = h_p(\alpha_1, \cdots, \alpha_q; \beta_1, \cdots, \beta_s; z) * f(z). \tag{1.9}$$

we observe that, for a function f of the form (1.1), we have

$$H_p(\alpha_1, \cdots, \alpha_q; \beta_1, \cdots, \beta_s)f(z) = z^p + \sum_{n=p+1}^{\infty} \Gamma_n(p) a_n z^n, \tag{1.10}$$

where

$$\Gamma_n(p) = \frac{(\alpha_1)_{n-p} \cdots (\alpha_q)_{n-p}}{(\beta_1)_{n-p} \cdots (\beta_s)_{n-p}(n-p)!}. \tag{1.11}$$

For convenience, we write

$$H_{p,q,s}(\alpha_1) = H_p(\alpha_1, \cdots, \alpha_q; \beta_1, \cdots, \beta_s). \tag{1.12}$$

Thus, after some calculations, we have

$$z(H_{p,q,s}(\alpha_1)f(z))' = \alpha_1 H_{p,q,s}(\alpha_1+1)f(z) - (\alpha_1 - p)H_{p,q,s}(\alpha_1)f(z). \tag{1.13}$$

It is well known that the linear operator $H_{p,q,s}(\alpha_1)$ is called the Dziok-Srivastava linear operator. Many interesting subclasses of analytic functions, associated with the Dziok-Srivastava linear operator $H_{p,q,s}(\alpha_1)$ and its many special cases, were investigated recently by Dziok and Srivastava [2,3,4], Gangadharan et al.[5], Liu [6], Liu and Srivastava [7] and others. In the present sequel to these earlier works, we shall use the method of differential subordination to derive several interesting properties of the Dziok-Srivastava linear operator $H_{p,q,s}(\alpha_1)$.

2. A Theorem Involving Differential Subordination

Theorem 1. *Let $f(z) \in A(p)$ and let $F(z)$ be defined by*

$$F(z) = [1 - \lambda(1 + \alpha_1 - p)]H_{p,q,s}(\alpha_1)f(z) + \lambda\alpha_1 H_{p,q,s}(\alpha_1+1)f(z). \tag{2.1}$$

If

$$\frac{(H_{p,q,s}(\alpha_1)f(z))^{(j)}}{z^{p-j}} \prec \frac{p!}{(p-j)!}h(1-2\gamma, -1; z) \quad (z \in U), \tag{2.2}$$

then we have

$$\frac{F^{(j)}(z)}{z^{p-j}} \prec \frac{p!(1-\lambda+\lambda p)}{(p-j)!}h(1-2\gamma, -1; z) \quad (|z| < \rho), \tag{2.3}$$

where $0 \le j \le p$, $0 \le \gamma < 1$ and

$$\rho = \left[1 + \left(\frac{\lambda}{1-\lambda+\lambda p}\right)^2\right]^{1/2} - \frac{\lambda}{1-\lambda+\lambda p}. \tag{2.4}$$

The bound $\rho \in (0,1)$ is best possible.

Proof. From (1.13) and (2.1), we obtain

$$\begin{aligned} F^{(j)}(z) &= [1-\lambda(1+\alpha_1-p)](H_{p,q,s}(\alpha_1)f(z))^{(j)} \\ &\quad +\lambda\alpha_1(H_{p,q,s}(\alpha_1+1)f(z))^{(j)} \\ &= (1-\lambda+\lambda j)(H_{p,q,s}(\alpha_1)f(z))^{(j)}+\lambda z(H_{p,q,s}(\alpha_1)f(z))^{(j+1)}. \end{aligned} \tag{2.5}$$

Consider the function

$$\varphi(z)=(1-\beta)\frac{z}{1-z}+\beta\frac{z}{(1-z)^2} \quad (z\in U),$$

where $\beta=\lambda/(1-\lambda+\lambda p)>0$. We first show that

$$Re\left\{\frac{\varphi(\rho z)}{\rho z}\right\}>\frac{1}{2} \quad (z\in U), \tag{2.6}$$

where $\rho=(1+\beta^2)^{1/2}-\beta$.

Let $1/(1-z)=Re^{i\theta}$ and $|z|=r<1$. In view of

$$cos\theta=\frac{1+R^2(1-r)}{2R}, \quad R\geq\frac{1}{1+r},$$

we have

$$\begin{aligned} 2Re\left\{\frac{\varphi(z)}{z}-\frac{1}{2}\right\} &= 2(1-\beta)Rcos\theta+2\beta R^2cos2\theta-1 \\ &= R^4\beta(1-r^2)^2+R^2((1-\beta)(1-r^2)-2\beta r^2) \\ &\geq R^2[\beta(1-r)^2+(1-\beta)(1-r^2)-2\beta r^2) \\ &= R^2(1-2\beta r-r^2)>0 \end{aligned}$$

for $|z|=r<\rho$, which gives (2.6). Thus the function φ has the integral representation

$$\frac{\varphi(\rho z)}{\rho z}=\int_{|x|=1}\frac{d\mu(x)}{1-xz} \quad (z\in U), \tag{2.7}$$

where $\mu(x)$ is a probability measure on $|x|=1$.

Now putting

$$g(z)=\frac{(p-j)!}{p!}\frac{(H_{p,q,s}(\alpha_1)f(z))^{(j)}}{z^{p-j}},$$

we see that $g(z) = 1 + b_1 z + \cdots$ is analytic in U and it follows from (2.2) that

$$Re g(z) > \gamma \quad (z \in U). \tag{2.8}$$

Since we can write

$$g(z) + \beta z g'(z) = \left(\frac{\varphi(z)}{z}\right) * g(z),$$

it follows from (2.7) and (2.8) that

$$\begin{aligned} \mathrm{Re}\{g(\rho z) + \beta \rho z g'(\rho z)\} &= \mathrm{Re}\left\{(\frac{\varphi(\rho z)}{\rho z}) * g(z)\right\} \\ &= \mathrm{Re}\left\{\int_{|x|=1} g(xz) d\mu(x)\right\} \\ &> \gamma \quad (z \in U). \end{aligned} \tag{2.9}$$

Note that

$$g(z) + \beta z g'(z) = \frac{(p-j)!}{p!(1-\lambda+\lambda p)} \frac{F^{(j)}(z)}{z^{p-j}}.$$

Hence, from (2.5) and (2.9), we conclude that (2.3) holds.

To show that the bound ρ is sharp we take $f(z) \in A(p)$ defined by

$$\frac{(p-j)!}{p!} \frac{(H_{p,q,s}(\alpha_1))^{(j)}}{z^{p-j}} = \gamma + (1-\gamma)\frac{1+z}{1-z}.$$

Noting that

$$\begin{aligned} \tfrac{(p-j)!}{p!(1-\lambda+\lambda p)} \tfrac{F^{(j)}(z)}{z^{p-j}} &= \gamma + (1-\gamma)\frac{1+z}{1-z} + \beta(1-\gamma) z \left(\frac{1+z}{1-z}\right)' \\ &= \gamma + (1-\gamma)\frac{1+2\beta z - z^2}{(1-z)^2} = \gamma \end{aligned}$$

for $z = \rho\, e^{i\pi}$ the proof is completed.

3. A Theorem Involving Subordinating Factor

Let $q, s \in N$ and suppose that the parameters $\alpha_1, \cdots, \alpha_q$ and $\beta_1, \cdots, \beta_s$ are positive real numbers. Also let $0 \le b \le 1$ and $-b \le a < b$. We denote by $V(q, s; a, b)$ the class of functions $f \in A(1)$ of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \quad (a_n \ge 0), \tag{3.1}$$

which also satisfy the following condition

$$\alpha_1 \frac{H_{1,q,s}(\alpha_1+1)f(z)}{H_{1,q,s}(\alpha_1)f(z)} + 1 - \alpha_1 \prec \frac{1+az}{1+bz}. \tag{3.2}$$

Dziok and Srivastava [2] proved the following result.

Lemma 1. ([2]). *A function f of the form* (3.1) *belongs to the class $V(q,s;a,b)$ if and only if*

$$\sum_{n=2}^{\infty}((b+1)n-(a+1))\Gamma_n(1) \leq b-a, \tag{3.3}$$

where $\Gamma_n(1)$ is defined by (1.11).

We shall also make use of the following definition and results.

Definition 1. (Subordinating Factor Sequence). A sequence $\{b_n\}_{n=1}^{\infty}$ of complex numbers is called a subordinating factor sequence if for every univalent function $f(z)$ in C, we have the subordination given by

$$\sum_{n=1}^{\infty} b_n a_n z^n \prec f(z) \quad (z \in U, a_1 = 1). \tag{3.4}$$

Lemma 2. *The sequence $\{b_n\}_{n=1}^{\infty}$ is a subordinating factor sequence if and only if*

$$Re\left\{1 + 2\sum_{n=1}^{\infty} b_n z^n\right\} > 0 \quad (z \in U). \tag{3.5}$$

This lemma due to Wilf [8].

Theorem 2. *Let $q = s+1, \alpha_{s+1} = 1$ and $0 \leq \beta_j \leq \alpha_j$ $(j = 1, 2, \cdots, s)$. If $f(z) \in V(q,s;a,b)$, then*

$$\frac{(1+2b-a)\Gamma_2(1)}{2[(1+2b-a)\Gamma_2(1)+(b-a)]}(f * g)(z) \prec g(z) \quad (z \in U) \tag{3.6}$$

for every function $g(z)$ in C, and

$$Ref(z) > -\frac{(1+2b-a)\Gamma_2(1)+(b-a)}{(1+2b-a)\Gamma_2(1)}. \tag{3.7}$$

The constant
$$\frac{(1+2b-a)\Gamma_2(1)+(b-a)}{(1+2b-a)\Gamma_2(1)}$$
cannot be replaced by a larger one.

Proof. Let $f(z) \in V(q,s;a,b)$ and suppose that $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ be any function in the class C. Then we readily have

$$\begin{aligned}(3.8)\qquad &\frac{(1+2b-a)\Gamma_2(1)}{2[(1+b)\Gamma_2(1)+(b-a)(1+\Gamma_2(1))]}(f*g)(z)\\ &= \frac{(1+2b-a)\Gamma_2(1)}{2[(1+b)\Gamma_2(1)+(b-a)(1+\Gamma_2(1))]}(z-\sum_{n=2}^{\infty} a_n b_n z^n).\end{aligned}$$

Thus, by Definition 1, the subordination result (3.6) will hold true if

$$(3.9)\qquad \left\{-\frac{(1+2b-a)\Gamma_2(1)}{2[(1+b)\Gamma_2(1)+(b-a)(1+\Gamma_2(1))]}a_n\right\}_{n=1}^{\infty}$$

is a subordinating factor sequence with $a_1 = 1$. In view of Lemma 2, this is equivalent to the following inequality

$$(3.10)\quad \operatorname{Re}\left\{1-\sum_{n=1}^{\infty}\frac{(1+2b-a)\Gamma_2(1)}{[(1+b)\Gamma_2(1)+(b-a)(1+\Gamma_2(1))]}a_n z^n\right\} > 0 \quad (z \in U).$$

Under the hypothesis of the theorem, the sequence $\{(1+2b-a)\Gamma_n(1)\}_{n=2}^{\infty}$ is nondecreasing. Thus we have

$$\begin{aligned}&\operatorname{Re}\left\{1-\sum_{n=1}^{\infty}\frac{(1+2b-a)\Gamma_2(1)}{[(1+b)\Gamma_2(1)+(b-a)(1+\Gamma_2(1))]}a_n z^n\right\}\\ &= \operatorname{Re}\left\{1-\frac{(1+2b-a)\Gamma_2(1)}{[(1+b)\Gamma_2(1)+(b-a)(1+\Gamma_2(1))]}z-\frac{\sum_{n=2}^{\infty}(1+2b-a)\Gamma_2(1)a_n z^n}{[(1+b)\Gamma_2(1)+(b-a)(1+\Gamma_2(1))]}\right\}\\ &\geq 1-\frac{(1+2b-a)\Gamma_2(1)}{[(1+b)\Gamma_2(1)+(b-a)(1+\Gamma_2(1))]}r-\frac{\sum_{n=2}^{\infty}[(b+1)n-(a+1)]\Gamma_n(1)a_n r^n}{[(1+b)\Gamma_2(1)+(b-a)(1+\Gamma_2(1))]}\\ &> 1-\frac{(1+2b-a)\Gamma_2(1)}{[(1+b)\Gamma_2(1)+(b-a)(1+\Gamma_2(1))]}r-\frac{b-a}{[(1+b)\Gamma_2(1)+(b-a)(1+\Gamma_2(1))]}r\\ &> 0 \quad (|z| = r < 1),\end{aligned}$$

where we have also made use of the assertion (2.3) of Lemma 1. This proves (3.6).

The inequality (3.7) follows from (3.6) upon setting

$$g(z) = \frac{z}{1-z} = z + \sum_{n=2}^{\infty} z^n \in C.$$

Next we consider the function

$$q(z) = z - \frac{b-a}{(1+2b-a)\Gamma_2(1)} z^2, \tag{3.11}$$

which is a member of the class $V(q, s; a, b)$. Then, by using (3.6),we have

$$\frac{(1+2b-a)\Gamma_2(1)}{2[(1+b)\Gamma_2(1)+(b-a)(1+\Gamma_2(1)} q(z) \prec \frac{z}{1-z} \quad (z \in U). \tag{3.12}$$

It is also easily verified for the function $q(z)$ defined by (3.11) that

$$\min\left\{ Re\left(\frac{(1+2b-a)\Gamma_2(1)}{2[(1+b)\Gamma_2(1)+(b-a)(1+\Gamma_2(1))]} q(z) \right) \right\} = -\frac{1}{2} \quad (z \in U),$$

which completes the proof of Theorem 2.

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